Online Appendix for "Lerner Symmetry: A Modern Treatment"

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Abstract

This Appendix provides the proofs of Theorem 1, Theorem 2, and Proposition 1.

1 Perfect Competition

For convenience, we first repeat the definition of a competitive equilibrium as well as assumptions A1-A3. We then offer a formal proof of Theorem 1.

1.1 Equilibrium

A competitive equilibrium with taxes, $t \equiv \{t_{ij}^k(n)\}$, subsidies, $s \equiv \{s_{ij}^k(n)\}$, and lump-sum transfers, $\tau \equiv \{\tau(h)\}$ and $T \equiv \{T_{ij}\}$, corresponds to quantities $c \equiv \{c(h)\}$, $l \equiv \{l(h)\}$, $m \equiv \{m(f)\}$, $y \equiv \{y(f)\}$, and prices $p \equiv \{p_{ij}^k\}$ such that:

(i) $(c(h), l(h))$ solves

$$\max_{(\hat{c}(h),\hat{l}(h))\in\Gamma(h)} u(\hat{c}(h), \hat{l}(h); h)$$

$$p(1 + t(h)) \cdot \hat{c}(h) = p(1 + s(h)) \cdot \hat{l}(h) + \pi \cdot \theta(h) + \tau(h), \text{ for all } h;$$

(ii) $(m(f), y(f))$ solves

$$\pi(f) \equiv \max_{(\hat{m}(f),\hat{y}(f))\in\Omega(f)} p(1 + s(f)) \cdot \hat{y}(f) - p(1 + t(f)) \cdot \hat{m}(f), \text{ for all } f;$$

(iii) markets clear:

$$\sum_f y(f) + \sum_h l(h) = \sum_h c(h) + \sum_f m(f);$$

(iv) government budget constraints hold:

$$\sum_{j,k} p_{ji}^k (\sum_h t_{ji}^k(h) c_{ji}^k(h) + \sum_f t_{ji}^k(f) m_{ji}^k(f)) + \sum_{j\neq i} T_{ji}$$
$$= \sum_{j,k} p_{ij}^k (\sum_h s_{ij}^k(h) l_{ij}^k(h) + \sum_f s_{ij}^k(f) y_{ij}^k(f)) + \sum_{h\in H_i} \tau(h) + \sum_{j\neq i} T_{ij}, \text{ for all } i;$$

1.2 Assumptions

A1. *For any firm f, production sets can be separated into*

$$\Omega(f) = \Omega_{i_0}(f) \times \Omega_{-i_0}(f),$$

where $\Omega_{i_0}(f)$ denotes the set of feasible production plans, $\{m_{ji_0}^k(f), y_{i_0 j}^k(f)\}$, in country i_0 and $\Omega_{-i_0}(f)$ denotes the set of feasible plans, $\{m_{ji}^k(f), y_{ij}^k(f)\}_{i\neq i_0}$, in other countries.

A2. *For any household h, consumption sets can be separated into*

$$\Gamma(h) = \Gamma_{i_0}(h) \times \Gamma_{-i_0}(h),$$

where $\Gamma_{i_0}(h)$ *denotes the set of feasible consumption plans,* $\{c^k_{ji_0}(f), l^k_{i_0 j}(f)\}$*, in country* i_0*;* $\Gamma_{-i_0}(h)$ *denotes the set of feasible plans,* $\{c^k_{ji}(f), l^k_{ij}(f)\}_{i \neq i_0}$*, in other countries; and* $\Gamma_{i_0}(h)$ *and* $\Gamma_{-i_0}(h)$ *are such that* $h \in H_{i_0} \Rightarrow \Gamma_{-i_0}(h) = \{0\}$ *and* $h \notin H_{i_0} \Rightarrow \Gamma_{i_0}(h) = \{0\}$.

A3. *For any foreign country* $j \neq i_0$*, the total value of assets held in country* i_0 *prior to the tax reform is zero,* $\pi_{i_0} \cdot \sum_{h \in H_j} \theta(h) = 0$.

1.3 Lerner Symmetry

Theorem 1 (Perfect Competition)**.** *Consider a reform of trade taxes in country* i_0 *satisfying*

$$\frac{1 + \tilde{t}^k_{ji_0}(n)}{1 + t^k_{ji_0}(n)} = \frac{1 + \tilde{s}^k_{i_0 j}(n)}{1 + s^k_{i_0 j}(n)} = \eta \text{ for all } j \neq i_0, k, \text{ and } n,$$

for some $\eta > 0$*; all other taxes are unchanged. If A1 and A2 hold, then* $\mathcal{E}(t,s) = \mathcal{E}(\tilde{t}, \tilde{s})$*; if A1, A2, and A3 hold, then* $\mathcal{E}(t,s,T) = \mathcal{E}(\tilde{t}, \tilde{s}, T)$.

Proof. ($\mathcal{E}(t,s) = \mathcal{E}(\tilde{t}, \tilde{s})$). It suffices to establish that $\mathcal{E}(t,s) \subseteq \mathcal{E}(\tilde{t}, \tilde{s})$, since then, reversing the notation, one also has $\mathcal{E}(\tilde{t}, \tilde{s}) \subseteq \mathcal{E}(t,s)$, yielding the desired equality. For any $(c,l,m,y) \in \mathcal{E}(t,s)$ with associated (p, τ, T), we show that $(c,l,m,y) \in \mathcal{E}(\tilde{t}, \tilde{s})$ by constructing a new $(\tilde{p}, \tilde{\tau}, \tilde{T})$ to verify the equilibrium conditions (i)-(iv).

For all h, i, j, and k set

$$\tilde{p}^k_{ij} = \begin{cases} p^k_{ij}\eta & \text{if } i = j = i_0, \\ p^k_{ij} & \text{otherwise,} \end{cases} \tag{1.1}$$

$$\tilde{\tau}(h) = \tilde{p}(1 + \tilde{t}(h)) \cdot c(h) - \tilde{p}(1 + \tilde{s}(h)) \cdot l(h) - \tilde{\pi} \cdot \theta(h), \tag{1.2}$$

$$\tilde{T}_{ij} = T_{ij} + [\pi_i - \tilde{\pi}_i] \cdot \sum_{h \in H_j} \theta(h), \tag{1.3}$$

with $\tilde{\pi} \equiv \{\tilde{\pi}(f)\}$ the vector of firms' total profits under the new tax schedule and $\tilde{\pi}_i \equiv$

$\{\tilde{\pi}_i(f)\}$ the vector of profits derived from transactions in country i,

$$\tilde{\pi}(f) = \sum_{i,j,k} [\tilde{p}_{ij}^k (1 + \tilde{s}_{ij}^k(f)) y_{ij}^k(f) - \tilde{p}_{ji}^k (1 + \tilde{t}_{ji}^k(f)) m_{ji}^k(f)],$$
$$\tilde{\pi}_i(f) = \sum_{j,k} [\tilde{p}_{ij}^k (1 + \tilde{s}_{ij}^k(f)) y_{ij}^k(f) - \tilde{p}_{ji}^k (1 + \tilde{t}_{ji}^k(f)) m_{ji}^k(f)].$$

Given the change in taxes from (t, s) to $(\tilde{t}, \tilde{s})$ that we consider, equation (1.1) implies that all after-tax prices faced by buyers and sellers from country i_0 are multiplied by η,

$$\tilde{p}_{ji_0}^k (1 + \tilde{t}_{ji_0}^k(n)) = \eta p_{ji_0}^k (1 + t_{ji_0}^k(n)), \tag{1.4}$$
$$\tilde{p}_{i_0 j}^k (1 + \tilde{s}_{i_0 j}^k(n)) = \eta p_{i_0 j}^k (1 + s_{i_0 j}^k(n)), \tag{1.5}$$

while other after-tax prices remain unchanged,

$$(1 + \tilde{t}_{ji}^k(n)) \tilde{p}_{ji}^k = (1 + t_{ji}^k(n)) p_{ji}^k, \tag{1.6}$$
$$(1 + \tilde{s}_{ij}^k(n)) \tilde{p}_{ij}^k = (1 + s_{ij}^k(n)) p_{ij}^k, \tag{1.7}$$

if $i \neq i_0$. In turn, profits in the proposed equilibrium satisfy

$$\tilde{\pi}_i = \begin{cases} \pi_i \eta & \text{if } i = i_0, \\ \pi_i & \text{otherwise.} \end{cases} \tag{1.8}$$

First, consider condition (i). Equation (1.2) implies that the household budget constraint still holds at the original allocation $(c(h), l(h))$ given the new prices, $\tilde{p}$, taxes, $\tilde{t}$ and $\tilde{s}$, and transfers, $\tilde{\tau}$. Under A2, equations (1.4) and (1.5) are therefore sufficient for condition (i) to hold in country i_0, whereas equations (1.6) and (1.7) are sufficient for it to hold in countries $i \neq i_0$. Next, consider condition (ii). Under A1, equations (1.4) and (1.5) are again sufficient for condition (ii) to hold in country i_0, whereas equations (1.6) and (1.7) are sufficient for it to hold in countries $i \neq i_0$. Since the allocation (c, l, m, y) is unchanged in the proposed equilibrium, the good market clearing condition (iii) continues to hold. Finally, we verify the government budget balance condition (iv). Let R_i and $\tilde{R}_i$ denote the net revenues of country i's government at the original and proposed equilibria,

$$R_i \equiv \sum_{j,k} p_{ji}^k \Big(\sum_h t_{ji}^k(h) c_{ji}^k(h) + \sum_f t_{ji}^k(f) m_{ji}^k(f)\Big) + \sum_{j \neq i} T_{ji}$$
$$- \sum_{j,k} p_{ij}^k \Big(\sum_h s_{ij}^k(h) l_{ij}^k(h) + \sum_f s_{ij}^k(f) y_{ij}^k(f)\Big) - \sum_{h \in H_i} \tau(h) - \sum_{j \neq i} T_{ij},$$

$$\tilde{R}_i \equiv \sum_{j,k} \tilde{p}_{ji}^k (\sum_h \tilde{t}_{ji}^k(h) c_{ji}^k(h) + \sum_f \tilde{t}_{ji}^k(f) m_{ji}^k(f)) + \sum_{j \neq i} \tilde{T}_{ji}$$
$$- \sum_{j,k} \tilde{p}_{ij}^k (\sum_h \tilde{s}_{ij}^k(h) l_{ij}^k(h) + \sum_f \tilde{s}_{ij}^k(f) y_{ij}^k(f)) - \sum_{h \in H_i} \tilde{\tau}(h) - \sum_{j \neq i} \tilde{T}_{ij}.$$

In any country $i \neq i_0$, equations (1.1)–(1.3) imply

$$\tilde{R}_i = R_i + \sum_{j \neq i} \sum_{h \in H_i} [\pi_j - \tilde{\pi}_j] \cdot \theta(h) + \sum_{h \in H_i} [\tilde{\pi} - \pi] \cdot \theta(h) - \sum_{j \neq i} \sum_{h \in H_j} [\pi_i - \tilde{\pi}_i] \cdot \theta(h).$$

Using the government budget constraint in country i at the original equilibrium, $R_i = 0$, and noting that

$$\sum_{j \neq i} \sum_{h \in H_i} [\pi_j - \tilde{\pi}_j] \cdot \theta(h) = \sum_{h \in H_i} [\pi - \tilde{\pi}] \cdot \theta(h) - \sum_{h \in H_i} [\pi_i - \tilde{\pi}_i] \cdot \theta(h),$$

we therefore arrive at

$$\tilde{R}_i = - [\pi_i - \tilde{\pi}_i] \cdot \sum_j \sum_{h \in H_j} \theta(h).$$

Together with equation (1.8), this implies government budget balance, $\tilde{R}_i = 0$, for all $i \neq i_0$.

Let us now turn to country i_0. Equation (1.2) and A2 imply

$$\tilde{R}_{i_0} = - \sum_{j,k} \tilde{p}_{ji_0}^k (\sum_h c_{ji_0}^k(h)) + \sum_{j,k} \tilde{p}_{i_0 j}^k (\sum_h l_{i_0 j}^k(h)) + \tilde{\pi} \cdot \sum_{h \in H_{i_0}} \theta(h)$$
$$- \sum_{j,k,f} [\tilde{p}_{i_0 j}^k \tilde{s}_{i_0 j}^k(f) y_{i_0 j}^k(f) - \tilde{p}_{ji_0}^k \tilde{t}_{ji_0}^k(f) m_{ji_0}^k(f)] + \sum_{j \neq i} \tilde{T}_{ji_0} - \sum_{j \neq i} \tilde{T}_{i_0 j}.$$

By equation (1.3), this is equivalent to

$$\tilde{R}_{i_0} = - \sum_{j,k} \tilde{p}_{ji_0}^k (\sum_h c_{ji_0}^k(h)) + \sum_{j,k} \tilde{p}_{i_0 j}^k (\sum_h l_{i_0 j}^k(h)) + \tilde{\pi} \cdot \sum_{h \in H_{i_0}} \theta(h)$$
$$- \sum_{j,k,f} [\tilde{p}_{i_0 j}^k \tilde{s}_{i_0 j}^k(f) y_{i_0 j}^k(f) - \tilde{p}_{ji_0}^k \tilde{t}_{ji_0}^k(f) m_{ji_0}^k(f)] + \sum_{j \neq i_0} [T_{ji_0} + [\pi_j - \tilde{\pi}_j] \cdot \sum_{h \in H_{i_0}} \theta(h)]$$
$$- \sum_{j \neq i_0} [T_{i_0 j} + [\pi_{i_0} - \tilde{\pi}_{i_0}] \cdot \sum_{h \in H_j} \theta(h)].$$

Together with the households' budget constraints, the government budget constraint in country i_0 in the original equilibrium implies

$$\sum_{j,k} p^k_{ji_0}(\sum_h c^k_{ji_0}(h)) + \sum_{j\neq i_0} T_{i_0 j} = \sum_{j,k} p^k_{i_0 j}(\sum_h l^k_{i_0 j}(h)) + \pi \cdot \sum_{h\in H_{i_0}} \theta(h) + \sum_{j\neq i_0} T_{ji_0}.$$

Combining the two previous observations, we get

$$\begin{aligned}\tilde{R}_{i_0} = & -\sum_{j,k}(\tilde{p}^k_{ji_0} - p^k_{ji_0})(\sum_h c^k_{ji_0}(h)) + \sum_{j,k}(\tilde{p}^k_{i_0 j} - p^k_{i_0 j})(\sum_h l^k_{i_0 j}(h)) \\ & - \sum_{j,k,f}[\tilde{p}^k_{i_0 j}\tilde{s}^k_{i_0 j}(f) y^k_{i_0 j}(f) - \tilde{p}^k_{ji_0}\tilde{t}^k_{ji_0}(f) m^k_{ji_0}(f)] + \sum_{j,k,f}[p^k_{i_0 j}s^k_{i_0 j}(f) y^k_{i_0 j}(f) - p^k_{ji_0}t^k_{ji_0}(f) m^k_{ji_0}(f)] \\ & + [\tilde{\pi}_{i_0} - \pi_{i_0}] \cdot \sum_j \sum_{h\in H_j} \theta(h).\end{aligned}$$

Using equation (1.1) and the definitions of π_{i_0} and $\tilde{\pi}_{i_0}$, this simplifies into

$$\tilde{R}_{i_0} = (1-\eta)\sum_k p^k_{i_0 i_0}[\sum c^k_{i_0 i_0}(h) + \sum_f m^k_{i_0 i_0}(f) - \sum_h l^k_{i_0 i_0}(h) - \sum_f y^k_{i_0 i_0}(f)].$$

Together with the good market clearing condition (iii), this proves government budget balance $\tilde{R}_{i_0} = 0$. This concludes the proof that $(c, l, m, y) \in \mathcal{E}(\tilde{t}, \tilde{s})$.

$(\mathcal{E}(t, s, T) = \mathcal{E}(\tilde{t}, \tilde{s}, T))$. As before, it suffices to establish $\mathcal{E}(t, s, T) \subseteq \mathcal{E}(\tilde{t}, \tilde{s}, T)$. Equations (1.3) and (1.8) imply

$$\tilde{T}_{ij} = \begin{cases} T_{ij} & \text{if } i \neq i_0 \text{ and } j \neq i, \\ T_{ij} + (1-\eta)\pi_i \cdot \sum_{h\in H_j} \theta(h) & \text{if } i = i_0 \text{ and } j \neq i_0. \end{cases}$$

Under A3, this simplifies into $\tilde{T}_{ij} = T_{ij}$ for all $i \neq j$. Together with the first part of our proof, this establishes that $(c, l, m, y) \in \mathcal{E}(\tilde{t}, \tilde{s}, T)$. □

2 Imperfect Competition

For convenience, we repeat the definition of an equilibrium under imperfect competition as well as assumption A1′. We then offer a formal proof of Theorem 2.

2.1 Equilibrium

An equilibrium requires households to maximize utility subject to budget constraint taking prices and taxes as given (condition i), markets to clear (condition iii), and government budget constraints to hold (condition *iv*), but it no longer requires firms to be price-takers. In place of condition (ii), each firm f chooses a correspondence $\sigma(f)$ that describes the set of quantities $(y(f), m(f)) \in \Omega(f)$ that it is willing to supply and demand at every price vector p. The correspondence $\sigma(f)$ must belong to a feasible set $\Sigma(f)$. For each strategy profile $\sigma \equiv \{\sigma(f)\}$, an auctioneer then selects a price vector $P(\sigma)$ and an allocation $C(\sigma) \equiv \{C(\sigma, h)\}$, $L(\sigma) \equiv \{L(\sigma, h)\}$, $M(\sigma) \equiv \{M(\sigma, f)\}$, and $Y(\sigma) \equiv \{Y(\sigma, f)\}$ such that the equilibrium conditions (i), (iii), and (iv) hold. Firm f solves

$$\max_{\sigma(f) \in \Sigma(f)} P(\sigma)(1 + s(f)) \cdot Y(\sigma, f) - P(\sigma)(1 + t(f)) \cdot M(\sigma, f), \tag{2.1}$$

taking the correspondences of other firms $\{\sigma(f')\}_{f' \neq f}$ as given.

2.2 Assumptions

A1'. *For any firm f, production sets can be separated into*

$$\Omega(f) = \Omega_{i_0}(f) \times \Omega_{-i_0}(f),$$

where $\Omega_{i_0}(f)$ and $\Omega_{-i_0}(f)$ are such that either $\Omega_{-i_0}(f) = \{0\}$ or $\Omega_{i_0}(f) = \{0\}$.

In line with the proof of Theorem (1), we define the function ρ_η mapping p into $\tilde{p}$ using (1.1), that is,

$$\rho_\eta(p^k_{ij}) = \begin{cases} p^k_{ij}\eta & \text{if } i = j = i_0, \\ p^k_{ij} & \text{otherwise.} \end{cases} \tag{2.2}$$

Its inverse ρ_η^{-1} is given by

$$\rho_\eta^{-1}(p^k_{ij}) = \begin{cases} p^k_{ij}/\eta & \text{if } i = j = i_0, \\ p^k_{ij} & \text{otherwise.} \end{cases}$$

For any $\eta > 0$, we assume that if $\sigma(f) \in \Sigma(f)$, then $\tilde{\sigma}(f) = \sigma(f) \circ \rho_\eta^{-1} \in \Sigma(f)$.

2.3 Lerner Symmetry

Theorem 2 (Imperfect Competition)**.** *Consider the tax reform of Theorem 1. If A1′ and A2 hold, then $\mathcal{E}(t, s) = \mathcal{E}(\tilde{t}, \tilde{s})$; if A1′, A2, and A3 hold, then $\mathcal{E}(t, s, T) = \mathcal{E}(\tilde{t}, \tilde{s}, T)$.*

Proof. Fix an equilibrium with strategy profile σ, taxes (t, s), auctioneer's choices $P(\sigma')$,

$C(\sigma')$, $L(\sigma')$, $M(\sigma')$ and $Y(\sigma')$, and realized prices $p = P(\sigma)$. Define a new strategy profile

$$\tilde{\sigma} = \sigma \circ \rho_\eta^{-1}.$$

We show that $\tilde{\sigma}$ is an equilibrium strategy, with taxes $(\tilde{t}, \tilde{s})$ and auctioneer choices, $\tilde{P}(\tilde{\sigma}') = \rho_\eta(P(\tilde{\sigma}' \circ \rho_\eta))$, $\tilde{C}(\tilde{\sigma}') = C(\tilde{\sigma}' \circ \rho_\eta)$, $\tilde{L}(\tilde{\sigma}') = L(\tilde{\sigma}' \circ \rho_\eta)$, $\tilde{M}(\tilde{\sigma}') = M(\tilde{\sigma}' \circ \rho_\eta)$, $\tilde{Y}(\tilde{\sigma}') = Y(\tilde{\sigma}' \circ \rho_\eta)$, and realized prices $\tilde{p} = \tilde{P}(\tilde{\sigma}) = \rho_\eta(p)$.

We focus on the profit maximization problem of a given firm f; the rest of the proof is identical to the perfect competition case. Define the set of feasible deviation strategies for firm f at the original and proposed equilibria

$$\mathcal{D}_{f,\sigma} = \{\sigma' \mid (\sigma'(f), \sigma(-f)) \text{ for all } \sigma'(f) \in \Sigma(f)\},$$

$$\mathcal{D}_{f,\tilde{\sigma}} = \{\tilde{\sigma}' \mid (\tilde{\sigma}'(f), \tilde{\sigma}(-f)) \text{ for all } \tilde{\sigma}'(f) \in \Sigma(f)\},$$

where $\sigma(-f) = \{\sigma(f')\}_{f' \neq f} \in \Pi_{f' \neq f} \Sigma(f')$ and $\tilde{\sigma}(-f) = \{\tilde{\sigma}(f')\}_{f' \neq f} \in \Pi_{f' \neq f} \Sigma(f')$.

By assumption, $\tilde{\sigma}(f) = \sigma(f) \circ \rho_\eta^{-1} \in \Sigma(f)$. We therefore need to prove that

$$\begin{aligned} &\tilde{P}(\tilde{\sigma})(1 + \tilde{s}(f)) \cdot \tilde{Y}(\tilde{\sigma}, f) - \tilde{P}(\tilde{\sigma})(1 + \tilde{t}(f)) \cdot \tilde{M}(\tilde{\sigma}, f) \\ &\qquad \geq \tilde{P}(\tilde{\sigma}')(1 + \tilde{s}(f)) \cdot \tilde{Y}(\tilde{\sigma}', f) - \tilde{P}(\tilde{\sigma}')(1 + \tilde{t}(f)) \cdot \tilde{M}(\tilde{\sigma}', f), \end{aligned} \tag{2.3}$$

for all $\tilde{\sigma}' \in \mathcal{D}_{f,\tilde{\sigma}}$.

By condition (2.1), σ satisfies

$$\begin{aligned} &P(\sigma)(1 + s(f)) \cdot Y(\sigma, f) - P(\sigma)(1 + t(f)) \cdot M(\sigma, f) \\ &\qquad \geq P(\sigma')(1 + s(f)) \cdot Y(\sigma', f) - P(\sigma')(1 + t(f)) \cdot M(\sigma', f), \end{aligned} \tag{2.4}$$

for all $\sigma' \in \mathcal{D}_{f,\sigma}$. Decompose

$$(M(\sigma', f), Y(\sigma', f)) = (M_{i_0}(\sigma', f), M_{-i_0}(\sigma', f), Y_{i_0}(\sigma', f), Y_{-i_0}(\sigma', f))$$

so that $(M_{i_0}(\sigma', f), Y_{i_0}(\sigma', f)) \in \Omega_{i_0}(f)$ and $(M_{-i_0}(\sigma', f), Y_{-i_0}(\sigma', f)) \in \Omega_{-i_0}(f)$. Decompose $P(\sigma')$, $t(f)$ and $s(f)$ in the same manner. With this notation, A1′ and (2.4) imply

$$\begin{aligned} &P_{i_0}(\sigma)(1 + s_{i_0}(f)) \cdot Y_{i_0}(\sigma, f) - P_{i_0}(\sigma)(1 + t_{i_0}(f)) \cdot M_{i_0}(\sigma, f) \\ &\qquad \geq P_{i_0}(\sigma')(1 + s_{i_0}(f)) \cdot Y_{i_0}(\sigma', f) - P_{i_0}(\sigma')(1 + t_{i_0}(f)) \cdot M_{i_0}(\sigma', f) \end{aligned} \tag{2.5}$$

and

$$P_{-i_0}(\sigma)(1+s_{-i_0}(f))\cdot Y_{-i_0}(\sigma,f)-P_{-i_0}(\sigma)(1+t_{-i_0}(f))\cdot M_{-i_0}(\sigma,f)$$
$$\geq P_{-i_0}(\sigma')(1+s_{-i_0}(f))\cdot Y_{-i_0}(\sigma',f)-P_{-i_0}(\sigma')(1+t_{-i_0}(f))\cdot M_{-i_0}(\sigma',f), \quad (2.6)$$

as one of the two inequalities holds trivially as an equality with zero on both sides.

For any $\tilde{\sigma}'\in\Pi_f\Sigma(f)$ and $\sigma'=\tilde{\sigma}'\circ\rho_\eta\in\Pi_f\Sigma(f)$, the new auctioneer's choices imply

$$\tilde{P}(\tilde{\sigma}')(1+\tilde{s}(f))\cdot\tilde{Y}(\tilde{\sigma}',f)-\tilde{P}(\tilde{\sigma}')(1+\tilde{t}(f))\cdot\tilde{M}(\tilde{\sigma}',f)$$
$$=\rho_\eta(P(\tilde{\sigma}'\circ\rho_\eta))(1+\tilde{s}(f))\cdot Y(\tilde{\sigma}'\circ\rho_\eta,f)-\rho_\eta(P(\tilde{\sigma}'\circ\rho_\eta))(1+\tilde{t}(f))\cdot M(\tilde{\sigma}'\circ\rho_\eta,f)$$
$$=\rho_\eta(P(\sigma'))(1+\tilde{s}(f))\cdot Y(\sigma',f)-\rho_\eta(P(\sigma'))(1+\tilde{t}(f))\cdot M(\sigma',f)$$

Equation (2.2) further implies,

$$\rho_\eta(P^k_{ij}(\sigma'))(1+\tilde{s}^k_{ij}(f))=\begin{cases}\eta P^k_{ij}(\sigma')(1+s^k_{ij}(f)) & \text{for all } j \text{ and } k \text{ if } i=i_0,\\ P^k_{ij}(\sigma')(1+s^k_{ij}(f)) & \text{for all } j \text{ and } k \text{ if } i\neq i_0,\end{cases}$$

$$\rho_\eta(P^k_{ji}(\sigma'))(1+\tilde{t}^k_{ji}(f))=\begin{cases}\eta P^k_{ji}(\sigma')(1+t^k_{ji}(f)) & \text{for all } j \text{ and } k \text{ if } i=i_0,\\ P^k_{ji}(\sigma')(1+t^k_{ji}(f)) & \text{for all } j \text{ and } k \text{ if } i\neq i_0.\end{cases}$$

Thus, it follows that

$$\tilde{P}_{i_0}(\tilde{\sigma}')(1+\tilde{s}_{i_0}(f))\cdot\tilde{Y}_{i_0}(\tilde{\sigma}',f)-\tilde{P}_{i_0}(\tilde{\sigma}')(1+\tilde{t}_{i_0}(f))\cdot\tilde{M}_{i_0}(\tilde{\sigma}',f)$$
$$=\eta\left(P_{i_0}(\sigma')(1+s_{i_0}(f))\cdot Y_{i_0}(\sigma',f)-P_{i_0}(\sigma')(1+t_{i_0}(f))\cdot M_{i_0}(\sigma',f)\right), \quad (2.7)$$

and

$$\tilde{P}_{-i_0}(\tilde{\sigma}')(1+\tilde{s}_{-i_0}(f))\cdot\tilde{Y}_{-i_0}(\tilde{\sigma}',f)-\tilde{P}_{-i_0}(\tilde{\sigma}')(1+\tilde{t}_{-i_0}(f))\cdot\tilde{M}_{-i_0}(\tilde{\sigma}',f)$$
$$=P_{-i_0}(\sigma')(1+s_{-i_0}(f))\cdot Y_{-i_0}(\sigma',f)-P_{-i_0}(\sigma')(1+t_{-i_0}(f))\cdot M_{-i_0}(\sigma',f). \quad (2.8)$$

Since for any $\tilde{\sigma}'\in\mathcal{D}_{f,\tilde{\sigma}}$, we have $\sigma'=\tilde{\sigma}'\circ\rho_\eta\in\mathcal{D}_{f,\sigma}$, (2.5)-(2.8) imply

$$\tilde{P}_{i_0}(\tilde{\sigma})(1+\tilde{s}_{i_0}(f))\cdot\tilde{Y}_{i_0}(\tilde{\sigma},f)-\tilde{P}_{i_0}(\tilde{\sigma})(1+\tilde{t}_{i_0}(f))\cdot\tilde{M}_{i_0}(\tilde{\sigma},f)$$
$$\geq\tilde{P}_{i_0}(\tilde{\sigma}')(1+\tilde{s}_{i_0}(f))\cdot\tilde{Y}_{i_0}(\tilde{\sigma}',f)-\tilde{P}_{i_0}(\tilde{\sigma}')(1+\tilde{t}_{i_0}(f))\cdot\tilde{M}_{i_0}(\tilde{\sigma}',f),$$

and

$$\tilde{P}_{-i_0}(\tilde{\sigma})(1+\tilde{s}_{-i_0}(f)) \cdot \tilde{Y}_{-i_0}(\tilde{\sigma}, f) - \tilde{P}_{-i_0}(\tilde{\sigma})(1+\tilde{t}_{-i_0}(f)) \cdot \tilde{M}_{-i_0}(\tilde{\sigma}, f)$$
$$\geq \tilde{P}_{-i_0}(\tilde{\sigma}')(1+\tilde{s}_{-i_0}(f)) \cdot \tilde{Y}_{-i_0}(\tilde{\sigma}', f) - \tilde{P}_{-i_0}(\tilde{\sigma}')(1+\tilde{t}_{-i_0}(f)) \cdot \tilde{M}_{-i_0}(\tilde{\sigma}', f),$$

for all $\tilde{\sigma}' \in \mathcal{D}_{f,\tilde{\sigma}}$. Adding up these last two inequalities gives (2.3). □

3 Nominal Rigidities

For convenience, we repeat the adjustment in prices before taxes,

$$\frac{\tilde{p}_{ij}^k}{p_{ij}^k} = \begin{cases} \eta & \text{if } i = j = i_0, \\ 1 & \text{otherwise.} \end{cases} \tag{3.1}$$

For parts of the proof of Proposition 1, we will use the fact that given the tax reform of Theorem 1, equation (3.1) is equivalent to

$$\frac{\tilde{p}_{ij}^k(1+\tilde{s}_{ij}^k(n))}{p_{ij}^k(1+s_{ij}^k(n))} = \frac{\tilde{p}_{ji}^k(1+\tilde{t}_{ji}^k(n))}{p_{ji}^k(1+t_{ji}^k(n))} = \begin{cases} \eta & \text{for all } j \text{ and } k, \text{ if } i = i_0, \\ 1 & \text{for all } j \text{ and } k, \text{ if } i \neq i_0. \end{cases} \tag{3.2}$$

Proposition 1. *Consider the tax reform of Theorem 1 with $\eta \neq 1$. Suppose $p \in \mathcal{P}(t,s)$ and $\tilde{p}$ satisfies (3.1). Then $\tilde{p} \in \mathcal{P}(\tilde{t},\tilde{s})$ holds if prices are rigid in the origin country's currency after sellers' taxes or the destination country's currency after buyers' taxes, but not if they are rigid before taxes. Likewise, $\tilde{p} \in \mathcal{P}(\tilde{t},\tilde{s})$ holds if prices are rigid in a dominant currency before taxes and country $i_0 \neq i_D$, but not if $i_0 = i_D$.*

Proof. We first consider the three cases for which $\tilde{p} \in \mathcal{P}(\tilde{t},\tilde{s})$.

Case 1: Prices are rigid in the origin country's currency after sellers' taxes,

$$\mathcal{P}(t,s) = \{\{p_{ij}^k\} | \exists \{e_l\} \text{ such that } p_{ij}^k(1+s_{ij}^k(n)) = \bar{p}_{ij}^{k,i}(1+\bar{s}_{ij}^k(n))/e_i \text{ for all } i, j, k, n\}.$$

Consider $p \in \mathcal{P}(t,s)$. Let us guess $\tilde{e}_{i_0}/e_{i_0} = 1/\eta$ and $\tilde{e}_i/e_i = 1$ if $i \neq i_0$. For any j, k, consider first $i \neq i_0$. From (3.2), we have

$$\tilde{p}_{ij}^k(1+\tilde{s}_{ij}^k(n)) = p_{ij}^k(1+s_{ij}^k(n)) = \bar{p}_{ij}^{k,i}(1+\bar{s}_{ij}^k(n))/e_i = \bar{p}_{ij}^{k,i}(1+\bar{s}_{ij}^k(n))/\tilde{e}_i.$$

Next consider $i = i_0$. From (3.2), we have

$$\tilde{p}^k_{i_0 j}(1 + \tilde{s}^k_{i_0 j}(n)) = \eta p^k_{i_0 j}(1 + s^k_{i_0 j}(n)) = \eta \bar{p}^{k,i_0}_{i_0 j}(1 + \bar{s}^k_{i_0 j}(n))/e_{i_0} = \bar{p}^{k,i_0}_{i_0 j}(1 + \bar{s}^k_{i_0 j}(n))/\tilde{e}_{i_0}.$$

This establishes that $\tilde{p} \in \mathcal{P}(\tilde{t}, \tilde{s})$.

Case 2: Prices are rigid in the destination country's currency after buyers' taxes,

$$\mathcal{P}(t,s) = \{\{p^k_{ij}\} | \exists \{e_l\} \text{ such that } p^k_{ij}(1 + t^k_{ij}(n)) = \bar{p}^{k,j}_{ij}(1 + \bar{t}^k_{ij}(n))/e_j \text{ for all } i, j, k, n\}.$$

Consider $p \in \mathcal{P}(t,s)$. Let us guess $\tilde{e}_{i_0}/e_{i_0} = 1/\eta$ and $\tilde{e}_i/e_i = 1$ if $i \neq i_0$. For any i, k, consider first $j \neq i_0$. From (3.2), we have

$$\tilde{p}^k_{ij}(1 + \tilde{t}^k_{ij}(n)) = p^k_{ij}(1 + t^k_{ij}(n)) = \bar{p}^{k,j}_{ij}(1 + \bar{t}^k_{ij}(n))/e_j = \bar{p}^{k,j}_{ij}(1 + \bar{t}^k_{ij}(n))/\tilde{e}_j.$$

Next consider $j = i_0$. From (3.2), we have

$$\tilde{p}^k_{ii_0}(1 + \tilde{t}^k_{ii_0}(n)) = \eta p^k_{ii_0}(1 + t^k_{ii_0}(n)) = \eta \bar{p}^{k,i_0}_{ii_0}(1 + \bar{t}^k_{ii_0}(n))/e_{i_0} = \bar{p}^{k,i_0}_{ij}(1 + \bar{t}^k_{ii_0}(n))/\tilde{e}_{i_0}.$$

This establishes that $\tilde{p} \in \mathcal{P}(\tilde{t}, \tilde{s})$.

Case 3: Prices are rigid in a dominant currency before taxes are imposed, and $i_0 \neq i_D$,

$$\mathcal{P}(t,s) = \{\{p^k_{ij}\} | \exists \{e_l\} \text{ such that } p^k_{ij} = \bar{p}^{k,i_D}_{ij}/e_{i_D} \text{ for all } i \neq j, k \text{ and } p^k_{ii} = \bar{p}^{k,i}_{ii}/e_i \text{ for all } k\}.$$

Consider $p \in \mathcal{P}(t,s)$. Let us guess $\tilde{e}_{i_0}/e_{i_0} = 1/\eta$ and $\tilde{e}_i/e_i = 1$ if $i \neq i_0$, including $\tilde{e}_{i_D}/e_{i_D} = 1$ since $i_0 \neq i_D$. For any k, j, consider first $i \neq j$. From (3.1), we have

$$\tilde{p}^k_{ij} = p^k_{ij} = \bar{p}^{k,i_D}_{ij}/e_{i_D} = \bar{p}^{k,i_D}_{ij}/\tilde{e}_{i_D}.$$

Next consider $i = j \neq i_0$. From (3.1), we have

$$\tilde{p}^k_{ii} = p^k_{ii} = \bar{p}^{k,i}_{ii}/e_i = \bar{p}^{k,i}_{ii}/\tilde{e}_i.$$

Finally, consider $i = j = i_0$. From (3.1), we have

$$\tilde{p}^k_{i_0 i_0} = \eta p^k_{i_0 i_0} = \eta \bar{p}^{k,i_0}_{i_0 i_0}/e_{i_0} = \bar{p}^{k,i_0}_{i_0 i_0}/\tilde{e}_{i_0}.$$

This establishes that $\tilde{p} \in \mathcal{P}(\tilde{t}, \tilde{s})$.

We now turn to the three cases for which $\tilde{p} \notin \mathcal{P}(\tilde{t}, \tilde{s})$.

Case 4: Prices are rigid in the origin country's currency before sellers's taxes,

$$\mathcal{P}(t,s) = \{\{p^k_{ij}\} | \exists \{e_l\} \text{ such that } p^k_{ij} = \bar{p}^{k,i}_{ij}/e_i \text{ for all } i,\, j,\, k,n\}.$$

Consider $p \in \mathcal{P}(t,s)$. Suppose $\tilde{p} \in \mathcal{P}(\tilde{t},\tilde{s})$. From (3.1), we have

$$\tilde{p}^k_{i_0 j} = p^k_{i_0 j} = \bar{p}^{k,i_0}_{i_0 j}/e_{i_0} = \bar{p}^{k,i_0}_{i_0 j}/\tilde{e}_{i_0} \text{if } j \neq i_0,$$
$$\tilde{p}^k_{i_0 i_0} = \eta p^k_{i_0 i_0} = \eta \bar{p}^{k,i_0}_{i_0 i_0}/e_{i_0} = \bar{p}^{k,i_0}_{i_0 i_0}/\tilde{e}_{i_0} \text{otherwise.}$$

The first equation gives $\tilde{e}_{i_0}/e_{i_0} = 1$; the second gives $\tilde{e}_{i_0}/e_{i_0} = 1/\eta$. A contradiction.

Case 5: Prices are rigid in the destination country's currency before buyers' taxes,

$$\mathcal{P}(t,s) = \{\{p^k_{ij}\} | \exists \{e_l\} \text{ such that } p^k_{ij} = \bar{p}^{k,j}_{ij}/e_j \text{ for all } i,\, j,\, k,n\}.$$

Start with $p \in \mathcal{P}(t,s)$. Suppose $\tilde{p} \in \mathcal{P}(\tilde{t},\tilde{s})$. From (3.1), we have

$$\tilde{p}^k_{i i_0} = p^k_{i i_0} = \bar{p}^{k,i_0}_{i i_0}/e_{i_0} = \bar{p}^{k,i_0}_{i i_0}/\tilde{e}_{i_0} \text{if } i \neq i_0,$$
$$\tilde{p}^k_{i_0 i_0} = \eta p^k_{i_0 i_0} = \eta \bar{p}^{k,i_0}_{i_0 i_0}/e_{i_0} = \bar{p}^{k,i_0}_{i_0 i_0}/\tilde{e}_{i_0} \text{otherwise.}$$

The first equation gives $\tilde{e}_{i_0}/e_{i_0} = 1$; the second gives $\tilde{e}_{i_0}/e_{i_0} = 1/\eta$. A contradiction.

Case 6: Prices are rigid in a dominant currency before taxes are imposed, and $i_0 = i_D$,

$$\mathcal{P}(t,s) = \{\{p^k_{ij}\} | \exists \{e_l\} \text{ such that } p^k_{ij} = \bar{p}^{k,i_0}_{ij}/e_{i_0} \text{ for all } i \neq j,\, k \text{ and } p^k_{ii} = \bar{p}^{k,i}_{ii}/e_i \text{ for all } k\}.$$

Start with $p \in \mathcal{P}(t,s)$. Suppose $\tilde{p} \in \mathcal{P}(\tilde{t},\tilde{s})$. From (3.1), we have

$$\tilde{p}^k_{i_0 j} = p^k_{i_0 j} = \bar{p}^{k,i_0}_{i_0 j}/e_{i_0} = \bar{p}^{k,i_0}_{i_0 j}/\tilde{e}_{i_0} \text{if } j \neq i_0,$$
$$\tilde{p}^k_{i_0 i_0} = \eta p^k_{i_0 i_0} = \eta \bar{p}^{k,i_0}_{i_0 i_0}/e_{i_0} = \bar{p}^{k,i_0}_{i_0 i_0}/\tilde{e}_{i_0} \text{otherwise.}$$

The first equation gives $\tilde{e}_{i_0}/e_{i_0} = 1$; the second gives $\tilde{e}_{i_0}/e_{i_0} = 1/\eta$. A contradiction. □