Web Appendix for
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This appendix examines a number of extensions and variations of the model. We start with an example where the signals have two realizations.

1. Two State Example

The state θ takes value 0 with probability $\frac{1}{2}$ and 1 with probability $\frac{1}{2}$. A binary public signal is equal to the true state with probability $q \in \left[\frac{1}{2}, 1\right]$; it is incorrect with probability $1-q$. Each of two players observes a private signal that is correct probability $p \in \left[\frac{1}{2}, 1\right]$, and incorrect with probability $1-p$. Player i will set his action equal to

$$(1-r) E_i(\theta) + r E_i(a_j).$$

It is useful to summarize the possible outcomes:

State ω	Public Signal y	1's Signal x_1	2's Signal x_2	Probability
0	0	0	0	$\frac{1}{2}qp^2$
0	0	0	1	$\frac{1}{2}qp(1-p)$
0	0	1	0	$\frac{1}{2}qp(1-p)$
0	0	1	1	$\frac{1}{2}q(1-p)^2$
0	1	0	0	$\frac{1}{2}(1-q)p^2$
0	1	0	1	$\frac{1}{2}(1-q)p(1-p)$
0	1	1	0	$\frac{1}{2}(1-q)p(1-p)$
0	1	1	1	$\frac{1}{2}(1-q)(1-p)^2$
1	0	0	0	$\frac{1}{2}(1-q)(1-p)^2$
1	0	0	1	$\frac{1}{2}(1-q)p(1-p)$
1	0	1	0	$\frac{1}{2}(1-q)p(1-p)$
1	0	1	1	$\frac{1}{2}(1-q)p^2$
1	1	0	0	$\frac{1}{2}q(1-p)^2$
1	1	0	1	$\frac{1}{2}qp(1-p)$
1	1	1	0	$\frac{1}{2}qp(1-p)$
1	1	1	1	$\frac{1}{2}qp^2$

Thus player 1's conditional probability of state 1 (expectation of θ) is:

Public Signal y	Private Signal x_1	Prob $\theta = 1$
0	0	$\frac{(1-q)(1-p)}{qp+(1-q)(1-p)}$
0	1	$\frac{(1-q)p}{q(1-p)+(1-q)p}$
1	0	$\frac{q(1-p)}{q(1-p)+(1-q)p}$
1	1	$\frac{qp}{qp+(1-q)(1-p)}$

Player 1's conditional probability that player 2 has observed private signal 1 is:

Public Signal y	Private Signal x_1	Prob $x_2 = 1$
0	0	$\frac{p(1-p)}{qp+(1-q)(1-p)}$
0	1	$\frac{(1-q)p^2+q(1-p)^2}{q(1-p)+(1-q)p}$
1	0	$\frac{p(1-p)}{q(1-p)+(1-q)p}$
1	1	$\frac{qp^2+(1-q)(1-p)^2}{qp+(1-q)(1-p)}$

Consider the strategy:

Public Signal y	Private Signal x_2	Action
0	0	$1-\overline{a}$
0	1	$1-\underline{a}$
1	0	$\underline{a}$
1	1	$\overline{a}$

When is there an equilibrium where both players follow this strategy? Suppose player 1 observed public signal 1 and private signal 1. His expectation of θ would be

$$\frac{qp}{qp+(1-q)(1-p)}$$

His expectation of player 2's action (if 2 was following the above strategy) would be

$$\left(\frac{qp^2+(1-q)(1-p)^2}{qp+(1-q)(1-p)}\right)\overline{a}+\left(1-\frac{qp^2+(1-q)(1-p)^2}{qp+(1-q)(1-p)}\right)\underline{a}$$

For equilibrium, we must have

$$\overline{a}=\left\{\begin{array}{c} (1-r)\frac{qp}{qp+(1-q)(1-p)} \\ +r\left(\begin{array}{c} \left(\frac{qp^2+(1-q)(1-p)^2}{qp+(1-q)(1-p)}\right)\overline{a} \\ +\left(1-\frac{qp^2+(1-q)(1-p)^2}{qp+(1-q)(1-p)}\right)\underline{a}\end{array}\right)\end{array}\right\} \tag{1.1}$$

Similarly, suppose player 1 observed public signal 1 and private signal 0. His expectation of θ would be

$$\frac{q(1-p)}{q(1-p)+(1-q)p}$$

His expectation of player 2's action (if 2 was following the above strategy) would be

$$\left(\frac{p(1-p)}{q(1-p)+(1-q)p}\right)\overline{a}+\left(1-\frac{p(1-p)}{q(1-p)+(1-q)p}\right)\underline{a}$$

For equilibrium, we must have

$$\underline{a}=\left\{\begin{array}{l}(1-r)\frac{q(1-p)}{q(1-p)+(1-q)p} \\ +r\left(\begin{array}{c}\left(\frac{p(1-p)}{q(1-p)+(1-q)p}\right)\overline{a} \\ +\left(1-\frac{p(1-p)}{q(1-p)+(1-q)p}\right)\underline{a}\end{array}\right)\end{array}\right\} \tag{1.2}$$

Solving (1.1) and (1.2) for $\overline{a}$ and $\underline{a}$,

$$\begin{aligned}\overline{a} &= \frac{qp\left[(1-r)(q(1-p)+p(1-q))+r(1-p)\right]}{(1-r)(q(1-p)+p(1-q))(qp+(1-q)(1-p))+r(1-p)p} \\ \underline{a} &= \frac{q(1-p)\left[(1-r)(qp+(1-q)(1-p))+rp\right]}{(1-r)(q(1-p)+p(1-q))(qp+(1-q)(1-p))+r(1-p)p}\end{aligned}$$

Observe that as $r\to 0$,

$$\begin{aligned}\overline{a} &\to \frac{qp}{qp+(1-q)(1-p)} \\ \underline{a} &\to \frac{q(1-p)}{q(1-p)+p(1-q)}\end{aligned}$$

This is the socially optimal strategy. But as $r\to 1$,

$$\begin{aligned}\overline{a} &\to q \\ \underline{a} &\to q\end{aligned}$$

Thus only public information is used.

What is welfare, i.e., the expected value of $-(a_1-\omega)^2$ under this strategy?

$$
\begin{aligned}
W(p,q,r) &= \left\{ \begin{array}{c} -\frac{1}{2}\left[qp(1-\overline{a})^2 + q(1-p)(1-\underline{a})^2 + (1-q)p(\underline{a})^2 + (1-q)(1-p)(\overline{a})^2\right] \\ -\frac{1}{2}\left[qp(\overline{a}-1)^2 + q(1-p)(\underline{a}-1)^2 + (1-q)p(-\underline{a})^2 + (1-q)(1-p)(-\overline{a})^2\right] \end{array} \right\} \\
&= -\left[qp(1-\overline{a})^2 + q(1-p)(1-\underline{a})^2 + (1-q)p(\underline{a})^2 + (1-q)(1-p)(\overline{a})^2\right] \\
&= -\left[\begin{array}{l} qp\left(1-\frac{qp[(1-r)(q(1-p)+p(1-q))+r(1-p)]}{(1-r)(q(1-p)+p(1-q))(qp+(1-q)(1-p))+r(1-p)p}\right)^2 \\ +q(1-p)\left(1-\frac{q(1-p)[(1-r)(qp+(1-q)(1-p))+rp]}{(1-r)(q(1-p)+p(1-q))(qp+(1-q)(1-p))+r(1-p)p}\right)^2 \\ +(1-q)p\left(\frac{q(1-p)[(1-r)(qp+(1-q)(1-p))+rp]}{(1-r)(q(1-p)+p(1-q))(qp+(1-q)(1-p))+r(1-p)p}\right)^2 \\ +(1-q)(1-p)\left(\frac{qp[(1-r)(q(1-p)+p(1-q))+r(1-p)]}{(1-r)(q(1-p)+p(1-q))(qp+(1-q)(1-p))+r(1-p)p}\right)^2 \end{array} \right]
\end{aligned}
$$

Lemma 1.1. *Public information is damaging (i.e., $\frac{dW(p,q,r)}{dq}<0$) if and only if*

$$
\begin{aligned}
\frac{1}{2} &< r < 1 \\
\text{and } \frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{r(3-2r)}} &< p < 1 \\
\text{and } \frac{1}{2} &< q < \frac{1}{2}(1+f(p,r))
\end{aligned}
$$

where

$$
f(p,r) = \frac{\sqrt{(2r-1)(1-r)(-4rp(1-p)(3-2r)+(2r-1)(1-r))}}{(2r-1)(1-r)(2p-1)}.
$$

If $r \le \frac{1}{2}$ or $p \le \frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{r(3-2r)}}$, then $\frac{dW(p,q,r)}{dq} \ge 0$ for all q.

Notice that the minimum value of $\sqrt{\frac{1}{r(3-2r)}}$ is $\sqrt{\frac{8}{9}} = \frac{2}{3}\sqrt{2}$ (this is realized when $r=\frac{3}{4}$). Thus if $\frac{dW(p,q,r)}{dq}<0$, we must have p equal to at least $\frac{1}{2}+\frac{1}{3}\sqrt{2}\approx 0.971$.

Sketch of Proof. The equation

$$
\frac{dW}{dq} = 0
$$

has three roots, $q=\frac{1}{2}$, $q=\frac{1}{2}(1-f(p,r))$ and $q=\frac{1}{2}(1+f(p,r))$. If $f(p,r)\in(0,1)$, then welfare is decreasing in q over the interval $\left(\frac{1}{2},\frac{1}{2}(1+f(p,r))\right)$ and increasing over the interval $\left(\frac{1}{2}(1+f(p,r)),1\right)$. If $f(p,r)$ is imaginary, then welfare must be everywhere increasing. Observe that $f(p,r)$ is a real number if

$$\begin{aligned}-4rp(1-p)(3-2r)+(2r-1)(1-r) &\geq 0\\ \text{or } p(1-p) &\leq \frac{(2r-1)(1-r)}{4r(3-2r)}\end{aligned}$$

For $\frac{1}{2}\leq p\leq 1$, this latter inequality will hold only if

$$p\geq\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{r(3-2r)}}.$$

Finally, observe that the following are equivalent if $f(p,r)$ is real:

$$\begin{aligned}f(p,r) &< 1\\ -4rp(1-p)(3-2r)+(2r-1)(1-r) &< (2r-1)(1-r)(2p-1)^2\\ 4rp(1-p)(3-2r) &> (2r-1)(1-r)4p(1-p)\\ r(3-2r) &> (2r-1)(1-r)\\ 0 &> -1\end{aligned}$$

2. Alternative Welfare Definitions

Let us now revert to the framework in the main body of the paper with normally distributed θ and a continuum of players. We will examine more general payoff functions for the players and the welfare functions that they give rise to. Suppose that player i seeks to maximize the general payoff:

$$u_i(\mathbf{a},\theta)\equiv\left\{\begin{array}{l}-r_1\int_0^1(a_j-a_i)^2\,dj\\ -r_2(a_i-\theta)^2\\ -r_3\left(a_i-\int_{j\in[0,1]}a_j dj\right)^2\\ +r_4\int_0^1\int_0^1(a_j-a_k)^2\,dj dk\\ -r_5\left(\int_{j\in[0,1]}a_j dj-\theta\right)^2\end{array}\right\}. \tag{2.1}$$

The specification of payoffs allows differing weights to the losses arising from the distances between a_i, θ, and the average actions. From the first order condition, the optimal action for i is given by

$$\widehat{r}\int_{j\in[0,1]} E_i\left(a_j\right)+\left(1-\widehat{r}\right)E_i\left(\theta\right).$$

where

$$\widehat{r}=\frac{r_1+r_3}{r_1+r_2+r_3}.$$

We can solve for the equilibrium in the same way as before, yielding equilibrium actions:

$$a_i=\frac{\alpha y+\beta\left(1-\hat{r}\right)x_i}{\alpha+\beta\left(1-\hat{r}\right)}$$

In deriving an expression for welfare, note that

$$\begin{aligned}
a_i-a_j &= \frac{1}{\beta\left(1-\widehat{r}\right)+\alpha}\beta\left(1-\widehat{r}\right)\left(\varepsilon_i-\varepsilon_j\right)\\
a_i-\int_{j\in[0,1]} a_j dj &= \frac{1}{\beta\left(1-\widehat{r}\right)+\alpha}\beta\left(1-\widehat{r}\right)\varepsilon_i\\
a_i-\theta &= \frac{1}{\beta\left(1-\widehat{r}\right)+\alpha}\left[\beta\left(1-\widehat{r}\right)\varepsilon_i+\alpha\eta\right]\\
\int_{i\in[0,1]} a_i di-\theta &= \frac{1}{\beta\left(1-\widehat{r}\right)+\alpha}\alpha\eta
\end{aligned}$$

Normalized welfare is then

$$\begin{aligned}
W &\equiv \frac{1}{1-\widehat{r}}\int_0^1 u_i\left(\mathbf{a},\theta\right)di\\
&= -\tfrac{1}{1-\widehat{r}}\left[\tfrac{1}{\beta(1-\widehat{r})+\alpha}\right]^2\left[\beta\left(1-\widehat{r}\right)^2\left(2\left(r_1-r_4\right)+r_2+r_3\right)+\alpha\left(r_2+r_5\right)\right]
\end{aligned}$$

Then, the derivative $\frac{dW}{d\alpha}$ is given by

$$\tfrac{1}{1-\widehat{r}}\left[\tfrac{1}{\beta(1-\widehat{r})+\alpha}\right]^3\left\{\begin{array}{l}-\beta\left(1-\widehat{r}\right)\left(r_2+r_5-2\left(1-\widehat{r}\right)\left(2\left(r_1-r_4\right)+r_2+r_3\right)\right)\\+\alpha\left(r_2+r_5\right)\end{array}\right\}$$

Thus public information is always valuable if $\beta=0$. Public information can sometimes be damaging (when $\beta>0$ and α is low) when

$$r_2+r_5\geq\frac{2r_2\left(2\left(r_1-r_4\right)+r_2+r_3\right)}{r_1+r_2+r_3}.$$

Our leading model in section 2 is the special case of this when $r_1 = r_4 = r$, $r_2 = 1 - r$ and $r_3 = r_5 = 0$. In this case, this condition reduces to $r \geq \frac{1}{2}$.

We note a variation on this example. Let each player i have the following payoff function:

$$u_i(\mathbf{a}, \theta) \equiv (1-\varepsilon) V(\theta, \bar{a}) - \varepsilon (1-r)(a_i - \theta)^2 - \varepsilon r \int_0^1 (a_j - a_i)^2 \, dj \qquad (2.2)$$

for some small $\varepsilon > 0$. Equilibrium is unaltered by this change in payoffs, since the first term is an externality that no individual player can influence. However, for small ε, social welfare will be approximately equal to $V(\theta, \bar{a})$. For some choice of $V(\cdot)$, public information may be damaging *even in the absence of private information*. For example, suppose that

$$V(\theta, \overline{a}) = \left\{ \begin{array}{l} 1, \text{ if } \overline{a} \geq a^* \\ 0, \text{ if } \overline{a} \leq a^* \end{array} \right\}$$

Suppose that the only information is the public signal $y = \theta + \eta$, where η is normally distributed with mean zero and precision α. Each player will set his action equal to y. So conditional on state θ, expected welfare (for small ε) is

$$1 - \Phi\left(\sqrt{\alpha}\,(a^* - \theta)\right).$$

This is decreasing in α if $a^* > \theta$. In other words, if players would do something socially inefficient if there were perfect information, then reducing the accuracy of public information should be expected to improve social welfare in some states.[1]

3. Correlated Private Signals

Consider a two player version of the model where the players can observe many signals, where the signals are multivariate normal with a general correlation structure. To fix ideas, suppose there are two players $i = 1, 2$. There is a public signal $y = \theta + \eta$, where η is normally distributed with mean 0 and precision α. In addition, each player i observes two private signals $x_{i1} = \theta + \varepsilon_{i1}$ and $x_{i2} = \theta + \varepsilon_{i2}$. While ε_{i1} and ε_{i2} are assumed to be independent, we assume that $(\varepsilon_{1j}, \varepsilon_{2j})$ are jointly normally distributed with zero means and covariance matrix

$$\begin{pmatrix} \frac{1}{\beta_j} & \frac{\rho_j}{\beta_j} \\ \frac{\rho_j}{\beta_j} & \frac{1}{\beta_j} \end{pmatrix}.$$

[1]This effect occurs in the analysis of transparency in currency markets in Metz (2000) and in the public good game of Teoh (1997).

Thus each signal j has precision β_j, and the correlation coefficient between the two players' jth signals is $\rho_j \in [0,1)$.

In this setting, we can show that while the socially optimal action for player i (minimizing $-(a_i - \theta)^2$) is

$$a_i = \frac{\alpha y + \beta_1 x_{i1} + \beta_2 x_{i2}}{\alpha + \beta_1 + \beta_2},$$

the equilibrium action for player i is

$$a_i = \frac{\alpha y + \beta_1 \left(\frac{1-r}{1-r\rho_1}\right) x_{i1} + \beta_2 \left(\frac{1-r}{1-r\rho_2}\right) x_{i2}}{\alpha + \left(\frac{1-r}{1-r\rho_1}\right) \beta_1 + \left(\frac{1-r}{1-r\rho_2}\right) \beta_2}.$$

This expression concretely captures the trade-off between the accuracy of a signal and its ability to coordinate the players. The corresponding expression for welfare is

$$-\frac{\alpha + \left(\frac{1-r}{1-r\rho_1}\right)^2 \beta_1 + \left(\frac{1-r}{1-r\rho_2}\right)^2 \beta_2}{\left(\alpha + \left(\frac{1-r}{1-r\rho_1}\right) \beta_1 + \left(\frac{1-r}{1-r\rho_2}\right) \beta_2\right)^2}.$$

Let us proceed to demonstrate this, and develop the general framework. A state θ is drawn from a uniform distribution on the real line. There is a public signal $y = \theta + \eta$. Each agent $i = 1, 2$ observes n private signals; the kth signal of agent i is $x_{ik} = \theta + \varepsilon_{ik}$. We write $\boldsymbol{\varepsilon}_i$ for the n-vector of agent i's noise terms. We assume that the $(2n+1)$ vector

$$\boldsymbol{\xi} = \begin{pmatrix} \eta \\ \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 \end{pmatrix}$$

is normally distributed with mean

$$\begin{pmatrix} 0 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

(we are writing $\mathbf{0}$ for a vector of 0s of arbitrary length) and covariance matrix

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{00} & \boldsymbol{\Sigma}_{01} & \boldsymbol{\Sigma}_{02} \\ \boldsymbol{\Sigma}_{10} & \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{20} & \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}.$$

We are interested in

$$\mathbf{z} = \begin{pmatrix} \theta \\ \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} - y\mathbf{1}.$$

(we are writing $\mathbf{1}$ for a vector of 1s of arbitrary length). Setting

$$\mathbf{A} = \begin{pmatrix} -1 & 0 & 0 & 0 & \cdot & \cdot \\ -1 & 1 & 0 & 0 & \cdot & \cdot \\ -1 & 0 & 1 & 0 & \cdot & \cdot \\ -1 & 0 & 0 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

we have

$$\mathbf{z} = \mathbf{A}\boldsymbol{\xi}$$

so z is normally distributed with mean

$$\begin{pmatrix} 0 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

and covariance matrix

$$\begin{aligned} \widehat{\boldsymbol{\Sigma}} &= \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}' \\ &= \begin{pmatrix} \widehat{\boldsymbol{\Sigma}}_{00} & \widehat{\boldsymbol{\Sigma}}_{01} & \widehat{\boldsymbol{\Sigma}}_{02} \\ \widehat{\boldsymbol{\Sigma}}_{10} & \widehat{\boldsymbol{\Sigma}}_{11} & \widehat{\boldsymbol{\Sigma}}_{12} \\ \widehat{\boldsymbol{\Sigma}}_{20} & \widehat{\boldsymbol{\Sigma}}_{21} & \widehat{\boldsymbol{\Sigma}}_{22} \end{pmatrix}. \end{aligned}$$

We will often be interested in the case where the public signal is conditionally independent of the various private signals. In this case,

$$\boldsymbol{\Sigma} = \begin{pmatrix} \tau^2 & 0 & 0 \\ 0 & \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ 0 & \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}$$

and

$$\widehat{\boldsymbol{\Sigma}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ 0 & \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix} + \tau^2\mathbf{M}$$

where $\mathbf{M}$ is a $(2n+1)\times(2n+1)$ matrix of 1's.

By standard properties of the multivariate normal (e.g., Spanos (1986), ch 14, p 317),

$$\begin{aligned} E_1(\theta - y) &= E_1(\theta - y\,|\mathbf{x}_1 - y\mathbf{1}) = \widehat{\boldsymbol{\Sigma}}_{01}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}(\mathbf{x}_1 - y\mathbf{1}) \\ E_2(\theta - y) &= E_2(\theta - y\,|\mathbf{x}_2 - y\mathbf{1}) = \widehat{\boldsymbol{\Sigma}}_{02}\widehat{\boldsymbol{\Sigma}}_{22}^{-1}(\mathbf{x}_1 - y\mathbf{1}) \end{aligned}$$

Thus player 1's *optimal action* is

$$\begin{aligned} a_1 &= E_1(\theta) \\ &= y + E_1(\theta - y) \\ &= y + \widehat{\boldsymbol{\Sigma}}_{01}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}(\mathbf{x}_1 - y\mathbf{1}) \end{aligned} \tag{3.1}$$

Also

$$\begin{aligned} E_1(\mathbf{x}_2 - y\mathbf{1}) &= E_1(\mathbf{x}_2 - y\mathbf{1}\,|\mathbf{x}_1 - y\mathbf{1}) = \widehat{\boldsymbol{\Sigma}}_{21}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}(\mathbf{x}_1 - y\mathbf{1}) \\ E_2(\mathbf{x}_1 - y\mathbf{1}) &= E_2(\mathbf{x}_1 - y\mathbf{1}\,|\mathbf{x}_2 - y\mathbf{1}) = \widehat{\boldsymbol{\Sigma}}_{12}\widehat{\boldsymbol{\Sigma}}_{22}^{-1}(\mathbf{x}_2 - y\mathbf{1}) \end{aligned}$$

Now

$$\begin{aligned} E_1E_2(\mathbf{x}_1 - y\mathbf{1}) &= E_1\left(\widehat{\boldsymbol{\Sigma}}_{12}\widehat{\boldsymbol{\Sigma}}_{22}^{-1}(x_2 - y)\right) \\ &= \widehat{\boldsymbol{\Sigma}}_{12}\widehat{\boldsymbol{\Sigma}}_{22}^{-1}(E_1(\mathbf{x}_2 - y\mathbf{1})) \\ &= \widehat{\boldsymbol{\Sigma}}_{12}\widehat{\boldsymbol{\Sigma}}_{22}^{-1}\widehat{\boldsymbol{\Sigma}}_{21}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}(\mathbf{x}_1 - y\mathbf{1}) \end{aligned}$$

and

$$\begin{aligned} E_2E_1E_2(\mathbf{x}_1 - y\mathbf{1}) &= E_2(E_1E_2(\mathbf{x}_1 - y\mathbf{1})) \\ &= E_2\left(\widehat{\boldsymbol{\Sigma}}_{12}\widehat{\boldsymbol{\Sigma}}_{22}^{-1}\widehat{\boldsymbol{\Sigma}}_{21}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}(\mathbf{x}_1 - y\mathbf{1})\right) \\ &= \widehat{\boldsymbol{\Sigma}}_{12}\widehat{\boldsymbol{\Sigma}}_{22}^{-1}\widehat{\boldsymbol{\Sigma}}_{21}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}(E_2(\mathbf{x}_1 - y\mathbf{1})) \\ &= \widehat{\boldsymbol{\Sigma}}_{12}\widehat{\boldsymbol{\Sigma}}_{22}^{-1}\widehat{\boldsymbol{\Sigma}}_{21}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}\widehat{\boldsymbol{\Sigma}}_{12}\widehat{\boldsymbol{\Sigma}}_{22}^{-1}(x_2 - y) \end{aligned}$$

Thus by induction

$$[E_1E_2]^n(\mathbf{x}_1 - y\mathbf{1}) = \left[\widehat{\boldsymbol{\Sigma}}_{12}\widehat{\boldsymbol{\Sigma}}_{22}^{-1}\widehat{\boldsymbol{\Sigma}}_{21}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}\right]^n(\mathbf{x}_1 - y\mathbf{1})$$

and

$$\begin{aligned} [E_1E_2]^n(E_1(\theta - y)) &= [E_1E_2]^n\left(\widehat{\boldsymbol{\Sigma}}_{01}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}(\mathbf{x}_1 - y\mathbf{1})\right) \\ &= \widehat{\boldsymbol{\Sigma}}_{01}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}\left[\widehat{\boldsymbol{\Sigma}}_{12}\widehat{\boldsymbol{\Sigma}}_{22}^{-1}\widehat{\boldsymbol{\Sigma}}_{21}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}\right]^n(\mathbf{x}_1 - y\mathbf{1}) \end{aligned}$$

Also

$$\begin{aligned}
E_2\left[E_1E_2\right]^n\left(E_1\left(\theta-y\right)\right) &= \widehat{\mathbf{\Sigma}}_{02}\widehat{\mathbf{\Sigma}}_{22}^{-1}\left[\widehat{\mathbf{\Sigma}}_{12}\widehat{\mathbf{\Sigma}}_{22}^{-1}\widehat{\mathbf{\Sigma}}_{21}\widehat{\mathbf{\Sigma}}_{11}^{-1}\right]^n\widehat{\mathbf{\Sigma}}_{12}\widehat{\mathbf{\Sigma}}_{22}^{-1}\left(\mathbf{x}_2-y\mathbf{1}\right)\\
\left[E_2E_1\right]^n\left(E_2\left(\theta-y\right)\right) &= \widehat{\mathbf{\Sigma}}_{02}\widehat{\mathbf{\Sigma}}_{22}^{-1}\left[\widehat{\mathbf{\Sigma}}_{21}\widehat{\mathbf{\Sigma}}_{11}^{-1}\widehat{\mathbf{\Sigma}}_{12}\widehat{\mathbf{\Sigma}}_{22}^{-1}\right]^n\left(\mathbf{x}_2-y\mathbf{1}\right)\\
E_1\left[E_2E_1\right]^n\left(E_2\left(\theta-y\right)\right) &= \widehat{\mathbf{\Sigma}}_{01}\widehat{\mathbf{\Sigma}}_{11}^{-1}\left[\widehat{\mathbf{\Sigma}}_{21}\widehat{\mathbf{\Sigma}}_{11}^{-1}\widehat{\mathbf{\Sigma}}_{12}\widehat{\mathbf{\Sigma}}_{22}^{-1}\right]^n\widehat{\mathbf{\Sigma}}_{21}\widehat{\mathbf{\Sigma}}_{11}^{-1}\left(\mathbf{x}_1-y\mathbf{1}\right)
\end{aligned}$$

Now player 1's equilibrium action is:

$$\begin{aligned}
a_1 &= (1-r)\,E_1(\theta)+(1-r)\,rE_1E_2(\theta)+(1-r)\,r^2E_1E_2E_1(\theta)+... \qquad (3.2)\\
&= y+\left\{\begin{array}{l}
(1-r)\,\widehat{\mathbf{\Sigma}}_{01}\widehat{\mathbf{\Sigma}}_{11}^{-1}\left(\mathbf{x}_1-y\mathbf{1}\right)\\
+(1-r)\,r\widehat{\mathbf{\Sigma}}_{01}\widehat{\mathbf{\Sigma}}_{11}^{-1}\widehat{\mathbf{\Sigma}}_{21}\widehat{\mathbf{\Sigma}}_{11}^{-1}\left(\mathbf{x}_1-y\mathbf{1}\right)\\
+(1-r)\,r^2\widehat{\mathbf{\Sigma}}_{01}\widehat{\mathbf{\Sigma}}_{11}^{-1}\left[\widehat{\mathbf{\Sigma}}_{12}\widehat{\mathbf{\Sigma}}_{22}^{-1}\widehat{\mathbf{\Sigma}}_{21}\widehat{\mathbf{\Sigma}}_{11}^{-1}\right]\left(\mathbf{x}_1-y\mathbf{1}\right)\\
+(1-r)\,r^3\widehat{\mathbf{\Sigma}}_{01}\widehat{\mathbf{\Sigma}}_{11}^{-1}\left[\widehat{\mathbf{\Sigma}}_{21}\widehat{\mathbf{\Sigma}}_{11}^{-1}\widehat{\mathbf{\Sigma}}_{12}\widehat{\mathbf{\Sigma}}_{22}^{-1}\right]\widehat{\mathbf{\Sigma}}_{21}\widehat{\mathbf{\Sigma}}_{11}^{-1}\left(\mathbf{x}_1-y\mathbf{1}\right)\\
+(1-r)\,r^4\widehat{\mathbf{\Sigma}}_{01}\widehat{\mathbf{\Sigma}}_{11}^{-1}\left[\widehat{\mathbf{\Sigma}}_{12}\widehat{\mathbf{\Sigma}}_{22}^{-1}\widehat{\mathbf{\Sigma}}_{21}\widehat{\mathbf{\Sigma}}_{11}^{-1}\right]^2\left(\mathbf{x}_1-y\mathbf{1}\right)\\
+....
\end{array}\right\}\\
&= y+(1-r)\,\widehat{\mathbf{\Sigma}}_{01}\widehat{\mathbf{\Sigma}}_{11}^{-1}\left[I+r^2\widehat{\mathbf{\Sigma}}_{12}\widehat{\mathbf{\Sigma}}_{22}^{-1}\widehat{\mathbf{\Sigma}}_{21}\widehat{\mathbf{\Sigma}}_{11}^{-1}+r^4\left[\widehat{\mathbf{\Sigma}}_{12}\widehat{\mathbf{\Sigma}}_{22}^{-1}\widehat{\mathbf{\Sigma}}_{21}\widehat{\mathbf{\Sigma}}_{11}^{-1}\right]^2+...\right]\\
&\quad\times\left(I+r\widehat{\mathbf{\Sigma}}_{21}\widehat{\mathbf{\Sigma}}_{11}^{-1}\right)\left(\mathbf{x}_1-y\mathbf{1}\right)\\
&= y+(1-r)\,\widehat{\mathbf{\Sigma}}_{01}\widehat{\mathbf{\Sigma}}_{11}^{-1}\left[I-r^2\widehat{\mathbf{\Sigma}}_{12}\widehat{\mathbf{\Sigma}}_{22}^{-1}\widehat{\mathbf{\Sigma}}_{21}\widehat{\mathbf{\Sigma}}_{11}^{-1}\right]^{-1}\left(I+r\widehat{\mathbf{\Sigma}}_{21}\widehat{\mathbf{\Sigma}}_{11}^{-1}\right)\left(\mathbf{x}_1-y\mathbf{1}\right)
\end{aligned}$$

In the symmetric case, where

$$\mathbf{N}=\widehat{\mathbf{\Sigma}}_{12}\widehat{\mathbf{\Sigma}}_{22}^{-1}=\widehat{\mathbf{\Sigma}}_{21}\widehat{\mathbf{\Sigma}}_{11}^{-1},$$

this formula becomes

$$a_1=y+(1-r)\,\widehat{\mathbf{\Sigma}}_{01}\widehat{\mathbf{\Sigma}}_{11}^{-1}\left[I-r\mathbf{N}\right]^{-1}\left(\mathbf{x}_1-y\mathbf{1}\right) \qquad (3.3)$$

In the original two signal example referred to in the text,

$$\mathbf{\Sigma}=\left(\begin{array}{lllll}
\frac{1}{\alpha} & 0 & 0 & 0 & 0\\
0 & \sigma_1^2 & 0 & \rho_1\sigma_1^2 & 0\\
0 & 0 & \frac{1}{\beta_2} & 0 & \rho_2\frac{1}{\beta_2}\\
0 & \rho_1\sigma_1^2 & 0 & \sigma_1^2 & 0\\
0 & 0 & \rho_2\frac{1}{\beta_2} & 0 & \frac{1}{\beta_2}
\end{array}\right)$$

and

$$\widehat{\mathbf{\Sigma}} = \begin{pmatrix} \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha} \\ \frac{1}{\alpha} & \frac{1}{\alpha}+\frac{1}{\beta_1} & \frac{1}{\alpha} & \frac{1}{\alpha}+\rho_1\frac{1}{\beta_1} & \frac{1}{\alpha} \\ \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha}+\frac{1}{\beta_2} & \frac{1}{\alpha} & \frac{1}{\alpha}+\rho_2\frac{1}{\beta_2} \\ \frac{1}{\alpha} & \frac{1}{\alpha}+\rho_1\frac{1}{\beta_1} & \frac{1}{\alpha} & \frac{1}{\alpha}+\frac{1}{\beta_1} & \frac{1}{\alpha} \\ \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha}+\rho_2\frac{1}{\beta_2} & \frac{1}{\alpha} & \frac{1}{\alpha}+\frac{1}{\beta_2} \end{pmatrix}$$

Now

$$\begin{aligned} \widehat{\mathbf{\Sigma}}_{01} &= \begin{pmatrix} \frac{1}{\alpha} & \frac{1}{\alpha} \end{pmatrix} \\ \widehat{\mathbf{\Sigma}}_{11} &= \begin{pmatrix} \frac{1}{\alpha}+\frac{1}{\beta_1} & \frac{1}{\alpha} \\ \frac{1}{\alpha} & \frac{1}{\alpha}+\frac{1}{\beta_2} \end{pmatrix} \end{aligned}$$

and

$$\mathbf{\Sigma}_{11}^{-1} = \frac{1}{\alpha+\beta_1+\beta_2}\begin{pmatrix} \beta_1(\alpha+\beta_2) & -\beta_1\beta_2 \\ -\beta_1\beta_2 & \beta_2(\beta_1+\alpha) \end{pmatrix}$$

so

$$\widehat{\mathbf{\Sigma}}_{01}\mathbf{\Sigma}_{11}^{-1} = \frac{1}{\alpha+\beta_1+\beta_2}\begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}.$$

Now by (3.1), player 1's optimal action is

$$\begin{aligned} a_1^* &= y + \widehat{\mathbf{\Sigma}}_{01}\mathbf{\Sigma}_{11}^{-1}\begin{pmatrix} x_{11}-y \\ x_{12}-y \end{pmatrix} \\ &= y + \frac{\beta_1 x_{11} - \beta_1 y + \beta_2 x_{12} - \beta_2 y}{\alpha+\beta_1+\beta_2} \\ &= \frac{\alpha y + \beta_1 x_{11} + \beta_2 x_{12}}{\alpha+\beta_1+\beta_2} \end{aligned}$$

Also

$$\mathbf{\Sigma}_{12} = \mathbf{\Sigma}_{21} = \begin{pmatrix} \frac{1}{\alpha}+\rho_1\frac{1}{\beta_1} & \frac{1}{\alpha} \\ \frac{1}{\alpha} & \frac{1}{\alpha}+\rho_2\frac{1}{\beta_2} \end{pmatrix}$$

so

$$\begin{aligned} \mathbf{N} &= \widehat{\mathbf{\Sigma}}_{21}\widehat{\mathbf{\Sigma}}_{11}^{-1} = \widehat{\mathbf{\Sigma}}_{12}\widehat{\mathbf{\Sigma}}_{22}^{-1} = \frac{1}{\alpha+\beta_1+\beta_2}\begin{pmatrix} \beta_1+\rho_1(\alpha+\beta_2) & \beta_2(1-\rho_1) \\ \beta_1(1-\rho_2) & \beta_2+\rho_2(\alpha+\beta_1) \end{pmatrix} \\ \mathbf{I}-r\mathbf{N} &= \frac{1}{\alpha+\beta_1+\beta_2}\begin{pmatrix} (1-r)\beta_1+(1-r\rho_1)(\alpha+\beta_2) & -r\beta_2(1-\rho_1) \\ -r\beta_1(1-\rho_2) & (1-r)\beta_2+(1-r\rho_2)(\alpha+\beta_1) \end{pmatrix} \end{aligned}$$

Thus by (3.3), player 1's equilibrium action is

$$\begin{aligned} a_1 &= y + (1-r)\,\widehat{\mathbf{\Sigma}}_{01}\widehat{\mathbf{\Sigma}}_{11}^{-1}\left[I - r\mathbf{N}\right]^{-1}\begin{pmatrix} x_{11} - y \\ x_{12} - y \end{pmatrix} \\ &= \frac{\alpha y + \beta_1\left(\frac{1-r}{1-r\rho_1}\right)x_{11} + \beta_2\left(\frac{1-r}{1-r\rho_2}\right)x_{12}}{\alpha + \beta_1\left(\frac{1-r}{1-r\rho_1}\right) + \beta_2\left(\frac{1-r}{1-r\rho_2}\right)} \end{aligned}$$