Generalized Poincaré-Hopf Theorem for Compact Nonsmooth Regions

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This paper presents an extension of the Poincaré-Hopf theorem to generalized critical points of a function on a compact region with nonsmooth boundary, M, defined by a finite number of smooth inequality constraints. Given a function $F: M \mapsto \mathbb{R}^n$, we define the generalized critical points of F over M, define the index for the critical point, and show that the sum of the indices of the critical points is equal to the Euler characteristic of M. We use the generalized Poincaré-Hopf theorem to present sufficient (local) conditions for the uniqueness of solutions to finite-dimensional variational inequalities and the uniqueness of stationary points in nonconvex optimization problems.

1. Introduction. The aim of this paper is to provide a generalization of the Poincaré-Hopf theorem for continuous functions over compact regions defined by a finite number of smooth inequality constraints. We use this generalized theorem to provide sufficient conditions for the uniqueness of solutions to finite-dimensional variational inequalities and the uniqueness of stationary points in nonconvex optimization problems for such regions.

Let M be a subset of the Euclidean set $\mathbb{R}^n$ and $k \leq n$ be a positive integer. M is a k-dimensional smooth manifold with boundary if for each $x \in M$ there exists an open set $U \subset \mathbb{R}^n$ and a continuously differentiable function $f: U \mapsto \mathbb{R}^n$ with a continuously differentiable inverse such that f maps the set $U \cap (\mathbb{R}^{k-1} \times \mathbb{R}_+ \times \{0\}^{n-k})$ to a neighborhood of x in M, where $\mathbb{R}_+$ denotes the set of nonnegative real numbers. Our definition of smooth manifold follows Milnor [22] and Mas-Colell [20], and focuses on smooth manifolds embedded in an ambient Euclidean space $\mathbb{R}^n$. For a more general definition (with or without boundary), see Guillemin and Pollack [14] or Hirsch [16].

For $M \subset \mathbb{R}^n$, $\chi(M)$ denotes its *Euler characteristic*. Euler characteristic is a useful *homotopy invariant*, in the sense that two sets that are *homotopy equivalent* (cf. Definition 4.3 and Lemma 4.5) have the same Euler characteristic. See Rotman [27], Chapter 1, for a definition and detailed discussion on homotopy equivalence. See Hirsch [16] or Guillemin and Pollack [14] for a definition and detailed discussion on the Euler characteristic of smooth manifolds, and Rotman [27] for the Euler characteristic of more general simplicial complexes. See Massey [21] for a proof of homotopy invariance of the Euler characteristic. The following is a list of the Euler characteristics of some familiar sets:

(i) Let n be a nonnegative integer and $B^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$, then $\chi(B^n) = 1$.

(ii) Let n be a nonnegative integer and $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$, then $\chi(S^n) = 2$ for n even and $\chi(S^n) = 0$ for n odd.

(iii) Let $M \subset \mathbb{R}^n$ be nonempty and convex (including $\mathbb{R}^n$ itself), then $\chi(M) = 1$.

The following is a version of the standard Poincaré-Hopf theorem applied to continuous vector fields on a smooth n-dimensional manifold $M \subset \mathbb{R}^n$ with boundary. The standard theorem claims that, when a vector field has a finite number of zeros over a smooth manifold, the algebraic sum of the zeros is equal to the Euler characteristic of the region (see Poincaré-Hopf Theorem in Milnor [22] for the smooth version and Hirsch [16], Chapter 5 for the continuous version). The following version uses a nondegeneracy assumption [(2) below] which guarantees that the vector field has a finite number of zeros.

POINCARÉ-HOPF THEOREM. *Let $M \subset \mathbb{R}^n$ be an n-dimensional compact smooth manifold with boundary. Let U be an open set containing M and $F: U \mapsto \mathbb{R}^n$ be a continuous function which is continuously differentiable at every $x \in Z$, where*

$$Z = \{x \in M \mid F(x) = 0\}$$

denotes the set of zeros of F over M. Assume the following:

ASSUMPTION 1 (A1). *F points outward on the boundary of M. In other words, given $x \in \mathrm{bd}(M)$, there exists a sequence $\epsilon_i \downarrow 0$ such that $x + \epsilon_i F(x) \notin M$ for all $i \in \mathbb{Z}^+$.*

ASSUMPTION 2 (A2). *For all $x \in Z$, $\det(\nabla F(x)) \neq 0$.*

Then, F has a finite number of zeros over M. Moreover, the algebraic sum of zeros (sum of indices corresponding to zeros) of F over M is equal to the Euler characteristic of M. In other words,

$$\sum_{x \in Z} \mathrm{sign}(\det(\nabla F(x))) = \chi(M),$$

where $\mathrm{sign}: \mathbb{R} \mapsto \{-1, 0, 1\}$ *is defined as*

$$\mathrm{sign}(x) = \begin{cases} -1 & \textit{if } x < 0, \\ 0 & \textit{if } x = 0, \\ 1 & \textit{if } x > 0. \end{cases}$$

The main contribution of this paper is to generalize the Poincaré-Hopf theorem in a number of directions. Our main result, Theorem 3.1, relaxes the smooth manifold assumption and allows M to be a region defined by a finite number of smooth inequality constraints, which is not necessarily a smooth manifold. Furthermore, we introduce the notion of a generalized critical point that encompasses both the interior zeros of the vector field and the zeros of the tangential vector field on the boundary used in Morse [23] and Gottlieb and Samaranayake [13] for the case of a smooth manifold. The approach enables us to drop (1) and prove a version of the Poincaré-Hopf theorem for generalized critical points over a region that is not necessarily a smooth manifold.

To illustrate the usefulness of the generalized Poincaré-Hopf theorem, we present two related applications: First, we derive sufficient (local) conditions for the uniqueness of solutions to variational inequalities. (Existing uniqueness results rely on global monotonicity conditions (Facchinei and Pang [12], Theorem 2.3.3).) Our generalization of the Poincaré-Hopf theorem is essential for this application since solutions to variational inequalities are typically on the boundary of the region defined by these inequalities (see Facchinei and Pang [12], p. 4). Second, we provide local sufficient conditions for the uniqueness of stationary points in constrained optimization problems. The most standard uniqueness results rely on convexity and quasiconvexity, and the generalized theorem is used to deal with stationary points that are on the boundary of the region defined by the constraints. Jongen et al. [18] also provide related uniqueness results, but they derive these from the generalization of Morse theory. Ben-El-Mechaiekh and Kryszewski [1], Cornet [7], and Cornet and Czarnecki [8, 9] use the axiomatic index (degree) theory (see Ortega and Rheinboldt [24], Chapter 6) to prove the existence of zeros of vector valued functions and correspondences, but they do not investigate uniqueness issues. Dierker [10], Mas-Colell [20], Varian [28], and Hildenbrand and Kirman [15] use the Poincaré-Hopf theorem to prove uniqueness results for general equilibrium economies with boundary conditions that restrict the equilibrium to be in the interior of the region (i.e., a zero of the vector field), and Eraslan and McLennan [11] use the axiomatic index theory to prove uniqueness in a bargaining game.

Our proof for the generalization relies on certain properties of the Euclidean projection, which we show in §4.1 and which could be of independent interest. We prove, among other things, that the projection on a region that can be represented by finitely many continuously differentiable inequalities and that satisfies a constraint qualification (linear independence constraint qualification [LICQ]; see Assumption 2.1), is a Lipschitz function. We further characterize the set over which the projection is continuously differentiable and derive an explicit formula for the Jacobian.

2. Notation and preliminaries. In this paper, all vectors are viewed as column vectors, and $x^T y$ denotes the inner product of the vectors x and y. We denote the two-norm as $\|x\| = (x^T x)^{1/2}$. For a given set X, we use $\mathrm{cl}(X)$ to denote the closure of X. When X is a finite set, we use $|X|$ to denote its cardinality. Given $x \in \mathbb{R}^n$ and $\delta > 0$, $B(x, \delta)$ denotes the open ball with radius δ centered at x, i.e.,

$$B(x, \delta) = \{y \in \mathbb{R}^n \mid \|y - x\| < \delta\}.$$

For a given function $f: A \mapsto B$, $f|_C: C \mapsto B$ denotes the restriction of f to $C \subset A$, and $f(C) \subset B$ denotes the image of C under f. If f is differentiable at x, then $\nabla f(x)$ denotes the gradient of f. If f is twice differentiable at x, then $H_f(x)$ denotes the Hessian of f at x. If $f: A \times B \mapsto C$ and $f(\cdot, y): A \to C$ is differentiable at $x \in A$, then $\nabla_x f(x, y)$ denotes the gradient of $f(\cdot, y)$ at $x \in A$.

We consider a compact region, M, defined by finitely many inequality constraints, i.e.,

$$M = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0,\ i \in I = \{1, 2, \ldots, |I|\}\}, \tag{1}$$

where the $g_i: \mathbb{R}^n \mapsto \mathbb{R}$, $i \in I$, are twice continuously differentiable. For some $x \in M$, let $I(x) = \{i \in I \mid g_i(x) = 0\}$ denote the set of active constraints. Throughout the paper, the following assumption will be in effect:

ASSUMPTION 2.1. *The set M is nonempty and every $x \in M$ satisfies the LICQ, i.e., for every $x \in M$, the vectors $\{\nabla g_i(x) \mid i \in I(x)\}$ are linearly independent (see Bertsekas et al. [4], §5.4).*

We define the *normal space at x* as the subspace of $\mathbb{R}^n$ spanned by the vectors $\{\nabla g_i(x) \mid i \in I(x)\}$. For $x \in M$, we denote the $n \times |I(x)|$ change-of-coordinates matrix from normal coordinates to standard coordinates as

$$G(x) = [\nabla g_i(x)|_{i \in I(x)}], \tag{2}$$

where columns ∇g_i are ordered in increasing order of i. We define the *tangent space at x* as the subspace that is the orthogonal complement in $\mathbb{R}^n$ of the normal space at x. Let $I^c(x) = \{|I(x)| + 1, \ldots, n\}$ and let $\{v_j, j \in I^c(x)\}$ be an arbitrary but fixed orthonormal basis of the tangent space at x. We denote the change-of-coordinates matrix from tangent coordinates to standard coordinates as

$$V(x) = [v_j|_{j \in I^c(x)}] \tag{3}$$

and note that

$$G(x)^T V(x) = V(x)^T G(x) = 0.$$

where 0 denotes the zero matrix with appropriate dimensions. (For notational convenience, when the normal space (respectively the tangent space) is zero dimensional, $G(x)$ (respectively $V(x)$) denotes the $n \times 0$ dimensional empty matrix.) We call the basis

$$\{g_i(x) \mid i \in I(x)\} \cup \{v_j,\ j \in I^c(x)\}$$

a *normal-tangent basis* for $x \in M$. We denote $C(x) = [G(x)\ V(x)]$ and note that $C(x)$ is a change-of-coordinates matrix from normal-tangent coordinates to standard coordinates.

We next recall the notion of a normal cone which will be used in our analysis (see Clarke [5] or Cornet [7]):

DEFINITION 2.1. Let M be given by (1). Let $x \in M$ with $I(x) \neq \varnothing$. The *normal cone of M at x*, $N_M(x)$, is defined by

$$N_M(x) = \{v \in \mathbb{R}^n \mid v = G(x)\lambda,\ \lambda \in \mathbb{R}^{|I(x)|},\ \lambda \geq 0\}.$$

We define the *boundary of the normal cone of M at x*, $\mathrm{bd}(N_M(x))$, by

$$\mathrm{bd}(N_M(x)) = N_M(x) - \mathrm{ri}(N_M(x)),$$

where $\mathrm{ri}(N_M(x))$ is the relative interior of the convex set $N_M(x)$, i.e.,

$$\mathrm{ri}(N_M(x)) = \{v \in \mathbb{R}^n \mid v = G(x)\lambda,\ \lambda \in \mathbb{R}^{|I(x)|},\ \lambda > 0\}.$$

If $I(x) = \varnothing$, we define $N_M(x) = \{0\}$ and $\mathrm{bd}(N_M(x)) = \varnothing$.

3. Generalized index theorem. We consider a region M given by (1) and present an extension of the Poincaré-Hopf theorem that applies for a generalized notion of critical points of a function over M.

DEFINITION 3.1. Let M be a region given by (1). Let U be an open set containing M and $F: U \mapsto \mathbb{R}^n$ be a function.

(a) We say that $x \in M$ is a *generalized critical point of F over M* if $-F(x) \in N_M(x)$. We denote the set of generalized critical points of F over M by $\mathrm{Cr}(F, M)$.
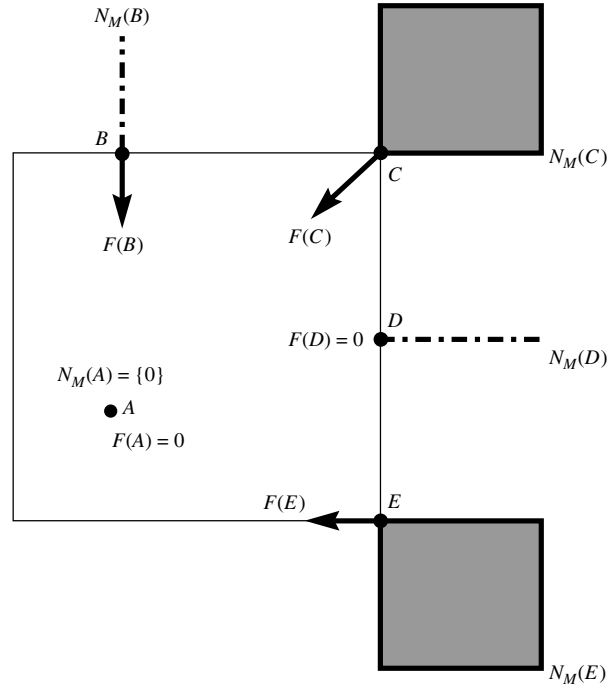
FIGURE 1. Illustration of critical points. Note that A, B, C are complementary critical points, whereas D and E are critical points that are not complementary.

(b) For $x \in \mathrm{Cr}(F, M)$, we define $\theta(x) \geq 0$ to be the unique vector in $\mathbb{R}^{|I(x)|}$ that satisfies

$$F(x) + G(x)\theta(x) = 0.$$

We say that $x \in M$ is a *complementary critical point* if $-F(x) \in \mathrm{ri}(N_M(x))$. In other words, $x \in \mathrm{Cr}(F, M)$ is complementary if and only if $\theta(x) > 0$ (see Figure 1).

(c) Let F be continuously differentiable at $x \in \mathrm{Cr}(F, M)$. We define

$$\Gamma(x) = V(x)^T \left(\nabla F(x) + \sum_{i \in I(x)} \theta_i(x) H_{g_i}(x) \right) V(x),$$

where $V(x)$ denotes the change-of-coordinates matrix from tangent coordinates to standard coordinates [cf. Equation (3)]. We say that x is a *nondegenerate critical point* if $\Gamma(x)$ is a nonsingular matrix. (For notational convenience, if A is the 0×0 dimensional empty matrix, we assume that A is nonsingular and $\det(A) = 1$. Thus, if the tangent space at x is zero dimensional, then $\Gamma(x)$ is nonsingular and x is a nondegenerate critical point of F.)

For an optimization problem, the local minima are generalized critical points of the gradient mapping of the objective function. To see this, let M be the region given by (1), U be an open set containing M, and $f: U \mapsto \mathbb{R}$ be a twice continuously differentiable function. Consider the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in M. \end{aligned} \tag{4}$$

Let x^* be a local optimum of (4). If $x^* \in \mathrm{int}(M)$, then by the unconstrained optimality conditions we have $\nabla f(x^*) = 0$. Since $N_M(x^*) = \{0\}$, we have $-\nabla f(x^*) \in N_M(x^*)$ and x^* is a generalized critical point of ∇f over M. For general (not necessarily interior) x^*, we have $-\nabla f(x^*) \in N_M(x^*)$, which is the optimality condition for minimization over an abstract set constraint (see Bertsekas et al. [4]). Therefore, every local minimum of f is a generalized critical point of ∇f over M. More generally, a generalized critical point is equivalent to a generalized equilibrium of Cornet and Czarnecki [9] and to a solution of the generalized equation of Robinson [25]. When M is convex, a generalized critical point is equivalent to a solution of the variational inequality problem (cf. Lemma 5.1).

We now define the notion of the index of a critical point and state our main result.

DEFINITION 3.2. Let M be a region given by (1). Let U be an open set containing M and $F: U \mapsto \mathbb{R}^n$ be a function that is continuously differentiable at $x \in \mathrm{Cr}(F, M)$, where x is complementary and nondegenerate. We

define the index of F at x as

$$\mathrm{ind}_F(x) = \mathrm{sign}(\det(\Gamma(x))).$$

We note that this definition is independent of the choice of $V(x)$. Changing $V(x)$ to another change-of-coordinates matrix $V'(x)$ corresponds to changing coordinates over the tangent space. Since $\Gamma(x)$ can be viewed as a linear map from the tangent space to itself, a change of coordinates does not change its determinant. Recall also that, for notational convenience, when A is the 0×0 dimensional empty matrix we assume $\det(A) = 1$. Thus, if the tangent space at x is zero dimensional, then $\mathrm{ind}_F(x) = 1$.

THEOREM 3.1.   *Let M be a compact region given by* (1). *Let U be an open set containing M and $F\colon U \mapsto \mathbb{R}^n$ be a continuous function which is continuously differentiable at every $x \in \mathrm{Cr}(F, M)$. Assume the following*:
   (A2) *Every $x \in \mathrm{Cr}(F, M)$ is complementary and nondegenerate.*
Then, F has a finite number of critical points over M. Moreover,

$$\sum_{x \in \mathrm{Cr}(F, M)} \mathrm{ind}_F(x) = \chi(M).$$

It is useful, especially for future reference, to note that the complementarity condition in this theorem corresponds to the strict complementary slackness condition used in the context of local sufficient optimality conditions in nonlinear programming (cf. Bertsekas [3]). The nondegeneracy condition is the extension of Assumption 2 in the Poincaré-Hopf theorem.

In §4, we will provide a proof of Theorem 3.1. Following Morse [23], we use an extension idea for this proof. The specific extension theorem, which is presented and proved in §4, may also be of independent interest.

4. The extension theorem and the proof of generalized Poincaré-Hopf theorem.   Let M be a compact region given by (1). Given $\epsilon > 0$, let

$$M^\epsilon = \{x \in \mathbb{R}^n \mid \|x - y\| < \epsilon \text{ for some } y \in M\}.$$

In other words, M^ϵ denotes the set of points with distance to M strictly less than ϵ. Our goal is to extend F to M^ϵ in a way that maps the generalized critical points of F to zeros of the extended function. Our extension relies on the properties of the projection function in a neighborhood of M.

4.1. Properties of the Euclidean projection.   In §4.1, we define the projection of a vector x in $\mathbb{R}^n$ on a closed possibly nonconvex set and explore its properties. We show that an inequality defined region that satisfies the LICQ condition displays local convexity properties. Consequently, the projection of proximal points on such nonconvex sets inherits many properties of projection on a convex set.

DEFINITION 4.1.   We define the projection correspondence $\pi\colon \mathbb{R}^n \rightrightarrows M$ as

$$\pi(y) = \arg\min_{x \in M} \|y - x\|.$$

We also define the distance function $d\colon \mathbb{R}^n \mapsto \mathbb{R}$ as

$$d(y) = \inf_{x \in M} \|y - x\|.$$

We note from Berge's maximum theorem (Berge [2]) that π is upper-semi-continuous and d is continuous. Also, since M is closed, we have $x \in M$ if and only if $d(x) = 0$. We can then characterize the sets M and M^ϵ as

$$M = \{x \in \mathbb{R}^n \mid d(x) = 0\},$$

and

$$M^\epsilon = \{x \in \mathbb{R}^n \mid d(x) < \epsilon\}.$$

We next show that for sufficiently small ϵ, the projection correspondence $\pi|_{M^\epsilon}$ is single-valued and Lipschitz continuous.

PROPOSITION 4.1.   *There exists $\epsilon > 0$ such that $\pi|_{M^\epsilon}$ is a globally Lipschitz function over M^ϵ. In other words, $\pi(x)$ is single-valued for all $x \in M^\epsilon$ and there exists $k > 0$ such that*

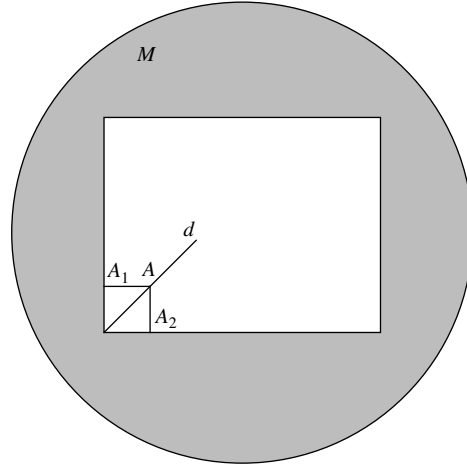$$\|\pi(y) - \pi(x)\| \le k\|y - x\|, \quad \forall x, y \in M^\epsilon.$$

FIGURE 2. *M* cannot be represented by finitely many continuously differentiable inequalities and the projection of a point *A* on the line segment *d* has two elements A_1 and A_2.

Two properties of *M* are essential for the preceding proposition: the first is that *M* can be represented by finitely many continuously differentiable inequalities, and the second is that *M* satisfies the LICQ condition. Figure 2 shows how the proposition fails for a region *M* that cannot be represented by finitely many inequalities. Similarly, the following example shows that the LICQ condition is essential for the proposition.

EXAMPLE 4.1. Let $g: \mathbb{R}^2 \mapsto \mathbb{R}$ be given by

$$g(x_1, x_2) = (x_1^2 - 1)^2 + x_2^2 - 1$$

and consider the set

$$M = \{x \in \mathbb{R}^2 \mid g(x) \leq 0\}.$$

We have, $g(0) = 0$ and $\nabla g(0) = 0$, therefore *M* does not satisfy the LICQ condition. It can be seen that $\pi((0, x_2))$ is not a singleton when $x_2 \neq 0$ (cf. Figure 3), so that the projection correspondence $\pi|_{M^\epsilon}$ is not a function for any $\epsilon > 0$.

We need some preliminary results to prove Proposition 4.1. We first note the following lemma that is a direct consequence of the LICQ condition:

LEMMA 4.1. *There exists some scalar $m > 0$ such that for all $x \in M$ with $I(x) \neq \varnothing$ and $\lambda \in \mathbb{R}^{|I(x)|}$, we have*

$$\|G(x)\lambda\| \geq m \sum_{i \in I(x)} |\lambda_i|, \tag{5}$$

where $G(x)$ is the change-of-coordinates matrix from normal to standard coordinates at x [cf. Equation (2)].
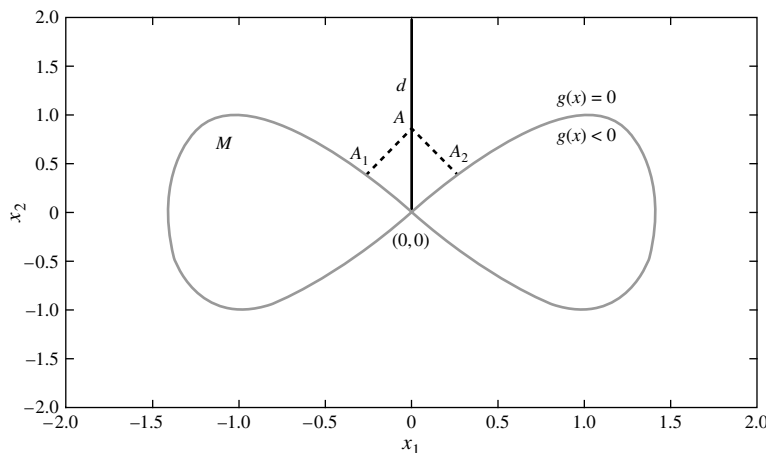

FIGURE 3. *M* does not satisfy the LICQ condition and the projection of a point *A* on the line *d* has two elements A_1 and A_2.

PROOF.   Let

$$\mathcal{P}(I) = \{S \subset I \mid S = I(x) \text{ for some } x \in M \text{ with } I(x) \neq \varnothing\}.$$

If $\mathcal{P}(I) = \varnothing$, then the statement is trivial and holds for any $m > 0$. Assume that $\mathcal{P}(I) \neq \varnothing$. For $S \in \mathcal{P}(I)$ define,

$$G_S = \{x \in M \mid g_i(x) = 0 \text{ if } i \in S\}.$$

Clearly, G_S is a compact and nonempty set. By the LICQ condition, for every $x \in G_S$ the vectors $[\nabla g_i(x)|_{i \in S}]$ are linearly independent. This implies that the following minimization problem has a positive solution:

$$m_S(x) = \min_{\|u\|=1,\, u \in \mathbb{R}^{|S|}} \left\| \sum_{i \in S} u_i \nabla g_i(x) \right\|.$$

By Berge's maximum theorem (Berge [2]), $m_S(x)$ is continuous in x. Then, since $m_S(x) > 0$ for all $x \in G_S$ and G_S is compact, there exists $\overline{m}_S > 0$ such that $m_S(x) \geq \overline{m}_S$ for all $x \in G_S$. Since the set $\mathcal{P}(I)$ is finite, $m' = \min_{S \in \mathcal{P}(I)} \overline{m}_S > 0$.

We claim that the result holds with $m = m'/|I|$. Let $x \in M$ with $I(x) \neq \varnothing$ and $\lambda \in \mathbb{R}^{|I(x)|}$. If $\lambda = 0$, the statement is trivial. When $\lambda \neq 0$, we have $I(x) \in \mathcal{P}(I)$ and $x \in G_{I(x)}$, which implies

$$\left\| \sum_{i \in I(x)} \frac{\lambda_i}{\|\lambda\|} \nabla g_i(x) \right\| \geq m',$$

and thus

$$\left\| \sum_{i \in I(x)} \lambda_i \nabla g_i(x) \right\| \geq m' \|\lambda\|$$

$$\geq \frac{m'}{\sqrt{|I(x)|}} \sum_{i \in I(x)} |\lambda_i| \geq \frac{m'}{|I|} \sum_{i \in I(x)} |\lambda_i|,$$

where to get the second inequality, we used the Cauchy-Schwartz inequality. We conclude that the result holds with $m = m'/|I|$. □

DEFINITION 4.2.   Given $\epsilon > 0$, we define the ϵ-normal correspondence $N^\epsilon \colon M \mapsto \mathbb{R}^n$ as

$$N^\epsilon(x) = (x + N_M(x)) \cap M^\epsilon.$$

We also define the correspondence $\text{ri}(N^\epsilon(x))$ as

$$\text{ri}(N^\epsilon(x)) = (x + \text{ri}(N_M(x))) \cap M^\epsilon.$$

The following lemma shows that for sufficiently small positive ϵ, the ϵ-normal correspondence satisfies a Lipschitzian property:

LEMMA 4.2.   *There exists $\epsilon > 0$ and $k > 0$ such that for all $x, y \in M$, and $s_x \in N^\epsilon(x)$, $s_y \in N^\epsilon(y)$, we have*

$$\|y - x\| \leq k \|s_y - s_x\|.$$

PROOF.   For some $i \in I$, we define the function $e_i \colon U \times U \mapsto \mathbb{R}$ as

$$e_i(x, y) = \begin{cases} \dfrac{g_i(y) - g_i(x) + \nabla g_i(x)^T(y - x) + (1/2)(y - x)^T H_{g_i}(x)(y - x)}{\|y - x\|^2} & \text{if } y \neq x, \\ 0 & \text{if } y = x. \end{cases} \tag{6}$$

Since g_i is twice continuously differentiable, the function e_i is continuous. Thus, the function $|e_i|$ has a maximum over the compact set $M \times M$, i.e., there exists some $\mu > 0$ such that

$$|e_i(x, y)| < \mu, \quad \forall x, y \in M. \tag{7}$$

Given $n \times n$ matrix A, we define the matrix norm of A as

$$\|A\| = \max_{\|u\|=1} \|Au\|. \tag{8}$$

We note, by Cauchy-Schwartz inequality, that

$$|u^T A u| \leq \|A\| \|u\|^2, \quad \forall \, u \in \mathbb{R}^n. \tag{9}$$

Let $H_i = \max_{x \in M} \|H_{g_i}(x)\|$. The maximum exists since $\|H_{g_i}\|$ is a continuous function over the compact region M. Let

$$H_m = \max_{i \in I} H_i. \tag{10}$$

Also, let $m > 0$ be a scalar that satisfies Equation (5) in Lemma 4.1.

We will prove that the result in Lemma 4.2 holds for

$$\epsilon = \frac{m}{2n(H_m + 2\mu)} > 0, \tag{11}$$

and $k = 2$. Let $x, y \in M$ and s_x, s_y in $N^\epsilon(x)$ and $N^\epsilon(y)$, respectively. If $y = x$ then we are done. Assume $y \neq x$. Then, by the definition of the ϵ-normal correspondence and the normal cone, there exist scalars $\lambda_i \geq 0$, $i \in I(x)$, and $\gamma_j \geq 0$, $j \in I(y)$ such that

$$s_x = x + \sum_{i \in I(x)} \lambda_i \nabla g_i(x), \tag{12}$$

$$s_y = y + \sum_{j \in I(y)} \gamma_j \nabla g_j(y). \tag{13}$$

We claim

$$\sum_{i \in I(x)} |\lambda_i| < \frac{1}{2n(H_m + 2\mu)} \leq \frac{1}{2(H_m + 2\mu)}. \tag{14}$$

Since $s_x \in N^\epsilon(x)$, we have $\|s_x - x\| < \epsilon$. If $I(x) = \varnothing$, the claim in (14) is trivial. Else if $I(x) \neq \varnothing$, by Lemma 4.1, we have

$$m \sum_{i \in I(x)} |\lambda_i| \leq \left\| \sum_{i \in I(x)} \lambda_i \nabla g_i(x) \right\| = \|s_x - x\| < \epsilon \leq \frac{m}{2n(H_m + 2\mu)},$$

which implies Equation (14). Similarly, we have

$$\sum_{j \in I(y)} |\gamma_j| < \frac{1}{2n(H_m + 2\mu)} \leq \frac{1}{2(H_m + 2\mu)}.$$

Using the definition of the function e_i [cf. Equation (6)], we have for all i,

$$g_i(y) = g_i(x) + \nabla g_i(x)^T (y - x) + \frac{1}{2}(y - x)^T H_{g_i}(x)(y - x) + e_i(y, x)\|y - x\|^2,$$

and

$$g_i(x) = g_i(y) + \nabla g_i(y)^T (x - y) + \frac{1}{2}(x - y)^T H_{g_i}(y)(x - y) + e_i(x, y)\|x - y\|^2.$$

Multiplying the preceding relations with λ_i and γ_j, respectively, and summing over $i \in I(x)$ and $j \in I(y)$, we obtain

$$
\sum_{i \in I(x)} \lambda_i g_i(y) + \sum_{j \in I(y)} \gamma_j g_j(x) = \left(\sum_{i \in I(x)} \lambda_i \nabla g_i(x) - \sum_{j \in I(y)} \gamma_j \nabla g_j(y) \right)^T (y - x)
$$
$$
+ \sum_{i \in I(x)} \lambda_i \left(\frac{1}{2}(y - x)^T H_{g_i}(x)(y - x) + \epsilon_i(y, x)\|y - x\|^2 \right)
$$
$$
+ \sum_{j \in I(y)} \gamma_j \left(\frac{1}{2}(x - y)^T H_{g_j}(y)(x - y) + \epsilon_j(x, y)\|x - y\|^2 \right). \tag{15}
$$

Since $x, y \in M$ and $\lambda_i \geq 0$, $\gamma_j \geq 0$, the term on the left-hand side of Equation (15) is nonpositive. By Equations (12) and (13), it follows that the first term on the right-hand side is equal to

$$(s_x - x - s_y + y)^T (y - x) = (s_x - s_y)^T (y - x) + \|y - x\|^2.$$

Combining Equation (7), Equation (9), and the definition of H_m in (10), it can be seen that the second term on the right of Equation (15) is bounded below by

$$\sum_{i \in I(x)} -|\lambda_i|(H_m/2 + \mu)\|y - x\|^2 \geq -\frac{1}{2(H_m + 2\mu)}|H_m/2 + \mu|\|y - x\|^2 \geq -1/4\|y - x\|^2,$$

where we used Equation (14) to get the second inequality. Similarly, the last term on the right-hand side of Equation (15) is bounded below by $-1/4\|y - x\|^2$. Combining the above relations, Equation (15) yields

$$0 \geq (s_x - s_y)^T(y - x) + \|y - x\|^2 - \frac{1}{4}\|y - x\|^2 - \frac{1}{4}\|y - x\|^2,$$

which implies that

$$-\frac{1}{2}\|y - x\|^2 \geq (s_x - s_y)^T(y - x) \geq -\|s_y - s_x\|\|y - x\|,$$

where we used the Cauchy-Schwartz inequality to get the second inequality. Finally, since $y \neq x$, we obtain

$$\|y - x\| \leq 2\|s_y - s_x\|.$$

Hence, the claim is satisfied with the ϵ given by Equation (11) and $k = 2$. □

For the rest of §4.1, let $\epsilon > 0$, $k > 0$ be fixed scalars that satisfy the claim of Lemma 4.2. The following is a corollary of Lemma 4.2 and shows that the ϵ-normal correspondence is injective.

COROLLARY 4.1. *Given $x, y \in M$, if $x \neq y$, then $N^\epsilon(x) \cap N^\epsilon(y) = \varnothing$.*

PROOF. Let $v \in N^\epsilon(x) \cap N^\epsilon(y)$. Then, by Lemma 4.2, $\|y - x\| \leq k\|v - v\| = 0$ implies that $\|y - x\| = 0$ and hence $x = y$ as desired. □

We next note the following lemma that shows that the correspondence N^ϵ is the inverse image of the projection correspondence.

LEMMA 4.3. *Given $y \in M^\epsilon$ and $p \in M$, $p \in \pi(y)$ if and only if $y \in N^\epsilon(p)$.*

PROOF. Let $y \in M^\epsilon$ and $p \in \pi(y)$. Then, by the optimality conditions, we have $y - p \in N_M(p)$ and thus $y \in N^\epsilon(p)$. Conversely, let $y \in N^\epsilon(p)$ for some $p \in M$. Assume that $p \notin \pi(y)$. Then there exists $p' \in \pi(y)$ such that $p' \neq p$. Then $y \in N^\epsilon(p') \cap N^\epsilon(p)$, which is a contradiction by Lemma 4.1. Therefore, we must have $p \in \pi(y)$, completing the proof. □

PROOF OF PROPOSITION 4.1. Assume that there is some $x \in M^\epsilon$ such that $\pi(x)$ is not single-valued. Then, there exist $p, q \in \pi(x) \subset M$ such that $p \neq q$. By Lemma 4.3, $x \in N^\epsilon(p)$ and $x \in N^\epsilon(q)$, therefore $x \in N^\epsilon(p) \cap N^\epsilon(q)$, contradicting Lemma 4.1. Therefore, we conclude that $\pi|_{M^\epsilon}$ is single-valued.

Let $x, y \in M^\epsilon$. Then, $x \in N^\epsilon(\pi(x))$ and $y \in N^\epsilon(\pi(y))$ (cf. Lemma 4.3), and it follows by Lemma 4.2 that

$$\|\pi(y) - \pi(x)\| \leq k\|y - x\|,$$

showing that π is globally Lipschitz. □

The next proposition shows that the distance function restricted to $M^\epsilon - M$, $d|_{M^\epsilon - M}$, is continuously differentiable. Since $\pi(x)$ is a singleton for all $x \in M^\epsilon - M$ (cf. Proposition 4.1), this result follows from Theorem 4.11 in Clarke et al. [6]. We present an alternative proof in the appendix for completeness.

PROPOSITION 4.2. *The distance function d is continuously differentiable for all $x \in M^\epsilon - M$ with derivative*

$$\nabla d(x) = \frac{x - \pi(x)}{d(x)}.$$

We note the following corollary to the preceding proposition:

COROLLARY 4.2. *There exists $\epsilon' > 0$ such that $\mathrm{cl}(M^{\epsilon'})$ is an n-dimensional smooth manifold. Moreover,*

$$\mathrm{bd}(M^{\epsilon'}) = \{x \in \mathbb{R}^n \mid d(x) = \epsilon'\}.$$

PROOF.   Let ϵ' such that $0 < \epsilon' < \epsilon$. From Proposition 4.2, $d: M^\epsilon - M \mapsto (0, \epsilon)$ is a continuously differentiable function with $\nabla d(y) = (y - \pi(y))/d(y)$ for every $y \in M^\epsilon - M$. Since $\nabla d(y) \neq 0$ for every y such that $d(y) = \epsilon'$, we have that ϵ' is a regular value of d in the sense of Sard (cf. Milnor [22]). Then, from the pre-image theorem, $\mathrm{cl}(M^{\epsilon'}) = \{x \in \mathbb{R}^n \mid d(x) \leq \epsilon'\}$ is an n-dimensional smooth manifold with boundary (cf. Lemma 3 in Milnor [22], Chapter 2). Furthermore, the boundary is characterized by

$$\mathrm{bd}(M^{\epsilon'}) = \{x \in \mathbb{R}^n \mid d(x) = \epsilon'\},$$

completing the proof.   □

For the rest of §4.1, we assume that $\epsilon > 0$ is a sufficiently small fixed scalar such that it also satisfies the result of Corollary 4.2.

Given $y \in M^\epsilon$, we define $\lambda(y) \in \mathbb{R}^{|I(\pi(y))|}$ to be the unique vector that satisfies

$$y - \pi(y) = G(\pi(y))\lambda(y). \tag{16}$$

We also define $H(y)$ as

$$H(y) = \sum_{i \in I(\pi(y))} \lambda_i(y) H_{g_i}(\pi(y)) \tag{17}$$

and we adopt the notation

$$[X]\|_Y = Y^T X Y$$

where X and Y are matrices with appropriate dimensions. If Y is invertible, we also adopt the notation

$$[X]|_Y = Y^{-1} X Y.$$

We next study the differentiability properties of the projection function. For the following proposition, we adopt the convention that the inverse of the 0×0 empty matrix is the 0×0 empty matrix. Then, when $V(\pi(y))$ is the $n \times 0$ empty matrix, i.e., when the tangent space at $\pi(y)$ is zero dimensional, the equation reads $\nabla \pi(y) = 0$.

PROPOSITION 4.3.   *Let y be a vector in $\mathrm{ri}(N^\epsilon(\pi(y)))$. Then, π is continuously differentiable at y. Moreover, the Jacobian of π in the tangent-normal coordinates of $\pi(y)$ is*

$$[\nabla \pi(y)]|_{C(\pi(y))} = \begin{bmatrix} 0 & 0 \\ 0 & (I + [H(y)]\|_{V(\pi(y))})^{-1} \end{bmatrix},$$

where $C(x) = [G(x) \ \ V(x)]$ denotes the change-of-coordinates matrix from normal-tangent coordinates to standard coordinates for $x \in M$ (cf. §2) and I denotes the identity matrix with appropriate dimension.

Proposition 4.3 agrees with similar formulas obtained by Holmes [17] with the gauge function when M is assumed to be convex. Note that this proposition could be proved using Theorem 5.3 of Robinson [26] after casting the problem of finding the Euclidean projection as a variational condition. We include a direct proof here that is more in line with the rest of our analysis and notation.

PROOF.   We assume without loss of generality that $I(\pi(y)) = \{1, 2, \ldots, I_y\}$ where $I_y = |I(\pi(y))|$. We will prove the proposition using the implicit function theorem. Let $f: M^\epsilon \times \mathbb{R}^n \times \mathbb{R}^{I_y} \mapsto \mathbb{R}^{n+I_y}$ be such that

$$f_{\{1,2,\ldots,n\}}(v, p, \gamma) = v - p - \sum_{i \in I(\pi(y))} \gamma_i \nabla g_i(p),$$

$$f_{n+j}(v, p, \gamma) = g_j(p) \quad \text{for } j \in I(\pi(y)).$$

Then f is a continuously differentiable function since the g_i are twice continuously differentiable. For $a = (y, \pi(y), \lambda(y))$, we have $f(a) = 0$. Denote by $J(y)$ the Jacobian $\nabla_{(p, \gamma)} f$ evaluated at a. Then

$$J(y) = \begin{bmatrix} J_{UL}(y) & G(\pi(y)) \\ G(\pi(y))^T & 0 \end{bmatrix}, \tag{18}$$

where $J_{UL}(y) = -I - H(y)$.

We first claim that $J_{UL}(y)$ is negative definite. From the proof of Lemma 4.2, we know that ϵ was chosen sufficiently small such that

$$\sum_{k \in I(\pi(y))} |\lambda_k(y)| \leq \frac{1}{2n(H_m + 2\mu)}, \tag{19}$$

where H_m is given by (10) and $\mu > 0$ is a scalar. For a matrix A, let A^{ij} denote its entry at the ith row and the jth column. Then, from the definition of the matrix norm in (8) and the definition of H_m in (10), we have

$$|H_{g_k}(\pi(y))^{ij}| \le H_m, \quad \forall i, j \in \{1, 2, \ldots, n\}.$$

Then for $i \in \{1, 2, \ldots, n\}$, we have

$$
\begin{aligned}
(J_{UL}(y))^{ii} - \sum_{j \in \{1,2,\ldots,n\}-\{i\}} |J_{UL}(y)^{ij}| &= -1 + \sum_{k \in I(\pi(y))} \lambda_k(y) H_{g_k}(\pi(y))^{ii} - \sum_{j \in \{1,2,\ldots,n\}-i} \left| \sum_{k \in I(\pi(y))} \lambda_k(y) H_{g_k}(\pi(y))^{ij} \right| \\
&\le -1 + \sum_{j \in \{1,2,\ldots,n\}} \sum_{k \in I(\pi(y))} |\lambda_k(y)| H_m \\
&\le -1 + \sum_{j \in \{1,2,\ldots,n\}} \frac{1}{2n(H_m + 2\mu)} H_m \\
&< -1 + \sum_{j \in \{1,2,\ldots,n\}} 1/n = 0,
\end{aligned}
$$

where we used Equation (19) to get the second inequality. Thus, $J_{UL}(y)$ is trictly diagonally negative dominant and hence is negative definite. We next claim that $J(y)$ is nonsingular. Assume, to obtain a contradiction, that there exists $(p, \gamma) \ne 0$ such that $J(y)(p, \gamma)^T = 0$. Then,

$$J_{UL}(y)p^T + G(\pi(y))\gamma^T = 0 \tag{20}$$

$$G(\pi(y))^T p^T = 0. \tag{21}$$

Premultiplying Equation (20) by p and using Equation (21), we obtain

$$p J_{UL}(y) p^T = 0.$$

Since $J_{UL}(y)$ is strictly negative definite, it follows that $p = 0$. Then, from Equation (20), $G(\pi(y))\gamma^T = 0$ and since the columns of $G(\pi(y))$ are linearly independent, we obtain $\gamma = 0$. Thus, $(p, \gamma) = 0$, which is a contradiction, establishing that $J(y)$ is nonsingular.

Then, the implicit function theorem applies to $f(v, p, \gamma)$ and there exist open sets $D^1 \subset M^\epsilon$, $D^2 \subset \mathbb{R}^n$, $D^3 \subset \mathbb{R}^{I(y)}$ such that

$$a = (y, \pi(y), \lambda(y)) \in (D^1 \times D^2 \times D^3)$$

and there exist unique continuously differentiable functions $p \colon D^1 \mapsto D^2$, $\gamma \colon D^1 \mapsto D^3$ such that

$$f(v, p(v), \gamma(v)) = 0$$

for all $v \in D^1$.

We next show that there exists an open set $D \subset \mathbb{R}^n$ containing y such that $p(v) = \pi(v)$ and $\gamma(v) = \lambda(v)$ for all $v \in D$. Note that $p(y) = \pi(y)$ and $\gamma(y) = \lambda(y)$. Since $y \in \mathrm{ri}(N^\epsilon(\pi(y)))$, we have $\lambda(y) > 0$. Then $\gamma(y) = \lambda(y) > 0$, and the continuity of γ implies that there exists an open set $D^\gamma \subset D^1$ containing y such that $\gamma(v) > 0$ for $v \in D^\gamma$. Similarly, $g_j(p(y)) = g_j(\pi(y)) < 0$ for all $j \notin I(\pi(y))$, thus there exists an open set $D^g \subset D^1$ containing y such that $g_j(p(v)) < 0$ for all $j \notin I(\pi(y))$ and $v \in D^g$. Let

$$D = D^\gamma \cap D^g.$$

Then $y \in D$. For $v \in D$ we have

$$v = p(v) + \sum_{i \in I(\pi(y))} \gamma_i(v) \nabla g_i(p(v)) \tag{22}$$

$$g_i(p(v)) = 0, \quad \forall i \in I(y)$$

$$g_j(p(v)) < 0, \quad \forall j \notin I(y).$$

Then we have $p(v) \in M$ and $I(p(v)) = I(\pi(y))$ and it follows from Equation (22) that $v \in N^\epsilon(p(v))$. Therefore, from Lemma 4.3 and the definition in (16), we have $p(v) = \pi(v)$ and $\gamma(v) = \lambda(v)$ for all $v \in D$.

Since p is a continuously differentiable function over a neighborhood D of y, we conclude that π is continuously differentiable at y as desired. Moreover, by the implicit function theorem, we have the following expression for the Jacobian of $(\pi; \lambda)$ at y:

$$\nabla(\pi, \lambda)(y) = -(\nabla_{(p, \gamma)} f(a))^{-1} \nabla_v f(a)$$
$$= -J(y)^{-1} I^{(n+I_y, n)},$$

where for positive integers n, m

$$I^{(n, m)} = \begin{cases} \text{the } n \times n \text{ identity matrix,} & \text{if } n = m, \\ \text{the } n \times m \text{ matrix } [I^{(n, n)} \quad 0], & \text{if } n < m, \\ \text{the } n \times m \text{ matrix } [I^{(m, m)} \quad 0]^T, & \text{if } n > m. \end{cases}$$

For $y \in \mathrm{ri}(N^\epsilon(\pi(y)))$, since $p = \pi$ in a neighborhood of y we have

$$\nabla \pi(y) = -I^{(n, n+I_y)} J(y)^{-1} I^{(n+I_y, n)}. \tag{23}$$

For notational simplicity, we fix y and denote $V = V(\pi(y))$, $G = G(\pi(y))$, $C = C(\pi(y))$, $H = H(y)$, $J = J(y)$, and $J_{UL} = J_{UL}(y)$. By Equation (23), we have

$$[\nabla \pi(y)]|_C = -C^{-1} I^{(n, n+I_y)} J^{-1} I^{(n+I_y, n)} C. \tag{24}$$

We first note that

$$I^{(n+I_y, n)} C = \begin{bmatrix} \nabla g_i(\pi(y)) & \cdots & v_j & \cdots \\ 0 & \cdots & 0 & \cdots \end{bmatrix}.$$

We next let e^k denote the kth unit vector in $\mathbb{R}^{n+I_y}$. Then

$$J e^{n+i} = \begin{bmatrix} \nabla g_i(\pi(y)) \\ 0 \end{bmatrix},$$

thus

$$J^{-1} \begin{bmatrix} \nabla g_i(\pi(y)) \\ 0 \end{bmatrix} = e^{n+i}. \tag{25}$$

Next, we let

$$J^{-1} \begin{bmatrix} v_j \\ 0 \end{bmatrix} = \begin{bmatrix} x_j \\ y_j \end{bmatrix}. \tag{26}$$

Then,

$$J \begin{bmatrix} x_j \\ y_j \end{bmatrix} = \begin{bmatrix} v_j \\ 0 \end{bmatrix}$$

and thus

$$J_{UL} x_j + G y_j = v_j, \tag{27}$$

and

$$G^T x_j = 0. \tag{28}$$

Since columns of V span the space of vectors orthogonal to each column of G, Equation (28) implies that $x_j = V \beta_j$ for some $\beta_j \in \mathbb{R}^{n-I_y}$. Then, premultiplying Equation (27) by V^T, we obtain

$$V^T J_{UL} V \beta_j + V^T G y_j = V^T v_j$$

and using the fact that $V^T G = 0$ and that $V^T J_{UL} V$ is invertible, we get

$$\beta_j = (V^T J_{UL} V)^{-1} e_j, \tag{29}$$

hence β_j is the jth column of $(V^T J_{UL} V)^{-1}$. (When V is not the empty matrix, since J_{UL} is strictly negative definite, $V^T J_{UL} V$ is strictly negative definite, and hence is invertible. When V is the $n \times 0$ dimensional empty

matrix, then $V^T J_{UL} V$ is the 0×0 dimensional empty matrix and it is invertible by our convention.) Let $\beta = [\beta_j|_{j \in \{1,2,\ldots,n-I_y\}}]$ and $Y = [y_j|_{j \in \{1,2,\ldots,n-I_y\}}]$. Then, using Equations (25), (26), and (29),

$$J^{-1}I^{(n+I_y,\,n)}C = \begin{bmatrix} 0 & V\beta \\ I^{(I_y,\,I_y)} & Y \end{bmatrix} = \begin{bmatrix} 0 & V(V^T J_{UL} V)^{-1} \\ I^{(I_y,\,I_y)} & Y \end{bmatrix}.$$

Substituting this expression in Equation (24) yields

$$
\begin{aligned}
[\nabla \pi(y)]|_C &= -C^{-1}I^{(n,\,n+I_y)} \begin{bmatrix} 0 & V(V^T J_{UL} V)^{-1} \\ I^{(I_y,\,I_y)} & Y \end{bmatrix} \\
&= -C^{-1}[0 \quad V(V^T J_{UL} V)^{-1}] \\
&= C^{-1}[G \quad V]\begin{bmatrix} 0 & 0 \\ 0 & (V^T(-J_{UL})V)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & (V^T(I+H)V)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & (I+V^THV)^{-1} \end{bmatrix},
\end{aligned}
$$

showing the desired relation. $\square$

4.2. The extension theorem.

We now state and prove the extension theorem that will subsequently be used in the proof of the generalized Poincaré-Hopf theorem.

THEOREM 4.1. *Let M be a region given by* (1). *Let U be an open set containing M and $F: U \mapsto \mathbb{R}^n$ be a continuous function. Let $F^{\max} = \max_{x \in M} \|F(x)\|$ and let $r \in \mathbb{R}$. Let $F_r: \mathrm{cl}(M^\epsilon) \mapsto \mathbb{R}^n$ be defined as*

$$F_r(y) = F(\pi(y)) + r(y - \pi(y)).$$

Then, we have the following:

(i) *F_r is a continuous function.*

(ii) *Let $y \in \mathrm{ri}(N^\epsilon(\pi(y)))$ and F be continuously differentiable at $\pi(y) \in M$. Then, F_r is continuously differentiable at y. Moreover, the Jacobian of F_r in the tangent-normal coordinates of $\pi(y)$ is*

$$[\nabla F_r(y)]|_{C(\pi(y))} = \begin{bmatrix} rI & S \\ 0 & [\nabla F(\pi(y)) + rH(y)]\|_{V(\pi(y))}(I + [H(y)]\|_{V(\pi(y))})^{-1} \end{bmatrix} \tag{30}$$

for some $I_y \times (n - I_y)$ matrix S. Furthermore,

$$\det(\nabla F_r(y)) = r^{I_y}\det([\nabla F(\pi(y)) + rH(y)]\|_{V(\pi(y))})\det(I + [H(y)]\|_{V(\pi(y))})^{-1}. \tag{31}$$

Also, if $r > 0$, then

$$\mathrm{sign}(\det(\nabla F_r(y))) = \mathrm{sign}(\det([\nabla F(\pi(y)) + rH(y)]\|_{V(\pi(y))})). \tag{32}$$

(iii) *If $r > F^{\max}/\epsilon$, then F_r points outward $\mathrm{cl}(M^\epsilon)$ on the boundary of $\mathrm{cl}(M^\epsilon)$. In other words, given $y \in \mathrm{bd}(M^\epsilon)$, there exists a sequence $\epsilon_i \downarrow 0$ such that $y + \epsilon_i F_r(y) \notin \mathrm{cl}(M^\epsilon)$ for all $i \in \mathbb{Z}^+$.*

PROOF. (i) Follows immediately since the projection function π is continuous.

(ii) Let $y \in \mathrm{ri}(N^\epsilon(\pi(y)))$. Then F_r is continuously differentiable at y since F is continuously differentiable at $\pi(y)$ and π is continuously differentiable at y (cf. Proposition 4.3). For notational simplicity, we fix y and denote $V = V(\pi(y))$, $G = G(\pi(y))$, $C = C(\pi(y))$, $H = H(y)$.

First consider the case in which V is the $n \times 0$ empty matrix, i.e., $I_y = n$. In this case, by the chain rule and the fact that $\nabla \pi(y) = 0$ (cf. Proposition 4.3), we have

$$\nabla F_r(y) = \nabla F(y)\nabla \pi(y) + r(I - \nabla \pi(y)) = rI,$$

showing Equation (30). Equations (31) and (32) also follow since the determinant of a 0×0 matrix is equal to 1 by our convention.

Next, consider the case in which V is not an empty matrix. Since $V^T G = 0$ and $V^T V = I$, we note that

$$[G \quad V]^{-1} = \begin{bmatrix} R \\ V^T \end{bmatrix} \tag{33}$$

for some matrix R. We can write the Jacobian of F_r in the tangent-normal coordinates as

$$
\begin{aligned}
[\nabla F_r(y)]|_C &= [\nabla F(\pi(y))]|_C [\nabla \pi(y)]|_C + r(I - [\nabla \pi(y)]|_C) \\
&= ([\nabla F(\pi(y))]|_C - rI)[\nabla \pi(y)]|_C + rI \\
&= [G \quad V]^{-1}(\nabla F(\pi(y)) - rI)[G \quad V] \begin{bmatrix} 0 & 0 \\ 0 & (I + V^T H V)^{-1} \end{bmatrix} + rI \\
&= \begin{bmatrix} \cdots & \cdots \\ \cdots & V^T(\nabla F(\pi(y)) - rI)V \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & (I + V^T H V)^{-1} \end{bmatrix} + \begin{bmatrix} rI & 0 \\ 0 & r(I + V^T H V)(I + V^T H V)^{-1} \end{bmatrix},
\end{aligned}
$$

where, to get the last equation, we used Equation (33) and the fact that $(V^T(I + H)V)$ is invertible (cf. Proof of Proposition 4.3). Then, for some matrix S, we have,

$$
\begin{aligned}
[\nabla F_r(y)]|_C &= \begin{bmatrix} 0 & S \\ 0 & V^T(\nabla F(\pi(y)) - rI)V(I + V^T H V)^{-1} \end{bmatrix} + \begin{bmatrix} rI & 0 \\ 0 & V^T(rI + rH)V(I + V^T H V)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} rI & S \\ 0 & V^T(\nabla F(\pi(y)) - rI + rI + rH)V(I + V^T H V)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} rI & S \\ 0 & V^T(\nabla F(\pi(y)) + rH)V(I + V^T H V)^{-1} \end{bmatrix}
\end{aligned}
$$

as desired. The determinant then can be calculated as

$$
\begin{aligned}
\det(\nabla F_r(y)) &= \det(rI^{l_y \times l_y})\det(V^T(\nabla F(\pi(y)) + rH)V(I + V^T H V)^{-1}) \\
&= r^{l_y}\det(V^T(\nabla F(\pi(y)) + rH)V)\det(I + V^T H V)^{-1}
\end{aligned}
$$

as desired. We have noted that $V^T(-I - H)V$ is negative definite (cf. Proof of Proposition 4.3). Then, $I + V^T H V$ is positive definite and thus $(I + V^T H V)^{-1}$ is positive definite. Then

$$\det(I + V^T H V)^{-1} > 0$$

and for $r > 0$ we have

$$\text{sign}(\det(\nabla F_r(y))) = \text{sign}(\det(V^T(\nabla F(\pi(y)) + rH)V))$$

as desired.

(iii) Let $y \in \text{bd}(M^\epsilon)$. By Corollary 4.2, we have

$$\text{bd}(M^\epsilon) = \{y \in \mathbb{R}^n \mid d(x) = \epsilon\},$$

which implies

$$d(y) = \|y - \pi(y)\| = \epsilon. \tag{34}$$

Since

$$\text{cl}(M^\epsilon) = \{x \in \mathbb{R}^n \mid d(x) \leq \epsilon\},$$

a vector $v \in \mathbb{R}^n$ points outward $\text{cl}(M^\epsilon)$ at y if

$$v^T \nabla d(y) = v^T \frac{y - \pi(y)}{d(y)} > 0. \tag{35}$$

We have

$$
\begin{aligned}
F_r(y)^T(y - \pi(y)) &= F(\pi(y))^T(y - \pi(y)) + r(y - \pi(y))^T(y - \pi(y)) \\
&\geq -\|F(\pi(y))\|\,\|y - \pi(y)\| + r\|y - \pi(y)\|\,\|y - \pi(y)\| \\
&\geq -F^{\max}\epsilon + r\epsilon^2 \\
&> \epsilon^2\left(r - \frac{F^{\max}}{\epsilon}\right),
\end{aligned}
$$

where we used the Cauchy-Schwartz inequality to get the first inequality and Equation (34) and the definition of $F^{\max}$ to get the second inequality. If $r > F^{\max}/\epsilon$, then $F_r(y)^T(y - \pi(y)) > 0$ and by Equation (35), F_r points outward at y as desired. This completes the proof of the theorem. $\square$

The following proposition establishes that there exists a one-to-one correspondence between the zeros of F_r over M^ϵ and the critical points of F over M.

PROPOSITION 4.4. *Let M be a region given by* (1). *Let U be an open set containing M and $F\colon U \mapsto \mathbb{R}^n$ be a continuous function. Let $F^{\max} = \max_{x \in M} \|F(x)\|$ and let $r > F^{\max}/\epsilon$ be a scalar. Let $F_r\colon \mathrm{cl}(M^\epsilon) \mapsto \mathbb{R}^n$ be defined as*

$$
F_r(y) = F(\pi(y)) + r(y - \pi(y)).
$$

and let

$$
Z = \{z \in M^\epsilon \mid F_r(z) = 0\}.
$$

Define the function $s\colon \mathrm{Cr}(F, M) \mapsto \mathbb{R}^n$ such that

$$
s(x) = x - \frac{F(x)}{r} = x + G(x)\frac{\theta(x)}{r}
$$

(cf. Definition 3.1.*) Then, s is a one-to-one and onto function from $\mathrm{Cr}(F, M)$ to Z, with inverse equal to $\pi|_Z$.*

PROOF. Clearly, $s(x) - x \in N_M(x)$. Also, since

$$
\|s(x) - x\| = \frac{\|F(x)\|}{r} \leq \frac{F^{\max}}{r} < \epsilon,
$$

we have

$$
s(x) \in N^\epsilon(x). \tag{36}
$$

Then it follows from Proposition 4.3 that $\pi(s(x)) = x$. Also, it follows from the definition of λ that $\lambda(s(x)) = \theta(x)/r$. We first show that $s(x) \in Z$ for all $x \in \mathrm{Cr}(F, M)$. Note that

$$
\begin{aligned}
F_r(s(x)) &= F(\pi(s(x))) + r(s(x) - x) \\
&= F(x) + r\left(\frac{1}{r}G(x)\theta(x)\right) \\
&= F(x) + G(x)\theta(x) = 0.
\end{aligned}
$$

Thus $s(x) \in Z$. We next show that $s\colon \mathrm{Cr}(F, M) \mapsto Z$ is onto. To see this, let $y \in Z$. Then,

$$
F_r(y) = F(\pi(y)) + r(y - \pi(y)) = 0
$$

implies

$$
-F(\pi(y)) = r(y - \pi(y)) = rG(\pi(y))\lambda(y).
$$

Since $r\lambda(y) \geq 0$, we have $-F(\pi(y)) \in N_M(\pi(y))$ and thus $\pi(y) \in \mathrm{Cr}(F, M)$. Moreover,

$$
s(\pi(y)) = \pi(y) - \frac{F(\pi(y))}{r} = \pi(y) + \frac{r(y - \pi(y))}{r} = y,
$$

and thus $s\colon \mathrm{Cr}(F, M) \mapsto Z$ is onto.

We finally show that s is one-to-one. Assume there exists $x, x' \in \mathrm{Cr}(F, M)$ such that $y = s(x) = s(x')$. Then $y \in N^\epsilon(x)$ and $y \in N^\epsilon(x')$ [cf. Equation (36)]. From Lemma 4.1, $x = x'$ and thus s is one-to-one. We conclude that s is a one-to-one and onto function from $\mathrm{Cr}(F, M)$ to Z (see Figure 4). $\square$
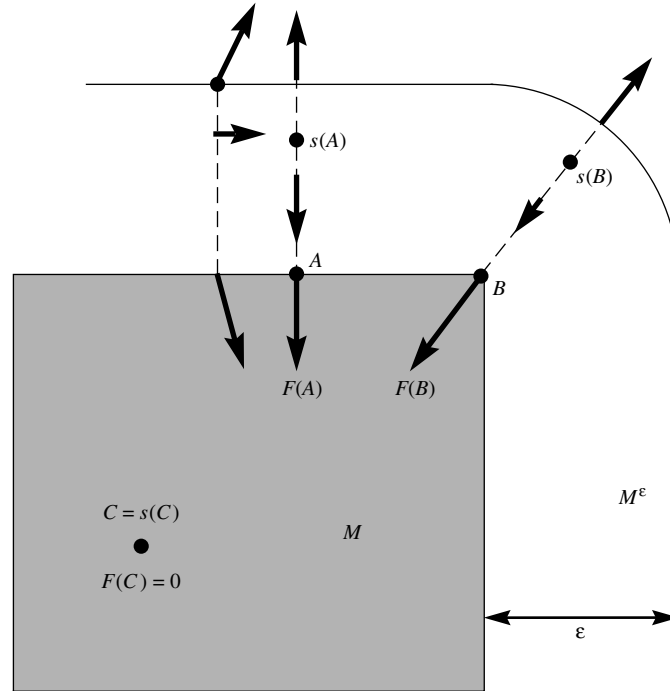
FIGURE 4. *A*, *B*, *C* are the generalized critical points of *F*, whereas $s(A)$, $s(B)$, and $s(C)$ are the critical points in the usual sense of the extended function F_r. In other words, *s* converts the generalized critical points of *F* to critical points of F_r.

The following lemma relates the index of a generalized critical point to the Poincaré-Hopf index of the zero of the extended function.

LEMMA 4.4. *Let $r > F^{\max}/\epsilon$ and $F_r: M^\epsilon \mapsto \mathbb{R}^n$ be the extension defined in Theorem* 4.1 *and Proposition* 4.4. *Let $x \in \mathrm{Cr}(F, M)$ be complementary and nondegenerate, and let F be continuously differentiable at x. Then, F_r is continuously differentiable at $s(x)$. Moreover,*

$$\mathrm{ind}_F(x) = \mathrm{sign}(\det(\nabla F_r(s(x)))). \tag{37}$$

PROOF. If $x \in \mathrm{Cr}(F, M)$ is complementary, then $\theta(x) > 0$. Thus $\lambda(s(x)) = \theta(x)/r > 0$. Therefore, $s(x) \in \mathrm{ri}(N^\epsilon(x))$. Then, F_r is continuously differentiable at $s(x)$ (cf. part (ii) of Proposition 4.1). Furthermore, we have

$$\mathrm{sign}(\det(\nabla F_r(s(x)))) = \mathrm{sign}(\det(V(x)^T(\nabla F(x) + rH(s(x)))V(x)))$$

$$= \mathrm{sign}\left(\det\left(V(x)^T\left(\nabla F(x) + \sum_{i \in I(x)} r\lambda_i(s(x))H_{g_i}(x)\right)V(x)\right)\right)$$

$$= \mathrm{sign}\left(\det\left(V(x)^T\left(\nabla F(x) + \sum_{i \in I(x)} \theta_i(x)H_{g_i}(x)\right)V(x)\right)\right)$$

$$= \mathrm{sign}(\det(\Gamma(x)))$$

$$= \mathrm{ind}_F(x)$$

as desired. □

4.3. Proof of the generalized Poincaré-Hopf theorem.

We are now ready to prove Theorem 3.1.
PROOF OF THEOREM 3.1. Let $r > F^{\max}/\epsilon$ and $F_r: \mathrm{cl}(M^\epsilon) \mapsto \mathbb{R}^n$ be the extension defined in Theorem 4.1 and Proposition 4.4. Let

$$Z = \{z \in \mathbb{R}^n \mid F_r(z) = 0\}.$$

By Corollary 4.2, $\mathrm{cl}(M^\epsilon)$ is a manifold with boundary. Moreover, by Proposition 4.1, F_r points outward on the boundary of M^ϵ and is continuously differentiable at every $y \in Z$ since $\pi(y) \in \mathrm{Cr}(F, M)$ is complementary and

nondegenerate and F is continuously differentiable at $\pi(y)$. Then, the Poincaré-Hopf theorem applies to F_r and Z has a finite number of elements. By Proposition 4.4, $\mathrm{Cr}(F, M)$ has a finite number of elements. Moreover,

$$\sum_{z \in Z} \mathrm{sign}(\det(\nabla F_r(z))) = \chi(\mathrm{cl}(M^\epsilon)).$$

Then using Proposition 4.4 and Lemma 4.4, we have

$$\sum_{x \in \mathrm{Cr}(F, M)} \mathrm{ind}_F(x) = \sum_{x \in \mathrm{Cr}(F, M)} \mathrm{sign}(\det(\Gamma(x)))$$

$$= \sum_{s(x) \in Z} \mathrm{sign}(\det(\nabla F_r(s(x)))) = \chi(\mathrm{cl}(M^\epsilon)). \tag{38}$$

To complete the proof, we need an additional lemma. First, recall the following definition:

DEFINITION 4.3.   Two continuous functions $a, b\colon X \mapsto Y$ are *homotopic* if there exists a continuous map $F\colon X \times [0, 1] \mapsto Y$ such that

$$F(x, 0) = a(x) \qquad \text{and} \qquad F(x, 1) = b(x), \quad \text{for all } x \in X.$$

Such a function F is called a *homotopy* between a and b.

Next, we have the following result:

LEMMA 4.5.   *The set* $\mathrm{cl}(M^\epsilon)$ *is homotopy equivalent to* M, *i.e., there exists continuous functions* $f\colon \mathrm{cl}(M^\epsilon) \mapsto M$ *and* $g\colon M \mapsto \mathrm{cl}(M^\epsilon)$ *such that* $f \circ g$ *is homotopic to* i_M *and* $g \circ f$ *is homotopic to* $i_{\mathrm{cl}}(M)$, *where* $i_X\colon X \mapsto X$ *denotes the identity function on some set* X. *In particular,* $\chi(\mathrm{cl}(M^\epsilon)) = \chi(M)$.

PROOF.   Let $f = \pi|_{\mathrm{cl}(M^\epsilon)}\colon \mathrm{cl}(M^\epsilon) \mapsto M$ and $g = i_M\colon M \mapsto M$. Then, f and g are continuous and

$$f \circ g(x) = f(g(x)) = \pi(x) = x, \quad \text{for all } x \in M.$$

Then, $f \circ g = i_M$ and thus is homotopic to i_M. We have

$$g \circ f(x) = g(f(x)) = \pi(x), \quad \text{for all } x \in \mathrm{cl}(M). \tag{39}$$

Let $F(\mathrm{cl}(M^\epsilon) \times [0, 1]) \mapsto \mathbb{R}^n$ such that

$$F(x, \xi) = (1 - \xi)\pi(x) + \xi x.$$

F is continuous since π is continuous over $\mathrm{cl}(M^\epsilon)$. We have

$$F(x, 0) = \pi(x) = g \circ f(x) \quad \text{for all } x \in \mathrm{cl}(M),$$

where we used Equation (39), and

$$F(x, 1) = x = i_{\mathrm{cl}(M)}(x), \quad \text{for all } x \in \mathrm{cl}(M).$$

Thus, F is a homotopy between $g \circ f$ and $i_{\mathrm{cl}(M)}$, which implies that $g \circ f$ is homotopic to $i_{\mathrm{cl}(M)}$. Then f and g satisfy the conditions for homotopy equivalence of $\mathrm{cl}(M^\epsilon)$ and M, which implies that $\mathrm{cl}(M^\epsilon)$ is homotopy equivalent to M as desired. In fact, this argument proves the stronger result that M is a strong deformation retract of M^ϵ, i.e., there exists a homotopy $F\colon M^\epsilon \times [0, 1] \mapsto M^\epsilon$ such that $F(x, 0) = x$ and $F(x, 1) \in M$ for all $x \in M^\epsilon$ (cf. Kosniowski [19]).

Since the Euler characteristic is invariant for sets that are homotopy equivalent to each other (see, for example, Massey [21]), we have $\chi(\mathrm{cl}(M^\epsilon)) = \chi(M)$, completing the proof of Lemma 4.5. $\square$

Finally, combining the result of Lemma 4.5 with Equation (38), we complete the proof of Theorem 3.1. $\square$

5. Applications of the generalized Poincaré-Hopf theorem.

In §5, we use Theorem 3.1 to present sufficient conditions for the uniqueness of solutions to finite-dimensional variational inequalities and the stationary points in nonconvex optimization problems.

5.1. Variational inequalities.

DEFINITION 5.1. Let $M \subset \mathbb{R}^n$ be a region and $F: M \mapsto \mathbb{R}^n$ be a function. The variational inequality problem is to find a vector $x \in M$ such that

$$(y - x)^T F(x) \geq 0, \quad \forall y \in M.$$

We denote the set of solutions to this problem with $\text{VI}(F, M)$.

The following lemma establishes that, when M is given by (1), every solution of the variational inequality problem corresponds to a generalized critical point of F over M and vice versa. The proof is straightforward and hence omitted.

LEMMA 5.1. *Let $M \subset \mathbb{R}^n$ be a closed convex region given by (1). Then,*

$$x \in \text{VI}(F, M) \iff x \in \text{Cr}(F, M).$$

The following proposition establishes sufficient conditions for the existence and the uniqueness of solutions to a variational inequality problem over a convex and compact region M given by (1). The proof is a direct application of Theorem 3.1 in view of Lemma 5.1.

PROPOSITION 5.1. *Let M be a compact convex region given by (1). Let U be an open set containing M and $F: U \mapsto \mathbb{R}^n$ be a continuous function that is continuously differentiable at every $x \in \text{Cr}(F, M)$. Assume that every $x \in \text{Cr}(F, M)$ is complementary and nondegenerate. Furthermore, assume that $\text{ind}_F(x) = 1$ for all $x \in \text{Cr}(F, M)$. Then, $\text{VI}(F, M)$ has a unique element.*

5.2. Karush-Kuhn-Tucker points in nonconvex optimization.

In the special case when F is the gradient of a scalar valued function, a generalized critical point of F corresponds to a Karush-Kuhn-Tucker (KKT) stationary point of a constrained optimization problem, which allows us to establish sufficient conditions for the uniqueness of the local optimum for nonconvex optimization problems.

In some works in the optimization literature, the word "nondegenerate solution" is used for what we define as a *complementary critical point* (cf. Definition 3.4.1 in Facchinei and Pang [12]). We choose to use the latter naming, since we use the former to denote a different concept (cf. Definition 5.2, part (c)).

DEFINITION 5.2. Consider a compact region M given by (1). Let U be an open set containing M and $f: U \mapsto \mathbb{R}$ be a twice continuously differentiable function.

(a) We say that $x \in M$ is a KKT point of f over M if there exists $\mu_i \geq 0$ such that

$$\nabla f(x) + \sum_{i \in I(x)} \mu_i \nabla g_i(x) = 0. \tag{40}$$

We denote the set of KKT points of f over M with $\text{KKT}(f, M)$.

(b) For $x \in \text{KKT}(f, M)$, we define $\mu(x) \geq 0$ to be the unique vector in $R^{|I(x)|}$ that satisfies Equation (40). We say that x is a complementary KKT point if $\mu(x) > 0$.

(c) We define

$$\Lambda(x) = V(x)^T \left(H_f(x) + \sum_{i \in I(x)} \mu_i(x) H_{g_i}(x) \right) V(x), \tag{41}$$

where $V(x)$ denotes the change-of-coordinates matrix from tangent coordinates to standard coordinates [cf. Equation (3)]. We say that x is a nondegenerate KKT point if $\Lambda(x)$ is a nonsingular matrix.

(d) Let x be a complementary and nondegenerate KKT point of f over M. We define the KKT index of f at x as

$$\text{ind}_f^{\text{KKT}}(x) = \text{sign}(\det(\Lambda(x))).$$

Note that the KKT points of f over M are precisely the generalized critical points of ∇f over M, with $\theta(x) = \mu(x)$ in Definition 3.1. Therefore, the following proposition and the subsequent corollary follow directly from Theorem 3.1.

PROPOSITION 5.2. *Consider a compact region M given by (1). Let U be an open set containing M and $f: U \mapsto \mathbb{R}$ be a twice continuously differentiable function. Assume that every KKT point of f over M is complementary and nondegenerate. Then, the set $\text{KKT}(F, M)$ has a finite number of elements. Moreover, we have*

$$\sum_{x \in \text{KKT}(f, M)} \text{ind}_f^{\text{KKT}}(x) = \chi(M).$$

COROLLARY 5.1. *Consider a compact region M given by* (1). *Let U be an open set containing M and $f: U \mapsto \mathbb{R}$ be a twice continuously differentiable function. Assume that $\chi(M) = 1$ and that every $x \in \mathrm{KKT}(f, M)$ is nondegenerate, complementary, and satisfies $\mathrm{ind}_f^{\mathrm{KKT}}(x) = 1$. Then, f has a unique local minimum over M which is also the global minimum.*

Traditional optimization results establish the uniqueness of the minimum under a global convexity assumption, i.e., when the Hessian of the objective function is positive definite for every vector in the constraint region. In contrast, Corollary 5.1 establishes the uniqueness of a local minimum of a nonconvex optimization problem by a sufficient local condition that requires that the determinant of the (generalized) Hessian is positive at every KKT point of the function over the region. In the following example, we use Corollary 5.1 to prove the uniqueness of a local minimum in a nonconvex problem:

EXAMPLE 5.1. Let n be a positive integer, $L = \{1, 2, \ldots, n\}$ be a set, $a, b \in \mathbb{R}$ be scalars such that $0 < a < 1/n$ and $a < b$. Let $f: M = [a, b]^n \mapsto \mathbb{R}^n$ be a function given by

$$f(p) = -\sum_{j \in L} \log\left(\frac{p_j}{p^{-j}}\right),$$

where $p^{-j} = 1 + \sum_{k \neq j;\, k \in L} p_k$. It can be seen that $f(p)$ is a nonconvex function.

We claim that every KKT point of f is complementary, nondegenerate, and has KKT index equal to 1. Let $p \in \mathrm{KKT}(f, M)$. By Definition 5.2, every KKT point satisfies

$$\nabla f_i(p) \geq 0, \quad \text{if } p_i = a \tag{42}$$

$$\nabla f_i(p) = 0, \quad \text{if } a < p_i < b \tag{43}$$

$$\nabla f_i(p) \leq 0, \quad \text{if } p_i = b. \tag{44}$$

If the inequalities in (42) and (44) are strict, p is complementary. If $p_i = a$, then

$$\nabla f_i(p) = -\frac{1}{a} + \sum_{j \neq i} \frac{1}{p^{-j}} \leq -\frac{1}{a} + (n - 1) < 0,$$

since $a < 1/(n-1)$, therefore Equation (42) does not hold for any $i \in L$. Let $I(p)$ denote the set of indices for which $p_i = b$. Assume, to get a contradiction, that $\nabla f_i(p) = 0$ for some $i \in I(p)$. Then, for $j \in I(p)$, $p_i = p_j = b$, and by symmetry, we have

$$\nabla f_j(p) = \nabla f_i(p) = 0.$$

Together with Equation (43), the preceding relation implies that $\nabla f_i(p) = 0$ for all $i \in L$. Then, using the inequality $\sum_{j \neq k;\, j \in L} p^j / p^{-k} < 1$, we obtain

$$0 = \sum_{j \in L} p_j \nabla f_j(p) = -n + \sum_{k \in L} \sum_{j \neq k;\, j \in L} \frac{p_j}{p^{-k}} < 0,$$

yielding a contradiction. Hence, $\nabla f_i(p) < 0$ for all $i \in I(p)$, implying that p is a complementary KKT point. Let $I^{NB}(p)$ denote the set of indices for which Equation (43) holds. For this example, $\Lambda(p)$ (cf. Equation (41)) is given by

$$\Lambda(p) = H_f(p)|_{I^{NB}(p)},$$

where $H_f(p)|_{I^{NB}(p)}$ denotes the principal submatrix of $H_f(p)$ corresponding to the indices in $I^{NB}(p)$. Let

$$C = P H_f(p) P,$$

where P is the $n \times n$ diagonal matrix with entries p_i in the diagonal. It can be seen that $C|_{I^{NB}(p)}$ is positive diagonally dominant. Consequently, $\det(\Lambda(p)) > 0$, which implies that p is nondegenerate and $\mathrm{ind}_f^{\mathrm{KKT}}(p) = 1$ as desired.

Since $\chi(M) = 1$ and every $p \in \mathrm{KKT}(f, M)$ is complementary, nondegenerate, and satisfies $\mathrm{ind}_f^{\mathrm{KKT}}(p) = 1$, by Corollary 5.1, f has a unique local minimum over M. It can be seen that the unique local minimum is located at $p = (b, b, \ldots, b)$. In essence, using the first-order conditions given by Equations (42), (43), and (44), we could prove that f is locally convex around each stationary point, which, by Corollary 5.1, implies that f has a unique stationary point and a unique local minimum over M. Note that the Poincaré-Hopf theorem cannot be applied to this problem since the region M is not a manifold and the unique critical point of the vector field ∇f is on the boundary of the region M.

Appendix. Proof of Proposition 4.2. For the proof, we need the following result:

LEMMA A.1. *For any $x \in M^\epsilon$ and $z \in M$, we have*

$$(x - z)^T (\pi(x) - z) \geq 0.$$

PROOF. For any $x \in M^\epsilon$ and $z \in M$, we can write $x - \pi(x) = (x - z) + (z - \pi(x))$, which implies that

$$\|x - \pi(x)\|^2 = \|x - z\|^2 + 2(x - z)^T (z - \pi(x)) + \|z - \pi(x)\|^2.$$

By the definition of $\pi(x)$, we have $\|x - \pi(x)\| \leq \|x - z\|$; therefore, the preceding implies that $(x - z)^T (\pi(x) - z) \geq 0$, completing the proof. $\square$

The following lemma proves that a function that has values close to another differentiable function is also differentiable. This lemma was inspired by the proof of Theorem 3.1 in Clarke et al. [6].

LEMMA A.2. *Let $A \subset \mathbb{R}^n$ be an open set, and $f, g: A \mapsto \mathbb{R}$ be scalar valued functions such that f is differentiable at $x \in A$. Assume that $f(x) = g(x)$ and there exists $r > 0$ such that*

$$\|f(y) - g(y)\| \leq r \|y - x\|^2, \quad \forall y \in A. \tag{A1}$$

Then, g is differentiable at x with derivative equal to $\nabla f(x)$.

PROOF. Define

$$e_x^f(v) = \frac{f(x + v) - f(x) - \nabla f(x)^T v}{\|v\|}.$$

The assumption that f is differentiable at $x \in A$ implies that

$$\lim_{v \to 0} e_x^f(v) = 0.$$

By (A1), for all $v \neq 0$ and sufficiently small such that $x + v \in A$, we have

$$e_x^g(v) = \frac{g(x + v) - g(x) - \nabla f(x)^T v}{\|v\|} \leq \frac{f(x + v) + r\|v\|^2 - f(x) - \nabla f(x)^T v}{\|v\|} = e_x^f(v) + r\|v\|.$$

Similarly,

$$e_x^g(v) \geq \frac{f(x + v) - r\|v\|^2 - f(x) - \nabla f(x)^T v}{\|v\|} = e_x^f(v) - r\|v\|.$$

Combining the preceding two relations, we obtain

$$e_x^f(v) - r\|v\| \leq e_x^g(v) \leq e_x^f(v) + r\|v\|.$$

By taking the limit as $v \to 0$,

$$0 = \lim_{v \to 0} e_x^f(v) - r\|v\| \leq \lim_{v \to 0} e_x^g(v) \leq \lim_{v \to 0} e_x^f(v) + r\|v\| = 0.$$

Thus, $\lim_{v \to 0} e_x^g(v) = 0$, showing that g is differentiable at x with derivative $\nabla f(x)$ as desired. $\square$

PROOF OF PROPOSITION 4.2. Let x be an arbitrary vector in $M^\epsilon - M$. Consider the function $f: M^\epsilon - M \mapsto \mathbb{R}$ given by

$$f(w) = \|w - \pi(x)\|.$$

Let $\delta \in \mathbb{R}$ be such that $0 < \delta < d(x)$. Then, f is differentiable on the ball $B(x, \delta)$ with derivative

$$\nabla f(w) = \frac{w - \pi(x)}{\|w - \pi(x)\|}.$$

Let $y \in B(x, \delta)$ (see Figure A.1). By the definition of $d(y)$ and δ, we have
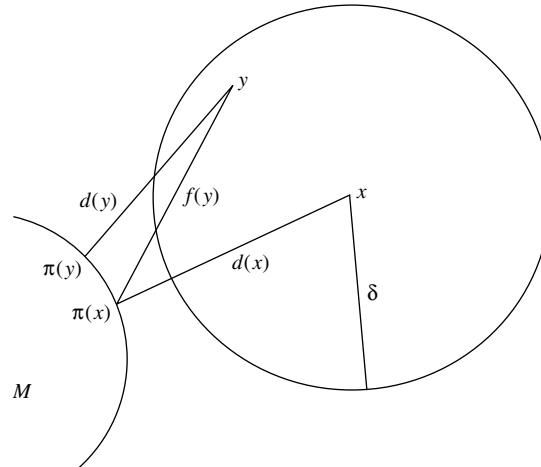
$$f(y) \geq d(y) \geq d(x) - \delta > 0. \tag{A2}$$

FIGURE A.1. The distance function d is close to the function f in a neighborhood of x. Then, since f is differentiable at x, d is also differentiable at x with derivative $\nabla f(x)$.

Using $y - \pi(x) = y - \pi(y) + (\pi(y) - \pi(x))$, we obtain

$$
\begin{aligned}
f(y)^2 &= \|y - \pi(x)\|^2 = \|y - \pi(y)\|^2 + \|\pi(y) - \pi(x)\|^2 + 2(y - \pi(y))^T(\pi(y) - \pi(x)) \\
&= d(y)^2 + \|\pi(y) - \pi(x)\|^2 + 2(y - x + x - \pi(y))^T(\pi(y) - \pi(x)) \\
&= d(y)^2 + \|\pi(y) - \pi(x)\|^2 + 2(y - x)^T(\pi(y) - \pi(x)) + 2(x - \pi(y))^T(\pi(y) - \pi(x)) \\
&\leq d(y)^2 + \|\pi(y) - \pi(x)\|^2 + 2\|y - x\|\|\pi(y) - \pi(x)\| \\
&\leq d(y)^2 + (k^2 + 2k)\|y - x\|^2,
\end{aligned}
$$

where we used Lemma A.1 (with $z = \pi(y)$) and the Cauchy-Schwartz inequality to get the first inequality and the fact that π is globally Lipschitz over M^ϵ (cf. Proposition 4.1) to get the second inequality. Using the preceding, we obtain

$$
f(y)^2 - d(y)^2 \leq (k^2 + 2k)\|y - x\|^2,
$$

which implies

$$
f(y) - d(y) \leq \frac{k^2 + 2k}{f(y) + d(y)}\|y - x\|^2.
$$

Using Equation (A2), we further obtain

$$
f(y) - d(y) \leq \frac{k^2 + 2k}{2(d(x) - \delta)}\|y - x\|^2.
$$

Then, for

$$
r = \frac{k^2 + 2k}{2(d(x) - \delta)} > 0,
$$

we have

$$
0 \leq f(y) - d(y) \leq r\|y - x\|^2
$$

for all $y \in B(x, \delta)$. Since $f(x) = d(x)$, we conclude by Lemma A.2 that d is differentiable at x with derivative

$$
\nabla d(x) = \nabla f(x) = \frac{x - \pi(x)}{\|x - \pi(x)\|} = \frac{x - \pi(x)}{d(x)}.
$$

Since x was an arbitrary point in $M^\epsilon - M$, and $\nabla d(x) = (x - \pi(x))/d(x)$ is a continuous function of x over $M^\epsilon - M$, we conclude that d is continuously differentiable over $M^\epsilon - M$, completing the proof. $\square$

References

[1] Ben-El-Mechaiekh, H., W. Kryszewski. 1997. Equilibria of set-valued maps on nonconvex domains. *Trans. Amer. Math. Soc.* **349** 4159–4179.

[2] Berge, C. 1963. *Topological Spaces*. Macmillan, New York.

[3] Bertsekas, D. 1995. *Nonlinear Programming*. Athena Scientific, Belmont, MA.

[4] Bertsekas, D., A. Nedic, A. Ozdaglar. 2003. *Convex Analysis and Optimization*. Athena Scientific, Belmont, MA.

[5] Clarke, F. H. 2003. *Optimization and Nonsmooth Analysis*. Wiley, New York.

[6] Clarke, F. H., R. J. Stern, P. R. Wolenski. 1995. Proximal smoothness and the lower-c^2 property. *J. Convex Anal.* **2**(1/2) 117–144.

[7] Cornet, B. 2002. Euler characteristic and fixed-point theorems. *Positivity* **63** 243–360.

[8] Cornet, B., M. Czarnecki. 2000. Necessary and sufficient conditions for the existence of (generalized) equilibria on a compact epi-Lipschitzian domain. *Comm. Appl. Nonlinear Anal.* **7** 21–53.

[9] Cornet, B., M. Czarnecki. 2001. Existence of generalized equilibria. *Nonlinear Anal.* **44** 555–574.

[10] Dierker, E. 1972. Two remarks on the number of equilibria of an economy. *Econometrica* **40** 951–953.

[11] Eraslan, H., A. McLennan. 2005. Uniqueness of stationary equilibrium payoffs in coalitional bargaining. Working paper, University of Pennsylvania, Philadelphia, PA. http://finance.wharton.upenn.edu/~eraslan/HulyaEraslanWebPage/Papers/bar_latex060107.pdf

[12] Facchinei, F., J.-S. Pang. 2003. *Finite-Dimensional Variational Inequalities and Complementarity Problems*, Vol. 1. Springer-Verlag, Berlin, Germany.

[13] Gottlieb, D. H., G. Samaranayake. 1995. The index of discontinuous vector fields. *New York J. Math.* **1** 130–148.

[14] Guillemin, V., A. Pollack. 1974. *Differential Topology*. Prentice Hall, Englewood Cliffs, NJ.

[15] Hildenbrand, W., A. P. Kirman. 1988. *Equilibrium Analysis and Variations on Themes by Edgeworth and Walras*. North-Holland, Amsterdam, The Netherlands.

[16] Hirsch, M. W. 1976. *Differential Topology, Graduate Texts in Mathematics*, Vol. 33. Springer-Verlag, Berlin, Germany.

[17] Holmes, R. B. 1973. Smoothness of certain metric projections on Hilbert space. *Trans. Amer. Math. Soc.* **184** 87–100.

[18] Jongen, Th. H., P. Jonker, F. Twilt. 2000. *Nonlinear Optimization in Finite Dimensions, Nonconvex Optimization and Its Applications*, Vol. 47. Kluwer, Boston, MA.

[19] Kosniowski, C. 1980. *A First Course in Algebraic Topology*. Cambridge University Press, Cambridge, UK.

[20] Mas-Colell, A. 1985. *The Theory of General Economic Equilibrium*. Cambridge University Press, Cambridge, UK.

[21] Massey, W. S. 1991. *A Basic Course in Algebraic Topology*. Springer-Verlag, Berlin, Germany.

[22] Milnor, J. W. 1997. *Topology from the Differential Viewpoint*. Princeton University Press, Princeton, NJ.

[23] Morse, M. 1929. Singular points of vector fields under general boundary conditions. *Amer. J. Math.* **51**(2) 165–178.

[24] Ortega, J. M., W. C. Rheinboldt. 1970. *Iterative Solution of Nonlinear Equations in Several Variables*. Academic Press, New York.

[25] Robinson, S. M. 1979. Generalized equations. *Math. Programming Stud.* **10** 128–141.

[26] Robinson, S. M. 2003. Constraint nondegeneracy in variational analysis. *Math. Oper. Res.* **28**(2) 201–232.

[27] Rotman, J. J. 1988. *An Introduction to Algebraic Topology*. Springer, Berlin, Germany.

[28] Varian, H. R. 1975. A third remark on the number of equilibria of an economy. *Econometrica* **43**(5–6) 985.