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Systemic credit freezes in financial lending networks

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Abstract
This paper develops a network model of interbank lending, in which banks decide to extend credit to their potential borrowers. Borrowers are subject to shocks that may force them to default on their loans. In contrast to much of the previous literature on financial networks, we focus on how anticipation of future defaults may result in ex ante "credit freezes," whereby banks refuse to extend credit to one another. We first characterize the terms of the interbank contracts and the patterns of interbank lending that emerge in equilibrium. We then study how shifts in the distribution of shocks can result in complex credit freezes that travel throughout the network. We use this framework to analyze the effects of various policy interventions on systemic credit freezes.

Keywords Systemic risk · Endogenous financial networks · Credit freezes · Contagion · Counterparty risk

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1 Introduction

By the onset of the financial crisis of 2008, the U.S. financial system had become greatly interconnected. This not only reflected complex relations in interbank and overnight lending, but also various kinds of securitized lending relations including in the repo market [31]. A distinguishing feature of many of these transactions was the need for the lenders to assess not just their borrowers' credit worthiness but also the creditworthiness of their borrowers' borrowers and so on. These variegated credit relations ground to a halt following the collapse of Lehman Brothers in September 2008, as many institutions found their access to credit to be frozen [3,14].

Although these events have triggered a growing literature investigating the possibility of contagion in financial networks, the main focus so far has been on ex post contagion, i.e., the possibility that the failure of an institution triggers financial distress for its counterparties or for other companies holding its shares.[1] However, an even more important dynamic during the crisis was driven by ex ante considerations: credit freezes induced by the fear that the future liquidity or profitability of borrowers would be compromised [3,4]. Such fears were visible even before the collapse of Lehman Brothers. The run on Bear Stearns, which started on March 12, 2008, was initiated by its inability to secure funding in the repo market [14]. This episode was followed by some hedge funds' inability to trade outstanding Bear Stearns debt [15,31,36], largely because institutions such as Goldman Sachs, Credit Suisse, and Deutsche Bank had "little or no interest to renew repos in the face of concerns over the dealer bank's solvency" [22]. Subsequently, the bankruptcy of the hedge fund Carlyle Corporation as well as the severe distress felt by Merrill Lynch, Washington Mutual, and Wachovia—which led to their acquisition by other institutions—were triggered by similar credit freezes, even though they did not have any direct counterparty exposure to Lehman Brothers. A similar credit freeze appears to have been important in the downfall of the UK bank Northern Rock [14]. Some authors, such as Allen and Babus [8], suggest interbank credit freezes may have begun as early as August 2007.

In this paper, we develop an elementary model of ex ante credit freezes. We consider an economy consisting of depositors with access to funds and entrepreneurs with access to profitable investment opportunities. The economy also consists of a collection of banks that can intermediate between the depositors and the entrepreneurs. We capture the possibility of financial intermediation by a network, according to which each bank can lend to any bank or entrepreneur it is connected to. The connections in this network may represent existing relationships or trust between the parties. Interbank contracts are determined by potential lenders making offers to potential borrowers. We focus on fixed-interest-rate contracts, according to which the lender commits to a pre-specified interest rate and the borrower can decide to borrow as much as it desires at that rate. A borrower that is unable to meet its borrowing obligations (say, due to a liquidity shock) defaults and repays nothing to any of it creditors. Therefore, in anticipation of such an event, a potential lender may alternatively decide to "freeze" the borrower's access to credit in order to avoid potential future losses.

We start by analyzing subgame perfect equilibria of the lending and borrowing game described above under sequential offers and borrowing, characterizing the terms of the interbank contracts and the patterns of interbank lending. Though there are many subgame perfect equilibria, we show that there exists a unique "strong" equilibrium in pure strategies, where decisions to offer and borrow are robust to other banks deviating slightly from their equi-

[1] For example, see Acemoglu et al. [2], Cabrales et al. [17], Elliott et al. [25], Gai and Kapadia [29], Jorion and Zhang [34], and Allen and Gale [9].

librium strategies. While the details of this unique equilibrium may differ depending on the order of offers made, we show that whether there is ultimately lending to entrepreneurs (who are located at the leaves of the network) is not sensitive to this choice. We then proceed to characterize the structure of equilibria and conditions for different types of credit freezes in response to changes in the distribution of future liquidity shocks, which may increase banks' default probabilities.

In networks with a single entrepreneur, we show that all credit freezes are *monotone* and *systemic*: an adverse shift in the distribution of shocks can only induce more credit freezes throughout the economy and, in the extreme, cuts all banks' access to credit. We further show that credit freezes in tree networks (where each bank can borrow from at most one other bank) are "simple" in the sense that they remain confined to the branch of the financial network that experienced the adverse shift. For a chain (where each bank borrows from and lends to at most one other bank), we provide a tight characterization of the likelihood of credit freezes in response to changes in the distribution of shocks.

With multiple entrepreneurs, however, the form of potential credit freezes becomes significantly richer. First, credit freezes may originate not with the affected bank but somewhere else in the network. Such complex freezes arise because an anticipated (future) liquidity shock to a bank affects the profitability of banks in very different parts of the network, pushing some of those from a safe into a precarious position. In particular, one risky bank may cause other banks to have their credit frozen, even though the afflicted bank, and all of its lenders and borrowers, do not lose access to credit. Second, the effects of adverse shifts can be non-monotonic, in the sense that greater risks for some banks can increase overall lending in the network. Such an outcome may arise when the worsening situation of a bank allows a competitor to take over some of its customers, improve its riskiness and creditworthiness, and then expand further.

We conclude the paper by considering the role of policy in reducing the extent of credit freezes in a lending network. Specifically, we allow the central bank to offer assistance to a subset of banks in the form of a discount window or through asset purchases. When a freeze occurs in a chain, we show that an *untargeted* policy, where the central bank improves financial conditions as a whole—for example, by subsidizing interest rates—does no worse than a *targeted* policy, which attempts to alleviate distress in the most vulnerable part of the network. Beyond chains, however, a targeted policy can be more effective. When freezes are simple, we show that the best targeted policy helps the branch of the network with banks experiencing a credit freeze. In contrast, with complex freezes, optimal targeted policies may need to be directed to parts of the network not suffering from credit freezes (because these seemingly unaffected banks may still be at the epicenter of the crisis). These results suggest that, as the network becomes more interconnected, the optimal policy response becomes increasingly more complex and is more sensitive to the underlying financial network structure.

Related literature In addition to the literature on ex post contagion in financial networks mentioned earlier, our paper is related to a growing literature that emphasizes how ex ante fears of declining asset values or fire sales can induce credit freezes.[2] For example, Diamond and Rajan [21] develop a model of market freeze based on fears of future fire sales, while Caballero and Simsek [16] provide a model of liquidity hoarding where banks that are uninformed about the health of their borrowers' borrowers may come to fear future contagion and start offloading risky assets for protection. Similar to our work, credit freezes in these models have their origins in the interconnections in the financial system. However, and in contrast to the

[2] Brunnermeier [14] and Duffie [22] for general discussions.

prior literature, our main focus is on how the structural properties of the financial network shape the fear of contagion and the nature of resulting credit freezes.

A related strand of literature focuses on endogenous formation of financial networks and the extent of systemic risk when banks strategically choose their trading partners [1,7,13,37].[3] We build on this literature by developing a framework in which banks endogenously choose the terms of interbank contracts, while taking into account the potential for future defaults. However, in contrast to this previous work, our focus is on how these considerations can lead to widespread credit freezes prior to the network formation stage.

Most immediately related to our work is Anand et al. [10], where lending decisions take the form of a coordination game: banks decide whether to rollover short-term credit when facing the risk of the borrower defaulting if it cannot secure enough funding from other lenders. Using the setting proposed by Allen and Babus [8], they show that an uptick in the risk of a few counterparties can lead to widespread credit freezes. Similar mechanisms are explored by Ahnert [6], Infante and Vardoulakis [32], Zhou [41] , and Liu [38]. In contrast to these papers, credit freezes in our model are driven entirely by fundamentals.

Our model of credit freeze combines many of the ideas from this literature, but provides the following new contribution: in an interconnected financial system, the fear of ex post default cascades can lead to ex ante credit freezes. These credit freezes negatively impact market liquidity and can prevent safe institutions from having access to short-term funding. Because of the interconnectedness of the financial system, decisions to reduce lending can invoke responses from other banks to do the same. As a result, the propagation of credit freeze throughout the system can destroy the many benefits of financial interconnectivity. For example, freezes in the interbank lending market can reduce efficiency for business loans due to monopolistic pricing (as in Corbae and Gofman [19]) or restrict the redistribution of liquidity to meet reserve/capital requirements [28]. Finally, in contrast to Anand et al. [10] and others, we link banks' ex ante lending decisions—including the possibility of credit freezes—to models of ex post contagion studied extensively in earlier literature.

2 Model

Consider an economy consisting of a collection of risk-neutral financial institutions denoted by $\mathcal{B} = \{1, \ldots, n\}$, a unit mass of identical depositors indexed 0, and a finite collection of entrepreneurs $\mathcal{E}$. The economy lasts for three periods, $t = 0, 1, 2$. At the initial period, agents can enter into pairwise lending agreements that specify the interest rates at which they can borrow from one another; borrowing and lending occur at $t = 1$ according to the terms specified at $t = 0$; and all debts are due at $t = 2$.

The representative depositor is endowed with an unlimited supply of funds at $t = 0$ and has access to a linear risk-free technology with a (gross) rate of return r_0, which is realized at $t = 2$. Each entrepreneur $j \in \mathcal{E}$, on the other hand, has access to a safe but "bulky" investment opportunity of size \$1 with a rate of return r_i^*, realized at $t = 2$. Thus, as long as $r_i^* > r_0$, there are gains from trade (for one unit of investment) between entrepreneur i and the representative depositor.

In addition to the depositors and the entrepreneurs, the economy comprises a collection of financial institutions $\mathcal{B}$ (banks, for short) that can serve as potential intermediaries between depositors and entrepreneurs. Each bank $i \in \mathcal{B}$ has an asset with random return η_i that

[3] Also see Zawadowski [40], Farboodi [27], and Erol [26], who study how endogenous formation of financial networks can shape systemic risk.

is realized at $t = 2$. These assets represent the uncertain returns of all outside projects undertaken by the banks that are not captured in our stylized lending market. Furthermore, every bank i has an outside liability with face value $v_i > 0$ due at $t = 2$, which is senior to its all other obligations. These liabilities may represent employee wages, operational costs, or any other form of senior debt. We refer to the difference $z_i = \eta_i - v_i$ as the (liquidity) *shock* to bank i and assume that $\mathbf{z} = (z_i)_{i \in \mathcal{B}}$ is distributed independently across banks according to some probability distribution $\mathcal{Q}$, which we refer to as the economy's *risk profile*.

While there are potential gains from trade between entrepreneurs and depositors, the parties may not be able to trade with one another directly. We assume that each agent can only enter into pairwise contracts with a subset of other agents in the economy. Such intermediation frictions may arise due to transaction costs, agency problems, search frictions, or regulatory restrictions. We represent these trading frictions by an exogenously-given directed network $\mathbf{G}$ of potential lending opportunities, or *(opportunity) network*, with each vertex corresponding to an agent (bank, depositor, or entrepreneur) in the economy. A directed edge is present from agent i to agent j (denoted by $i \to j$) if i and j can enter into a bilateral contract, with i serving as a lender to j. Given the network of possible trading relationships $\mathbf{G}$, we define $\mathcal{N}_{in}(j) = \{i : i \to j \in \mathbf{G}\}$ and $\mathcal{N}_{out}(j) = \{k : j \to k \in \mathbf{G}\}$ as the sets of potential lenders and borrowers of j, respectively. We impose the natural assumption that $\mathcal{N}_{out}(i) = \varnothing$ for all entrepreneurs $i \in \mathcal{E}$ and $\mathcal{N}_{in}(0) = \varnothing$ for the representative depositor.

2.1 Timing and interbank contracts

At $t = 0$, each agent can offer take-it-or-leave-it fixed-interest rate lending contracts to its potential borrowers in network $\mathbf{G}$, whereby the lender commits to provide the borrower with as much funds as desired at the offered interest rate. These offers are made sequentially according to a pre-specified order, but can be withdrawn at the end of the period.

Formally, we assume that period $t = 0$ consists of $2n + 2$ sub-periods denoted by $\tau = 1, \ldots, 2n + 2$. In sub-period $\tau \leq n + 1$, agent $j = \mathcal{O}(\tau)$ has the option to make an offer with a constant interest rate $R_{j \to k}$ to any potential borrower $k \in \mathcal{N}_{out}(j)$, where $\mathcal{O} : \{1, \ldots, n+1\} \to \mathcal{B} \cup \{0\}$ is a mapping that specifies the order at which agents can make offers to one another at $t = 0$. The contract with face value $R_{j \to k}$ is a commitment by j to lend to k at the fixed interest rate $R_{j \to k}$. We use $R_{j \to k} = \varnothing$ to denote the scenario in which j refuses to make any offer to k. While lenders cannot revise the terms of the contracts they offer to their potential borrowers, we assume they can opt out of any contract in the second half of the period once all offers are made. More specifically, in sub-period $\tau > n + 1$, bank $j = \mathcal{O}(\tau - n - 1)$ can choose to withdraw any of the contracts $R_{j \to k}$ made to its potential borrowers $k \in \mathcal{N}_{out}(j)$, in which case, $R_{j \to k} = \varnothing$ (which take place in the same order $\mathcal{O}$ with which offers are made). Otherwise, bank j remains committed to lending to k at interest rate $R_{j \to k}$.[4]

Once the contracting stage at $t = 0$ is over, each agent can borrow as much as desired from its potential lenders at $t = 1$. The borrowing decisions are made sequentially according to a pre-specified order $\mathcal{L}$. More specifically, we assume that period $t = 1$ consists of $n + |\mathcal{E}|$ sub-periods and that, at sub-period τ, agent $j = \mathcal{L}(\tau)$ chooses to borrow $x_{i \to j}$ units of funds from each bank $i \in \mathcal{N}_{in}(j)$ at the pre-specified rate $R_{i \to j}$, provided that $R_{i \to j} \neq \varnothing$.

[4] This stage is introduced to rule out equilibria that may arise due to coordination failures: banks may refuse to extend credit to others if they worry that no bank will subsequently extend them a credit line with sufficiently favorable terms. The withdrawal stage in the model rules out the possibility of such miscoordinations. See Di Maggio and Tahbaz-Salehi [20] for a discussion.

Throughout, we assume that if the lender i cannot meet its commitments to deliver the funds to all its borrowers at $t = 1$, it faces a prohibitively large cost (imposed, say, by a regulator). This assumption therefore guarantees that, in any equilibrium,

$$\sum_{i \in \mathcal{N}_{in}(j)} x_{i \to j} \geq \sum_{k \in \mathcal{N}_{out}(j)} x_{j \to k}. \tag{1}$$

The final period, $t = 2$, corresponds to the time period at which the value of all outside investments are realized and all debts are due. More specifically, we assume that after the realization of $\mathbf{z}$, each bank j chooses an amount $y_{j \to i}$ to repay its obligation $R_{i \to j} x_{i \to j}$ to any lender i that it has borrowed from. To make these repayments, j may use funds generated from its net outside investments $z_j = \eta_j - \upsilon_j$ and its own receivable payments, $\sum_{k \in \mathcal{N}_{out}(j)} y_{k \to j}$. Throughout, we assume that j's failure to meet its $t = 2$ obligations results in two types of costs. First, any shortfall in j's payments to its creditors results in a costly liquidation process, which prevents j from paying anything to any of its creditors, that is, $y_{j \to i} = 0$ if $z_j + \sum_k y_{k \to j} < \sum_k R_{k \to j} x_{k \to j}$.[5] Second, we assume that if the borrower j defaults on its obligation to i, it faces an exogenous bankruptcy cost $F \geq 0$, which may correspond to reputational costs and legal fees associated with bankruptcy.

Taken together, the net profit of bank $j \in \mathcal{B}$ at the end of $t = 2$ is given by

$$\pi_j = \begin{cases} z_j + \sum_k y_{k \to j} - \sum_i R_{i \to j} x_{i \to j} & \text{if } z_j + \sum_k y_{k \to j} \geq \sum_i R_{i \to j} x_{i \to j} \\ -F & y_{j \to i} < R_{i \to j} x_{i \to j} \quad \text{for any } i \in \mathcal{N}_{in}(j) \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

with the convention that $x_{i \to j} = 0$ (and $R_{i \to j} x_{i \to j} = 0$) if $R_{i \to j} = \varnothing$.

2.2 Financial networks

The interest rate and borrowing decisions at $t = 0$ and $t = 1$ can be summarized by the pair $(\mathbf{R}, \mathbf{x})$, where $\mathbf{R}$ and $\mathbf{x}$ denote the vectors of interest rates and borrowing decisions of all agents, respectively. Throughout, and with some abuse of terminology, we refer to the tuple $(\mathbf{R}, \mathbf{x})$ as the economy's *financial network*. Note that while the underlying (opportunity) network $\mathbf{G}$ is assumed to be exogenous, the financial network $(\mathbf{R}, \mathbf{x})$ is an endogenous equilibrium object and depends on the lenders' offered contracts as well as the borrowers' borrowing decisions.

Any financial network $(\mathbf{R}, \mathbf{x})$ can alternatively be represented by a pair of directed, weighted subnetworks of $\mathbf{G}$, capturing the pairwise interest rates and quantities. More specifically, we define the *interest rate network* $\mathbf{R}$ by removing all potential lender-borrower pairs $i \to j$ from $\mathbf{G}$ such that $R_{i \to j} = \varnothing$. Hence, while $\mathbf{G}$ consists of all agents that can trade with one another at $t = 0$, the interest rate network defined by $\mathbf{R}$ consists of agents that can trade with each other at $t = 1$. Similarly, we define the *borrowing network* by removing all potential lender-borrower pairs $i \to j$ from $\mathbf{G}$ such that $x_{i \to j} = 0$. Thus, the borrowing network captures the set of agents that end up trading with one another at $t = 1$. Note that, by definition, the lending network is necessarily a subnetwork of the interest rate network.

[5] This assumption thus rules out the possibility of "fractional defaults" as in Eisenberg and Noe [23] and Acemoglu et al. [2], whereby banks may only default on a fraction of their obligations to their creditors.

We say that two financial networks are equivalent if (i) their corresponding borrowing networks coincide and (ii) the corresponding interest rate networks coincide wherever there is an edge in their (common) borrowing network. Put differently, $(\mathbf{R}, \mathbf{x})$ and $(\mathbf{R}', \mathbf{x}')$ are equivalent if $\mathbf{x} = \mathbf{x}'$ and $R_{i\to j} = R'_{i\to j}$ whenever $x_{i\to j} > 0$. Note that two financial networks are equivalent even if their interest rate networks differ, provided that these differences occur along edges of $\mathbf{G}$ where there is no borrowing.

2.3 Solution concept

We conclude this section by defining our solution concept. Recall that all interest rate offers are made at $t = 0$, the borrowing decisions are made at $t = 1$, and all repayments occur at $t = 2$. We therefore proceed by defining and characterizing the equilibrium recursively using backward induction.

We start by focusing on the economy at $t = 2$, when the financial network and hence all interest rates $\mathbf{R}$ and borrowing decisions $\mathbf{x}$ are already determined. Recall that each bank j is committed to repay $R_{i\to j}x_{i\to j}$ to each of its lender $i \in \mathcal{N}_{in}(j)$. The bank, however, may not be able to meet its obligations, in which case it defaults. More specifically, if $y_{k\to j}$ denotes the amount that j receives from its borrower k, then j defaults if $z_j + \sum_{k\in\mathcal{N}_{out}(j)} y_{k\to j} < \sum_{i\in\mathcal{N}_{in}(j)} R_{i\to j}x_{i\to j}$. Furthermore, recall that, by assumption, any shortfall in j's payments to its creditors results in a costly liquidation process that prevents j from paying anything to any of its creditors. Thus, the amount $y_{j\to i}$ that j is able to repay bank i satisfies

$$y_{j\to i} = \begin{cases} R_{i\to j}x_{i\to j} & \text{if } z_j + \sum\limits_k y_{k\to j} \geq \sum\limits_i R_{i\to j}x_{i\to j} \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where z_j denotes the shock to bank j. Since all repayments occur simultaneously, we can define the following concept:

Definition 1 Given financial network $(\mathbf{R}, \mathbf{x})$ and vector of realized shocks $\mathbf{z}$, a *repayment equilibrium* is a collection of interbank repayments $\boldsymbol{y} = (y_{j\to i})_{(j\to i)\in\mathbf{G}}$ that satisfies the system of Eq. (3) for all pairs of banks i and j.

With the above notion in hand, we can now proceed to the borrowing stage at $t = 1$, when the quantities are determined.

Definition 2 Given vector of interbank interest rates $\mathbf{R}$, a *borrowing equilibrium* is a collection of interbank borrowing decisions $\mathbf{x}$ and repayments $\boldsymbol{y}(\mathbf{R}, \mathbf{x}, \mathbf{z})$ such that

(i) $\boldsymbol{y}(\mathbf{R}, \mathbf{x}, \mathbf{z})$ is a repayment equilibrium for financial network $(\mathbf{R}, \mathbf{x})$ and shock realization $\mathbf{z}$;
(ii) each bank j makes its borrowing decisions $(x_{i\to j})_{i\in\mathcal{N}_{in}(j)}$ to maximize its expected profits in (2).[6]

Borrowing equilibria have two important properties. First, in any borrowing equilibrium, banks borrow exactly as much as they lend out, that is, inequality (1) holds as an equality for all banks i. This is consequence of the fact that both underborrowing and overborrowing are

[6] This statement assumes that, given financial network $(\mathbf{R}, \mathbf{x})$, the repayment equilibrium at $t = 2$ is unique for all realizations of $\mathbf{z}$. We show in the Appendix that, for all $\mathbf{z}$, the repayment equilibrium is indeed unique for any financial network emerging in equilibrium.

unprofitable: the former results in a cost imposed by the regulator, whereas the latter requires the bank to pay interest on funds that are not invested. Second, in any borrowing equilibrium, banks borrow this entire amount from lenders with the best terms (i.e., lowest interest rate) and split their demand amongst multiple lenders only if they offer the same exact interest rate.

We are now ready to define the economy's full equilibrium, which endogenizes the terms of the contracts at $t = 0$.

Definition 3 A (subgame perfect) *equilibrium* is a collection of interest rates $\mathbf{R}$, borrowing decisions $\mathbf{x}(\mathbf{R})$, and repayments $\boldsymbol{y}(\mathbf{R}, \mathbf{x}, \mathbf{z})$ such that

(i) $\boldsymbol{y}(\mathbf{R}, \mathbf{x}, \mathbf{z})$ is a repayment equilibrium at $t = 2$ given the financial network $(\mathbf{R}, \mathbf{x})$ and any $\mathbf{z}$;
(ii) the tuple $(\mathbf{x}, \boldsymbol{y})$ is a borrowing equilibrium at $t = 1$ given the interest rates $\mathbf{R}$;
(iii) each bank i chooses the interest rates $(R_{i\to j})_{j\in\mathcal{N}_{out}(i)}$ at $t = 0$ to maximize its expected profits.

According to the above definition, each agent chooses an optimal interest rate for every observable history in the sequential offering stage at $t = 0$, anticipating that the borrowing decisions and repayments will be determined via borrowing and repayment equilibria, respectively.

Unlike borrowing equilibria, the interest rate offers made in equilibrium can be quite complex. For instance, the interest rate offered by bank i to a potential borrower j depends not only on j's counterparty risk, but also on the default risk of j's potential borrowers, that of its borrowers' borrowers, and so on. Furthermore, the face value of the interest rates also depends on the nature of the competition induced by the network. Last but not least, there may be multiple subgame perfect equilibria, as banks could play weakly dominated strategies as a best response.

To rule out such economically uninteresting equilibria, we consider a refinement of our solution concept defined in Definition 3. This refinement, which we refer to as *strong equilibrium*, is a variant of agent-form trembling-hand perfect equilibrium, with the set of trembles restricted to thick-tailed distributions.[7] Importantly, our equilibrium notion implies that, at the sub-period with the option to make an offer, each bank makes arbitrarily small trembles around its equilibrium offer.[8] As we will show in the subsequent sections, this refinement ensures essential uniqueness of equilibrium in our game.

3 Equilibrium characterization

In this section, we first establish the existence of an equilibrium in our environment and show that the equilibrium financial network is generically unique. We then provide a characterization of financial networks that are formed in equilibrium. These results will serve as the basis of our comparative statics analyses in Sect. 4.

[7] This restriction is introduced in order to ensure that the best response of banks when offering interest rates to their potential borrowers converge to the equilibrium point in question as we take the limit of the trembles towards zero.

[8] See Appendix A for a formal definition of strong equilibrium and more details on its implications for equilibrium refinement. This concept is closely related to "trembling-hand perfect equilibrium" in extensive-form games.

3.1 Existence and uniqueness

We start with a general existence result.

Theorem 1 *Let* $\mathbf{G}$ *denote an arbitrary network.*

(a) *There exists a repayment equilibrium for any financial network* $(\mathbf{R}, \mathbf{x})$ *and any vector of shocks* $\mathbf{z}$*.*
(b) *There exists a borrowing equilibrium for any given vector of interest rates* $\mathbf{R}$*.*
(c) *There exists a strong equilibrium in pure strategies.*

While Theorem 1 guarantees the existence of a strong equilibrium for any $\mathbf{G}$, in general, the equilibrium may not be unique. For instance, for any equilibrium in which bank i does not make an offer to bank j, there are many other equilibria in which bank i makes an offer to bank j, but with a prohibitively large interest rate; in either case, j will not borrow from i. To rule out such economically uninteresting multiplicity, we define the following concept:

Definition 4 An equilibrium is *essentially unique* if the financial networks corresponding to all equilibria are equivalent.

Theorem 2 *For any network* $\mathbf{G}$ *and a generic probability distribution* $\mathcal{Q}(\mathbf{z})$*, there is an essentially unique strong equilibrium in pure strategies.*[9]

The above result thus establishes that, unlike many models of endogenous network formation, the equilibrium financial network in our environment is essentially unique. In addition to providing sharp predictions, this uniqueness result enables us to perform meaningful comparative statics on how changes in the network structure and the economy's risk profile impact pairwise interest rates, the extent of borrowing and lending, and defaults in the financial system.

We note that the essential uniqueness result in Theorem 2 only applies to strong equilibria, and indeed, there are often multiple subgame perfect equilibria: if interest rate trembles are ruled out, there may be multiple best-response offers in weakly-dominated strategies. For instance, if a bank anticipates that its contract will be undercut by a competing bank, it would be indifferent between not offering any contracts and offering a contract at or above the equilibrium interest rate of its competitor (including contracts that may be unprofitable). The resulting equilibrium rates and flow of funds in the financial network depend on how such banks break these indifferences, which is pinned-down only in a strong equilibrium.

As a final remark, we note that the restriction that the probability distribution $\mathcal{Q}$ is generic cannot be dispensed with. For instance, if two banks i and j with identical return distributions compete over the same potential borrower k, any division of k's borrowing decisions between i and j corresponds to a different equilibrium. The genericity restriction on $\mathcal{Q}$ rules out such knife-edge indifference situations that entail multiplicity of equilibria.

3.2 Equilibrium financial networks

With Theorems 1 and 2 in hand, we now proceed to characterize the financial networks that are formed in equilibrium.

[9] Because the standard Lebesgue measure is not well-defined over the space of continuous probability distributions, we use the notion of generic probability distribution from [39]. This notion is based on the use of "probes," such as polynomial functions of order k as approximations to smooth probability distributions. Generic properties are those that hold for almost all order k polynomials. See Appendix C for more details.

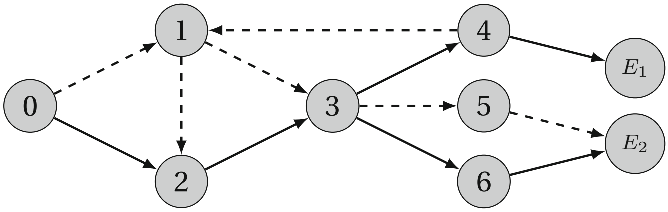


Fig. 1 This figure depicts the opportunity network $\mathbf{G}$. Vertices E_1 and E_2 repressent two entrepreneurs and vertex 0 represents to the depositor. Solid lines depict pairwise relationships with interest rate offers and positive borrowing in equilibrium; dashed lines indicate relationships in $\mathbf{G}$ with no interest rates offered in equilibrium

Theorem 3 *Given any network* $\mathbf{G}$ *and a generic probability distribution* $\mathcal{Q}(\mathbf{z})$*, any strong equilibrium is equivalent to a strong equilibrium* $(\mathbf{R}_*, \mathbf{x}_*)$ *such that*

(i) $\mathbf{R}_*$ *and* $\mathbf{x}_*$ *agree, in the sense that* $R^*_{i\to j} \neq \varnothing$ *if and only if* $x^*_{i\to j} > 0$ *for all pairs* i *and* j*;*
(ii) *the common network of* $\mathbf{R}_*$ *and* $\mathbf{x}_*$ *is a directed tree.*

Recall from Theorem 2 that, generically, all strong equilibria are equivalent to one another. Theorem 3 provides a characterization of this equivalence class: all strong equilibria are the same as an equilibrium in which the interest rates offered in equilibrium and all the borrowing occur along the same directed tree that connects the representative depositor to the entrepreneurs, as depicted in Fig. 1. This is the case irrespective of whether the underlying network $\mathbf{G}$ is a directed tree or not.

To see the intuition for Theorem 3, first observe that the equilibrium financial network cannot contain any directed cycles. Suppose to the contrary that there is strictly positive lending along a (directed) cycle over the financial network. If so, all equilibrium interest rates on such a cycle must be identical, as otherwise a bank would be lending at a rate that is strictly less than the rate it is borrowing at. On the other hand, because there is a positive probability of default for all banks, any bank that borrows and lends at the same exact interest rate is necessarily making negative expected profits. Thus, the interest rates offered in any equilibrium have to induce an acyclic subnetwork over $\mathbf{G}$.

The fact that the equilibrium financial network has a tree-like structure (and is therefore acyclic) then follows from the fact that banks tremble around their interest rate offers. These trembles guarantee that, no matter the structure of the underlying $\mathbf{G}$, there always exists exactly one "most competitive" lender for each bank, thus implying that the outcome that all banks except one withdraw their offers is always an equilibrium.

3.3 Robustness

Recall from Sect. 2 that agents make offers to and borrow from one another sequentially according to the exogenously-specified orders $\mathcal{O}$ and $\mathcal{L}$, respectively. Even though Theorem 2 establishes that the financial networks formed in the strong equilibria corresponding to a given pair of orders $(\mathcal{O}, \mathcal{L})$ all coincide, in general, the equilibrium financial network may depend on the sequence at which banks are able to take actions. Our next result establishes that, for networks with a single entrepreneur, even though this dependence may matter for equilibrium interest rates and the patterns of interbank lending, it does not impact whether the entrepreneur is eventually funded or not.

Theorem 4 *Consider a financial network with a single entrepreneur and let* $(\mathbf{R}, \mathbf{x})$ *and* $(\mathbf{R}', \mathbf{x}')$ *denote the financial networks in the essentially unique strong equilibria corresponding to order pairs* $(\mathcal{O}, \mathcal{L})$ *and* $(\mathcal{O}', \mathcal{L}')$ *, respectively. Then, the entrepreneur is funded in* $(\mathbf{R}, \mathbf{x})$ *if and only if it is funded in* $(\mathbf{R}', \mathbf{x}')$.

Recall that the entrepreneur has access to a bulky investment project, and so is either fully funded or not at all in equilibrium. Theorem 4 thus establishes that the realized gains from trade between the representative depositor and an entrepreneur, intermediated through the banking system, do not depend on the order at which various agents can make or accept offers. As a result, whether an entrepreneur is funded or not only depends on the nature of intermediation frictions (captured via the network $\mathbf{G}$) and the underlying distribution of shocks $\mathcal{Q}(\mathbf{z})$. However, the order at which various agents can take their actions may impact how gains from trade are distributed in the economy and which banks are more resilient to liquidity shocks at $t = 2$.

4 Credit freezes in single-entrepreneur economies

Having established the basic equilibrium properties, we now turn to investigating how the interaction between the financial network architecture and distribution of shocks $\mathcal{Q}$ determines the possibility of a credit freeze in the financial network, formally defined as follows:

Definition 5 Bank or entrepreneur j experiences a *credit freeze* in financial network $(\mathbf{R}, \mathbf{x})$ if all of j's potential lenders refuse to extend credit to j, i.e., if $R_{i\to j} = \varnothing$ for all $i \in \mathcal{N}_{in}(j)$. A credit freeze is *systemic* if all entrepreneurs experience a credit freeze.[10]

It is immediate that for any bank j experiencing a credit freeze, equilibrium borrowing satisfies $x_{i\to j} = x_{j\to k} = 0$ for all $i \in \mathcal{N}_{in}(j)$ and all $k \in \mathcal{N}_{out}(j)$. Therefore, such a bank j would be frozen out of the borrowing network entirely, despite the possibility that there may be positive gains from trade.

4.1 Network architecture

We start our analysis by providing comparative static results on how the economy's network can shape the likelihood of credit freezes. To simplify the analysis, we restrict our attention to networks with a single entrepreneur. While real-world financial networks are significantly more complex, focusing on such networks enables us to demonstrate banks' ex ante incentives to borrow to and lend from one another as well as the ex post consequences of such decisions in the most transparent manner. We consider economies with multiple entrepreneurs in Sect. 5.

As a first observation, note that Theorem 3(b) implies that in an economy with a single entrepreneur, as long as there is no systemic credit freeze, the common network of $\mathbf{R}_*$ and $\mathbf{x}_*$ is necessarily in the form of a directed chain network from the depositor to the entrepreneur. Our next result then establishes when such an economy experiences a credit freeze.

Proposition 1 *Let* $\mathbf{G}$ *contain a single entrepreneur. The entrepreneur experiences a credit freeze if and only if it experiences a credit freeze for all chain subnetworks* $\mathbf{H} \subset \mathbf{G}$.

10 Throughout we refer to credit freezes in order to emphasize that following a change in the distribution of shocks $\mathcal{Q}$, the decision not to lend by some banks leads to stoppages in credit flows.

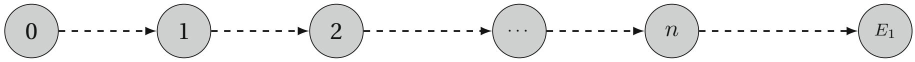


Fig. 2 Chain network with n banks

The importance of Proposition 1 is twofold. First, it establishes that, to determine whether the economy's single entrepreneur experiences a credit freeze, it is sufficient to restrict attention to the chain subnetworks that connect the depositor to the entrepreneur in $\mathbf{G}$—as depicted in Fig. 2. Second, it also implies that addition of new financial intermediation opportunities (in the form of new edges in the network $\mathbf{G}$) reduces the likelihood that the entrepreneur experiences a credit freeze, as such a change can only increase the number of chain subnetworks through which credit can flow. The following corollary formalizes this observation:[11]

Corollary 1 *Let $\underline{\mathbf{G}} \subseteq \bar{\mathbf{G}}$ denote two networks, each consisting of a single entrepreneur. If the entrepreneur experiences a credit freeze in $\bar{\mathbf{G}}$, then it also experiences a credit freeze in $\underline{\mathbf{G}}$.*

In view of Proposition 1 and Corollary 1, we next turn our attention to chain networks similar to the one depicted in Fig. 2.[12] It is easy to see that, in any such economy, all credit freezes are systemic, in the sense that either the banking system functions as normal and the depositor (indirectly) funds the entrepreneur, or all banks refuse to extend credit to their respective borrowers.[13]

To express our next result, we say a risk profile $\mathcal{Q}$ is *symmetric* if $\mathcal{Q}(z_i) = \mathcal{Q}(z_j)$ for all pairs of banks i and j. We have the following result:

Theorem 5 *Let $\mathbf{G}$ be a chain network. Then, for generic set of risk profiles $\mathcal{Q}$,*

(a) *there exists $\bar{r}_0 < r^*$ such that the economy experiences a systemic freeze if and only if $r_0 > \bar{r}_0$;*
(b) *there exists $\underline{r}^* > r_0$ such that the economy experiences a systemic freeze if and only if $r^* < \underline{r}^*$;*
(c) *furthermore, if $\mathcal{Q}$ is symmetric, there exists $\bar{n}$ such that the economy experiences a systemic freeze if and only if $n \geq \bar{n}$.*

Taken together, the three parts of Theorem 5 indicate that, even when there are positive gains from trade, the financial system may not be able to allocate depositors' excess funds to the entrepreneurs if there are significant "intermediation frictions". In the context of the financial network in Fig. 2, intermediation frictions are captured by a long credit chain: adding one more bank to the credit chain implies that there needs to be one more bank to intermediate funds between the depositor and the borrower. Since each bank in the credit chain must be compensated for the risk of default in the system—and this compensation needs to take place via a gap between their borrowing and lending rates—a long enough credit chain exhausts

[11] This result only holds for economies with a single entrepreneur. As we show in Sect. 5, the impact of increased competition on lending is ambiguous when there are multiple entrepreneurs in the network.

[12] Note, however, that the chain subnetwork along which lending takes place is endogenously determined, as it depends on the structure of $\mathbf{G}$ and the shock distribution $\mathcal{Q}$. Hence, limiting attention to arbitrary chain networks is not without loss of generality.

[13] Formally, there is always a strong equilibrium where either (i) \$1 flows from the depositor to the entrepreneur or (ii) there is a systemic credit freeze. However, there may be other equivalent strong equilibria, for instance, where bank 1 offers a prohibitively large interest rate to bank 2, but with no flow of funds anywhere in the chain.

the gains from trade between the depositor and the entrepreneur.[14] Note that this is true even when all banks are almost perfectly safe. The following example clarifies the working of this mechanism.

Example 1 Let **G** denote the n-bank chain with representative depositor 0, banks $\{1, \ldots, n\}$, and a single entrepreneur, as in Fig. 2. Assume every bank is subject to i.i.d. shocks $z_i \in \{-M, \zeta\}$, where M is some large positive constant, $\zeta \in (0, 1)$, and return $z_i = \zeta$ occurs with probability $1 - \varepsilon$, for some small $\varepsilon > 0$.[15] Thus, with a high probability, the bank has a moderate and positive return, but with some small probability ε, the shock wipes out the bank. To simplify the analysis, we set the default cost F to 0.

Given the simple structure of the chain, we can solve for equilibrium interest rates recursively. First, observe that if bank n lends to the entrepreneur, it demands an interest rate $R_{n\to E} = r^*$, where r^* is the rate of return on the entrepreneur's project. Also note that if bank $n-1$ lends to bank n, it also charges r^*, i.e., $R_{n-1\to n} = r^*$. This is because if $n-1$ charges an interest rate above r^*, bank n prefers not to engage in interbank lending at all. Next, consider the problem of bank $n-i-1$ lending to bank $n-i$. The former does not receive a repayment from the latter if any of the banks indexed $n-i+1$ through n have a bad return, an event that occurs with probability $1-(1-\varepsilon)^i$. Therefore, bank $n-i-1$ lends to bank $n-i$ if and only if

$$(1-\varepsilon)^{i+2}(\zeta + R_{n-i-1\to n-i} - R_{n-i-2\to n-i-1}) \geq (1-\varepsilon)\zeta.$$

The left-hand side of the above equation is the expected profit of $n-i-1$ of lending to $n-i$, whereas the right-hand side is equal to the bank's expected profit if it does not engage in interbank lending and borrowing. Consequently, the equilibrium interest rates satisfy the recursion

$$R_{n-i-1\to n-i} = R_{n-i\to n-i+1} - \frac{1-(1-\varepsilon)^i}{(1-\varepsilon)^i}\zeta,$$

which, coupled with the initial condition $R_{n-1\to n} = r^*$, leads to the following closed-form expression for equilibrium interest rates

$$\begin{aligned} R_{n-i-1\to n-i} &= r^* + \zeta i - \frac{\zeta(1-(1-\varepsilon)^i)}{\varepsilon(1-\varepsilon)^{i-1}} \\ &= r^* - i(i-1)\zeta\varepsilon/2 + o(\varepsilon). \end{aligned} \tag{4}$$

Equation (4) illustrates that, for small values of ε, the interest rate markups needed to support interbank lending grow quadratically in the length of the chain. This is because counterparty risk intensifies with the length of the chain due to fears of downstream defaults. Therefore, holding the aggregate gains from trade $r^* - r_0$ fixed, a credit freeze arises for any interbank lending chain exceeding length $\bar{n} = \sqrt{2(r^* - r_0)/(\zeta\varepsilon)}$.

We conclude this discussion by noting that, while the breakdown of intermediation in long chains predicted by Theorem 5 is similar to the results of Di Maggio and Tahbaz-Salehi [20], the freezes in the two models are driven by fundamentally different forces. As we argued already, in our model, banks' refusal to extend credit lines to potential borrowers is

[14] Notice the contrast with Corollary 1: while the corollary considers the addition of a link to a network (with a given set of banks), this theorem considers adding a new bank to a chain network (which thus removes a link and adds two new links to the new bank).

[15] We restrict ζ to be in (0, 1) so that no bank can fully absorb a counterparty loss. This assumption guarantees that any default cascade that begins at some agent j propagates upstream to all its direct and indirect lenders.

driven by counterparty risk and the fear of defaults by their direct or indirect borrowers. In contrast, credit freezes in Di Maggio and Tahbaz-Salehi [20] are due to the build up of moral hazard over intermediation chains: if intermediation chains are long enough, the volume or distribution of collateralizable assets may not be sufficient to counteract the agency problems.

4.2 Risk profile

In our next set of results, we study how changes in the economy's risk profile—that is, distribution $\mathcal{Q}$ of shocks $\mathbf{z}$—shapes the likelihood and nature of credit freezes.

Definition 6 Risk profile $\mathcal{Q}'$ *stochastically dominates* $\mathcal{Q}$ if $\mathcal{Q}_i'(z_i)$ first-order stochastically dominates $\mathcal{Q}_i(z_i)$ for all i. If, in addition, $\mathcal{Q}_i'(z_i)$ strictly dominates $\mathcal{Q}_i(z_i)$, we say bank i *experiences an adverse shift in the distribution of shocks* (or adverse shift for short) in response to a change from $\mathcal{Q}'$ to $\mathcal{Q}$.

The notion of stochastic dominance defined above is weaker than the more restrictive notion of *statewise dominance*, according to which, for any realized state of the world at $t = 2$, the liquidity shocks under $\mathcal{Q}'$ are always more favorable than those under $\mathcal{Q}$ for all banks. Furthermore, note that following an adverse shift in the distribution of shocks, no bank has more liquidity under $\mathcal{Q}$ than under $\mathcal{Q}'$ (in the sense of first-order stochastic dominance) and every bank subject to an adverse shift has strictly less liquidity in some states of the world at $t = 2$.

Proposition 2 *Let* $\mathbf{G}$ *be a chain network with risk profile* $\mathcal{Q}$. *If there is no systemic freeze, then there exists* $\bar{F} > 0$ *such that for all* $F > \bar{F}$, *whenever* $\mathcal{Q}'$ *stochastically dominates* $\mathcal{Q}$, *there is no systemic freeze under* $\mathcal{Q}'$.

Proposition 2 captures the intuitive result that systemic credit freezes are tightly linked to the risk faced by the banks: a deterioration in the banks' returns (in the sense of Definition 6) can result in more systemic freezes.

We remark that the requirement of a large default cost F in Proposition 2 cannot be dispensed with. On the one hand, a shift in the distribution of shocks towards a dominated distribution decreases the profitability of bank i (holding the contracts and the borrowing decisions constant), which makes bank i more likely to default. This, in turn, decreases the profitability of the loans made by i's direct and indirect lenders, making lending on the whole less attractive. On the other hand, however, in the response to such a shift in the risk profile, bank i's risk attitudes also change: bank i becomes less averse to potentially risky interbank lending. Although its direct and indirect borrowers are now more likely to default, the limited liability constraint leads to an increase in i's risk appetite. When the bank faces a large default cost F, the first effect dominates the second.

Definition 7 Risk profile $\mathcal{Q}'$ *has more tail risks* than risk profile $\mathcal{Q}$ if

(i) $\mathcal{Q}_i(z_i) - \mathcal{Q}_i'(z_i)$ is constant over $z_i \in [-r^*, r^*]$;
(ii) $\mathcal{Q}_i(z_i)$ single-crosses $\mathcal{Q}_i'(z_i)$ at some $\lambda_i \geq r^*$;[16]

for all i.

The first part of the definition ensures that the two distributions are similar "in the middle". The second part imposes that all corresponding marginal distributions single-cross at $\lambda_i \geq r^*$

[16] See Chateauneuf et al. [18].

and thus guarantees that, while the likelihood of being at or below the single-crossing point λ_i is the same for both $\mathcal{Q}_i(z_i)$ and $\mathcal{Q}'(z_i)$, the liquidity shocks are more likely to take extreme values under $\mathcal{Q}'$ than under $\mathcal{Q}$. Furthermore, the requirement that the single-crossing points λ_i are sufficiently positive guarantees that an increase in the tail risk in sense of Definition 7 does not increase bank i's likelihood of survival, even in the event of downstream default. We have the following result:

Proposition 3 *Let* $\mathbf{G}$ *be a chain network and suppose* $\mathcal{Q}'$ *has more tail risks than* $\mathcal{Q}$*. If there is a systemic freeze under* $\mathcal{Q}$*, then there is a systemic freeze under* $\mathcal{Q}'$*.*

The intuition underlying this result is straightforward. Limited liability implies that an increase in the tail risk of a bank's investment (in the sense of Definition 7) (i) increases the bank's upside risk conditional on survival, (ii) raises the likelihood of default, but (iii) has only a small impact on its expected losses in case of default. As a result, an increase in the bank's tail risk makes lending to this bank less attractive, while also increasing the likelihood of default cascades to its direct and indirect lenders. Proposition 3 therefore suggests that any change in market conditions that raises tail risks—such as greater volatility in the values of the assets held by the banks—will increase the likelihood of credit freezes.[17]

So far we have assumed that liquidity shocks across banks are independent. Next, we study how the nature of credit freezes depends on the correlation across banks' liquidity shocks. To simplify the analysis, we assume that liquidity shocks $\mathbf{z} = (z_1, \ldots, z_n)$ are jointly normally distributed with common mean $\mathbb{E}[z_i] = \mu > 0$, common variance $\text{var}(z_i) = \sigma^2$, and pairwise correlations $\rho > -1/(n-1)$. By Proposition 3, credit freezes become more likely as σ increases. This is a consequence of the fact that the probability of a tail event that leads to a default is growing in σ. Our next result relates the likelihood of a credit freeze to the correlation parameter ρ.

Proposition 4 *Suppose that banks' liquidity shocks are jointly normally distributed. Then, there exists* $\underline{F} > 0$ *and* $\bar{\rho} < 1$ *such that there is no credit freeze if* $\rho > \bar{\rho}$ *and* $F < \underline{F}$*.*

The above result is related to the risk-stacking mechanism of Elliot et al. [24] and Jackson and Pernoud [33]. When banks' liquidity shocks are highly correlated, interbank lending is less risky: all banks fail in the same states of the world, irrespective of whether they enter into interbank lending contracts or not. Default cascades are therefore immaterial in the sense that they do not pose any extra risk on the banks. Banks will then be willing to extend lending to their potential borrowers. Conversely, as the asset returns become less correlated (or negatively correlated), interbank loans become less profitable, as this increases the likelihood of a default contagion in the states of the world where bank i's returns are positive. Consequently, a sufficient reduction in correlation ρ results in a credit freeze.[18]

We conclude this discussion by going beyond the chain network structure and considering credit freezes in the more general class of economies with a single entrepreneur. To this end, we focus on adverse shifts to a specific subset of banks, which enables us to isolate credit freezes arising from network effects from those driven entirely by immediate counterparty concerns.

[17] These observations also imply that if we allow banks to choose the riskiness of their outside investments, limited liability may push them towards riskier assets, but with significant negative systemic implications.

[18] A high bankruptcy cost F encourages banks to diversify in order to avoid costly default. Our assumption that $F < \underline{F}$ ensures that the lack of diversification as shocks become more correlated does not dominate the increase in expected profits from making the loans.

Proposition 5 *Suppose the economy consists of a single entrepreneur and consider adverse shifts to a subset of banks $\mathcal{R} \subseteq \mathcal{B}$. Then, there exists $\bar{F}$ such that for all $F > \bar{F}$ credit freezes are monotone in the sense that*

(a) *if all banks $j \in \mathcal{R}$ experienced a credit freeze before the adverse shift, all banks in $\mathcal{R}$ continue to experience a credit freeze;*
(b) *total lending to the entrepreneur never increases.*

Statement (a) of the above proposition establishes the intuitive result that a deterioration in a bank's distribution of liquidity shocks cannot result in access to new credit: the bank's potential creditors can only face higher risks and hence will be less likely to extend it a credit line. Statement (b) of Proposition 5 then illustrates that the consequences of such deterioration may propagate further downstream in the credit chain and potentially lead to a credit freeze for the entrepreneur. This result is a consequence of the fact that lending in a single-entrepreneur economy is always in the form of a single intermediation chain from the depositor to the entrepreneur, irrespective of the structure of the network (Theorem 3). As a result, adverse shifts in the distribution of shocks in the sense of Definition 6 can only divert incentives away from lending along this path of the financial network.[19]

5 Credit freezes with multiple entrepreneurs

In Sect. 4, we focused on economies with a single entrepreneur and showed that credit freezes are systemic (Proposition 1) and monotone (Proposition 5). In this section, we show that in economies with multiple entrepreneurs, credit freezes may take more complex forms. In particular, we show that, in the presence of multiple entrepreneurs, credit freezes are not necessarily systemic (in the sense that only some part of the financial network may come to a standstill), they may occur in the part of the network not affected by adverse shifts, and that the response to an adverse shift may be non-monotone. We establish these results by means of a series of examples.

5.1 Simple freezes

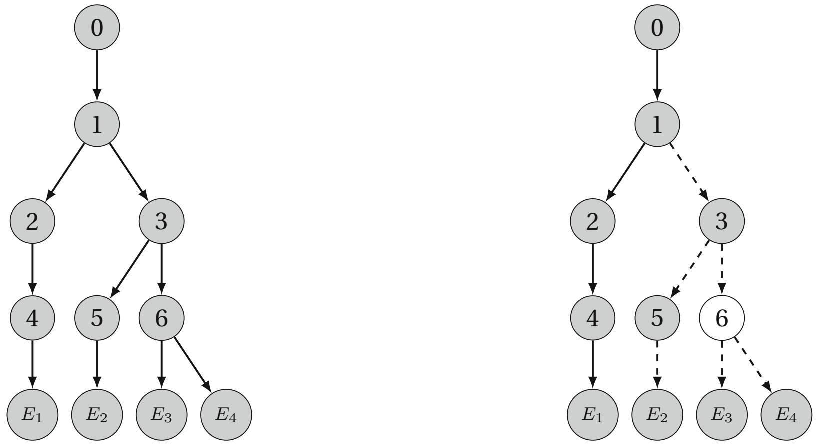
We first focus on networks $\mathbf{G}$ in the form of directed trees by assuming that every bank has exactly one potential lender, though it may have multiple potential borrowers.[20] This structural restriction shuts down any effect arising from competition between banks over lending contracts. We investigate the effect of competition in the next subsection.

Definition 8 Consider adverse shifts to a subset of banks $\mathcal{R} \subset \mathcal{B}$. We say any resulting freeze is *simple* if, for each bank $j \in \mathcal{R}$ there exists a bank $j^* \in \mathcal{B}$ such that:

(i) bank j is a direct or indirect borrower of j^*;
(ii) all banks experiencing a credit freeze after the adverse shift are also (direct or indirect) borrowers of j^*.

[19] Note that, in this proposition, we assume large values of F to control for risk attitudes, as in Proposition 2 (see the discussion in Sect. 4.2).

[20] Recall from Theorem 3 that while, in equilibrium, interbank borrowing and lending always occurs in a tree structure, the opportunity network $\mathbf{G}$ need not be a tree. We now separately consider the implications of a tree opportunity network.



(a) before the distributional shock **(b)** after the distributional shock

Fig. 3 The financial network before (**a**) and after (**b**) an adverse distributional shock to bank 6. Solid lines depict relationships in interbank lending in equilibrium, whereas dashed lines represent relationships in **G** (i.e., opportunities) which are not used in equilibrium (i.e., credit freezes)

In the context of the directed tree networks we consider in this subsection, a simple freeze corresponds to a scenario in which all banks belong to the same subtree of the network. From Proposition 1, since all credit freezes with a single entrepreneur are systemic, they are also simple.

Proposition 6 *If* **G** *is a directed tree, adverse shifts in the distribution of shocks induce only simple freezes.*

To illustrate the nature of credit freezes and how they may propagate in directed trees, we next provide two examples. Our first example illustrates how an adverse shift to bank i can cause credit freezes to initiate at bank i, and then propagate upstream and downstream to its potential lenders and borrowers, leading to a credit freeze in an entire subtree of the network. Our second example shows why, even though freezes are simple in tree networks, an adverse shift may lead to an increase in total lending (regardless of F), an outcome that is impossible in single-entrepreneur economies (Proposition 5).

Example 2 (Propagation of simple freezes) Consider the network in Fig. 3a and suppose the parameters are such that all banks lend to their designated borrowers in equilibrium. Next, consider an adverse shift to bank 6 that increases the bank's likelihood of default. A sufficient increase in bank 6's default likelihood would make it unprofitable for bank 3 to lend to bank 6, thus resulting in a credit freeze for entrepreneurs E_3 and E_4.

But note that the adverse shift to bank 6 may also result in a credit freeze for entrepreneur E_2, as depicted in Fig. 3b, even though there is no direct or indirect lending relationship between bank 6 and E_2. To see this possibility, note that before the shift in the shock distribution, bank 3 (indirectly) funded the three entrepreneurs and all these loans were profitable. After the adverse shift, however, the only profitable lending available to bank 3 would be to fund E_2 via bank 5. However, this reduction in bank 3's profitability reduces bank 1's incentive to lend to bank 3: bank 3 is less profitable, while facing the same or even perhaps higher default risk. This may make the loan to bank 3 unprofitable at any interest rate,

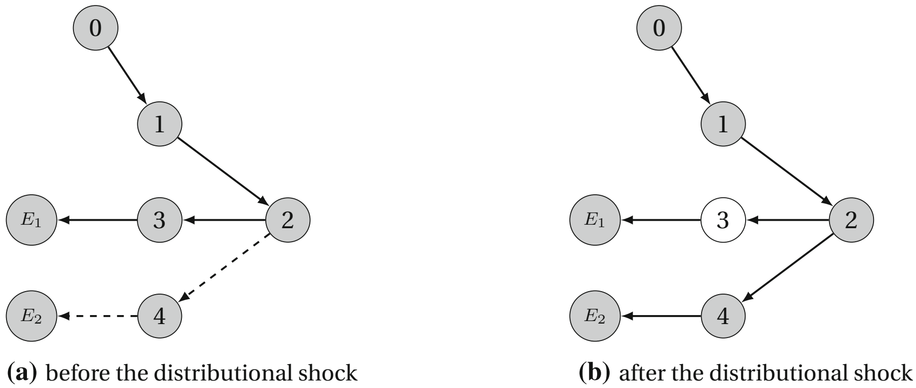


Fig. 4 The financial network before (**a**) and after (**b**) a shock to bank 3. Solid lines depict relationships in interbank lending in equilibrium, while dashed lines represent relationships in **G** that are not used in equilibrium (i.e., credit freezes)

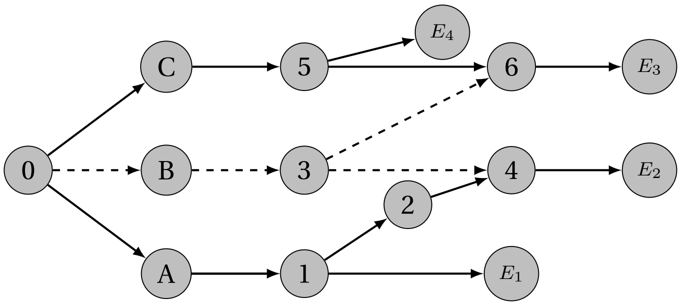
thereby creating a freeze from bank 1 to bank 3. As bank 3 loses access to credit, bank 5 and entrepreneur E_2 also experience a credit freeze.

Example 3 (Non-monotone freeze in trees) Consider the economy depicted in Fig. 4. To directly contrast with Proposition 2, suppose that $F \to \infty$, so that default is very costly. Let the liquidity shock to each bank i be $z_i \in \{-\zeta_i, +\kappa\}$ for some $\kappa \gg 0$, where both outcomes are equally likely. Furthermore, let us assume that $\zeta_1 = \zeta_3 = 0$, and that $0 < \zeta_2 < \zeta_4 < 1$.

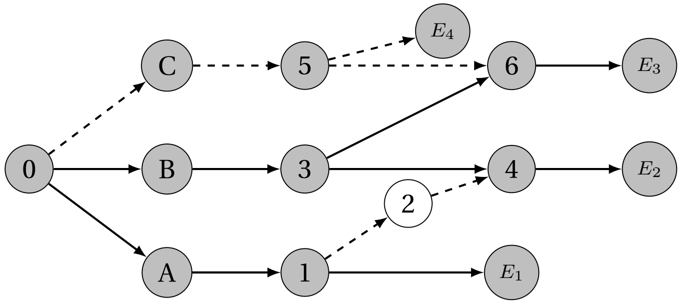
In such an economy, bank 1 faces a trade-off between the volume of the loan and the interest rate it can charge. On the one hand, if bank 1 charges $R_{1\to 2} = r^* - \zeta_2$, then bank 2 charges $R_{2\to 3} = r^*$ to bank 3 but does not offer a credit line to bank 4. On the other hand, if bank 1 charges $R_{1\to 2} = r^* - \zeta_4$, then bank 2 will still charge bank 3 an interest rate of $R_{2\to 3} = r^*$ but in addition offers a contract with interest rate $R_{2\to 4} = r^* - \zeta_4$ to bank 4, thus, effectively, doubling the loan amount from bank 1 to bank 2. Hence, if the gains from trade satisfy $(r^* - r_0) < 2\zeta_4 - \zeta_2$, then bank 1 will charge $R_{1\to 2} = r^* - \zeta_2$ and bank 2 only makes an offer to bank 3, with the resulting equilibrium financial network depicted in Fig. 4a.

Now suppose we introduce a shift in the distribution of shocks for bank 3 that increases the magnitude of the negative shock from $\zeta_3 = 0$ to $\zeta_3 = \zeta_4 - \zeta_2 > 0$. As long as $\zeta_4 < (r^* - r_0) < 2\zeta_4 - \zeta_2$, bank 1 will offer the contract $R_{1\to 2} = r^* - \zeta_4$ (or infinitesimally less) and bank 2 offers the contracts of $R_{2\to 3} = r^* - \zeta_3$ and $R_{2\to 4} = r^* - \zeta_4$ to banks 3 and 4, respectively. Because the lending path through bank 3 is more risky, bank 1 must charge a lower interest rate to bank 2 to support lending along any path, which now makes the larger loan volume more attractive. This results in the equilibrium financial network depicted in Fig. 4b.

To summarize, even though freezes in trees are simple, the propagation of adverse shifts is substantially richer than in networks with a single entrepreneur. First, credit freezes can spread both upstream and downstream in the network. Second, adverse shifts can increase, rather than reduce, lending, because they change the relative profitability of different banks in the network, potentially shifting funding towards banks that can then significantly expand their own lending.



(a) before the distributional shock



(b) after the distributional shock

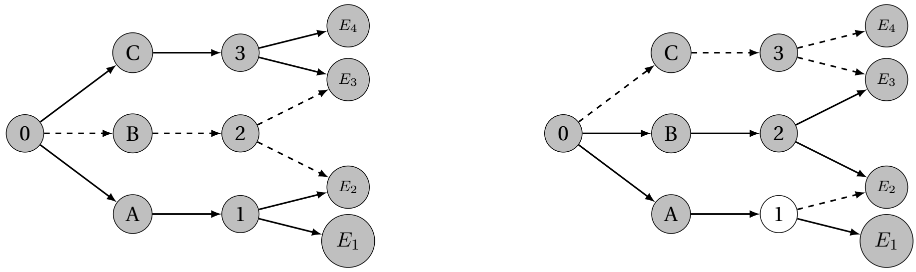
Fig. 5 The financial network before (**a**) and after (**b**) an adverse distributional shock to bank 2. Solid lines depict relationships in interbank lending in equilibrium, whereas dashed lines represent relationships in **G** that are not used in equilibrium (i.e., credit freezes)

5.2 Complex freezes

We now turn our attention to more general network structures and show that freezes can be *complex* in non-tree-like economies, in the sense that properties (i) or (ii) of Definition 8 may no longer be satisfied.

We illustrate such a possibility with three examples. First, we demonstrate how an adverse shift in the distribution of shocks in one part of the network can lead to a freeze in an entirely different segment. Second, we provide an example where a bank experiencing an adverse shift may not lose credit but can cause a freeze for other banks. And lastly, much like in Example 3, we show how an adverse shift in one part of the network can induce more lending somewhere else.

Example 4 (Freezes in multiple branches) Consider the economy depicted in Fig. 5. Similarly to Example 1, assume that $z_i = \zeta > 0$ with probability p_i and $z_i = -M$ with probability $1 - p_i$, where M is some large positive constant. We assume that $p_1 = p_2 = p_4 = p_6 = 1$, thus implying that the corresponding banks are always safe (conditional on no downstream defaults). Additionally, we assume that banks A, B, and C never default so that $R_{0\to A} = R_{0\to B} = R_{0\to C} = r_0$. Finally, we assume that $p_5 < p_3 < 1$, so that there is only a small probability banks 3 and 5 experience a bad liquidity shock, but bank 5 is riskier than bank 3.



(a) before the distributional shock **(b)** after the distributional shock

Fig. 6 The financial network before (**a**) and after (**b**) an adverse distributional shock to bank 1

Suppose the equilibrium financial network before the adverse shift is pictured in Fig. 5a. The chain from bank A to entrepreneur 2 poses no risk, so bank A would be willing to charge $r_0 + \delta$ for any $\delta > 0$ which will undercut bank B given the risk from bank 3. As long as bank 5 is not too risky, bank C has an incentive to lend to bank 5 and capture the profits from both entrepreneurs (E_3 and E_4), despite the competition from bank 3 who only offers a loan to E_3. As long as bank 3 presents some risk, it can be shown that bank C will be willing to undercut bank B's offer to bank 3 because the existence of an additional entrepreneur boosts profits.

Now suppose we introduce an adverse shift in the distribution of shocks to bank 2. For simplicity, suppose the asset it is holding is revealed as very toxic, so it is believed that $p_2 \approx 0$. The equilibrium lending network is now given by Fig. 5b. The reasoning is as follows. Bank 1 will never lend to bank 2 since it will almost certainly default. This implies that bank B no longer faces competition in its lending along the chain to E_2 via banks 3 and 4. Since the risk of bank 3 is lower than that of bank 5, and bank B has (indirect) monopolistic access to E_2, the loan is profitable enough that it can now compete with bank C over bank 6 (and indirectly, E_3) as well. Given that bank 5 will not be able to compete with bank 3 over bank 6, bank C may find the loan to bank 5 no longer profitable, resulting in a credit freeze for E_4.

In summary, unlike the simple freezes in Definition 8, an adverse shift for bank 2 results not only in credit freezes in the branch of the financial network that bank 2 belongs to (i.e., A), but also in a credit freeze in branch C, with E_4 losing access to funding as a result.

Example 5 (Freeze only in an unaffected branch) Consider the economy depicted in Fig. 6a, where the larger entrepreneur denotes a more profitable one (i.e., a larger r_i^*). Let us again use the setup of the previous example, with $z_i \in \{-M, \zeta\}$ and $z_i = \zeta$ with probability p_i , all independently distributed. Also suppose that banks A, B, and C never default (so that $R_{0\to A} = R_{0\to B} = R_{0\to C} = r_0$) and that bank 1 is perfectly safe conditional on no downstream defaults (i.e., $p_1 = 1$). Finally, suppose $p_3 < p_2 < 1$.

Because bank 2 is safer than bank 3, bank B may be able to undercut the interest rate bank C charges to bank 3, thereby making bank 2 more competitive over E_3. On the other hand, it cannot undercut the interest rate bank A charges bank 1, therefore bank B will provide at most a \$1 loan, which indirectly funds E_3. However, for sufficiently low p_2, it may be the case that such a loan is not profitable enough to warrant lending given the risk from bank 2. Therefore, bank B will freeze credit to bank 2. Because the branch funded by bank C has access to two entrepreneurs (one of which is monopolistic, E_4), lending is still profitable to E_3. All entrepreneurs receive access to funding, as pictured in Fig. 6a.

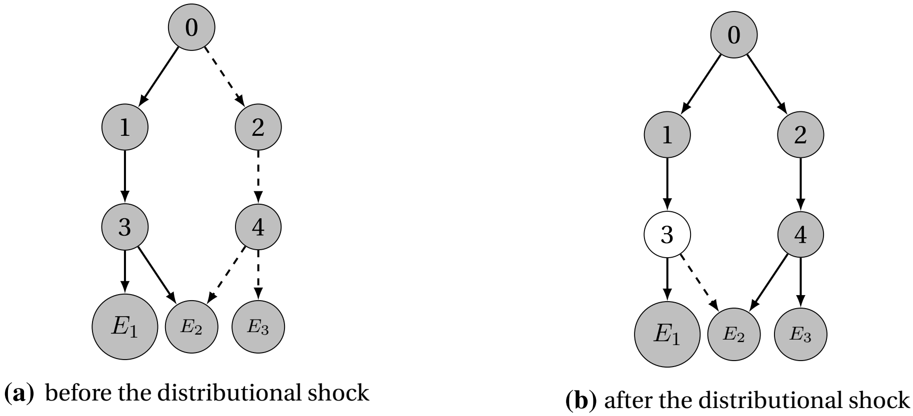


Fig. 7 The financial network before (**a**) and after (**b**) an adverse distributional shock to bank 3

Now suppose we introduce a small adverse shift in the distribution of shocks to bank 1, say, by increasing its probability of a bad return so that $p_1' < p_3 < p_2 < 1$. The branch funded by bank B may now be able to compete over E_2, which increases the profitability of the loan. Since bank 2 is also less risky than bank 3, this implies that it can compete over E_3, which may in turn induce a freeze from bank C. As a consequence, bank 3 (and E_4) lose access to credit, despite the fact that the adverse shift occurs at bank 1, who continues to have access to credit from bank A, as seen in Fig. 6b.

We observe that after an adverse shift to bank 1, bank A continues to lend to bank 1 and entrepreneur E_1 still obtains funding along the branch with bank A. However, entrepreneur E_4 and banks C and 3 lose access to credit, despite experiencing no change in riskiness along their branch.

Example 6 (Non-monotone freezes) This example illustrates another type of non-monotone freezes, whereby adverse shifts, by removing previous lending opportunities, actually improve overall lending. As such, it is also a bridge to our discussion in the next subsection on the relationship between competition and credit.

Consider the economy depicted in Fig. 7. As before, let $z_i \in \{-M, \zeta\}$ for some large positive constant M and $z_i = \zeta$ with probability p_i. Suppose banks 1 and 2 never default and that $p_4 < p_3 < 1$ initially. In equilibrium, bank 3 will be more competitive than bank 4: $R_{1\to3} \leq R_{2\to4}$. With only E_3, it may be unprofitable for bank 2 to lend to bank 4 altogether, so entrepreneur 3 is not funded. Now suppose bank 3 experiences and adverse shift, which allows bank 4 to be more competitive over E_2 because it will receive $R_{1\to3} \leq R_{2\to4}$. This may be sufficient to make the loan profitable (which now funds both E_2 and E_3). As long as the risk of bank 3 does not increase too much, E_1 will still be funded (because this loan is initially sufficiently profitable). Hence, the increase in risk to bank 3 increases the total amount of lending in the system, and in particular allows bank 4 to gain access to credit. In other words, introducing greater risk into the system may lead to a counterintuitive increase in lending.

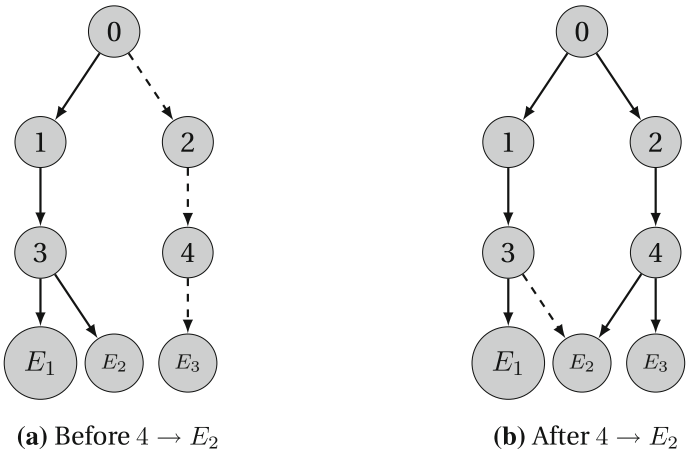


Fig. 8 The opportunity network before (**a**) and after (**b**) adding the opportunity $4 \to E_2$, under risk profile $\mathcal{Q}^*$

5.3 Competition and freezes

In this subsection, we study the consequences of increasing competition in the network. As demonstrated in Sect. 4, with a single entrepreneur, increasing the number of links within a given network of banks makes credit freezes less likely. Similarly, the same logic implies that in an economy with multiple entrepreneurs, reducing intermediation frictions can create less risky intermediation chains between a depositor and an entrepreneur, thereby alleviating credit freezes. However, there is also a counteracting effect from adding new links, which arises from competition. As a consequence of this latter effect, a reduction in the intermediation frictions between the depositor and entrepreneurs—in the form of adding additional lending opportunities—does not guarantee an increase in aggregate lending. The following example illustrates such a possibility.

Example 7 (Effects from competition) Consider the economy depicted in Fig. 8, where we define risk profile $\mathcal{Q}^*$ as the one where bank 4 is more likely to default than bank 3, and banks 1 and 2 never default. By adding a link from bank 4 to E_2, E_3 gains access to credit and no agent loses access to credit. The mechanism for this expansion of credit is different than the one we saw in Sect. 4, which was to shorten the chain along which credit travel to the entrepreneur. In contrast, here the new link to entrepreneur E_2 makes bank 4 more profitable and this then enables it to also fund entrepreneur E_3.

Conversely, Fig. 9 shows how competition can reduce lending. Now, under risk profile $\mathcal{Q}^{**}$ bank 3 is slightly more likely to default than bank 4, and banks 1 and 2 never default. Consequently, before the new link is added, bank 4 has monopolistic access over E_2 and E_3. However, after, bank 3 is able to undercut bank 4, which reduces the profits from the loans made by bank 4. This may induce bank 2 to freeze credit to bank 4, with E_3 losing credit. In this case, adding a new link made total lending decrease.

Therefore, even though adding new links only decreases the intermediation frictions between depositors and entrepreneurs, the impact on total lending is ambiguous because competition impacts the profitability of different banks' loans.

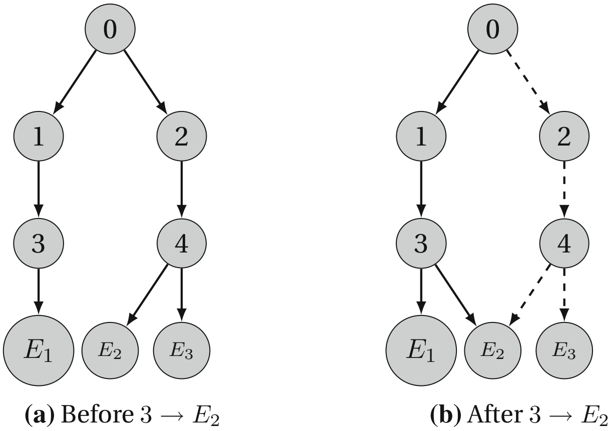


Fig. 9 The opportunity network before (**a**) and after (**b**) adding the opportunity $3 \to E_2$, under risk profile $\mathcal{Q}^{**}$

6 Extensions

We consider two short extensions of interest. First, we show that the fear of future liquidity problems can trigger additional financial intermediation in order to insulate small institutions from counterparty risk. This endogenous intermediation takes the form of a large institution bearing the risk from smaller banks' potential defaults. However, this redistribution of risk in the network can lead to a systemic credit freeze if the large bank's future solvency becomes questionable. Second, we show that more complex financial networks may emerge when banks are allowed to offer more complex lending contracts. However, our earlier qualitative results on credit freezes remain robust to such changes.

6.1 Risk-bearing capacity

Recall from Theorem 5c that, as long as the risk profile $\mathcal{Q}$ is symmetric, the likelihood of a systemic credit freeze increases in the length of the chain. This, however, is no longer true if the risk profile is not symmetric. To illustrate such a possibility, we define a *risk-bearing bank* as a bank that is always safe, regardless of whether its borrowers repay the loan. That is, bank i is risk-bearing if $z_i \geq r^*$ with high probability, so that bank i will almost never default.

Proposition 7 *Consider the chain network* $\mathbf{G}$ *with n banks and generate a new chain network* $\mathbf{G}'$ *via the* subdivision *of an arc* $i \to (i+1)$ *in* $\mathbf{G}$ *by adding a risk-bearing bank* j *between* i *and* $i+1$ *(i.e.,* $i \to j \to (i+1)$*). For sufficiently large* F,[21] *there is a credit freeze in* $\mathbf{G}'$ *only if there is a credit freeze in* $\mathbf{G}$.

Therefore, Proposition 7 offers an alternative perspective to Theorem 5, in that additional intermediation can reduce the likelihood of a credit freeze. This is a consequence of the fact

[21] We require sufficiently large F for the same reason as in Propositions 2 and 5: bank Ω with high risk-bearing capacity is more risk-averse to lending; this guarantees that there is no shift in risk attitudes by channeling funds through the additional intermediary.

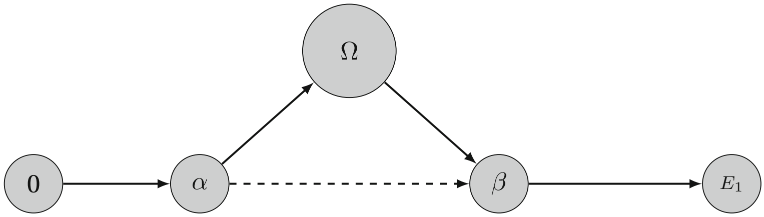


Fig. 10 Intermediation network

that risk-bearing banks act as firebreaks in the cascade of defaults. Channeling funds through such banks thus reduces systemic risk and allows banks to borrow and lend profitably at lower interest rates.[22]

For illustration, consider the network shown in Fig. 10 , where the initial network is a chain consisting of two banks (α and β). We now add a new bank Ω with high risk-bearing capacity, so that bank α, instead of providing funds directly to bank β, has to go through Ω. Though this lengthens the intermediation chain, Ω offers bank α protection against the potential default of β, making a systemic freeze less likely. Thus, bank Ω here plays a role akin to the role played by dealer banks during times of credit distress. As emphasized by Duffie [22]:

> Other dealer banks are increasingly being asked to enter derivatives trades, called 'novations,' that have the effect of inserting the other dealers between Beta and its original derivatives counterparties, insulating those counterparties from Beta's default risk.

Viewed through the prism of the above quote, Proposition 7 shows the possibility for novations, or channeling funds through an additional intermediary, can reduce the extent of credit freeze in the network. This can also provide an alternative rationale for the endogenous emergence of core-periphery structures in financial networks (Afonso et al. [5], Gofman [30], Bech and Atalay [11]).[23] However, as the number of novations from other banks (other than α) to bank Ω increases, or the solvency of bank Ω itself is called into question, its risk-bearing capacity may drop, potentially triggering a systemic freeze.

6.2 Quantity restrictions

We conclude this section by allowing banks to write contracts that not only specify the interest rate but also the maximum amount they are willing to lend to each borrower. Formally, instead of each bank i offering an interest rate $R_{i\to j}$ and allowing bank j to decide how much to borrow, bank i may also specify an upper bound $\bar{x}_{i\to j}$ for every potential borrower

22 In the context of Example 1, inserting a risk-bearing bank resets the compensating interest rate differential between the borrower and lender back to 0. Hence, if a fraction of the banks have risk-bearing capacity, then these differentials do not grow unboundedly as the chain gets longer.

23 Note that this effect is distinct from the one emphasized by Farboodi [27] . In Farboodi [27], core banks have higher-return but riskier projects, allowing peripheral banks to obtain intermediation rents using their own source of funds, which in turn creates inefficient levels of systemic risk. In our case, we obtain essentially the opposite result: voluntary intermediation comes from the fact that peripheral banks can insulate themselves and reduce potential default cascades by channeling funds through larger "safer" intermediaries who are unlikely to default.

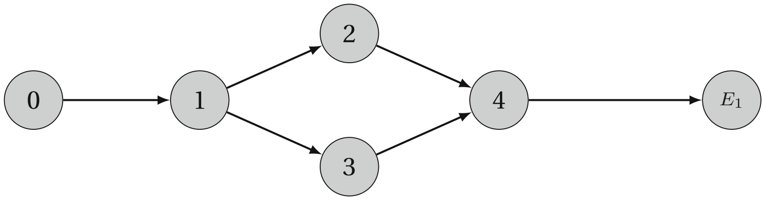


Fig. 11 Directed acyclic network

$j \in \mathcal{N}_{in}(i)$. At $t = 2$, each bank j may borrow as much as it desires up to the limit specified by the contract, i.e., $x_{i\to j} \leq \bar{x}_{i\to j}$.

Proposition 8 *Suppose banks may limit the amount of lending to each borrower. Then,*

(a) *the equilibrium financial network is directed acyclic (and $\mathbf{x}_*$ and $\mathbf{R}_*$ agree);*
(b) *there is systemic freeze in the economy with quantity-restricted contracts only if there is systemic freeze in the original economy, without quantity-restricted contracts.*

Proposition 8(a) shows that under quantity-restricted contracts, the equilibrium financial network is directed acyclic. This contrasts with our baseline framework, where the equilibrium financial network is generically a tree. This is because, in general, one of the paths to an entrepreneur has a lower cost than all other paths, ruling out the possibility that two banks are simultaneously supplying credit to the entrepreneur or to a bank supplying credit to the entrepreneur, and so on. Even though the form of the equilibrium financial network is more general under quantity restrictions, the major properties of credit freezes, including examples of non-monotonicity and complex freeze, still hold with contracts of this form.

Example 8 (Directed acyclic networks with quantity-restrictions) Consider the network in Fig. 11. Suppose bank 1 is perfectly safe (i.e., $z_1 = \kappa \gg 0$), banks 2 and 3 have $z_2 = \zeta_2 < 1$ and $z_3 = \zeta_3 < 1$ (but with $\zeta_2 + \zeta_3 > 1$), and bank 4 has $z_4 \in \{-M, \kappa\}$ with equal probabilities, where M is some large positive constant and $\kappa \gg 0$. Then, bank 4 defaults with probability 1/2, and bank 2 (resp. bank 3) defaults if bank 4 defaults when $R_{1\to 2}x_{1\to 2} > \zeta_2$ (resp. $R_{1\to 3}x_{1\to 3} > \zeta_3$). By setting $\bar{x}_{1\to 2} < \zeta_2$ and $\bar{x}_{1\to 3} < \zeta_3$, bank 1 ensures that it gets repaid with probability 1, which is strictly more profitable than lending to only bank 2 or bank 3, who would then repay with probability only $1/2$. However, when no quantity restrictions are in place, bank 4 borrows from the bank offering a lower interest rate, thus forcing either bank 2 or bank 3 to borrow the entire amount, which in turn generates repayment risk to bank 1. This argument thus clarifies that the equilibrium financial network is not a directed tree, but instead given by Fig. 11.

7 Policy responses

By definition, a credit freeze occurs when banks, despite the presence of gains from trade, refuse to extend credit to their corresponding borrowers. As a result, credit freezes are in general inefficient. In this section, we investigate potential policy responses by a regulator aimed at reducing inefficiencies arising from freezes throughout the financial network.

As illustrated by our various results and examples in Sect. 5, the extent and nature of credit freezes can be quite complex. This makes characterizing the optimal policy response for a general economy quite challenging. Instead of providing a detailed characterization of the optimal policy, we take the approach of showing that as the financial system becomes more interconnected, the policymaker must implement ever more sophisticated policies to handle credit freezes. Our results thus indicate that knowledge of the underlying risks and the lending network is of critical importance when conducting policy.

We also demonstrate through analytical results and examples that naive policies can sometimes exacerbate the likelihood of systemic credit freezes. Furthermore, we show that implementing the wrong policy (e.g., one that treats a complex freeze as a simple freeze) can be worse than doing nothing. We additionally show that, generally, the optimal policy can be significantly cheaper than the overall amount of lending it restores, and even sometimes *costless*. This makes a strong case for central bank involvement in the event of systemic freezes.[24]

Our results follow the thread of Sects. 4 and 5 by considering optimal policies in networks with a single entrepreneur and then multiple entrepreneurs. Throughout, we assume that the regulator's main policy instrument is *liquidity injection*, either in the form of asset purchases or a discount window. In the context of our framework, we model such a liquidity injection policy by assuming that the regulator can provide additional liquidity to bank j through a higher z_j; that is, a positive shift in the distribution of shocks affecting bank j.

7.1 General findings

Consider a central bank with a budget $B > 0$ and suppose the space of available policy options is a vector of interventions $\boldsymbol{\epsilon} = \{\epsilon_0, \ldots, \epsilon_n\}$ such that $\sum_{i=0}^{n} \epsilon_i \leq B$. By intervening, the central bank introduces a shift in the distribution of shocks, where $z_i' = z_i + \epsilon_i$ for every bank i.[25] A positive ϵ_i can be interpreted as providing funds directly to bank i, either to be lent out or to insulate the bank from default, whereas $\epsilon_i < 0$ corresponds to a policy that absorbs liquidity at bank i (i.e., an asset sell-off).[26] Similarly, $\epsilon_0 > 0$ represents a cash injection at the depositor, who is then required to invest in the interbank market (and not the outside risk-free technology).[27] Given the set of feasible policies, we say the central bank implements an *untargeted policy* if it provides assistance to the economy only through the depositor (i.e., $\epsilon_0 = B$). Otherwise, we say it implements a *targeted policy*, providing assistance directly to some banks in the network.

Throughout, we assume the central bank's objective is to maximize the realized gains from trade, given by $\sum_{j \in \mathcal{E}^*}(r_j^* - r_0)$, where $\mathcal{E}^*$ is the set of entrepreneurs that are able to fund their projects.

Proposition 9 *An untargeted policy is optimal in networks with a single entrepreneur.*

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24 For discussions of optimal policies in models based on ex post contagion, see Bernard et al. [12] and Kanik [35].

25 For simplicity, we are modeling this policy intervention as a direct liquidity injection or transfer. It is equivalent to a subsidized loan from the central bank. In particular, if the bank has to repay the central bank an amount $r_i \epsilon_i$ (where r_i is the discount interest rate from the central bank) at time $t = 2$, provided that doing so does not put the bank in default, then all of our results apply identically.

26 While in reality asset purchases do not target a single bank, we think of such a policy as targeting the distressed assets composing this bank's balance sheet. For instance, the Fed's policy to purchase mortgage-backed securities (MBS) during the crisis was in-part designed to target large dealer banks whose balance sheets comprised of sizable MBS positions.

27 Providing the depositor with liquidity does not change her incentives for lending, so the central bank must condition these funds on their use for interbank lending. The policy is equivalent to one where the central bank acts as a "depositor" itself, and directly lends to banks connected to the depositor in **G** (but not others).

The intuition for this result is as follows. An untargeted policy that allocates all the funds to the depositor allows her to charge lower interest rates profitably, as these funds are provided at little opportunity cost. This in turn induces all other lenders (which are direct and indirect borrowers of the depositor) to also charge lower interest rates in equilibrium, facilitating lending throughout the chain. Because interest rates at $t = 0$ can be used as a tool to redistribute future liquidity at $t = 2$, such an untargeted policy can necessarily mimic a targeted policy, as interest rates adjust in equilibrium to account for the differences. However, the converse is not necessarily true: by providing funds further downstream, a bank cannot leverage the interest rate as an instrument to redistribute liquidity further upstream. This is because the interest rate payment of a downstream bank is conditional on its solvency at $t = 2$, whereas an adjustment of an upstream interest rate is equivalent to a cash transfer at $t = 0$. Therefore, with a single entrepreneur, an untargeted policy outperforms all targeted policies.

Our next result focuses on economies with multiple entrepreneurs and considers credit freezes that arise in response to adverse shifts for a single bank. While the policymaker only observes the realized equilibrium financial network and not the underlying opportunity network **G**, we identify an effective rescue policy as if the central banker knew which bank experienced an adverse shift, or what the underlying (opportunity) network **G** was.

Proposition 10 *Suppose that a financial network experiences an adverse shift of the form* $z_j = z_j - \delta$ *for a single bank that leads to a simple freeze.*[28] *Moreover, assume no bank linked to the depositor has a credit freeze. Then there exists a budget* B^* *and some bank* j^* *that is a (direct or indirect) lender to all banks with frozen credit, such that:*

(a) *A targeted policy which targets only (direct or indirect) borrowers of bank* j^* *can restore all lending without introducing any additional credit freezes.*
(b) *Any untargeted policy restoring lending requires some budget* $B^{**} > B^*$.

The above result thus establishes that, when freezes are simple, there is a very natural policy to restore full lending: the central bank should spend its entire budget on rescuing banks in distressed parts of the network. While this policy is not necessarily optimal, it nonetheless outperforms the untargeted policy and will not inadvertently lead to credit freeze elsewhere in the network.

One consequence of the above result is that, even in the event of a simple freeze, having network knowledge is crucial for conducting policy, though a policymaker may limit his or her scope to banks without access to credit.[29]

7.2 Other policy features

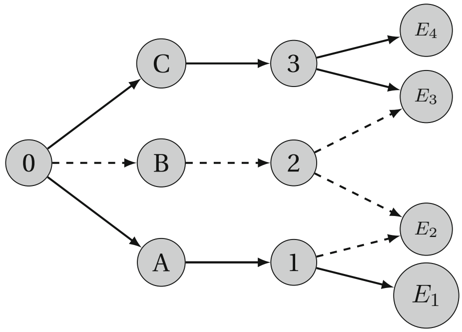
In the case of complex freezes, it may be impossible to relieve all credit freezes by using the class of policies in Proposition 10. We illustrate this insight below by revisiting Example 5.

Example 9 (Ineffective policy with complex freeze) Consider Fig. 6b from Example 5, which is the financial network after an adverse shift to bank 1. Recall that banks A, B, and C are

[28] This adverse shift corresponds to a leftward shift of the distribution function $\mathcal{Q}(z_j)$. The amount of the shift, δ, is the anticipated liquidity shock bank j now faces.

[29] This result is in the same spirit as Jackson and Pernoud [33], but relates to ex ante rescue policies (before the realization of liquidity shocks) to ensure lending markets continue to function when future solvencies are in-question.

Fig. 12 Example 9 after Policy



always safe, whereas banks 1, 2, and 3 receive a positive liquidity shock $z_i = \zeta$ with probabilities p_1, p_2, and p_3, where $p_1 < p_3 < p_2 < 1$. In this network, bank 3 and E_4 lose access to credit after an adverse shift to bank 1, despite the fact that bank 1 still has access to credit (i.e., Fig. 6b). The only policy of the form described in Proposition 10 is one that targets the distressed bank 3. When $\epsilon_3 > 0$ is small, the financial network remains as in Fig. 6b. When ϵ_3 is large, bank C becomes insulated from a negative shock to bank 3, which allows it to lend profitably to bank 3 at a lower interest rate, who will then also undercut bank 2 in lending to E_3 . However, such a policy may make the chain from bank B to bank 2 to E_2 unprofitable. Furthermore, given the added risk at bank 1, the chain from bank A abstains from lending to E_2 altogether, resulting in the financial network pictured in Fig. 12. Therefore, under any policy that targets bank 3, either E_2 or E_4 does not have access to credit. A better policy is to alleviate the risk at bank 1, which would obtain lending for *all* entrepreneurs, as pictured in Fig. 6a.

The above example illustrates that it may be necessary for the policymaker to intervene in counter-intuitive ways, for example by targeting a bank with access to credit than one without. Because central bank intervention can impact competition and the flow of profits from lending elsewhere in the network, these interventions can have non-trivial impacts on credit freezes across the entire system.

Another policy option is to directly lend to entrepreneurs to ensure that their projects are funded. In general, policy-makers may not have the know-how to identify high-quality entrepreneurs or may lack the ability to monitor their post-borrowing behavior. In addition, the same economic mechanism that makes optimal interventions sometimes take place away from the source of an adverse shift also implies that it may be more efficient to intervene in the financial network as opposed to lend directly to the entrepreneurs. The next example illustrates this point by showing that lending directly to entrepreneurs may be much more expensive than optimally targeting part of the financial network, as the latter strategy exploits the equilibrium responses of other banks following the intervention.

Example 10 (Direct lending to entrepreneurs) Consider the tree opportunity network depicted in Fig. 13, consisting of $m + 1$ banks and m entrepreneurs. We assume that shock to bank 1 is given by $z_1 = M \gg 0$ with probability 1, $z_2 \in \{-\zeta, 0\}$ where $z_2 = 0$ with probability p, while $z_i = 0$ for banks $i \in \{3, \dots, m + 2\}$. Also, let us assume $F = 0$, $pr^* < r_0$, and $m(r^* - r_0) < \zeta$. It is easy to verify that, absent any interventions, no entrepreneur receives access to credit. This is because bank 2 defaults if it is hit with a negative shock and bank 1 cannot be sufficiently compensated for this risk.

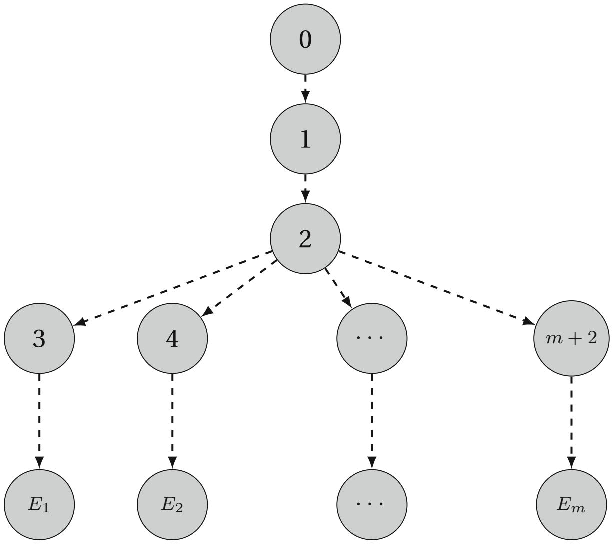
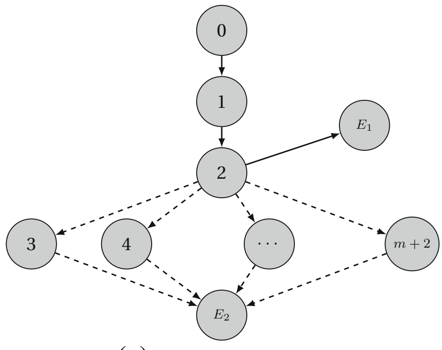


Fig. 13 Opportunity network for Example 10

Now consider a policy that provides funding directly to the entrepreneurs. The key observation is that the central bank must provide the entire loan of \$1 to an entrepreneur in order to ensure the entrepreneur can invest in the project. To see this, note that if the central bank lends less than \$1 to entrepreneur j, then the entrepreneur must raise the shortfall from bank $j+1$. Clearly, the total amount of the loans from each bank $j+1$ to its corresponding entrepreneur j is equal to the amount of loans from bank 2, which does not affect the probability that bank 2 will default on its repayment to bank 1, as bank 2 defaults as long as it is hit with a negative shock. But since $pr^* < r_0$, and $m(r^* - r_0) < \zeta$, bank 1 still does not find it profitable to lend to bank 2 (in any amount). Taken together, these observations imply that, to alleviate all credit freezes by targeting entrepreneurs directly, the central bank must spend a budget of m.

Instead, consider a policy where the central bank provides funding to bank 2 for ζ to be repaid at no interest (i.e., $r_2 = 1$). Then banks $1, \ldots, m+2$ are safe almost surely, and so all entrepreneurs have access to credit following this policy, given that $r^* - r_0 > 0$. Moreover, when $r^* - r_0 \ll 1, \zeta \ll m$, which implies that a policy targeting bank 2 does much better than one that provides the entrepreneurs with funding directly. This example therefore shows that an intervention in parts of the network with the bottleneck may be more cost-effective than directly lending to entrepreneurs because it encourages additional lending by other banks.

Finally, we end this subsection by comment on two (at first) counterintuitive aspects of policy. First, when there are non-monotone freezes, optimal policy can increase lending in a costless manner. Recall from Example 3 that removing liquidity from bank 3 actually *increased* the total amount of lending in the system because it created incentives for banks to lower interest rates and increase loan volume. Second, Example 6, also shows that policy interventions aimed at increasing lending can backfire and reduce overall financial intermediation. In particular, a policy that provides a positive shock to bank 3 prevents bank 4 from funding additional entrepreneurs due to competition effects. While stylized, these examples indicate that certain policies that decrease the likelihood of survival of certain banks (from, say, asset sales driving the price of assets down) can actually increase total lending in the



(a) before the regulation **(b)** after the regulation

Fig. 14 The financial network before (**a**) and after (**b**) large exposure limit regulations

system. Liquidity injections by the central bank can lead to perverse effects, reduce total lending, and exacerbate or even cause a credit freeze.

7.3 Large exposure limits

We conclude this section by briefly discussing a different form of policy, namely, exposure limits chosen by means of prudential regulation. For simplicity, we assume exposure limit policies allow the regulator to restrict the exposure between any pair of institutions, such as the current 15% limits for G-SIB (globally systemically important financial institutions) to G-SIB exposures outlined by the Basel Committee.[30] By the means of an example, we show that an exposure limit imposed by a banking regulator can result in more lending. Importantly, this example demonstrates that, even if individual banks can adopt such limits themselves, they may not have the incentives for doing so, thus indicating the importance of imposing exposure limits by an outside regulator.

Example 11 (Exposure limits) We suppose that banks use contracts of the form in Sect. 6.2, which specify both an interest rate R and a quantity limit $\bar{x}$ on lending. Consider the opportunity network shown in Fig. 14, where bank 1 has shock distribution $z_1 = M \gg 0$ almost surely, bank 2 has $z_2 = 0$ almost surely, and banks 3 through $m + 2$ have $z_i \in \{-M, 0\}$ where $z_i = 0$ with probability p. For simplicity, suppose $F = 0$.

For any interest rate $R_{1\to 2} < r^*$ offered to bank 2, bank 2's best response (without any quantity restrictions) is to charge E_1 exactly r^* and offer loans at the rate r^* only to one bank out of $3, \ldots, m + 2$ (say, bank 3) who lends to E_2. If bank 3 experiences a negative shock, then it fails to repay bank 2, which in turn would fail to repay bank 1. Bank 1 may consider this unprofitable, thereby restricting bank 2's contract to $\bar{x}_{1\to 2} = 1$, so bank 2 only lends to entrepreneur E_1, and entrepreneur E_2 experiences a credit freeze.

Now suppose a regulator imposes exposure limits so that $\bar{x}_{2\to j} = 1/m$ for all $j \geq 3$. Then the only way E_2 can receive access to funding is if bank 2 lends exactly $x_{2\to j} = 1/m$ for all j. When m is sufficiently large, this diversification guarantees that bank 2 will be repaid almost exactly pr^*, which decreases the likelihood that bank 2 defaults. In this situation, bank 1 will offer $\bar{x}_{1\to 2} = 2$ because the additional loan reaching E_2 no longer (severely) increases

[30] See https://www.bis.org/publ/bcbs283.pdf.

the default risk of bank 2. Therefore, after imposing the exposure limits, E_2 receives access to funding.

When bank 1 extends credit to bank 2, it faces the following moral hazard problem: because bank 2 does not internalize the loss to bank 1 in its default, bank 2 maximizes its expected profit by exposing itself to counterparty risk from only one bank. Because of this, bank 1 must discipline bank 2 by restricting the size of the loan. However, exposure limits set by the regulator provide this discipline exogenously: because bank 2 is now forced to diversify, she reduces her expected profits but decreases her default likelihood at the same time. This allows bank 1 to extend additional credit to bank 2 knowing these regulatory limits will prevent bank 2 from taking excessive risk.

8 Conclusion

In this paper, we provided a model of ex ante credit freezes caused by fears of ex post contagion over financial networks. Our model is motivated by recent credit market turbulences. For example, at the beginning of the 2008 financial crisis, many financial institutions had difficulty raising short-term funding due to uncertainty about their and their counterparties' future solvency, which made potential lenders stop lending or demand greater risk premia and haircuts. Fear of lenders about future contagion also played a central role in the financial troubles of Bear Stearns even prior to the collapse of Lehman Brothers in September 2008.

In our model, a set of banks are connected to each other via an opportunity network. The leaves of this network represent entrepreneurs in need of funding and at the root is a depositor with sufficient funds. The network thus intermediates between the depositor and the entrepreneurs, and the structure of the network captures both opportunities for intermediation and various types of intermediation frictions (which preclude certain direct paths from being used because of lack of reputation or working relationship between banks). Crucially, the structure of the network determines both the interest rates that banks charge each other and to the entrepreneurs, and the exact path of credit in equilibrium.

We characterize the subgame perfect equilibria and a refinement thereof, strong equilibrium, in this setup. We show that adverse shifts in the distribution of bank returns can cause ex ante credit freezes. At the root of these freezes is the fear that negative shocks will lead to bank failures and thus contagion. The nature of these freezes depends intricately on the structure of the financial network. This is not only because the path of lending is determined by the financial network, but also because interest rate markups, and thus bank profitability and likelihood of future collapse, depend on the competition that financial interconnections induce among banks. These two channels together lead to potentially complex credit freezes.

We show that in networks with a single entrepreneur, all credit freezes are simple, in the sense that they originate with banks that are directly affected by adverse shifts and impact only the single branch of the network that was exposed to the adverse shift. However, in networks with a richer set of interconnections, complex credit freezes can emerge. These may have their epicenters not with the banks that are directly affected, but elsewhere in the network and may lead to a spillovers from branch of the financial network to the other. Such complex freezes arise because adverse shifts in the distribution of shocks change markups and the likelihood of survival of banks in different parts of the network.

We also show that complex freezes necessitate more nuanced policy interventions. In the case of credit chains, untargeted policies are optimal. In more general networks, as long as credit freezes are simple, targeted policies that directly help affected banks are optimal. If, however, credit freezes are complex, targeted policies may need to be directed to different parts of the network.

An interesting area of future research is to consider more general lending contracts as well as dynamic lending relationships. Though in related models, more sophisticated lending contracts can create greater resilience to shocks (by preventing inefficient liquidation), in our network setting more sophisticated contracts can also open the way to even more complex financial freezes, because new forces of competition and risk emerge. Our analysis highlights the need for future work focused on empirically and theoretically investigating the nature of complex freezes and optimal policy responses.

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A Appendix: Strong equilibrium

In this appendix, we provide a refinement of the economy's (subgame perfect) equilibrium in Definition 3 by considering a variant of agent-form trembling-hand perfect equilibrium, according to which banks may tremble around the interest rates offered in equilibrium, with the set of trembles restricted to thick-tailed distributions.

To formalize this concept, let $\epsilon_m = (\epsilon_{m,ij})_{(i,j)\in\mathbf{G}}$ denote a vector of random variables with distribution H_m, where each $\epsilon_{m,ij}$ is drawn independently from an atomless distribution with full support over $\mathbb{R}_+$ and cumulative distribution function $H_{m,ij}$. We say the sequence $\{H_m\}_{m=1}^{\infty}$ *generates a sequence of thick-tailed trembles* if (i) $\lim_{m\to\infty}\epsilon_{m,ij} = 0$ almost surely for all i, (ii) $\lim_{m\to\infty}(1-H_m(x))/H_m'(x) = 0$, and (iii) $\lim_{m\to\infty} H_m''(x)/H_m'(x) < \infty$ for all $x > 0$.

Definition 9 Let $\{H_m\}_{m=1}^{n}$ denote any sequence of distribution functions generating a sequence of thick-tailed trembles $\{\epsilon_m\}_{m=1}^{\infty}$. A *strong equilibrium* is a collection of interest rate offers $\bar{\mathbf{R}}$, borrowing decisions $\mathbf{x}(\mathbf{R})$, and repayments $\boldsymbol{y}(\mathbf{R}, \mathbf{x}, \mathbf{z})$, such that there exists a sequence $(\bar{\mathbf{R}}_m, \mathbf{x}_m, \mathbf{y}_m)$ where (i) $(\bar{\mathbf{R}}_m, \mathbf{x}_m, \mathbf{y}_m)$ is a subgame perfect equilibrium subject to the trembles $\tilde{R}_{m,ij} = \bar{R}_{m,ij} + \epsilon_{m,ij}$ for all m, and (ii) $\lim_{m\to\infty} ||(\bar{\mathbf{R}}_m, \mathbf{x}_m, \mathbf{y}_m) - (\bar{\mathbf{R}}, \mathbf{x}, \boldsymbol{y})||_\infty = 0$.

Recall from our discussion in Sect. 2 that there may be multiple subgame perfect equilibria, as banks could play weakly dominated strategies as best responses. Allowing for trembles in the strong equilibrium then rules out such equilibria. To see the role of thick-tailed trembles, note that, in general, banks face a tradeoff whenever they offer a higher interest rate to a potential borrower. On the one hand, conditional on being the most competitive lender, a higher rate ensures a higher profit margin for the bank. On the other hand, the higher rate also increases the likelihood that the bank is undercut by any of its competitors. Fat-tailed trembles ensure that the latter effect always dominates the former. As a result, less competitive banks (i.e., those with higher borrowing costs themselves) elect to charge just enough of a premium to break-even in expectation (accounting for the the risk of lending).

B Appendix: Proofs

Auxiliary Lemmas

Lemma 1 *In every (strong) borrowing equilibrium:*

$$\{i \in \mathcal{N}_{in}(j) : x_{i\to j} > 0\} = \arg\min_{i\in\mathcal{N}_{in}(j)} R_{i\to j}$$

and $\sum_{i \in \mathcal{N}_{in}(j)} x_{i \to j} = \sum_{k \in \mathcal{N}_{out}(j)} x_{j \to k}$.

Proof We prove this by backward induction on j according to order. For the last bank j according to order $\mathcal{L}$ to borrow, suppose that bank j borrows some amount $x^*_{i \to j} > 0$ from a lender i with $R_{i \to j} > \min_{i \in \mathcal{N}_{in}(j)} R_{i \to j}$. Since there are no restrictions on borrowing, bank j could borrow $x^*_{i \to j}$ from some bank $i^* = \arg\min_{i \in \mathcal{N}_{in}(j)} R_{i \to j}$ and increase its profit by $x^*_{i \to j}(R_{i \to j} - R_{i^* \to j})$. This dominates borrowing from a more expensive lender, so this cannot be the case in equilibrium. Similarly, if $\sum_{i \in \mathcal{N}_{in}(j)} x_{i \to j} < \sum_{k \in \mathcal{N}_{out}(j)} x_{j \to k}$, then bank j pays the prohibitive shortfall cost, whereas if $\sum_{i \in \mathcal{N}_{in}(j)} x_{i \to j} + \delta = \sum_{k \in \mathcal{N}_{out}(j)} x_{j \to k}$ for some $\delta > 0$, bank j loses $(R_{i^* \to j} - 1)\delta$ whenever it does not default and nothing when it does. Since the former occurs with positive probability (see Lemma 2), doing such is not profitable.

Now consider some bank j borrowing at time τ in $\mathcal{L}$. By the inductive hypothesis, it is clear that no bank borrowing after j conditions its borrowing decision on who bank j borrows from. Via the same logic as before, it is clear then that bank j borrows entirely from bank $i = \arg\min_{i \in \mathcal{N}_{in}(j)} R_{i \to j}$. Similarly, by the inductive hypothesis, the borrowing decisions of any banks $k \in \mathcal{N}_{out}(j)$ are not affected by bank j's borrowing decision, except possibly if both $j \to k$ and $k \to j$. Since in the perturbed game we have $R_{j \to k} \neq R_{k \to j}$ almost surely, it cannot be that both $j = \arg\min R_{j \to k}$ and $k = \arg\min R_{k \to j}$, so either $x_{j \to k} = 0$ always or k does not condition its borrowing on the decision of j. Therefore, just by the same reasoning as before, we must have $\sum_{i \in \mathcal{N}_{in}(j)} x_{i \to j} = \sum_{k \in \mathcal{N}_{out}(j)} x_{j \to k}$ for bank j, completing the inductive step. □

Lemma 2 *If $\mathcal{Q}(\mathbf{z})$ is generic then for any $\mathcal{K} \subset \mathcal{B}$, the probability the set of banks $\mathcal{K}$ default and the set of banks $\mathcal{B} \backslash \mathcal{K}$ do not default is always positive and never equal to 1.*

Proof By Example 3.9 in Ott and Yorke [39], $\mathcal{Q}(z_i)$ must be unbounded for all z_i. Since profits from interbank lending for bank j, $\sum_{k \in \mathcal{N}_{out}(j)} y_{k \to j}$ are bounded above by $(n + |\mathcal{E}|)r^*$, for every bank j we know there exists probability $p_j > 0$ such that $z_j < (n + |\mathcal{E}|)r^*$, and so bank j defaults. By independence, the probability banks $\mathcal{K}$ default is at least $(\min p_j)^{|\mathcal{K}|}$. Similarly, the most bank j could owe (even without repayments) is $(n + |\mathcal{E}|)r^*$, and for every bank j we know there exists probability $p_j > 0$ such that $z_j > (n + |\mathcal{E}|)r^*$, and so bank j does not default. By independence, the probability some bank $i \in \mathcal{K}$ does not default is p_i, so the set of banks $\mathcal{K}$ do not default with probability at least $p_i > 0$. □

Lemma 3 *In any single-entrepreneur network $\mathbf{G}$, there is a systemic freeze if and only if there exists no path $P = 0 \to i_1 \to \cdots \to i_k \to E$ (where E is an entrepreneur) with interest rates $\{R_{0 \to i_1}, R_{i_1 \to i_2}, \cdots, R_{i_k \to E}\} \equiv \mathbf{R}_P$ such that $\mathbb{E}[\pi_j] \geq \mathbb{E}[(z_j)_+]$ for all agents j on P, given $R_{k \to \ell} = \varnothing$ for all $k \to \ell$ not on P (where $\mathbb{E}$ is over the realizations of $\mathbf{z}$).*

Informally, this condition says there is a systemic freeze (i.e., no interbank lending) if and only if we cannot construct a path from the depositor to the entrepreneur, such that all banks prefer to lend at these interest rates than not engage in interbank lending at all.

Proof For the "if" direction, we prove the contrapositive: if there is no systemic freeze, then there must exist a path $P = 0 \to i_1 \to \cdots \to i_k \to E$ where the interest rates $\mathbf{R}_P$ give us $\mathbb{E}[\pi_j] \geq \mathbb{E}[(z_j)_+]$. By Theorem 3, we know the financial network $\mathbf{x}_*$ is an intermediation path P from the depositor to entrepreneur. Assume, however, this path has at least one bank j with $\mathbb{E}[\pi_j] < \mathbb{E}[(z_j)_+]$. By definition of the equilibrium, bank j is aware that no other bank in P acting later will withdraw its offer conditional on j not withdrawing, and moreover, all banks will borrow and lend so that P is the financial network. Therefore, bank j has a profitable one-shot deviation to withdraw its offer, contradicting that this is an equilibrium.

For the "only if" direction, suppose there exists a path $P = 0 \to i_1 \to \cdots \to i_k \to E$ where some interest rates $\mathbf{R}_P$ give us $\mathbb{E}[\pi_j] \geq \mathbb{E}[(z_j)_+]$ for all $j \in P$. By means of contradiction, suppose there is a systemic freeze. Consider the last agent i^* to act in $\mathcal{O}$ on the path P. Conditional on $R_{k\to\ell} = \varnothing$ for all $k \to \ell$, and given the interest rates $\mathbf{R}_P^*$ up until agent i^* (not necessarily equal to $\mathbf{R}_P$) such that agent i^* can offer some R_{i^*} and satisfy $\mathbb{E}[(z_{i^*})_+] \leq \mathbb{E}[\pi_{i^*}]$ and $\mathbb{E}[(z_j)_+] \leq \mathbb{E}[\pi_j]$ for all banks j on P. Then it is a best-response for bank i^* to offer some R_{i^*}, and no bank on P to withdraw (which does at least as well as $\mathbb{E}[(z_{i^*})_+]$). By backward induction, we see that every bank i on this path P can offer some R_i such that $\mathbb{E}[(z_j)_+] \leq \mathbb{E}[\pi_j]$ for all banks $j \in P$, and that conditional on offering R_i, bank j does (weakly) better than offering $\varnothing$. Since these offers do affect those banks outside of P, it is still an equilibrium for these banks to offer $\varnothing$. However, repeating this argument, we see that the first bank to offer according to $\mathcal{O}$ in P would prefer to offer to the next bank in P as opposed to not offer (i.e., offer $\varnothing$), and then not withdraw the contract. This contradicts the assumption that a systemic freeze was the equilibrium. □

Proof of Theorem 1

To prove part (a), we construct a repayment equilibrium for every realization of $\mathbf{z}$, iteratively (let τ be the τth iteration). Let $\mathcal{D}_\tau \subset \mathcal{B} \cup \mathcal{E}$ be the set of entrepreneurs in default at iteration τ. At $\tau = 0$, assume $\mathcal{D}_\tau = \mathcal{B} \cup \mathcal{E}$. At each $\tau \geq 1$, for each bank j, if $z_j + \sum_{k\in\mathcal{N}_{out}(j)\backslash\mathcal{D}_{\tau-1}} R_{j\to k}x_{j\to k} \geq \sum_{i\in\mathcal{N}_{in}(j)} R_{i\to j}x_{i\to j}$, then do not include j in $\mathcal{D}_\tau$, otherwise do. For entrepreneur k, at each $\tau \geq 1$, if $r^* \geq \sum_{j\in\mathcal{N}_{in}(k)} R_{j\to k}x_{j\to k}$, then do not include k in $\mathcal{D}_\tau$, otherwise do.

We prove this algorithm constructs a repayment equilibrium. First notice that if $j \notin \mathcal{D}_\tau$ then $j \notin \mathcal{D}_{\tau'}$ for all $\tau' \geq \tau$. This can be shown by induction: in the base case, the set of non-defaulting banks is empty, so this set can only increase. At the inductive step, we note that $\sum_{k\in\mathcal{N}_{out}(j)\backslash\mathcal{D}_{\tau-1}} R_{j\to k}x_{j\to k} \geq \sum_{k\in\mathcal{N}_{out}(j)\backslash\mathcal{D}_{\tau-2}} R_{j\to k}x_{j\to k}$, so each bank (or entrepreneur) will be able to meet its obligations in all $\tau' \geq \tau$ if it can at τ. Since $\mathcal{D}_\tau$ is a decreasing set, and there are finitely many banks, we are guaranteed this algorithm terminates at some τ^* with either $\mathcal{D}_{\tau^*} = \mathcal{D}_{\tau^*-1}$ or $\mathcal{D}_{\tau^*} = \emptyset$. In the latter case, we know $\mathcal{D}_{\tau^*+1} = \mathcal{D}_{\tau^*}$ so it is without loss of generality to consider only the former. We claim this admits a repayment equilibrium. For each bank $j \notin \mathcal{D}_{\tau^*}$ we have:

$$z_j + \sum_{k\in\mathcal{N}_{out}(j)\backslash\mathcal{D}_{\tau^*}} R_{j\to k}x_{j\to k} = z_j + \sum_{k\in\mathcal{N}_{out}(j)\backslash\mathcal{D}_{\tau^*-1}} R_{j\to k}x_{j\to k} \geq \sum_{i\in\mathcal{N}_{in}(j)} R_{i\to j}x_{i\to j}$$

and for each bank $j \in \mathcal{D}_{\tau^*}$ we have:

$$z_j + \sum_{k\in\mathcal{N}_{out}(j)\backslash\mathcal{D}_{\tau^*}} R_{j\to k}x_{j\to k} = z_j + \sum_{k\in\mathcal{N}_{out}(j)\backslash\mathcal{D}_{\tau^*-1}} R_{j\to k}x_{j\to k} < \sum_{i\in\mathcal{N}_{in}(j)} R_{i\to j}x_{i\to j}$$

(and similar for entrepreneurs), which proves the claim.

For any $\mathbf{R}$, and given the existence of a repayment equilibrium, the borrowing stage is a finite extensive-form game with perfect information [where the terminal nodes represent "random" payoffs, but where the banks maximize according to expected utility of Eq. 2]. By Zermelo's theorem, there exists a pure strategy borrowing equilibrium that can always be derived through backward induction, which establishes part (b).

Taking the borrowing equilibrium as given, the weak (subgame) perfect equilibrium also exists in pure strategies by Zermelo's theorem. In the strong equilibrium, for every perturbed game, after each offer, nature makes a move which perturbs the offer randomly (see Appendix A). To show this, we amend Zermelo's theorem for every node in the game-tree. For agent $j = \mathcal{O}(n+1)$ offering last, she simply chooses:

$$\mathbf{R}_j^* \in \arg \max_{\mathbf{R}\in\mathbb{R}^{|\mathcal{N}_{out}(j)|}} \mathbb{E}[\pi_j(\mathbf{R})|\mathcal{H}_j]$$

where utility π_j is given in Eq. 2, $\mathcal{H}_j$ is the entire history of offers, and expectation is over the interest rate trembles of agent j. For agent $i = \mathcal{O}(t)$ offering at time t, she simply chooses:

$$\mathbf{R}_i^* \in \arg \max_{\mathbf{R}\in\mathbb{R}^{|\mathcal{N}_{out}(i)|}} \mathbb{E}[\pi_i(\mathbf{R}, \mathbf{R}_{-i})|\mathcal{H}_i]$$

where utility π_i is given in Eq. 2 taking the actions $\mathbf{R}_{-i}$ of all future agents $\{k : \mathcal{O}^{-1}(k) > \mathcal{O}^{-1}(i)\}$ as given by backward-induction, h_i is the history of offers for agents $\{k : \mathcal{O}^{-1}(k) < \mathcal{O}^{-1}(i)\}$, and the expectation is over the interest rate trembles of agent i and all future offering agents $\{k : \mathcal{O}^{-1}(k) \geq \mathcal{O}^{-1}(i)\}$. Both the interest rate offers and borrowing decisions are in pure strategies. Therefore, every perturbed game has a (subgame perfect) equilibrium in pure strategies. By the convergence and uniqueness of these equilibria in Theorem 2 with trembles given in Appendix A, we see there exists a strong equilibrium in pure strategies. □

Proof of Theorem 2

By Theorem 3,[31] we know that the financial network $(\mathbf{R}_*, \mathbf{x}_*)$ is a directed tree. Let $\mathcal{T}$ be a strong topological order on this network. Then working from the agents closest to the depositor, we can solve for the unique repayment equilibrium via backward induction. In particular, for bank j at topological index $\mathcal{T}(j)$, we know that $y_{j\to i} = 0$ for some bank $i \in \mathcal{N}_{in}(j)$ if $z_j + \sum_{k\in\mathcal{N}_{out}(j)} y_{k\to j} - \sum_{i\in\mathcal{N}_{in}(j)} R_{i\to j}x_{i\to j} < 0$, otherwise $y_{j\to i} = R_{i\to j}x_{i\to j}$, where $\sum_{k\in\mathcal{N}_{out}(j)} y_{k\to j}$ is known because $\mathcal{T}(k) > \mathcal{T}(j)$ for all $k \in \mathcal{N}_{out}(j)$. Therefore, we can iteratively compute the repayment equilibrium for any $\mathbf{z}$, which is uniquely determined.

For any set of interest rates $\mathbf{R}$ in a perturbed game, we know that with probability 1 no two interest rates are identical, so borrowing takes the form given in Lemma 1 (i.e., a directed tree) . Let us consider the set $\mathbf{X}_*$, the lim $\sup_{n\to\infty}$ of borrowing networks (i.e., the set of all equilibrium borrowing networks which appear infinitely often as $n \to \infty$). Such a set $\mathbf{X}_*$ is necessarily non-empty. Suppose there are two distinct lending trees T, T' appearing in $\mathbf{X}_*$. Consider some bank j that lies at the intersection of these trees but borrows from different lenders i and i' in T and T', respectively. By construction of $\mathbf{X}_*$, as $\boldsymbol{\epsilon}_m \overset{a.s.}{\to} 0$, bank i lends to bank j^* with positive probability and bank i' lends to bank j^* also with positive probability. It clearly cannot be the case that i and i' make positive profits as $\boldsymbol{\epsilon}_m \overset{a.s.}{\to} 0$, given the (strong) equilibrium interest rates $\mathbf{R}_*, \mathbf{R}_*'$, respectively. Otherwise, whichever bank makes positive profits can reduce its interest rate by an arbitrarily small amount, which as $\boldsymbol{\epsilon}_m \overset{a.s.}{\to} 0$ would guarantee that it has the unique lowest interest rate and makes arbitrarily close to the same (positive) profits. Let $\partial\pi_{mi}$ (resp. $\partial\pi_{mi'}$) denote the marginal profit of lending to bank j^* for bank i (resp. bank i'). Therefore, $\lim_{m\to\infty} \mathbb{E}^{\mathcal{Q}}[\partial\pi'_{mi'}] = \mathbb{E}^{\mathcal{Q}}[\partial\pi_{mi}] = 0$ is a necessary

[31] Note that we cite Theorem 2 in the proof of Theorem 3 to show it is a directed tree, but we are leveraging only uniqueness of the interest rate and borrowing stages, and not the unique repayment equilibrium, which is the only time we use Theorem 3 in this proof.

condition.[32] For a generic $\mathcal{Q}$ and a generic tuple of interest rates $R_{i\to j}$, $R'_{i'\to j}$ (holding others constant) this will not be satisfied (see Appendix C, Proposition 12(c)). Thus, the tuple of $R_{i\to j}$, $R'_{i'\to j}$ where (marginal) profits for bank i and bank i' are zero, lie on a set of measure zero. Let us induct on the path from j^* to the depositor in both T and T'. To do so, replace i with the unique lender of i in T (call it ℓ) and replace i' with the unique lender of i' in T' (call it ℓ'). If these agents are not distinct, then we can replace j^* from before with $\ell^* = \ell = \ell'$ and repeat the above argument. Otherwise, by the same reasoning as the above, it must be the case that ℓ and ℓ' do not make positive (marginal) profits when lending to i and i', respectively. We can repeat this argument as needed until we either: (i) reach the depositor or (ii) reach an intermediation chain from the depositor to some bank β^* which is the same in both T and T'. In the former case, for generic $\mathcal{Q}$ and a given risk-free rate r_0, the only interest rates charged to banks in T and T' that allow both to be strong equilibria lie on a set of measure zero. By similar reasoning as before (using Ott and Yorke [39]), the expected profit of the depositor is generically larger either under T or T', and by charging an arbitrarily small difference in interest rate, can change the equilibrium to either T or T' with probability 1. Similarly, for the intermediation chain from the depositor to bank β^* can be replaced by an "equivalent depositor" with a different risk-free rate $\tilde{r}_0$ of the outside technology. Therefore, both $\mathcal{Q}$ (on the rest of the network) and $\tilde{r}_0$ are generic, so the same argument applies. Finally, either β^* or the depositor is better of deviating to a marginally different offer (which has an arbitrarily small impact on profits in either T or T'), but necessarily induces either T or T' to never be the borrowing network. This means in a strong equilibrium, there will be a unique lending network $\mathbf{x}_*$ (i.e., the set $\mathbf{X}_*$ is a singleton).

To show $\mathbf{R}_*$ is essentially unique in the strong equilibrium, it is enough to prove that no bank is indifferent between offering any two interest rates whenever $x_{i\to j} > 0$ (the result then follows from Zermelo's theorem and that other offers do not affect payoffs). We do this by backward induction on the offer order $\mathcal{O}$. Consider some bank j who takes as given its interest rate offers and chooses $\bar{\mathbf{R}}_j$. Agent j maximizes its (marginal) profit of lending to bank $k \in \mathcal{N}_{out}(j)$, taking as given offers to banks $k' \in \mathcal{N}_{out}(j)\backslash\{k\}$. Then, it chooses $\bar{R}^*_{j\to k} \in \arg\max_{\tilde{R}_{j\to k}} \mathbb{E}[\partial\pi_{j\to k}(\tilde{R}_{j\to k})]$. If $x^*_{j\to k} > 0$ with positive probability (bounded away from zero) as $\boldsymbol{\epsilon}_m \to \mathbf{0}$ it is clear by Lemma 1 that $\bar{R}^*_{j\to k} \to \min_{j'} R_{j'\to k}$ (where the min includes competing banks j' over k who do not immediately withdraw in the following stage). Otherwise, as we concluded before, bank j's offer to bank k does not affect the essential uniqueness of $\mathbf{R}_*$. For the inductive step, consider some other bank j' that offers, taking as given the history all interest rate offers, and all (relevant) future offers as known with certainty, given $\mathbf{R}_j$ (by the inductive assumption, since no bank is indifferent when its offer is relevant). As before, agent j maximizes its (marginal) profit of lending to bank $k \in \mathcal{N}_{out}(j)$, taking as given offers to banks $k' \in \mathcal{N}_{out}(j)\backslash\{k\}$. If $x^*_{j\to k} > 0$ with positive probability (bounded away from zero) as $\boldsymbol{\epsilon}_m \to \mathbf{0}$, then $\bar{R}^*_{j\to k} \to R^*_{j\to k} = \arg\max_{R_{j\to k}} \mathbb{E}^{\mathcal{Q}}[\partial\pi_{j\to k}(R_{j\to k})]$, which is unique by genericity of $\mathcal{Q}$ (and uniqueness of future "relevant" offers), see Proposition 12(d) in Appendix C. Otherwise, bank j's offer solves:

$$\begin{aligned}\bar{R}^*_{j\to k} &= \arg\max_{\bar{R}_{j\to k}} \mathbb{E}^{\mathcal{Q}}\left[\partial\pi_{j\to k}\left(\bar{R}_{j\to k} + \epsilon_{j\to k,m}\right)\right]\\ &= \arg\max_{\bar{R}_{j\to k}} \mathbb{E}^{\mathcal{Q}}\left[\partial\pi_{j\to k}\left(\bar{R}_{j\to k} + \epsilon_{j\to k,m}\right)\middle|\bar{R}_{j\to k} + \epsilon_{j\to k,m}\right.\end{aligned}$$

[32] Note that these expectations depend on the offer order $\mathcal{O}$, but are simply integrals over realizations of liquidity shocks, as is the form in Ott and Yorke [39], given that banks are not indifferent between making multiple offers for $R_{i\to j}$ when $x_{i\to j} > 0$ as shown in the following paragraph.

$$\leq \min_{j',j''}\{\tilde{R}_{j'\to k}, \bar{R}_{j''\to k}\left(\tilde{R}_{j\to k}\right) + \epsilon_{j'\to k,m}\}\Big]$$
$$\cdot \mathbb{P}\left[\bar{R}_{j\to k} + \epsilon_{j\to k,m} \leq \min_{j',j''}\left\{\tilde{R}_{j'\to k}, \bar{R}_{j''\to k}\left(\tilde{R}_{j\to k}\right) + \epsilon_{j'\to k,m}\right\}\right]$$

As $\boldsymbol{\epsilon}_m \to \mathbf{0}$, the above converges to:

$$\bar{R}^*_{j\to k} = \arg\max_{\bar{R}_{j\to k}} \mathbb{E}^{\mathcal{Q}}\left[\partial\pi_{j\to k}(\bar{R}_{j\to k})\middle|\bar{R}_{j\to k} \leq \min_{j',j''}\left\{\tilde{R}_{j'\to k}, \bar{R}_{j''\to k}(\tilde{R}_{j\to k})\right\}\right]$$
$$\cdot \mathbb{P}\left[\bar{R}_{j\to k} \leq \min_{j',j''}\left\{\tilde{R}_{j'\to k}, \bar{R}_{j''\to k}(\bar{R}_{j\to k})\right\}\right]$$
$$= \arg\max_{\bar{R}_{j\to k}} \int_{\mathcal{Q}} \partial\pi_{j\to k}(\bar{R}_{j\to k}) \prod_{j''}\left(1 - H_m\left(\bar{R}_{j''\to k}(\bar{R}_{j\to k}) - \bar{R}_{j\to k}\right)\right)\, d\mathcal{Q}$$
$$\implies \int_{\mathcal{Q}} \frac{\partial\ \partial\pi_{j\to k}}{\partial\bar{R}_{j\to k}} \prod_{j''}\left(1 - H_m(\bar{R}_{j''\to k} - \bar{R}_{j\to k})\right)$$
$$+ \partial\pi_{j\to k}(\bar{R}_{j\to k}) \sum_{j''}\left[1 - \frac{\partial\bar{R}_{j''\to k}}{\partial\bar{R}_{j\to k}}\right] H'_m\left(\bar{R}_{j''\to k}(\bar{R}_{j\to k})\right.$$
$$\left.-\bar{R}_{j\to k}\right) \prod_{j'''\neq j''}\left(1 - H_m\left(R_{j'''\to k} - \bar{R}_{j\to k}\right)\right)\, d\mathcal{Q} = 0$$

By the assumption on H_m in Appendix A, it is clear that $\mathbb{E}^{\mathcal{Q}}[\partial\pi_{j\to k}(\bar{R}_{j\to k})] \to 0$ as $m \to \infty$ in equilibrium if $\partial\bar{R}_{j''\to k}/\partial\bar{R}_{j\to k}$ remains (sufficiently) bounded away from 1. For this consider j'''s problem:

$$\bar{R}^*_{j''\to k}(\tilde{R}_{j'\to k}) = \arg\max_{\bar{R}_{j''\to k}} \mathbb{E}^{\mathcal{Q}}\left[\partial\pi_{j''\to k}(\bar{R}_{j''\to k} + \epsilon_{j\to k,m})\middle|\bar{R}_{j''\to k} + \epsilon_{j''\to k,m} \leq \min_{j'} \tilde{R}_{j',k}\right]$$
$$\cdot \mathbb{P}\left[\bar{R}_{j''\to k} + \epsilon_{j''\to k,m} \leq \min_{j'} \tilde{R}_{j',k}\right]$$
$$= \arg\max_{\bar{R}_{j''\to k}} \int_{\mathcal{Q}} \int_{-\infty}^{\min_{j'} \tilde{R}_{j'\to k} - \bar{R}_{j''\to k}} \partial\pi_{j''\to k}(\bar{R}_{j''\to k} + \alpha)\, dH(\alpha)\, d\mathcal{Q}$$

By the fundamental theorem of calculus, our first-order condition reduces to:

$$\implies \int_{\mathcal{Q}} H'\left(\min_{j'} \tilde{R}_{j'\to k} - \bar{R}^*_{j''\to k}\right)\left(\partial\pi_{j''\to k}\left(\min_{j'} \tilde{R}_{j'\to k}\right)\right) = 0$$

By the implicit function theorem, we observe that:

$$\int_{\mathcal{Q}} \left[H''\left(\min_{j'} \tilde{R}_{j'\to k} - \bar{R}^*_{j''\to k}\right)\left(\partial\pi_{j''\to k}\left(\min_{j'} \tilde{R}_{j'\to k}\right)\right)\left(1 - \frac{\partial\bar{R}^*_{j''\to k}}{\partial\min_{j'} \tilde{R}_{j'\to k}}\right)\right.$$
$$\left. + H'\left(\min_{j'} \tilde{R}_{j'\to k} - \bar{R}^*_{j''\to k}\right) \frac{\partial\ \partial\pi_{j''\to k}}{\partial\min_{j'} \tilde{R}_{j'\to k}}\right] d\mathcal{Q} = 0$$

which implies $\partial\bar{R}_{j''\to k}/\partial\min_{j'} \tilde{R}_{j'\to k}$ is (sufficiently) bounded away from 1 as $m \to \infty$, given that $\lim_{m\to\infty} H''_m/H'_m < \infty$, as assumed in Appendix A.

Finally, as we saw before, this implies by genericity there is a unique offer $\bar{R}_{j\to k}$ that gives bank j zero (expected) profits, via the inductive step and given the history of offers. Therefore, the interest rates $\mathbf{R}$ are unique in the strong equilibrium. □

Proof of Theorem 3

(i): Clearly if $x_{i\to j} > 0$ in the borrowing equilibrium, then $R_{i\to j} \neq \varnothing$. Otherwise if $x_{i\to j} = 0$ and $R_{i\to j} = R^* \neq \varnothing$ (otherwise, we are done), consider the withdrawal decision of bank i in the offer stage. Because the remaining subgame is perfect information, bank i's information set assigns probability 1 to bank i choosing $x_{i\to j} = 0$ in the borrowing stage. This means bank i is indifferent to offering R^* and offering $R_{i\to j} = \varnothing$ (i.e., withdrawing). Moreover, by Lemma 1 this deviation does not affect the future withdrawal decisions of banks $k \neq i$ or the borrowing decisions of banks $k \neq j$. By induction, it can therefore be established there exists a strong equilibrium where $R_{i\to j} = \varnothing$ if and only if $x_{i\to j} = 0$ (which is the contrapositive of the statement).

(ii): We first claim the financial network cannot contain any directed cycles. Suppose to the contrary we have a cycle of banks $i_0 \to i_2 \to i_k \to i_0$ such that $x_{i_\alpha\to i_{\alpha+1}} > 0$ (with mod k). Take $\underline{x} = \min_\alpha x_{i_\alpha\to i_{\alpha+1}} > 0$. Consider the case where $R_{i_\alpha\to i_{\alpha+1}} \geq R_{i_{\alpha+1}\to i_{\alpha+2}}$ for some α. Then after observing all interest rate offers, bank $i_{\alpha+1}$'s decision to not withdraw the offer to $i_{\alpha+2}$ is dominated by withdrawing. If bank $i_{\alpha+1}$ withdraws, by Lemma 1 it can borrow $x_{i_\alpha\to i_{\alpha+1}} > \underline{x}$ less from its lenders and lend less to bank $i_{\alpha+1}$ by the same amount. In the event that bank $i_{\alpha+1}$ is insolvent, both give the same payoff; in the event that bank $i_{\alpha+1}$ is solvent, bank $i_{\alpha+1}$ gets at least as much payoff when bank $i_{\alpha+2}$ is solvent (and strictly more when $R_{i_\alpha\to i_{\alpha+1}} > R_{i_{\alpha+1}\to i_{\alpha+2}}$), and gains at least $R_{i_\alpha\to i_{\alpha+1}} \cdot \underline{x} > 0$ when bank $i_{\alpha+2}$ is insolvent. Since the latter event occurs with positive probability by Lemma 2, withdrawing dominates not withdrawing bank $i_{\alpha+2}$'s offer, so in equilibrium we must have $R_{i_\alpha\to i_{\alpha+1}} < R_{i_{\alpha+1}\to i_{\alpha+2}}$ for all α. But because this is a cycle starting and ending at the same bank i_0, this cannot be.

Now by definition of strong equilibrium (Appendix A), for any perturbed game, no distinct interest rate offers are identical with probability 1. By Lemma 1, the borrowing equilibrium consists of every bank and entrepreneur borrowing from its cheapest lender. Therefore, every bank borrows from at most one other bank, which implies $\mathbf{x}_*$ (and by part (a), $\mathbf{R}_*$ as well), is a directed tree. By Theorem 2, the (unique) financial network of the strong equilibrium (which is the limit of perturbed games) must also be a directed tree. □

Proof of Theorem 4

In a single-entrepreneur network, this is a direct consequence of Lemma 3, since the existence of a systemic freeze depends only the risk profile $\mathcal{Q}$ and the network $\mathbf{G}$ and not the order of actions $(\mathcal{O}, \mathcal{L})$. For multiple entrepreneurs, identical reasoning as Lemma 3 can be extended to the case of trees, which are guaranteed to be the structure of the financial network in Theorem 3, except where we replace the profitable path P in Lemma 3 with profitable tree T. □

Proof of Proposition 1

First, we show that if the entrepreneur has a credit freeze in $\mathbf{G}$, then it has a credit freeze in every chain subnetwork $\mathbf{H} \subset \mathbf{G}$. We prove the contrapositive: if there is lending to the entrepreneur in some chain $\mathbf{H}$, then there must be lending in $\mathbf{G}$. By Lemma 3, we know there exist some interest rates $\mathbf{R}_P$ along the path $P = \mathbf{H}$ with $\mathbb{E}[\pi_j] \geq \mathbb{E}[(z_j)_+]$ for all j on this path. In $\mathbf{G}$, because $P \subset \mathbf{G}$, the same set of interest rates $\mathbf{R}_P$ along P does not change $\mathbb{E}[\pi_j]$ (because $R_{k\to\ell} = \varnothing$ for all $(k \to \ell) \in \mathbf{G}\backslash\mathbf{H}$). Therefore, applying Lemma 3 again, we see there is no systemic credit freeze in $\mathbf{G}$, so the sole entrepreneur does not experience a credit freeze in $\mathbf{G}$.

Next, we show if the entrepreneur has no credit freeze in $\mathbf{G}$, then there exists some chain subnetwork $\mathbf{H} \subset \mathbf{G}$ where the entrepreneur does not experience a credit freeze. Consider the path P guaranteed by Lemma 3 such that $\mathbb{E}[\pi_j] \geq \mathbb{E}[(z_j)_+]$ for some set of interest rates along this path. Taking $\mathbf{H} = P$, we note that these inequalities still hold in $\mathbf{H}$ for the same set of interest rates (neither $\mathbb{E}[\pi_j]$ nor $\mathbb{E}[(z_j)_+]$ change), so by Lemma 3, there is no credit freeze in $\mathbf{H}$. □

Proof of Corollary 1

By Proposition 1, if the entrepreneur experiences a credit freeze in $\bar{\mathbf{G}}$, then it experiences a credit freeze for every chain subnetwork. Since every chain subnetwork in $\underline{\mathbf{G}}$ is present in $\bar{\mathbf{G}}$ because $\underline{\mathbf{G}} \subset \bar{\mathbf{G}}$, there is a credit freeze for every chain subnetwork of $\underline{\mathbf{G}}$, so once again by Proposition 1, there is a (systemic) credit freeze in $\underline{\mathbf{G}}$. □

Proof of Theorem 5

For (a), we show that if $\mathbf{G}$ has no credit freeze with r_0, then it has no credit freeze in $\mathbf{G}'$ with $1 \leq r_0' \leq r_0$. Again, by Lemma 3 we have interest rates $R_{0\to1}, R_{1\to2}, \ldots, R_{n\to(n+1)}$ in $\mathbf{G}$ such that $\mathbb{E}[\pi_j] \geq \mathbb{E}[(z_j)_+]$ for all $j \in \{0, \ldots, n+1\}$. If we consider this same set of interest rates in $\mathbf{G}'$, then it is clear that $\mathbb{E}[\pi_j'] = \mathbb{E}[\pi_j] \geq \mathbb{E}[(z_j)_+] = \mathbb{E}[(z_j)_+]$ for all $j \in \{1, \ldots, n\}$. Then:

$$0 = \mathbb{E}[(z_0)_+] = \mathbb{E}[(z_0)_+] \leq \mathbb{E}[\pi_0] = \mathbb{E}\left[(z_0 + y_{1\to0} - r_0)_+\right] \\ \leq \mathbb{E}\left[(z_0 + y_{1\to0} - r_0')_+\right] = \mathbb{E}[\pi_0']$$

By Lemma 3, there is no credit freeze in $\mathbf{G}'$ with $r_0' \leq r_0$. Of course, setting $r_0 = r^*$ leads to a credit freeze, so therefore there exists some $\bar{r}_0$ where $r_0 > \bar{r}_0$ leads to credit freeze. Finally, to note that $\bar{r}_0 < r^*$, by Lemma 2 bank 1 defaults *and* bank 2 survives with positive probability, so bank 2 must make positive rents.

For (b), we show that if $\mathbf{G}$ has no credit freeze with r^*, then there is no credit freeze in $\mathbf{G}'$ with $r^{*'} \geq r^*$. We utilize Lemma 3 again; we have interest rates $R_{0\to1}, R_{1\to2}, \ldots, R_{n\to(n+1)}$ in $\mathbf{G}$ such that $\mathbb{E}[\pi_j] \geq \mathbb{E}[(z_j)_+]$ for all $j \in \{0, \ldots, n+1\}$. Because $R_{n\to(n+1)} \leq r^*$ in equilibrium, we know that $R_{n\to(n+1)} \leq r^{*'}$, so the entrepreneur is still solvent with probability 1 and has $\mathbb{E}[\pi_{n+1}] = r^{*'} - R_{n\to(n+1)} \geq 0 = \mathbb{E}[(z_{n+1})_+]$. Therefore, it is easy to see $\mathbb{E}[\pi_j'] = \mathbb{E}[\pi_j] \geq \mathbb{E}[(z_j)_+] = \mathbb{E}[(z_j)_+']$ for all $j \in \{0, \ldots, n+1\}$. By Lemma 3, there is no credit freeze in $\mathbf{G}'$ with $r^{*'} \geq r^*$. For the same reason as (a), it is clear that $\underline{r}^* > r_0$, as bank 2 must make positive rents from lending to bank 1.

For part (c), consider a chain of length n with no credit freeze. We first show that the chain of length $n-1$ will also not experience a credit freeze. By Lemma 3, in the n-bank chain $\mathbf{G}$, there exist interest rates $R_{0\to1}, R_{1\to2}, \ldots, R_{n\to(n+1)}$ such that $\mathbb{E}[\pi_j] \geq \mathbb{E}[(z_j)_+]$ for all $j \in \{0, \ldots, n+1\}$. In the $(n-1)$-bank chain $\mathbf{G}'$, let us consider the same set of interest rates $R_{0\to1}, R_{1\to2}, \ldots, R_{(n-1)\to n}$ (less the final offer, which does not exist in the shorter chain). We first claim $y_{j\to j-1}'$ (under $\mathbf{G}'$) FOSD $y_{j\to j-1}$ (under $\mathbf{G}$) for all $j \in \{1, \ldots, n+1\}$ for these interest rates. It is sufficient to show probability of repayment in $\mathbf{G}'$ exceeds that in $\mathbf{G}$. We prove this by induction. Since the entrepreneur repays with probability 1 when

$R_{(n-1)\to n} \le r^*$, the probability bank $n-1$ repays to bank $n-2$ is:

$$\mathbb{P}[z_{n-1} + R_{(n-1)\to n} \ge R_{(n-2)\to(n-1)}] \ge \mathbb{P}[z_{n-1} + y_{n\to(n-1)} \ge R_{(n-2)\to(n-1)}]$$

Suppose that $y'_{j\to j-1}$ FOSD $y_{j\to j-1}$. Then:

$$\mathbb{P}[z_{j-1} + y'_{j\to j-1} \ge R_{(j-2)\to(j-1)}] \ge \mathbb{P}[z_{j-1} + y_{j\to j-1} \ge R_{(j-2)\to(j-1)}]$$

so $y'_{j-1\to j-2}$ FOSD $y_{j-1\to j-2}$. Finally, we see that for all $j \in \{0, \ldots, n\}$:

$$\begin{aligned}\mathbb{E}[(z_j)'_+] = \mathbb{E}[(z_j)_+] \le \mathbb{E}[\pi_j]\\ &= \mathbb{E}\left[\left(z_j + y_{(j+1)\to j} - R_{(j-1)\to j}\right)_+\right] - F\cdot\mathbb{P}\left[z_j < R_{(j-1)\to j} - y_{(j+1)\to j}\right]\\ &\le \mathbb{E}\left[\left(z_j + y'_{(j+1)\to j} - R_{(j-1)\to j}\right)_+\right] - F\cdot\mathbb{P}\left[z_j < R_{(j-1)\to j} - y'_{(j+1)\to j}\right] = \mathbb{E}[\pi'_j]\end{aligned}$$

Therefore, by Lemma 3, there is no credit freeze in $\mathbf{G}'$, the $(n-1)$-bank chain. Finally, by Lemma 2 note there exist $p, q > 0$ (independent of i) such that probability that any bank $(i-1)$ on this chain defaults is at least $p > 0$ and the probability bank i does not default is at least $q > 0$ (by symmetry). The probability that both of these events occur simultaneously is at least pq by independence. Therefore, risk premia in the chain must satisfy $R_{i\to(i+1)} \ge R_{(i-1)\to i}/(1-pq)$ to make nonnegative profits. Therefore, for large enough $\bar{n}$, given the depositor is lending at least r_0, it is clear the (minimum) interest rate needed to charge the entrepreneur exceeds r^*, which implies by Lemma 3 there will be a credit freeze for all $n \ge \bar{n}$. □

Proof of Proposition 2

Suppose there is no systemic credit freeze in $\mathcal{Q}$, so by Lemma 3 there exist $R_{0\to1}, \ldots, R_{n\to(n+1)}$ such that:

$$\mathbb{E}[(z_j)_+] \le \mathbb{E}\left[\left(z_j + y_{(j+1)\to j} - R_{(j-1)\to j}\right)_+\right] - F\cdot\mathbb{P}\left[z_j < R_{(j-1)\to j} - y_{(j+1)\to j}\right]$$

for all j. If $\mathcal{Q}'$ FOSD $\mathcal{Q}$, we prove that $y'_{j+1\to j}$ FOSD $y_{j+1\to j}$. We do so by induction. Note the entrepreneur always repays in equilibrium regardless of the risk profile. Bank j repays if and only if $z_j \ge R_{(j-1)\to j} - y_{(j+1)\to j}$. It is straightforward to see $\mathbf{z}_{-j}$ is a sufficient statistic for $y_{(j+1)\to j}$, and $y'_{(j+1)\to j}$ FOSD $y_{(j+1)\to j}$ (by assumption) so we know that:

$$\mathbb{P}\left[z_j \ge R_{(j-1)\to j} - y_{(j+1)\to j}\right] \le \mathbb{P}\left[z'_j \ge R_{(j-1)\to j} - y'_{(j+1)\to j}\right] \tag{5}$$

which implies that $y'_{j\to(j-1)}$ FOSD $y_{j\to(j-1)}$ by rearranging. It is clear the inequality is strict if the conditional distribution $\mathbf{z}_j|\mathbf{z}_{-j}$ under $\mathcal{Q}'$ is different than under $\mathcal{Q}$ for some $\mathbf{z}_{-j}$ (i.e., if bank j experiences an adverse shift). For all banks j without an adverse shift, we have:

$$\begin{aligned}\mathbb{E}[(z'_j)_+] = \mathbb{E}[(z_j)_+]\\ &\le \mathbb{E}\left[\left(z_j + y_{(j+1)\to j} - R_{(j-1)\to j}\right)_+\right] - F\cdot\mathbb{P}\left[z_j < R_{(j-1)\to j} - y_{(j+1)\to j}\right]\\ &\le \mathbb{E}\left[\left(z'_j + y'_{(j+1)\to j} - R_{(j-1)\to j}\right)_+\right] - F\cdot\mathbb{P}\left[z'_j < R_{(j-1)\to j} - y'_{(j+1)\to j}\right]\end{aligned}$$

For all banks j with an adverse shift, the inequality in (5) is strict, so we have for some $\varepsilon > 0$:

$$\begin{aligned}\mathbb{E}[(z'_j)_+] - \varepsilon &\le \mathbb{E}[(z_j)_+] \\ &\le \mathbb{E}\left[\left(z_j + y_{(j+1)\to j} - R_{(j-1)\to j}\right)_+\right] - F \cdot \mathbb{P}\left[z_j < R_{(j-1)\to j} - y_{(j+1)\to j}\right] \\ &< \mathbb{E}\left[\left(z'_j + y'_{(j+1)\to j} - R_{(j-1)\to j}\right)_+\right] - F \cdot \mathbb{P}\left[z'_j < R_{(j-1)\to j} - y'_{(j+1)\to j}\right] - \varepsilon\end{aligned}$$

for sufficiently large F. Killing the ε expressions on both sides of the inequality, we see by Lemma 3, we see there is no credit freeze under $\mathcal{Q}'$. □

Proof of Proposition 3

First, consider the case of $F = 0$. Consider some risk profile $\mathcal{Q}$ that has more tail risks than $\mathcal{Q}'$. If $\mathcal{Q}$ has no credit freeze, then by Lemma 3, there exist $\{R_{0\to 1}, \ldots, R_{n\to(n+1)}\}$ such that for all $j \in \{0, \ldots, n\}$:

$$\mathbb{E}[(z_j)_+] \le \mathbb{E}\left[\left(z_j + y_{(j+1)\to j} - R_{(j-1)\to j}\right)_+\right]$$

Let us consider the same set of interest rates in the network with $\mathcal{Q}'$. First, we show by induction that the single-crossing property means $y'_{(j+1)\to j}$ FOSD $y_{(j+1)\to j}$. Again, in equilibrium, the entrepreneur always repays. Assuming $y'_{(j+1)\to j}$ FOSD $y_{(j+1)\to j}$, then:

$$\begin{aligned}\mathbb{E}[y'_{j\to(j-1)}] &= R_{(j-1)\to j}\mathbb{P}\left[z'_j \ge R_{(j-1)\to j} - y'_{(j+1)\to j}\right] \\ &= R_{(j-1)\to j}\Big(\mathbb{P}\left[z'_j \ge R_{(j-1)\to j} - y'_{(j+1)\to j} \middle| z'_j \ge r^*\right]\mathbb{P}[z'_j \ge r^*] \\ &\quad + \mathbb{P}\left[z'_j \ge R_{(j-1)\to j} - y'_{(j+1)\to j} \middle| z'_j < r^*\right]\mathbb{P}[z'_j < r^*]\Big) \\ &= R_{(j-1)\to j}\left(\mathbb{P}[z'_j \ge r^*] + \mathbb{P}\left[z'_j \ge R_{(j-1)\to j} - y'_{(j+1)\to j} \middle| z'_j < r^*\right]\mathbb{P}[z'_j < r^*]\right) \\ &\ge R_{(j-1)\to j}\left(\mathbb{P}[z_j \ge r^*] + \mathbb{P}\left[z_j \ge R_{(j-1)\to j} - y'_{(j+1)\to j} \middle| z_j < r^*\right]\mathbb{P}[z_j < r^*]\right) \\ &\ge R_{(j-1)\to j}\left(\mathbb{P}[z_j \ge r^*] + \mathbb{P}\left[z_j \ge R_{(j-1)\to j} - y_{(j+1)\to j} \middle| z_j < r^*\right]\mathbb{P}[z_j < r^*]\right) \\ &= \mathbb{E}[y_{j\to(j-1)}]\end{aligned}$$

where the first inequality follows from single-crossing at $\lambda \ge r^*$ and the second inequality follows from the inductive hypothesis. Because $y_{j\to(j-1)}$ is binary, this is sufficient for FOSD. We have the following realized values for $(z_j + y_{(j+1)\to j} - R_{(j-1)\to j})_+ - (z_j)_+$ for bank j:

$$\begin{cases} R_{j\to(j+1)} - R_{(j-1)\to j}, \text{ if } z_j \ge 0;\ y_{(j+1)\to j} = R_{j\to(j+1)} \\ R_{j\to(j+1)} - R_{(j-1)\to j} + z_j, \text{ if } R_{(j-1)\to j} - R_{j\to(j+1)} \le z_j < 0;\ y_{(j+1)\to j} = R_{j\to(j+1)} \\ -R_{(j-1)\to j}, \text{ if } z_j \ge R_{(j-1)\to j};\ y_{(j+1)\to j} = 0 \\ -z_j, \text{ if } 0 \le z_j < R_{(j-1)\to j};\ y_{(j+1)\to j} = 0 \end{cases}$$

Note $\mathbf{z}_{-j}$ is a sufficient statistic for $y_{(j+1)\to j}$ and $y'_{(j+1)\to j}$. We can break the above into three regions: (i) $z_j \ge R_{(j-1)\to j}$, (ii) $0 \le z_j \le R_{(j-1)\to j}$, and (iii) $R_{(j-1)\to j} - R_{j\to(j+1)} \le z_j < 0$. In the first region, we have:

$$(R_{j\to(j+1)} - R_{(j-1)\to j})\mathbb{P}\left[z_j \ge R_{(j-1)\to j} \middle| y_{(j+1)\to j} = R_{j\to j+1}\right]\mathbb{P}\left[y_{(j+1)\to j} = R_{j\to j+1}\right]$$

$$
\begin{aligned}
&- R_{(j-1)\to j}\mathbb{P}\left[z_j \geq R_{(j-1)\to j}\middle| y_{(j+1)\to j} = 0\right]\mathbb{P}\left[y_{(j+1)\to j} = 0\right] \\
&= \mathbb{P}\left[z_j \geq R_{(j-1)\to j}\right]\left\{(R_{j\to(j+1)} - R_{(j-1)\to j})\mathbb{P}\left[y_{(j+1)\to j} = R_{j\to j+1}\right]\right. \\
&\quad \left. -R_{(j-1)\to j}\mathbb{P}\left[y_{(j+1)\to j} = 0\right]\right\} \\
&\leq \mathbb{P}\left[z'_j \geq R_{(j-1)\to j}\right]\left\{(R_{j\to(j+1)} - R_{(j-1)\to j})\mathbb{P}\left[y_{(j+1)\to j} = R_{j\to j+1}\right]\right. \\
&\quad \left. -R_{(j-1)\to j}\mathbb{P}\left[y_{(j+1)\to j} = 0\right]\right\} \\
&\leq \mathbb{P}\left[z'_j \geq R_{(j-1)\to j}\right]\left\{(R_{j\to(j+1)} - R_{(j-1)\to j})\mathbb{P}\left[y'_{(j+1)\to j} = R_{j\to j+1}\right]\right. \\
&\quad \left. -R_{(j-1)\to j}\mathbb{P}\left[y'_{(j+1)\to j} = 0\right]\right\}
\end{aligned}
$$

where the equality is from independence, the first inequality is from single-crossing, and the second inequality is from the previous intermediate result about repayment. For the second region, we have:

$$
\begin{aligned}
&(R_{j\to(j+1)} - R_{(j-1)\to j})\mathbb{P}\left[0 \leq z_j \leq R_{(j-1)\to j}\middle| y_{(j+1)\to j} = R_{j\to j+1}\right]\mathbb{P}\left[y_{(j+1)\to j} = R_{j\to j+1}\right] \\
&\quad - \mathbb{E}\left[z_j\middle|0 \leq z_j \leq R_{(j-1)\to j};\, y_{(j+1)\to j} = 0\right]\mathbb{P}\left[0 \leq z_j \leq R_{(j-1)\to j}\middle| y_{(j+1)\to j} \right. \\
&\quad \left. = 0\right]\mathbb{P}\left[y_{(j+1)\to j} = 0\right] \\
&\quad = \mathbb{P}\left[0 \leq z_j \leq R_{(j-1)\to j}\right]\left\{(R_{j\to(j+1)} - R_{(j-1)\to j})\mathbb{P}\left[y_{(j+1)\to j} = R_{j\to j+1}\right]\right. \\
&\quad \left. - \mathbb{E}\left[z_j\middle|0 \leq z_j \leq R_{(j-1)\to j}\right]\mathbb{P}\left[y_{(j+1)\to j} = 0\right]\right\} \\
&\quad = \mathbb{P}\left[0 \leq z'_j \leq R_{(j-1)\to j}\right]\left\{(R_{j\to(j+1)} - R_{(j-1)\to j})\mathbb{P}\left[y_{(j+1)\to j} = R_{j\to j+1}\right]\right. \\
&\quad \left. - \mathbb{E}\left[z'_j\middle|0 \leq z'_j \leq R_{(j-1)\to j}\right]\mathbb{P}\left[y_{(j+1)\to j} = 0\right]\right\} \\
&\quad \leq \mathbb{P}\left[0 \leq z'_j \leq R_{(j-1)\to j}\right]\left\{(R_{j\to(j+1)} - R_{(j-1)\to j})\mathbb{P}\left[y'_{(j+1)\to j} = R_{j\to j+1}\right]\right. \\
&\quad \left. - \mathbb{E}\left[z'_j\middle|0 \leq z'_j \leq R_{(j-1)\to j}\right]\mathbb{P}\left[y'_{(j+1)\to j} = 0\right]\right\}
\end{aligned}
$$

where the first equality follows from independence, the second equality follows from condition (i), and the inequality follows from the intermediate result. Finally, in the third region:

$$
\begin{aligned}
&\left(R_{j\to(j+1)} - R_{(j-1)\to j} + \mathbb{E}\left[z_j\middle|R_{(j-1)\to j} - R_{j\to(j+1)} \leq z_j < 0\right]\right) \\
&\quad \cdot \mathbb{P}\left[R_{(j-1)\to j} - R_{j\to(j+1)} \leq z_j < 0\right]\mathbb{P}\left[y_{(j+1)\to j} = R_{j\to(j+1)}\right] \\
&= \left(R_{j\to(j+1)} - R_{(j-1)\to j} + \mathbb{E}\left[z'_j\middle|R_{(j-1)\to j} - R_{j\to(j+1)} \leq z'_j < 0\right]\right) \\
&\quad \cdot \mathbb{P}\left[R_{(j-1)\to j} - R_{j\to(j+1)} \leq z'_j < 0\right]\mathbb{P}\left[y_{(j+1)\to j} = R_{j\to(j+1)}\right] \\
&\leq \left(R_{j\to(j+1)} - R_{(j-1)\to j} + \mathbb{E}\left[z'_j\middle|R_{(j-1)\to j} - R_{j\to(j+1)} \leq z'_j < 0\right]\right) \\
&\quad \cdot \mathbb{P}\left[R_{(j-1)\to j} - R_{j\to(j+1)} \leq z'_j < 0\right]\mathbb{P}\left[y'_{(j+1)\to j} = R_{j\to(j+1)}\right]
\end{aligned}
$$

These together imply that $\mathbb{E}[\pi'_j] - \mathbb{E}[(z'_j)_+] \geq \mathbb{E}[\pi_j] - \mathbb{E}[(z_j)_+] \geq 0$, so by Lemma 3, there is no credit freeze under $\mathcal{Q}'$. To generalize to any F, simply note that because $y'_{(j+1)\to j}$ FOSD $y_{(j+1)\to j}$, the default probability of any bank j is less with risk profile $\mathcal{Q}'$ (less tail risks), so there continues to be no systemic freeze even when $F > 0$. □

Proof of Proposition 4

Consider some risk profile $\mathcal{Q}$ that is a normal distribution with common mean $\mu > 0$, variance $\sigma > 0$, and correlation ρ for all banks. It is sufficient by continuity in the default cost F to take $F = 0$ and note the result will still hold for all small values of F. Let us consider interest rates $R_{i\to(i+1)} = r_0 + (i+1)\cdot\frac{r^*-r_0}{n+1}$ for all $i \in \{0,\ldots,n\}$. Then the payoff of bank i is given by:

$$\begin{aligned}\mathbb{E}[\pi_j] &= \mathbb{E}[(z_j + y_{(j+1)\to j} - R_{(j-1)\to j})_+]\\ &= \mathbb{E}\left[(z_j + R_{j\to(j+1)} - R_{(j-1)\to j})_+\middle| y_{(j+1)\to j} = R_{j\to(j+1)}\right]\mathbb{P}\left[y_{(j+1)\to j} = R_{j\to(j+1)}\right]\\ &\quad + \mathbb{E}\left[(z_j - R_{(j-1)\to j})_+\middle| y_{(j+1)\to j} = 0\right]\mathbb{P}\left[y_{(j+1)\to j} = 0\right]\end{aligned}$$

For every $\epsilon > 0$, there exists $\rho^* < 1$ such that with correlation $\rho > \rho^*$, $\mathbb{P}[\min_j z_j > 0|z_1 > 0] \geq 1-\epsilon$ and $\mathbb{P}[\max_j z_j > 0|z_1 > 0] \geq 1-\epsilon$. It is clear that when $\min_j z_j > 0$, then $y_{(j+1)\to j} = R_{j\to(j+1)}$ and when $\max_j z_j < 0$, then $y_{(j+1)\to j} = 0$ for all $j \in \{0,\ldots,n\}$. Therefore, the above expression reduces to:

$$\mathbb{E}[\pi_j] \geq (1-\epsilon)\mathbb{P}[z_1 > 0]\left(\mathbb{E}[z_j|z_j > 0] + \frac{r^* - r_0}{n+1}\right)$$

whereas

$$\mathbb{E}[(z_j)_+] \geq \mathbb{E}[z_j|z_j > 0]\mathbb{P}[z_j > 0] = \mathbb{E}[z_j|z_j > 0]\mathbb{P}[z_1 > 0]$$

Taking ϵ close enough to zero (by taking ρ^* close enough to 1), we obtain that $\mathbb{E}[\pi_j] \geq \mathbb{E}[(z_j)_+]$. By Lemma 3, there is no credit freeze with $\mathcal{Q}$ with $\rho > \rho^*$. □

Proof of Proposition 5

For part (a), suppose lending path P gives us the borrowing network $\mathbf{x}_*$ before the adverse shifts. By assumption of (a), $\mathcal{Q}(z_i) = \mathcal{Q}'(z_i)$ for all banks i along the path P, as all banks experiencing an adverse shift experienced a credit freeze before the shift. We know the current lending path P satisfies the conditions of Lemma 3 both before and after the adverse shift, in that $\mathbb{E}[\pi_j^P] - F\cdot\mathbb{P}[\pi_j^P < 0]$ is the same before and after the adverse shifts for all banks $j \in P$, for any interest rates $\mathbf{R}_P$ (as is $\mathbb{E}[(z_j)_+]$ for all $j \in P$). For any other path P', the same logic as in Proposition 2 shows that $\mathbb{E}[\pi_j^{P'}] - F\cdot\mathbb{P}[\pi_j^{P'} < 0]$ is no greater than before the adverse shifts for all $j \in P'$, and that $\mathbb{E}[\pi_j^{P'}] - F\cdot\mathbb{P}[\pi_j^{P'} < 0] - \mathbb{E}[(z_j)_+]$ does not increase after the adverse shifts, given sufficiently large F. This establishes that $\mathbf{x}_*$ is the same before and after the adverse shifts; in particular, no bank on P loses access to credit after the adverse shift.

For (b), if the entrepreneur does not experience a credit freeze after the adverse shift, then by Lemma 3, there exists a path in P isomorphic to the chain network with interest rates $\mathbf{R}$ such that $\mathbb{E}[(z_j)_+] \leq \mathbb{E}[\pi_j]$ along this chain. Note that the chain network $\mathbf{H} \subset \mathbf{G}$ thus does not experience a credit freeze. By Proposition 2, when F is sufficiently large, there is no credit freeze in $\mathbf{H}$ before the adverse shifts. By Proposition 1, this implies there is no systemic freeze in $\mathbf{G}$ after the adverse shifts, and in particular Theorem 3 guarantees the entrepreneur still borrows \$1. So lending does not decrease before the adverse shifts. □

Proof of Proposition 6

We prove this result by induction. Suppose there is just a single bank in $j \in \mathcal{R}$. Let j^* be a borrower of the depositor who also lends (directly or indirectly) to j. The choice of j^* is unique because $\mathbf{G}$ is a tree; to see this, if there were multiple j_1^*, j_2^*, consider two paths P_1, P_2 that both have bank j on it, and by taking the first (topologically from the depositor) bank $k \in P_1 \cap P_2$, we see that k has at least two (potential) lenders, which is a contradiction. Consider all banks $\mathcal{B}^*$ borrowing from the depositor. Let d_j denote the (random) binary variable of whether bank j defaults. For all $k \in \mathcal{B}$ we have that:

$$\mathbb{E}[\pi_0] = \begin{cases} \max_{\{R_{0\to k}\}_{k\in\mathcal{B}^*}} \mathbb{E}\left[\sum_{\{k\in\mathcal{B}^*: R_{0\to k}\neq\varnothing\}} (R_{0\to k}(1 - d_k(\{R_{0\to\ell}\}_{\ell\in\mathcal{B}^*})) - r_0)x_{0\to k}\right] \\ \text{subject to } \ \mathbb{E}\left[\sum_{\{k\in\mathcal{B}^*: R'_{0\to k}\neq\varnothing\}} (R'_{0\to k}(1 - d_k(\{R'_{0\to\ell}\}_{\ell\in\mathcal{B}^*})) - r_0)x'_{0\to k}\right] \\ \leq 0 \ \forall\, R'_{0\to k} \leq R_{0\to k} \text{ for all } k \end{cases}$$

Note that because $\mathbf{G}$ is a tree, $d_k(\{R_{0\to\ell}\}_{\ell\in\mathcal{B}^*}) = d_k(R_{0\to k})$. By linearity of expectation, we have:

$$\mathbb{E}[\pi_0] = \begin{cases} \max_{\{R_{0\to k}\}_{k\in\mathcal{B}^*}} \sum_{\{k\in\mathcal{B}^*: R_{0\to k}\neq\varnothing\}} (R_{0\to k}(1 - \mathbb{E}[d_k(R_{0\to k})]) - r_0)x_{0\to k} \\ \text{subject to } \ \sum_{\{k\in\mathcal{B}^*: R'_{0\to k}\neq\varnothing\}} (R'_{0\to k}(1 - \mathbb{E}[d_k(R'_{0\to k})]) - r_0)x'_{0\to k} \leq 0 \ \forall\, R'_{0\to k} \\ \leq R_{0\to k} \text{ for all } k \end{cases}$$

This is a separable problem because removing the depositor would disconnect the graph, and so the interest rates charged to one bank have no bearing on the payoffs of the other banks linked to the depositor. So, in particular:

$$\mathbb{E}[\pi_0] = \begin{cases} \sum_{\{k\in\mathcal{B}^*: R_{0\to k}\neq\varnothing\}} \max_{R_{0\to k}} (R_{0\to k}(1 - \mathbb{E}[d_k(R_{0\to k})]) - r_0)x_{0\to k} \\ \text{subject to } \ (R'_{0\to k}(1 - \mathbb{E}[d_k(R'_{0\to k})]) - r_0)x'_{0\to k} \leq 0 \ \forall\, R'_{0\to k} \leq R_{0\to k} \text{ for all } k \end{cases}$$

Since no adverse shifts occurred for any banks in the subtrees of $k \neq j^*$, we know that $\mathbb{E}[d_k(R_{0\to k})]$ is the same before and after the adverse shift at bank j (for all $R_{0\to k}$). Because all of the above problems are separable over k, it is clear the financial network $(\mathbf{R}_*, \mathbf{x}_*)$ in all subtrees except possibly the one at bank j^* remains the same. In particular, any of these banks experience a credit freeze if and only if they did so before the adverse shift. Iteratively adding any banks j to $\mathcal{R}$ who experience an adverse shift, and repeating the above argument gives the desired result. □

Proof of Proposition 7

By Lemma 2, consider some set of contracts $R_{0\to 1}, \ldots, R_{n\to n+1}$ such that $\mathbb{E}[\pi_k] \geq \mathbb{E}[(z_k)_+]$ for all banks k before the addition of the risk-bearing bank. Note that because $z_j \geq r^*$, then $z_j + y_{(i+1)\to j} - R_{i\to j} \geq 0$, so bank j never defaults, even if $(i+1)$ does not repay j. Consider the set of contracts $R'_{0\to 1}, \ldots, R'_{(i-1)\to i}, R'_{i\to j}, R'_{j\to(i+1)}, \ldots, R'_{n\to(n+1)}$ with $R'_{k\to k+1} = R_{k\to k+1}$ for all $k \neq i$ and $R'_{i\to j} = R_{(i-1)\to i}, R'_{j\to(i+1)} = R_{i\to(i+1)}$. As in Proposition 2, when F is large it is sufficient to check default probabilities under these contracts are lower with risk-bearing bank i, then Lemma 2 guarantees there will no systemic freeze.

We prove $y'_{(k+1)\to k}$ FOSD $y'_{(k+1)\to k}$ for all $k \neq i$, that $y'_{j\to i}$ FOSD $y_{(i+1)\to i}$, and that $\mathbb{E}[(z_j)_+] \leq \mathbb{E}[\pi_j]$. It is clear that $y_{(k+1)\to k} = y'_{(k+1)\to k}$ for all $k \in \{1, \ldots, i-1\}$. Since

$y'_{j\to i} = R'_{i\to j}$ almost surely, it FOSD all other y, including $y_{(i+1)\to i}$. We prove for $k \in \{i+1, \ldots, 0\}$ by induction. We see that:

$$\begin{aligned}\mathbb{E}[y_{(k+1)\to k}] &= R_{k\to(k+1)}\mathbb{P}[z_{k+1} + y_{(k+2)\to(k+1)} - R_{k\to(k+1)} \geq 0] \\ &\leq R_{k\to(k+1)}\mathbb{P}[z_{k+1} + y'_{(k+2)\to(k+1)} - R_{k\to(k+1)} \geq 0] \\ &= \mathbb{E}[y'_{(k+1)\to k}]\end{aligned}$$

where the inequality follows from the inductive step. Lastly, we know that $\mathbb{E}[(z_i + y_{(i+1)\to i} - R_{(i-1)\to i})_+] - F \cdot \mathbb{P}[z_i + y_{(i+1)\to i} - R_{(i-1)\to i} < 0] \geq 0$. This implies that for large enough F, we have:

$$\mathbb{E}[\pi_j] = \mathbb{E}[z_j + y_{(i+1)\to j} - R'_{i\to j}] \geq \mathbb{E}[z_j + y_{(i+1)\to j} - R_{(i-1)\to i}] \geq \mathbb{E}[z_j]$$

Thus, $\mathbb{E}[\pi_j] \geq \mathbb{E}[(z_j)_+]$ for bank j, and we have showed there is no systemic freeze. □

Proof of Proposition 8

Part (a) of the result follows by the exact same reasoning as Theorem 3 for why in the original economy, the network cannot contain directed cycles. For part (b), note that the contracts offered with quantity-restrictions are a superset of those offered without them. Therefore, extending Lemma 3, if there exists a path P and interest rates $\mathbf{R}_P$ along this path such that the conditions of Lemma 3 hold, then for the same path there exist a set of interest rates and quantity-restrictions given by $(\mathbf{R}_P, |\mathcal{E}| + 1)$ such that the same conditions (i.e., willingness to lend) hold as well, as none of quantity restrictions bind. Thus, there cannot be systemic freezes with quantity restrictions if there is not a systemic freeze in the original economy. □

Proof of Proposition 9

It is enough to prove that if there exists a budget B that restores lending, then giving B to the depositor restores lending. Without loss of generality, suppose $\mathbf{G}$ is a chain. If B restores lending, then by Lemma 3 there exists $\sum_{i=0}^{n} \epsilon_i \leq B$ and $(R_{0\to 1}, \ldots, R_{n\to(n+1)})$ such that $\mathbb{E}[(z_i + \epsilon_i)_+] \leq \mathbb{E}[(z_i + \epsilon_i + y_{(i+1)\to i} - R_{(i-1)\to i})_+]$ for all $i \in \{1, \ldots, n+1\}$, with $\mathbb{E}[\epsilon_0 + y_{1\to 0} - r_0] \geq 0$. Instead, consider giving B entirely to the depositor. Similarly, consider interest rates $R'_{i\to(i+1)} = R_{i\to(i+1)} - \sum_{k=0}^{n-i-1} \epsilon_{n-k}$ for all $i \in \{0, \ldots, n-1\}$. Then:

$$\begin{aligned}\mathbb{E}[(z_i)_+] \leq \mathbb{E}[(z_i + \epsilon_i)_+] &\leq \mathbb{E}[(z_i + \epsilon_i + y_{(i+1)\to i} - R_{(i-1)\to i})_+] \\ &= \mathbb{E}[(z_i + y'_{(i+1)\to i} - R'_{(i-1)\to i})_+]\end{aligned}$$

for all $i \in \{1, \ldots, n\}$, where the equality follows from the fact that $(y'_{(i+1)\to i} - R'_{(i-1)\to i}) - (y_{(i+1)\to i} - R_{(i-1)\to i}) = \epsilon_i$ (and by simple induction, i.e., $\mathbb{P}[y_{(i+1)\to i} = 0] = \mathbb{P}[y'_{(i+1)\to i} = 0]$). Finally, note that for the depositor:

$$0 \leq \mathbb{E}[\epsilon_0 + y_{1\to 0} - r_0] \leq \mathbb{E}[B + y'_{1\to 0} - r_0]$$

which then implies by Lemma 3 there is no systemic freeze. □

Proof of Proposition 10

By Definition 8, we know if bank j is hit with an adverse shift and the freeze is simple, there exists a (direct or indirect) lender j^* of j such that all banks with frozen credit are a (direct or indirect) borrower of bank j^*. Consider the distribution $z'_j - z_j$, where z'_j is the (random) liquidity shock at bank j after the adverse shift and z_j is the liquidity shock before the distribution shift. Then setting $\epsilon_j = z'_j - z_j$ (which requires budget $B^* = \epsilon_j$) reverses the effects of the shock and restores full lending.

Consider B^* to be the smallest budget needed to restore full lending in a targeted policy of the form from (a). Let j^* be the only bank the depositor lends to with (direct or indirect) borrowers whose credit is frozen. Finally, let $x^* > 0$ be the amount lent to all other banks connected to the depositor, other than j^*. We claim that $B^{**} \geq B^* + x^* > B^*$ is the minimum budget required to restore lending in the untargeted policy (if it is possible). Because the freeze is simple, the depositor still uses funds x^* from the central bank to lend to banks other than j^* (i.e., the depositor does not change its lending decisions after the intervention). Thus, for any $B < B^* + x^*$, the depositor would set $R'_{0\to j^*} x'_{0\to j^*} < \max\{0, B - x^*\} + R_{0\to j^*} x_{0\to j^*}$, where $'$ denotes quantities after the rescue policy. By assumption, $\sum_{k\in\mathcal{B}^*(j^*)} \epsilon_k \geq B^*$ is a necessary condition to restore full lending to j^*'s (direct or indirect) borrowers, where $\mathcal{B}^*(j^*)$ is the set of (direct or indirect) borrowers of j^*. By the same reasoning as in Proposition 9, there exists no set of interest rates $\mathbf{R}_{\mathcal{B}^*(j^*)}$ in $\mathcal{B}^*(j^*)$ that mimic such a policy given that $\epsilon_{j^*} = \max\{0, B - x^*\} < B^*$. Thus, no untargeted policy that restores full lending with budget B^* exists. □

C Appendix: Prevalence Theory

This section is dedicated to explaining the relevant details of Ott and Yorke [39] needed for our work. This is of importance when we discuss *generic* risk profiles $\mathcal{Q}$, because the usual definition of "genericity" does not extend well to infinite-dimensional spaces (such as probability distribution functions). The rich theory of Ott and Yorke [39] allows us to handle a wide range of risk profiles (both discrete and continuous) throughout this paper.

We begin with the following discussion from Ott and Yorke [39] on the desirable properties of *geniricity*. If X is a topological vector space, a sound theory of genericity for topological vector spaces should satisfy the following *genericity axioms*.

(i) A generic subset of X is dense in X.
(ii) If $L \supset G$ and G is generic, then L is generic.
(iii) A countable intersection of generic sets is generic.
(iv) Every translate of a generic set is generic.
(v) A subset G of $\mathbb{R}^n$ is generic if and only if G is a set of full Lebesgue measure in $\mathbb{R}^n$.

In standard measure-theoretic terms, a subset $G \subset \mathbb{R}^n$ is said to be generic if $\mathbf{R}^n \backslash G$ has zero Lebesgue measure. This has problems in infinite-dimensional spaces: every separable Banach space with a translation-invariant Boreal measure (which is not identically zero) must assign infinite measure to all open sets. The example provided is the following: take the open ball $B(x, \epsilon)$. We can construct infinitely many disjoint open balls of radius $\epsilon/4$ containing with $B(x, \epsilon)$. Each of the balls has the same measure, and if the measure of $B(x, \epsilon)$ is finite, these balls of radius $\epsilon/4$ must have zero measure. But then the entire space can be covered by $(\epsilon/4)$-radius balls, so the space must have measure 0 (which fails to satisfy Axiom 5).

Definition 10 *(Definition 3.1 in Ott and Yorke* [39]). Let X be a completely metrizable topological vector space. A Borel set $E \subset X$ is said to be *prevalent* if there exists a Borel measure μ on X such that:

(a) $0 < \mu(C) < \infty$ for some compact subset C of X, and

(b) the set $E + x$ has full μ-measure (that is, the complement of $E + x$ has measure zero) for all $x \in X$.

More generally, a subset $F \subset X$ is prevalent if F contains a prevalent Borel set; we say that *almost every* element of X lies in F or that F is generic.

Proposition 11 (Proposition 3.3 in Ott and Yorke [39]) *The theory of prevalence satisfies Axioms 1–5.*

Therefore, when we refer to a property holding for a generic risk profile $\mathcal{Q}$, we mean the set of $\mathcal{Q}$ where this property holds is prevalent in the space of probability distribution functions. We present some useful facts which are useful and leveraged throughout the paper:

Proposition 12 *The following are true:*

(a) *For any constant c, for almost all discrete probability distributions $\mathcal{Q}$, $\mathbb{E}^{\mathcal{Q}}[z_j] \neq c$.*
(b) *For any constant c, for almost all continuous (and differentiable) probability distributions $\mathcal{Q}$, $\mathbb{E}^{\mathcal{Q}}[z_j] \neq c$ (consequence of Example 3.6).*
(c) *For almost all continuous (or countably discrete) probability distributions, $\mathcal{Q}$ is unbounded above and below (consequence of Example 3.9).*
(d) *For almost all continuous probability distributions $\mathcal{Q}$ and continuous (and sufficiently differentiable) functions f, $\mathbb{E}^{\mathcal{Q}}[f(\alpha)]$ has a unique global maximum in α.*

For each of these, the trick is to find a finite-dimensional subspace $P \subset X$ which is known as *probe* for a set $F \subset X$. This holds whenever there exists a Borel set $E \subset F$ such that $E + x$ has full λ_P-measure for all $x \in X$. This is a sufficient condition for a set F to be prevalent. Many genericity conditions in infinite-dimensional spaces (such as those probability distributions) can be proven using prevalence. See the paper Ott and Yorke [39] for examples.

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