12-2003

Payoff Continuity in Incomplete Information Games: A Comment

Casey G. Rothschild
MIT

Citation

Casey Rothschild (2005). Payoff continuity in incomplete information games: a comment. Journal of Economic Theory, 120, 270-274.

Payoff Continuity in Incomplete Information Games: A Comment

Casey G. Rothschild [1]

Department of Economics, Massachusetts Institute of Technology
Cambridge, Massachusetts 02142

Abstract

Kajii and Morris (*J. Econ. Theory* 1998, 267-276) provide necessary and sufficient conditions for two priors to be strategically close. The restrictiveness of these conditions establishes that strategic behavior can be highly sensitive to the assumed prior. Their results thus recommend care in the use of priors in economic modelling. Unfortunately, their proof of a central proposition fails for zero probability types. This comment corrects their proof to account for these cases.

Key words: Journal of Economic Literature Classification Numbers: C72, D82.

1 Introduction

In "Payoff Continuity in Incomplete Information Games," Kajii and Morris (KM) make a key contribution to our understanding of these games by showing that a change in priors will have a small effect on equilibrium play precisely when (1) the prior probability of every event changes little and (2) the set on which it becomes nearly common knowledge that posterior beliefs remain close has high measure. These tight requirements for continuity of equilibrium play highlight the importance of verifying the sensitivity of predictions to the assumed prior. KM's results are thus quite important, but their proof of a central proposition fails for zero probability types—and these may matter when multiple priors are considered. I amend the proof to avoid this problem, showing their results to be correct and incidentally tightening them and correcting a minor error in the proof of an earlier lemma. I conclude by suggesting that starting with posteriors rather than priors would ease the analysis.

Email address: caseyr@mit.edu (Casey G. Rothschild).

[1] Many thanks to Muhamet Yildiz for invaluable advice.

2 Correcting their proof

KM consider a static game with I ($2 \leq I < \infty$) players, each with a finite action set A_i ($\ni a_i$) and a countable type set T_i ($\ni t_i$). The state space is $\Omega = T_1 \times \cdots \times T_I \times S$ ($\ni \omega$), where S is a countable space of basic uncertainty. The (common) prior μ is a probability distribution over Ω, and posterior beliefs $\mu(\cdot|t_i)$ are defined via Bayes' law when possible and otherwise left unspecified. Player i has the utility function $u_i : A \times \Omega \to \Re$ which extends in the usual way to mixed strategies σ_i. Un-subscripted variables denote the vector with each player's corresponding variable. The subscript $-i$ has its usual meaning.

KM define $v_i[a_i, \sigma_{-i}; \mu, u; t_i] = \sum_{T_{-i}\times S} \mu(((t_i, t_{-i}), s)|t_i)u((a_i, \sigma_{-i}(t_{-i})), \omega)$ and $V_i[\sigma; \mu, u] = \sum_{\Omega} \mu(\omega)u_i(\sigma(t), \omega)$. σ is an ε-equilibrium iff: $\forall\, i$ and $\forall\, t_i$ with $\mu(t_i) > 0$, $\sigma_i(t_i)[a_i] > 0$ implies interim payoffs from a_i and t_i's best response are within ε; the term "equilibrium" is shorthand for 0-equilibrium. Beliefs (hence common p-beliefs C^p_μ and p-evident events) are defined via

$$B^p_\mu(F) = \{\omega \in \Omega | \forall i, \mu(t_i) > 0 \Rightarrow \mu(F|t_i) \geq p\}.$$

With two priors μ and μ', the set of events with "close" posteriors is:

$$\mathcal{A}_{\mu,\mu'}(\delta) \equiv \left\{ (t,s) \in \Omega \,\middle|\, \begin{array}{l} \forall i, \mu(t_i) > 0, \mu'(t_i) > 0, \text{ and} \\ |\mu_i(F|t_i) - \mu'_i(F|t_i)| \leq \delta \;\forall\; F \subseteq \Omega \end{array} \right\}.$$

A game is *bounded* by M if $|u_i(a,\omega) - u_i(a',\omega)| \leq M \;\forall\, a,\, a' \in A$, $\forall\, \omega \in \Omega$ and $\forall\, i$. The "distance" between priors is defined via

$$d^*(\mu, \mu') = \max\{d_1(\mu, \mu'), d_1(\mu', \mu), \sup_{E \subseteq \Omega} |\mu(F) - \mu'(F)|\},$$

where $d_1(\mu, \mu') = \inf\{\delta | \mu'(C^{1-\delta}_{\mu'}(\mathcal{A}_{\mu,\mu'}(\delta))) \geq 1 - \delta\}$.

The following proposition is central to Kajii and Morris' results:

KM Proposition 5:*Suppose that $d^*(\mu, \mu') \leq \delta$. Then if σ is an equilibrium of (μ, u) and u is bounded by M, there exists a $6\delta M$-equilibrium σ' of (μ', u) with $|V_i[\sigma; \mu, u] - V_i[\sigma'; \mu', u]| \leq 3\delta M$ for all i.*

In their proof of this proposition, KM assert: "[B]y Lemma 4 (with $E = C^{1-\delta}_{\mu'}(\mathcal{A}_{\mu,\mu'}(\delta)))$ and $\varepsilon_1 = \varepsilon_2 = \delta$ there exists a $6\delta M$-equilibrium of (μ', u)..." But Lemma 4 may not apply to $C^{1-\delta}_{\mu'}(\mathcal{A}_{\mu,\mu'}(\delta))$. To wit:

KM Lemma 4: *Suppose that event $E \subseteq \mathcal{A}_{\mu,\mu'}(\varepsilon_1)$ and E is $(1 - \epsilon_2)$-evident under μ'. If σ is an equilibrium of (μ, u) and u is bounded by M, there exists a $(4\varepsilon_1 + 2\varepsilon_2)M$-equilibrium σ' of (μ', u) with $\sigma'(t) = \sigma(t)$ at all $(t, s) \in E$.*

Applying Lemma 4 here thus requires $C^{1-\delta}_{\mu'}(\mathcal{A}_{\mu,\mu'}(\delta)) \subseteq \mathcal{A}_{\mu,\mu'}(\delta)$. This may not hold: take a three-state, two-player, three-type (per player) game and two priors μ and μ' with $\mu((1,1),1) = \mu'((1,1),1) = 1-\delta$, $\mu((2,2),2) = \mu'((3,3),3) = \delta$ and $\mu(\omega) = \mu'(\omega) = 0$ for all other ω. Then $\mathcal{A}_{\mu,\mu'}(\delta) = \{\omega_1\}$, where $\omega_i \equiv ((i,i),i)$. But $C^{1-\delta}_{\mu'}(\mathcal{A}_{\mu,\mu'}(\delta)) = \{\omega_1, \omega_2\}$, so $C^{1-\delta}_{\mu'} \not\subseteq \mathcal{A}_{\mu,\mu'}$. The key to the flaw is in the definitions of B^p_μ and $\mathcal{A}_{\mu,\mu'}$. ω is only in the latter if the types in state ω have positive probability, while $B^p_{\mu'}$ (and hence $C^p_{\mu'}$) has zero probability types believing *all* events.

The following lemma can be substituted for Lemma 4 in KM's proof of Proposition 5 to make it valid (and tighter).

Lemma 1 *If σ is an equilibrium of (μ, u) and u is bounded by M, $\exists$ a $5M\delta$-equilibrium σ' of (μ', u) with $\sigma'(t) = \sigma(t)$ at all $(t,s) \in E \equiv C'^{1-\delta}_{\mu}(\mathcal{A}_{\mu,\mu'}(\delta))$.*

Proof. Let $\hat{T}_i = \{t_i \in T_i | \exists (t_{-i}, s) \text{ s.t. } ((t_i, t_{-i}), s) \in E\}$. Require that $\sigma' = \sigma$ on E, and take σ' at any other state to be an equilibrium of the restricted game that results from imposing this requirement (as in KM). It remains to show that $\sigma(t_i)$ is a $5M\delta$ best response for any $t_i \in \hat{T}_i$ with $\mu'(t_i) > 0$.[2] So suppose that $\mu'(t_i) > 0$ and $\mu(t_i) > 0$. Then, clearly,

$$\mu'(E|t_i) \geq 1 - \delta. \tag{1}$$

Also, for all $F \subseteq \Omega$,

$$|\mu(F|t_i) - \mu'(F|t_i)| \leq \delta, \tag{2}$$

for, otherwise, $t_i \notin \hat{T}_i$, as $((t_i, t_{-i}), s) \notin B^p_{\mu'}(\mathcal{A}_{\mu,\mu'}(\delta))$ for any (t_{-i}, s) since type t_i never believes *everyone's* posteriors to be close when his *own* aren't. Take any action a_i with $\sigma'_i(t_i)[a_i] > 0$ and any action b_i. Let $\Delta v = v[a_i, \sigma'_{-i}; \mu', u; t_i] - v[b_i, \sigma'_{-i}; \mu', u; t_i]$, and let S^c denote the complement of a set S. Then:

$$\begin{aligned} \Delta v = \quad & \textstyle\sum_{\omega \in E} \mu'(\omega|t_i) \left(u(a_i, \sigma'_{-i}(t_{-i}), \omega) - u(b_i, \sigma'_{-i}(t_{-i}), \omega) \right) \\ & + \textstyle\sum_{\omega \in E^c} \mu'(\omega|t_i) \left(u(a_i, \sigma'_{-i}(t_{-i}), \omega) - u(b_i, \sigma'_{-i}(t_{-i}), \omega) \right). \end{aligned} \tag{3}$$

Boundedness and (1) show the second sum in (3) to be at least $-M\delta$. For the first sum, note that $\sigma'(t_i) = \sigma(t_i)$ on E. Since σ is an equilibrium of (μ, u):

$$\begin{aligned} & \textstyle\sum_{\omega \in E} \mu(\omega|t_i) \left(u(a_i, \sigma_{-i}(t_{-i}), \omega) - u(b_i, \sigma_{-i}(t_{-i}), \omega) \right) \\ & \quad \geq \textstyle\sum_{\omega \in E^c} \mu(\omega|t_i) \left(u(b_i, \sigma_{-i}(t_{-i}), \omega) - u(a_i, \sigma_{-i}(t_{-i}), \omega) \right) \geq -2M\delta\,, \end{aligned} \tag{4}$$

where I have used $\mu(E^c|t_i) \leq 2\delta$, which follows from equations (1) and (2).

[2] KM's equilibrium notion allows any $\sigma'(t_i)$ when $\mu'(t_i) = 0$. With a notion requiring t_i have *some* beliefs, note that $\sigma(t_i)$ is a $2M\delta$ best response for $\mu'(\cdot|t_i) = \mu(\cdot|t_i)$.

Using (4) and the fact[3] that

$$|\sum_{\omega\in E}(\mu'(\omega|t_i)-\mu(\omega|t_i))\,(u(a_i,\sigma_{-i}(t_{-i}),\omega)-u(b_i,\sigma_{-i}(t_{-i}),\omega))\,|\leq 2M\delta$$

gives $\sum_{\omega\in E}\mu'(\omega|t_i)\,(u(a_i,\sigma_{-i}(t_{-i}),\omega)-u(b_i,\sigma_{-i}(t_{-i}),\omega))\geq -4M\delta$. Hence, the right hand side of equation (3) is no smaller than $-5M\delta$. ■

Incidentally, in proving Lemma 4, KM claim: "If $t_i\in\hat{T}_i$, for any $a_i\in A_i$, $v_i[a_i,\sigma_{-i};\mu',u;t_i]-v_i[a_i,\sigma_{-i};\mu,u;t_i]\leq 2\varepsilon_1 M$, because $E\subseteq\mathcal{A}_{\mu,\mu'}(\varepsilon_1)$." The claim is false, so this proof is also incorrect. To see why, take a two player game with states $\omega_1=((1,1),1)$ and $\omega_2=((1,2),2)$, with $\mu(\omega_1)=\frac{1}{2}=\mu(\omega_2)$, $\mu'(\omega_1)=\frac{1}{2}+\varepsilon_1$ and $\mu'(\omega_2)=\frac{1}{2}-\varepsilon_1$. Take $M>0$ and u such that $u_1(a,\omega_1)=\frac{3M}{\varepsilon_1}$ and $u_1(a,\omega_2)=0\ \forall\ a\in A$. The game is bounded, as $|u_i(a,\omega)-u_i(a',\omega)|=0$ $\forall\,\omega,\ a,\ a'$. For $E=\{\omega_1,\omega_2\}\subseteq\mathcal{A}_{\mu,\mu'}(\varepsilon_1)$, the left side of their inequality reads: $|\sum_{T_{-i}\times S}(\mu'(t_{-i},s|t_i)-\mu(t_{-i},s|t_i))\,(u_i((a_i,\sigma_{-i}(t_{-i})),(t,s)))\,|$. For player 1, this equals $\left|\varepsilon_1(3\frac{M}{\varepsilon_1}-0)\right|=3M$, contradicting their claim. But the lemma is true and can be proved using the reasoning in Lemma 1 above.

3 Discussion and conclusions

The exposition of the hole in KM's proof highlights a difficulty which can arise in analysis that starts from priors—namely how to deal with the beliefs of zero probability types. KM's definitions suffice for proving their results, but in an earlier version of this paper I showed that the analysis is eased by starting from posteriors. While such an approach raises the question of when and how to introduce a common prior (needed later in KM), the recent literature on the connection between posterior beliefs and the common prior assumption (e.g. Samet (1998) and Feinberg (2000)) suggests that it is wise to take posteriors as the starting point and introduce a (consistent) prior only when necessary.

References

[1] Feinberg, Y. Characterizing common priors in the form of posteriors, *J. Econ. Theory* **91** (2000), 127-179.

[2] Kajii, A. and S. Morris, Payoff continuity in incomplete information games, *J. Econ. Theory* **82** (1998), 267-276.

[3] Samet, D. Common priors and separation of convex sets, *Games Econ. Behav.* **24** (1998), 172-174.

[3] Break into sums over $G\equiv\{\omega\in E|(\mu'(\omega|t_i)-\mu(\omega|t_i))>0\}$ and G^c and use (2).