Supplemental Appendix to

Factor models with many assets: strong factors, weak factors, and the two-pass procedure

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Abstract: This Supplemental Appendix contains proofs of some theoretical results stated in Anatolyev and Mikusheva "Factor models with many assets: strong factors, weak factors, and the two-pass procedure." In particular, in Section S1 we prove sufficient conditions for Assumption ERRORS stated in Example 1 and Lemma 1 of the paper. In Section S2 we prove validity of Assumptions GAUSSIANITY and COVARIANCE as stated in Lemmas 2 and 3 of the paper. Auxiliary statements are placed in Section S3.

Note on notation: All statements, assumptions, definitions stated in the paper "Factor models with many assets:strong factors, weak factors, and the two-pass procedure" by Anatolyev and Mikusheva keep the notations and labels from the paper, for example, Lemma 2. All new lemmas and equations stated in the Supplementary Appendix have labels starting with S, for example, Lemma S1 or equation (S2).

S1 Relation between different assumptions on idiosyncratic errors

S1.1 Example 1: conditional heteroskedasticity

Example 1. Here we restate it for proper reference. Assume that errors e_{it} have the following weak latent factor structure:

$$e_{it} = \pi_i' w_t + \eta_{it},$$

where (w_t, F_t) is stationary, w_t is a $k_w \times 1$ serially independent, conditional on $\mathcal{F}$, times series with $E(w_t|\mathcal{F}) = 0$ and $E(w_t w_t') = I_{k_w}$ (normalization without loss of generality). Assume $E\left[(\|F_t\|^4 + 1)(\|w_t\|^4 + 1)\right] < \infty$. We assume that the loadings satisfy the conditions

$\sum_{i=1}^N \pi_i\pi_i' \to \Gamma_\pi$ (the factors w_t are weak) and $N^{-1/2}\sum_{i=1}^N \pi_i\gamma_i' \to \Gamma_{\pi\gamma}$. Assume that the random variables η_{it} are independent both cross-sectionally and across time, are independent from w_t and F_t, have mean zero and finite fourth moments and variances σ_i^2 that are bounded above and such that $N^{-1}\sum_{i=1}^N \sigma_i^2 \to \sigma^2$.

Proof that Example 1 satisfies Assumption ERRORS. Assumption ERRORS(i) follows from w_t being serially uncorrelated conditionally on $\mathcal{F}$, and time series independence of η_{it}. For Assumption ERRORS(ii), note that for $t \neq s$,

$$\rho(t,s) = \sum_{i=1}^N \pi_i' \frac{w_s w_t'}{\sqrt{N}} \pi_i + \frac{1}{\sqrt{N}}\sum_{i=1}^N \eta_{it}\eta_{is} + \frac{1}{\sqrt{N}}\sum_{i=1}^N \left(\pi_i' w_t \eta_{is} + \pi_i' w_s \eta_{it}\right).$$

By assumptions made we have $E(w_t w_s'|\mathcal{F}) = 0$ and η_{it}'s independent from $\mathcal{F}$ and w_t's with mean zero, so $E(\rho(s,t)|\mathcal{F}) = 0$. This also implies that in $E(\rho(s,t)^2|\mathcal{F})$ all interaction terms are zero, so we have:

$$\begin{aligned} E(\rho(s,t)^2|\mathcal{F}) = {} & E\left[\left(\sum_{i=1}^N \pi_i' \frac{w_s w_t'}{\sqrt{N}} \pi_i\right)^2 |\mathcal{F}\right] + E\left(\frac{1}{\sqrt{N}}\sum_{i=1}^N \eta_{it}\eta_{is}\right)^2 \\ & + E\left[\left(\frac{1}{\sqrt{N}}\sum_{i=1}^N \pi_i' w_t \eta_{is}\right)^2 |\mathcal{F}\right] + E\left[\left(\frac{1}{\sqrt{N}}\sum_{i=1}^N \pi_i' w_s \eta_{it}\right)^2 |\mathcal{F}\right]. \end{aligned}$$

Note that

$$\begin{aligned} \left|\sum_{i=1}^N \pi_i' w_s w_t' \pi_i\right| = \left|\mathrm{tr}\left(w_s w_t' \sum_{i=1}^N \pi_i\pi_i'\right)\right| & \leq k_w \max \mathrm{ev}\left(w_s w_t' \sum_{i=1}^N \pi_i\pi_i'\right) \\ & \leq k_w \|w_s\|\|w_t\| \max \mathrm{ev}\left(\sum_{i=1}^N \pi_i\pi_i'\right) \leq C\|w_s\|\|w_t\|. \end{aligned}$$

Here we used that the scalar product can be represented as a trace, $\mathrm{tr}(ABC) = \mathrm{tr}(BCA)$, and the trace is equal to a sum of eigenvalues and hence is bounded by its dimension times the maximal eigenvalue. In the last inequality we used that the loadings π_i imply only a weak factor structure. By mutual independence of η_{it}'s, it is easy to see that

$$E\left(\frac{1}{\sqrt{N}}\sum_{i=1}^N \eta_{it}\eta_{is}\right)^2 = \frac{1}{N}\sum_{i=1}^N \sigma_i^2 < C$$

and

$$\operatorname{var}\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\pi_i' w_t \eta_{is}|\mathcal{F}\right) = \frac{1}{N}\sum_{i=1}^{N}\pi_i' E(w_t w_t'|\mathcal{F})\pi_i\sigma_i^2 < \frac{CE(\|w_t\|^2|\mathcal{F})}{N}.$$

Thus,

$$\begin{aligned}E\left[(\|F_t\|^4+1)\rho(s,t)^2\right] &= E\left[(\|F_t\|^4+1)E(\rho(s,t)^2|\mathcal{F})\right]\\ &\leq E\left[(\|F_t\|^4+1)\left(C\frac{(\|w_t\|^2+1)(\|w_s\|^2+1)}{N}+C\right)\right]<\infty,\end{aligned}$$

which proves validity of Assumption ERRORS(ii).

Now consider

$$S_t = \frac{1}{N}\sum_{i=1}^{N}e_{it}^2 = \sum_{i=1}^{N}\pi_i'\frac{w_t w_t'}{N}\pi_i + 2w_t'\frac{\sum_{i=1}^{N}\pi_i\eta_{it}}{N} + \frac{1}{N}\sum_{i=1}^{N}(\eta_{it}^2-\sigma_i^2) + \frac{1}{N}\sum_{i=1}^{N}\sigma_i^2.$$

Denote $\Phi_t = (1, F_t', vec(F_tF_t')')'$. First, let us prove that

$$\frac{\sqrt{N}}{T}\sum_{t=1}^{T}\left(S_t - \frac{1}{N}\sum_{i=1}^{N}\sigma_i^2\right)\Phi_t = o_p(1). \tag{S1}$$

The only non-trivial parts are $\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=1}^{T}\Phi_t w_t'\pi_i\eta_{it} = o_p(1)$ and $\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=1}^{T}\Phi_t(\eta_{it}^2 - \sigma_i^2) = o_p(1)$. For them, we use Chebyshev's inequality:

$$\begin{aligned}E\left[\left\|\frac{1}{\sqrt{N}T}\sum_{i=1}^{N}\sum_{t=1}^{T}\Phi_t w_t'\pi_i\eta_{it}\right\|^2\right] &= \frac{1}{NT^2}\sum_{i=1}^{N}\sigma_i^2 E\left(\sum_{t,s=1}^{T}\Phi_t'\Phi_s\pi_i' w_t w_s'\pi_i\right)\\ &= \frac{1}{NT^2}\sum_{i=1}^{N}\sigma_i^2\pi_i' E\left(\sum_{t=1}^{T}\|\Phi_t\|^2 E(w_t w_t'|\mathcal{F})\right)\pi_i\\ &\leq \frac{k_w}{NT^2}\max\mathrm{ev}\left(\sum_{i=1}^{N}\pi_i\pi_i'\sigma_i^2\right)\\ &\qquad\times\max\mathrm{ev}\left(E\left(\sum_{t=1}^{T}\|\Phi_t\|^2 E(w_t w_t'|\mathcal{F})\right)\right).\end{aligned}$$

For the first equality we used that η_{it}'s are independent from each other and from all F_t's and w_t's; for the second – that w_t's are conditionally serially uncorrelated and have conditional mean zero. By the moment assumptions,

$$\max \operatorname{ev}\left(E\left(\sum_{t=1}^{T}\|\Phi_t\|^2 E(w_t w_t'|\mathcal{F})\right)\right) = \max \operatorname{ev}\left(E\left(\sum_{t=1}^{T}\|\Phi_t\|^2 w_t w_t'\right)\right)$$
$$\leq TE\left[(\|F_t\|^4+\|F_t\|^2+1)\|w_t\|^2\right] \leq CT.$$

The variances σ_i^2 are all bounded and the factors are weak, which leads to

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\Phi_t w_t' \pi_i \eta_{it} = o_p(1).$$

Similarly,

$$E\left[\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\Phi_t(\eta_{it}^2-\sigma_i^2)\right\|^2\right] = \frac{1}{NT^2}\sum_{i=1}^{N}\sum_{t=1}^{T}E\left(\eta_{it}^2-\sigma_i^2\right)E\left[\|\Phi_t\|^2\right] \to 0.$$

This yields validity of statement (S1).

Let us now prove the first statement in Assumption ERRORS(iii):

$$\frac{\sqrt{N}}{T}\sum_{t=1}^{T}\widetilde{F}_t S_t = \frac{\sqrt{N}}{T}\sum_{t=1}^{T}F_t\left(S_t - \frac{1}{N}\sum_{i=1}^{N}\sigma_i^2\right) + \frac{1}{T}\sum_{t=1}^{T}F_t\cdot\sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^2 - \frac{1}{T}\sum_{t=1}^{T}S_t\right).$$

The first term is $o_p(1)$ according to statement (S1) as F_t is a part of Φ_t. Statement (S1) also implies that $\sqrt{N}(\frac{1}{N}\sum_{i=1}^{N}\sigma_i^2 - \frac{1}{T}\sum_{t=1}^{T}S_t) = o_p(1)$, which gives negligibility of the second term.

Now consider the second statement in Assumption ERRORS(iii):

$$\frac{1}{T}\sum_{t=1}^{T}\widetilde{F}_t\widetilde{F}_t' S_t = \frac{1}{T}\sum_{t=1}^{T}F_tF_t'S_t - \overline{F}\frac{1}{T}\sum_{t=1}^{T}\widetilde{F}_t'S_t - \frac{1}{T}\sum_{t=1}^{T}F_tS_t\overline{F}'.$$

We have already proved above that the second term is $o_p(1)$. By equation (S1), the first term equals $\frac{1}{T}\sum_{t=1}^{T}F_tF_t'\frac{1}{N}\sum_{i=1}^{N}\sigma_i^2 + o_p(1) \to^p \sigma^2 E(F_tF_t')$, while the third term equals $-\overline{F}\overline{F}'\frac{1}{N}\sum_{i=1}^{N}\sigma_i^2$ $+ o_p(1) \to^p -\sigma^2 EF_t(EF_t)'$. So, the second statement in Assumption ERRORS(iii) holds with $\Sigma_{SF^2} = \sigma^2\operatorname{var}(F_t)$.

Finally, for Assumption ERRORS(iv), consider

$$W_t = w_t\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\pi_i\gamma_i + \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\gamma_i\eta_{it}.$$

Thus,

$$E\left[(1+\|F_t\|^2)\|W_t\|^2\right] = E\left[(1+\|F_t\|^2)\left\|\frac{\sum_{i=1}^N \gamma_i\pi_i' w_t}{\sqrt{N}}\right\|^2\right] + E\left[(1+\|F_t\|^2)\right] E\left[\left\|\frac{\sum_{i=1}^N \gamma_i\eta_{it}}{\sqrt{N}}\right\|^2\right].$$

Notice that $E(w_t|\mathcal{F}) = 0$ and that η_{it}'s are independent from all other variables, thus there are no terms containing the first power of η_{it}. Now,

$$\begin{aligned}
\left\|\frac{\sum_{i=1}^N \gamma_i\pi_i' w_t}{\sqrt{N}}\right\|^2 &= \left(\frac{1}{\sqrt{N}}\sum_{i=1}^N \gamma_i\pi_i'\right) w_t w_t' \left(\frac{1}{\sqrt{N}}\sum_{i=1}^N \gamma_i\pi_i'\right)' \\
&= \operatorname{tr}\left(w_t w_t' \left(\frac{1}{\sqrt{N}}\sum_{i=1}^N \gamma_i\pi_i'\right)' \left(\frac{1}{\sqrt{N}}\sum_{i=1}^N \gamma_i\pi_i'\right)\right) \\
&\le k_w\|w_t\|^2 \max\operatorname{ev}\left(\left(\frac{1}{\sqrt{N}}\sum_{i=1}^N \gamma_i\pi_i'\right)' \left(\frac{1}{\sqrt{N}}\sum_{i=1}^N \gamma_i\pi_i'\right)\right) < C\|w_t\|^2.
\end{aligned}$$

The assumptions on the loadings guarantee that the maximal eigenvalue is bounded above by a constant. Next,

$$E\left[\left\|\frac{\sum_{i=1}^N \gamma_i\eta_{it}}{\sqrt{N}}\right\|^2\right] = \frac{1}{N}\sum_{i=1}^N \|\gamma_i\|^2\sigma_i^2 < C.$$

Thus, Assumption ERRORS (iv) is valid as well. □

S1.2 Case of independence between errors and factors. Proof of Lemma 1

Assumption ERRORS*

(i) The factors $\{F_t,\ t=1,...,T\}$ are independent from errors $\{e_{it},\ i=1,...,N,\ t=1,...,T\}$; the error terms $e_t = (e_{1t},...,e_{Nt})'$ are serially independent and identically distributed for different t with $Ee_{it}=0$ and $\sup_{i,t} Ee_{it}^4 < \infty$.

(ii) Let $\mathcal{E}_{N,T} = E\left[e_t e_t'\right]$ be the $N\times N$ cross-sectional covariance matrix. For some positive constants a, c and C, we have $\lim_{N,T}\frac{1}{N}\operatorname{tr}(\mathcal{E}_{N,T}) = a$ and

$$c < \liminf_{N,T\to\infty} \min\operatorname{ev}\left(\mathcal{E}_{N,T}\right) < \limsup_{N,T\to\infty} \max\operatorname{ev}\left(\mathcal{E}_{N,T}\right) < C.$$

(iii) $E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(e_{it}^2 - Ee_{it}^2)\right|^2 < C.$

Lemma 1 Assumption LOADINGS and Assumption ERRORS* imply Assumption ERRORS.

Proof of Lemma 1. Assumption ERRORS*(i) implies Assumption ERRORS(i). Given the independence between the two groups of variables and the moment condition for F_t stated in Assumption FACTORS, in order to prove Assumption ERRORS(ii) we need to show that $\sup_{s\neq t} E\rho(s,t)^2$ is bounded from above. Indeed,

$$E\rho(s,t)^2 = \frac{1}{N}\sum_{i,j=1}^{N} E[e_{it}e_{is}e_{jt}e_{js}] = \frac{1}{N}\sum_{i,j=1}^{N} E[e_{it}e_{jt}]E[e_{is}e_{js}] = \frac{1}{N}\mathrm{tr}(\mathcal{E}_{N,T}\mathcal{E}_{N,T}).$$

Here we used serial independence in Assumption ERRORS*(i) and the definition of covariance matrices. For any positive definite $N\times N$ matrix A we have $\mathrm{tr}(A^2) = \sum_{i=1}^{N}\lambda_i(A)^2 \leq N\left(\max \mathrm{ev}(A)\right)^2$, where $\lambda_i(A)$, $i = 1,...,N$, are eigenvalues of A. Thus, due to Assumption ERRORS*(ii), we have $\mathrm{tr}(\mathcal{E}_{N,T}\mathcal{E}_{N,T}) \leq NC^2$. Thus, the right hand side of the last displayed equation is bounded from above.

Assumption ERRORS (iii): Notice that since $\sum_{t=1}^{T}\widetilde{F}_t = 0$, we have

$$\frac{\sqrt{N}}{T}\sum_{t=1}^{T}\widetilde{F}_t S_t = \frac{\sqrt{N}}{T}\sum_{t=1}^{T}\widetilde{F}_t\big(S_t - \overline{\sigma_N^2}\big),$$

where we denote $\overline{\sigma_N^2} = N^{-1}\sum_{i=1}^{N} Ee_{it}^2$. Let us check that the second moment of the last expression converges to zero:

$$E\left\|\frac{\sqrt{N}}{T}\sum_{t=1}^{T}\widetilde{F}_t\big(S_t - \overline{\sigma_N^2}\big)\right\|^2 = \frac{N}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}E\left[\widetilde{F}_t'\widetilde{F}_s\big(S_t - \overline{\sigma_N^2}\big)\big(S_s - \overline{\sigma_N^2}\big)\right]$$

Given Assumption ERRORS*(i), only those terms survive that have $s = t$:

$$\frac{N}{T^2}\sum_{t=1}^{T}E[\widetilde{F}_t\widetilde{F}_t']E\left[\big(S_t - \overline{\sigma_N^2}\big)^2\right].$$

Notice that

$$NE\left[\big(S_t - \overline{\sigma_N^2}\big)^2\right] = E\left[\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(e_{it}^2 - Ee_{it}^2)\right|^2\right] < C,$$

using Assumption ERRORS*(iii). Thus, the first statement in Assumption ERRORS(iii) holds.

For the second statement, note that

$$E\left(\frac{1}{T}\sum_{t=1}^{T}\widetilde{F}_t\widetilde{F}_t'S_t\right)=\Sigma_F\frac{1}{N}\mathrm{tr}(\mathcal{E}_T)\to a\Sigma_F=\Sigma_{SF^2}.$$

In order to prove the second statement in Assumption ERRORS(iii) we will show that

$$T^{-1}\sum_{t=1}^{T}\widetilde{F}_t\widetilde{F}_t'(S_t-\overline{\sigma_N^2})\to^p 0.$$

Following the same steps as in the proof of the first statement,

$$E\left\|\frac{1}{T}\sum_{t=1}^{T}\widetilde{F}_t\widetilde{F}_t'(S_t-\overline{\sigma_N^2})\right\|^2=\frac{1}{T^2}\sum_{t=1}^{T}E[\|\widetilde{F}_t\|^4]E\left[(S_t-\overline{\sigma_N^2})^2\right],$$

and we showed before $E\left[(S_t-\overline{\sigma_N^2})^2\right]\to 0$. This finishes a proof of validity of Assumption ERRORS(iii).

Lastly,

$$E\left[\|W_t\|^2\right]=\frac{1}{N}\gamma'\mathcal{E}_{N,T}\gamma\le\frac{1}{N}\|\gamma\|^2\max\mathrm{ev}(\mathcal{E}_{N,T})<C.$$

Thus, Assumption ERRORS (iv) holds as well. □

S2 Statements about gaussianity. Proofs of Lemmas 2 and 3

S2.1 Re-statement of the Lemmas

For a set of vectors a_j, we denote by $(a_j)_{j=1}^4=(a_1',...,a_4')'$ a long vector consisting of the four vectors stacked upon each other; we denote by $(a_{jj^*})_{j<j^*}$ the vectors a_{jj^*} stacked together.

Assumption GAUSSIANITY Assume that the following convergence holds:

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\begin{pmatrix}\sqrt{T}\gamma_i\bar{e}_i\\ (\sqrt{T}\gamma_i u_i^{(j)})_{j=1}^{4}\\ (T\bar{e}_i u_i^{(j)})_{j=1}^{4}\\ (Tu_i^{(j)}u_i^{(j^*)})_{j<j^*}\end{pmatrix}=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\xi_i\Rightarrow\xi=\begin{pmatrix}\xi_{\gamma e}\\ (\xi_{\gamma j})_{j=1}^{4}\\ (\xi_{ej})_{j=1}^{4}\\ (\xi_{j,j^*})_{j<j^*}\end{pmatrix},$$

where ξ is a Gaussian vector with mean zero and covariance Σ_ξ.

Assumption COVARIANCE Assume that $\frac{1}{N}\sum_{i=1}^{N}\xi_i\xi_i' \to^p \Sigma_\xi$, where ξ_i and Σ_ξ are defined in Assumption GAUSSIANITY.

The assumptions we maintained in the previous sections are enough to guarantee that $\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\xi_i$ is $O_p(1)$. Assumption GAUSSIANITY establishes the asymptotic distribution of that quantity, while Assumption COVARIANCE allows one to construct valid standard errors. Below we provide sufficient conditions for the two new assumptions in the two leading examples discussed before: one where the observed factors are independent from the errors and the example of factor-driven conditional heteroskedasticity.

Lemma 2 Assume that Assumption ERRORS* holds and additionally,

(i) $E\|F_t\|^8<\infty$; $E\left\|\frac{1}{|T_j|}\sum_{t\in T_j}F_tF_t'-\Sigma_F\right\|\to 0$;

(ii) $\max_i\|\gamma_i\|<C$;

(iii) $\frac{1}{N}\mathrm{tr}(\mathcal{E}_{N,T}^2)\to a_2$ and $\frac{1}{N}\gamma'\mathcal{E}_{N,T}\gamma\to\Gamma_\sigma$, where Γ_σ is a full rank matrix;

(iv) $\frac{1}{N^2}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\sum_{i_3=1}^{N}\sum_{i_4=1}^{N}|Ee_{i_1t}e_{i_2t}e_{i_3t}e_{i_4t}|<C$;

then Assumption GAUSSIANITY holds. If in addition

$$\|\mathcal{E}_{N,T}-\mathrm{dg}(\mathcal{E}_{N,T})\|\to 0\quad\text{as } N,T\to\infty,$$

then Assumption COVARIANCE holds as well.

Lemma 3 Assume we have a setting as in Example 3. Assume additionally that conditions (i) and (ii) of Lemma S2.1 hold and the following is true:

(i) $E\left[(\|F_t\|^8+1)\|w_t\|^8\right]<\infty$;

(ii) $\frac{1}{N}\sum_{i=1}^{N}\sigma_i^4 \to \mu_4$ and $\frac{1}{N}\sum_{i=1}^{N}\sigma_i^2\gamma_i\gamma_i' \to \Gamma_\sigma$, where Γ_σ is a full rank matrix.

Then Assumption GAUSSIANITY holds. If in addition $\Gamma_{\pi\gamma} = 0$, then Assumption COVARIANCE holds as well.

Proof of common part of Lemmas 2 and 3. First, rewrite different components of ξ_i:

$$\frac{\sqrt{T}}{\sqrt{N}}\sum_{i=1}^{N}\gamma_i\bar{e}_i = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{s=1}^{T}\gamma_i e_{is} = \frac{1}{\sqrt{T}}\sum_{s=1}^{T}\frac{\gamma' e_s}{\sqrt{N}};$$

$$\frac{\sqrt{T}}{\sqrt{N}}\sum_{i=1}^{N}\gamma_i \otimes u_i^{(j)} = \frac{1}{\sqrt{T}}\sum_{s=1}^{T}\left(\frac{\gamma' e_s}{\sqrt{N}}\right)\otimes\left(\Sigma_F^{-1}\widetilde{F}_s^{(j)}\right)\mathbb{I}\{s\in T_j\}.$$

Consider the following sum:

$$\begin{aligned}\frac{T}{\sqrt{N}}\sum_{i=1}^{N}\bar{e}_i u_i^{(j)} &= \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t_1=1}^{T}\sum_{t_2\in T_j}\Sigma_F^{-1}\widetilde{F}_{t_2}^{(j)}e_{it_1}e_{it_2}\\ &= \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t\in T_j}\Sigma_F^{-1}\widetilde{F}_t^{(j)}e_{it}^2 + \frac{1}{T}\sum_{t_1=1}^{T}\sum_{t_2\in T_j, t_1\neq t_2}\Sigma_F^{-1}\widetilde{F}_{t_2}^{(j)}\frac{e_{t_1}'e_{t_2}}{\sqrt{N}}.\end{aligned}$$

Assumption ERRORS(iii) guarantees that $\left(T\sqrt{N}\right)^{-1}\sum_{i=1}^{N}\sum_{t\in T_j}\Sigma_F^{-1}\widetilde{F}_t^{(j)}e_{it}^2 = o_p(1)$. Thus, we are only interested in gaussianity of the second sum. Thus,

$$\frac{T}{\sqrt{N}}\sum_{i=1}^{N}\bar{e}_i u_i^{(j)} = \frac{1}{T}\sum_{s=1}^{T}\sum_{t<s}\Sigma_F^{-1}\left(\widetilde{F}_t^{(j)}\mathbb{I}\{t\in T_j\} + \widetilde{F}_s^{(j)}\mathbb{I}\{s\in T_j\}\right)\frac{e_t'e_s}{\sqrt{N}} + o_p(1).$$

Now assume that $j^* > j$ and consider the following sum:

$$\begin{aligned}\mathrm{vec}\left(\frac{T}{\sqrt{N}}\sum_{i=1}^{N}u_i^{(j^*)}u_i^{(j)\prime}\right) &= \frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\sum_{t\in T_j}\sum_{s\in T_{j^*}}\mathrm{vec}\left(\Sigma_F^{-1}\widetilde{F}_s^{(j^*)}\widetilde{F}_t^{(j)\prime}\Sigma_F^{-1}e_{it}e_{is}\right)\\ &= \sum_{s\in T_{j^*}}\sum_{t\in T_j}\frac{1}{T}\left(\Sigma_F^{-1}\widetilde{F}_s^{(j^*)}\right)\otimes\left(\Sigma_F^{-1}\widetilde{F}_t^{(j)}\right)\frac{e_t'e_s}{\sqrt{N}}.\end{aligned}$$

Define

$$v_s = \begin{pmatrix} 1 \\ \left\{\left(\Sigma_F^{-1}\widetilde{F}_s^{(j)}\right)\mathbb{I}\{s\in T_j\}\right\}_{j=1}^{4}\end{pmatrix};$$

$$w_{st}^{(j)} = \Sigma_F^{-1}\left(\widetilde{F}_s^{(j)}\mathbb{I}\{s\in T_j\} + \widetilde{F}_t^{(j)}\mathbb{I}\{t\in T_j\}\right);$$

$$w_{st}^{(j,j^*)} = \left(\Sigma_F^{-1}\widetilde{F}_s^{(j^*)}\right) \otimes \left(\Sigma_F^{-1}\widetilde{F}_t^{(j)}\right)\mathbb{I}\{t \in T_j, s \in T_{j^*}\}.$$

Now we can stack all the vectors $w_{st}^{(j,j^*)}$ and $w_{st}^{(j)}$ together. Call the resulting vector $w_{st} = (w_{st}^{(1)}, ..., w_{st}^{(1,2)\prime}, w_{st}^{(1,3)\prime}, ...)'$. Notice that

$$\frac{1}{\sqrt{N}}\sum_i \xi_i = \left(\begin{array}{c} \frac{1}{\sqrt{T}}\sum_{s=1}^T v_s \frac{1}{\sqrt{N}}\sum_{i=1}^N \gamma_i e_{is} \\ \frac{1}{T}\sum_{s=1}^T \sum_{t<s} w_{st}\frac{1}{\sqrt{N}}\sum_{i=1}^N e_{it}e_{is} \end{array}\right).$$

We will use the results established in the paper by Anatolyev and Mikusheva (2020); namely Theorems 3.1 and 4.1 from Anatolyev and Mikusheva (2020) will be used to prove Lemma 2, and Theorems 3.2 and 4.2 from Anatolyev and Mikusheva (2020) will be used to prove Lemma 3.

Now we check that Assumptions C, L and E from Anatolyev and Mikusheva (2020) hold in both Lemma 2 and Lemma 3.

For Assumption C(i), taking into account the structure of $v_s = \left(v_s^{0\prime}, \{v_s^{(j)\prime}\}_{j=1}^4\right)'$ and that $|T_j|/T = \frac{1}{4}$, we compute that

$$\frac{1}{T}\sum_{s=1}^T E(v_s v_s') \to \left(\begin{array}{cc} 1 & 0 \\ 0 & \frac{1}{4}I_4 \otimes \Sigma_F^{-1} \end{array}\right).$$

Consider now various sub-blocks of $\Omega_w = \lim \frac{1}{T^2}\sum_{s=1}^T \sum_{t<s} E(w_{st}w_{st}')$ and show that it is a full rank matrix. First, notice that $w_{st}^{(j,j^*)}w_{st}^{(j_1,j_1^*)\prime}$ is a zero matrix when $(j,j^*) \neq (j_1, j_1^*)$. In the same way, $w_{st}^{(j,j^*)}w_{st}^{(j_1)\prime}$ is a zero matrix when $j_1 \notin (j, j^*)$. Notice that

$$\begin{aligned} E\left[\left(\widetilde{F}_t^{(j)}\widetilde{F}_t^{(j)\prime}\right) \otimes \left(\widetilde{F}_s^{(j^*)}\widetilde{F}_s^{(j^*)\prime}\right)\right] &= \left(E\widetilde{F}_t^{(j)}\widetilde{F}_t^{(j)\prime}\right) \otimes \left(E\widetilde{F}_s^{(j^*)}\widetilde{F}_s^{(j^*)\prime}\right) \\ &+ E\left[\left(\widetilde{F}_t^{(j)}\widetilde{F}_t^{(j)\prime} - E\widetilde{F}_t^{(j)}\widetilde{F}_t^{(j)\prime}\right) \otimes \left(\widetilde{F}_s^{(j^*)}\widetilde{F}_s^{(j^*)\prime} - E\widetilde{F}_s^{(j^*)}\widetilde{F}_s^{(j^*)\prime}\right)\right]. \end{aligned}$$

For the $(j,j^*)^{\text{th}}$ block, we take $T^{-2}\sum_{s\in T_{j^*}}\sum_{t\in T_j}$ of the last displayed expression. Due to summability of covariances, the double average of covariances coming from the second term becomes negligible in the limit. Thus,

$$\frac{1}{T^2}\sum_{s=1}^T \sum_{t<s} E\left[w_{st}^{(j,j^*)}w_{st}^{(j,j^*)\prime}\right] \to \lim\left(\frac{|T_j||T_{j^*}|}{T^2}\right)\Sigma_F^{-1} \otimes \Sigma_F^{-1} = \frac{1}{16}\Sigma_F^{-1} \otimes \Sigma_F^{-1}.$$

Let us now focus on the block corresponding to the variance of the j^{th} term:

$$
\begin{aligned}
&\frac{1}{T^2}\sum_{s=1}^{T}\sum_{t<s} E\left[\left(\widetilde{F}_s^{(j)}\mathbb{I}\{s\in T_j\}+\widetilde{F}_t^{(j)}\mathbb{I}\{t\in T_j\}\right)\left(\widetilde{F}_s^{(j)}\mathbb{I}\{s\in T_j\}+\widetilde{F}_t^{(j)}\mathbb{I}\{t\in T_j\}\right)'\right]\\
&=\frac{1}{T^2}\sum_{s=1}^{T}\sum_{t<s}\Big(E\big[\widetilde{F}_s^{(j)}\widetilde{F}_s^{(j)\prime}\big]\mathbb{I}\{s\in T_j\}+E\big[\widetilde{F}_t^{(j)}\widetilde{F}_t^{(j)\prime}\big]\mathbb{I}\{t\in T_j\}\\
&\qquad\qquad\qquad+\Big(E\big[\widetilde{F}_s^{(j)}\widetilde{F}_t^{(j)\prime}\big]+E\big[\widetilde{F}_t^{(j)}\widetilde{F}_s^{(j)\prime}\big]\Big)\,\mathbb{I}\{s,t\in T_j\}\Big)\\
&=\frac{1}{T^2}\sum_{t_1=1}^{T}\sum_{t_2\in T_j}E\big[\widetilde{F}_{t_2}^{(j)}\widetilde{F}_{t_2}^{(j)\prime}\big]+\frac{1}{T^2}\sum_{t_1\in T_j}\sum_{t_2\in T_j}E\big[\widetilde{F}_{t_1}^{(j)}\widetilde{F}_{t_2}^{(j)\prime}\big].
\end{aligned}
$$

Again, due to stationarity and summability of covariances, the second term is negligible, so we have, in the limit,

$$
\frac{1}{T^2}\sum_{s=1}^{T}\sum_{t<s}E\big[w_{st}^{(j)}w_{st}^{(j)\prime}\big]\to\lim\left(\frac{|T_j|T}{T^2}\right)\Sigma_F^{-1}E\left[\widetilde{F}_t\widetilde{F}_t'\right]\Sigma_F^{-1}=\frac{1}{4}\Sigma_F^{-1}.
$$

By similar arguments, $\frac{1}{T^2}\sum_{s=1}^{T}\sum_{t<s}E\big[w_{st}^{(j)}w_{st}^{(j^*)\prime}\big]\to 0$ and $\frac{1}{T^2}\sum_{s=1}^{T}\sum_{t<s}E\big[w_{st}^{(j,j^*)}w_{st}^{(j)\prime}\big]\to 0$ when $j\neq j^*$. To summarize, we have shown that Ω_w is a block diagonal matrix, with each block being a full-rank matrix.

Assumptions C(ii) and C(iv) follow from condition (i) of Lemma 2. For Assumption C(iii), notice that the proof of convergence of all blocks follows the same outline as above, and condition (i) of Lemma 2 is sufficient. Assumption L is the same as condition (ii) of Lemma 2. Validity of Assumption E has been shown before in Lemma 1.

Proof of Lemma 2. Assumption I(i) was stated as a part of Assumption ERRORS*. Assumptions I(ii)-(iv) of Anatolyev and Mikusheva (2020) have been formulated as conditions (iii) and (iv) of Lemma 2.

Proof of Lemma 3. Assumption HC(i) of Anatolyev and Mikusheva (2020) comes from condition (i) of Lemma 3 and assumptions of Example 3. Assumption HC(ii) is implied by assumptions of Example 3. Assumption HC(iii) follows from condition (ii) of Lemma 3. Finally, Assumption HC(iv) of Anatolyev and Mikusheva (2020) follows from the assumption in Example 3 that w_t is, conditionally on $\mathcal{F}$, time-series independent, and hence $(w_t w_t)\otimes(v_t v_t)$ is not autocorrelated; thus, the law of large numbers applies coming from the moment condition stated in condition (i) of Lemma 3.

S3 Auxiliary lemma

Lemma S1 Under Assumptions FACTORS, LOADINGS and ERRORS, we have the following convergence as $N, T \to \infty$:

(1) $\frac{1}{\sqrt{N|T_k||T_j|}} \sum_{i=1}^N \sum_{t \in T_j} \sum_{s \in T_k} \widetilde{F}_t e_{it} \widetilde{F}_s' e_{is} = O_p(1)$ for $T_j \cap T_k = \varnothing$,

(2) $\frac{1}{\sqrt{N|T_k||T_j|}} \sum_{i=1}^N \sum_{t \in T_j} \sum_{s \in T_k} \widetilde{F}_t e_{it} e_{is} = O_p(1)$ for $T_j \cap T_k = \varnothing$,

(3) $\Sigma_F^{-1} \left(\frac{1}{NT} \sum_{i=1}^N \sum_{t=1}^T \widetilde{F}_t \widetilde{F}_t' e_{it}^2 \right) \Sigma_F^{-1} \to^p \Sigma_u = \Sigma_F^{-1} \Sigma_{SF^2} \Sigma_F^{-1}$,

(4) $\frac{1}{T\sqrt{N}} \sum_{i=1}^N \left(\sum_{t=1}^T \sum_{s=1, s \neq t}^T \widetilde{F}_t (\widetilde{F}_s', 1) e_{it} e_{is} \right) = O_p(1)$,

(5) $\sqrt{\frac{|T_j|}{N}} \sum_{i=1}^N \gamma_i \otimes \begin{pmatrix} u_i^{(j)} \\ \frac{1}{|T_j|} \sum_{t \in T_j} e_{it} \end{pmatrix} = O_p(1)$.

Proof of Lemma S1.

Preamble. Notice that due to the absence of serial correlation of idiosyncratic errors stated in Assumption ERRORS(i), for $t \neq s$ and $t_1 \neq s_1$ we have

$$E(\rho(s,t)\rho(s_1,t_1)|\mathcal{F}) = 0$$

unless $t = t_1$ and $s = s_1$ or $t = s_1$ and $s = t_1$.

Part (1). Note that

$$\frac{1}{\sqrt{N|T_k||T_j|}} \sum_{i=1}^N \sum_{t \in T_j} \sum_{s \in T_k} \widetilde{F}_t e_{it} \widetilde{F}_s' e_{is} = \frac{1}{\sqrt{|T_k||T_j|}} \sum_{t \in T_j} \sum_{s \in T_k} \widetilde{F}_t \widetilde{F}_s' \rho(t,s).$$

The expectation of the square of the last expression is equal to

$$\frac{1}{|T_k||T_j|} \sum_{t \in T_j} \sum_{t_1 \in T_j} \sum_{s \in T_k} \sum_{s_1 \in T_k} E\left(\widetilde{F}_t \widetilde{F}_s' \widetilde{F}_{t_1} \widetilde{F}_{s_1}' E\big(\rho(t,s)\rho(t_1,s_1)|\mathcal{F}\big) \right).$$

Using the preamble statement, we reduce four summation signs to only two, with each summand bounded above by Assumption ERRORS(ii). This implies that the second moment of the last sum is bounded, and hence implies statement (1).

Part (2). Analogously to Part (1).

Part (3). Note that

$$\frac{1}{NT}\sum_{i=1}^{N}\sum_{t=1}^{T}\widetilde{F}_t\widetilde{F}_t'e_{it}^2 = \frac{1}{T}\sum_{t=1}^{T}\widetilde{F}_t\widetilde{F}_t'S_t.$$

Then, Part (3) follows from Assumption ERRORS(iii).

Part (4). Note that

$$\frac{1}{T\sqrt{N}}\sum_{i=1}^{N}\left(\sum_{t=1}^{T}\sum_{s=1,s\neq t}^{T}\widetilde{F}_t e_{it}e_{is}\right) = \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1,s\neq t}^{T}\widetilde{F}_t\rho(s,t).$$

The second moment of this expression contains four summations – over t, $s \neq t$, t_1 and $s_1 \neq t_1$. However, by the preamble statement many terms are zero and the expression can be written as a double sum. Assumption ERRORS(ii) guarantees that all summands are bounded by the same constant, which leads to boundedness of the second moment of the expression of interest. Chebyshev's inequality delivers statement (4).

Part (5). Observe that

$$\begin{aligned}\sqrt{\frac{|T_j|}{N}}\sum_{i=1}^{N}\gamma_i\otimes\begin{pmatrix}u_i^{(j)}\\ \frac{1}{|T_j|}\sum_{t\in T_j}e_{it}\end{pmatrix} &= \frac{1}{\sqrt{|T_j|}}\sum_{t\in T_j}\sum_{i=1}^{N}\frac{1}{\sqrt{N}}\left(\gamma_i\otimes\begin{pmatrix}\Sigma_F^{-1}\widetilde{F}_t^{(j)}\\ 1\end{pmatrix}\right)e_{it}\\ &= \frac{1}{\sqrt{|T_j|}}\sum_{t\in T_j}W_t\otimes\begin{pmatrix}\Sigma_F^{-1}\widetilde{F}_t^{(j)}\\ 1\end{pmatrix},\end{aligned}$$

where W_t is defined in Assumption ERRORS(iv). Given serial independence of e_t conditional on $\mathcal{F}$, we obtain that W_t is conditionally serially independent and mean zero, and hence

$$\begin{aligned}E\left[\left\|\frac{1}{\sqrt{|T_j|}}\sum_{t\in T_j}W_t\otimes\begin{pmatrix}\Sigma_F^{-1}\widetilde{F}_t^{(j)}\\ 1\end{pmatrix}\right\|^2\right] &= \frac{1}{|T_j|}\sum_{t\in T_j}E\left[\left\|W_t\otimes\begin{pmatrix}\Sigma_F^{-1}\widetilde{F}_t^{(j)}\\ 1\end{pmatrix}\right\|^2\right]\\ &\leq CE\left[(1+\|F_t\|^2)\|W_t\|^2\right] < C.\end{aligned}$$

□

References

Anatolyev, S. and Mikusheva, A. (2020): "Limit Theorems for Factor Models," *Econometric Theory*, forthcoming. Available at `https://pages.nes.ru/sanatoly/Papers/CLTfm.htm`