Cascades in Networks and Aggregate Volatility *

Daron Acemoglu[†], Asuman Ozdaglar[‡], and Alireza Tahbaz–Salehi[§]

First version: August 2010
This version: October 2010

Abstract

We provide a general framework for the study of cascade effects created by interconnections between sectors, firms or financial institutions. Focusing on a multi-sector economy linked through a supply network, we show how structural properties of the supply network determine both whether aggregate volatility disappears as the number of sectors increases (i.e., whether the law of large numbers holds) and when it does, the rate at which this happens. Our main results characterize the relationship between first-order interconnections (captured by the weighted degree sequence in the graph induced by the input-output relations) and aggregate volatility, and more importantly, the relationship between higher-order interconnections and aggregate volatility. These higher-order interconnections capture the cascade effects, whereby low productivity or the failure of a set of suppliers propagates through the rest of the economy as their downstream sectors/firms also suffer and transmit the negative shock to their downstream sectors/firms. We also link the probabilities of tail events (large negative deviations of aggregate output from its mean) to sector-specific volatility and to the structural properties of the supply network.

Keywords: Aggregate volatility, risk, cascades, input-output structure, supply networks, Laws of large numbers.

JEL Classification: C67, D57, E32

*We thank Stefana Stantcheva for excellent research assistance, and Vasco Carvalho, Xavier Gabaix, Alp Simsek, Jean Tirole and seminar participants at Microsoft Research New England, Toulouse Network on Information Technology, and The International School and Conference on Network Science for helpful discussions and comments. We thank Bill Kerr and Kaushik Ghosh for initial help with the NBER productivity database.

†Department of Economics, Massachusetts Institute of Technology, 50 Memorial Drive, Cambridge MA, 02139 (e-mail: daron@mit.edu).

‡Laboratory for Information and Decision Systems, and Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge MA, 02139 (e-mail: asuman@mit.edu).

§Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge MA, 02139 (e-mail: alirezat@mit.edu).

1 Introduction

The recent economic crisis has further highlighted the importance of interconnections between firms and sectors in the economy. Both the spread of the risks emanating from the so-called "toxic" assets on the balance sheets of several financial institutions to the rest of the financial sector, and the transmission of the economic problems of the financial sector to the rest of the economy have been linked to such interconnections. In addition, both government policies aimed at shoring up several key financial institutions and the assistance to General Motors and Chrysler in the midst of the crisis were justified not so much because these institutions were "too big to fail" but because they were "too interconnected to fail". This was also the view of some industry insiders. In the fall of 2008, rather than asking for government assistance for Ford, Alan R. Mulally, the chief executive of Ford Motor Co., requested that the government supports General Motors and Chrysler. His reasoning for asking government support for his company's traditional rivals was that the failure of either GM or Chrysler would lead to the potential failure of their suppliers, and because Ford depended on many of the same suppliers as the other two automakers, it would also find itself in perilous territory.[1] Notably, this reasoning highlights that what might be important is not *first-order interconnections* (the fact that General Motors and Chrysler are highly connected firms), but *higher-order interconnections* resulting from the fact that General Motors and Chrysler were connected to suppliers that were in turn connected to another major company, Ford.

In this paper, we provide a mathematical framework for systematically evaluating how idiosyncratic shocks are translated into aggregate volatility because of interconnections. Though the general framework we develop is applicable to a variety of settings including firm-level interconnections resulting from producer-supplier relationships and financial linkages, we focus, for concreteness, on a multi-sector economy with input-output linkages between sectors, which we refer to as the *supply network* of the economy. The economy consists of n sectors and the supply network captures the input requirements of each sector. Mathematically, the supply network is represented by an $n \times n$ matrix W_n , with entry w_{ij} capturing the share of sector j's product in sector i's production technology. We refer to the sum of all w_{ij}'s in any column

[1] In a congressional testimony to the Senate Banking Committee on November 18, 2008, Mulally stated:

> "If any one of the domestic companies should fail, we believe there is a strong chance that the entire industry would face severe disruption. Ours is in some significant ways an industry that is uniquely interdependent — particularly with respect to our supply base, with more than 90 percent commonality among our suppliers. Should one of the other domestic companies declare bankruptcy, the effect on Ford's production operations would be felt within days — if not hours. Suppliers could not get financing and would stop shipments to customers. Without parts for the just-in-time inventory system, Ford plants would not be able to produce vehicles.
>
> "Our dealer networks also have substantial overlap. Approximately 400 of our dealers also have a GM or Chrysler franchise at their dealership, and we estimate that as many as 25% of our top 1500 dealers also own GM or Chrysler franchises. The failure of one of the companies would clearly have a great impact on our dealers with exposure to that company. In short, a collapse of one of our competitors here would have a ripple effect across all automakers, suppliers, and dealers — a loss of nearly 3 million jobs in the first year, according to an estimate by the Center for Automotive Research." (Mulally (2008a), see also Mulally (2008b)).

j as the *outdegree* or simply the *degree* of sector j. We define *aggregate volatility* as the standard deviation of log output in the economy, and study the relationship between aggregate volatility and the structure of the supply network.[2]

How do we determine whether and how much the supply network contributes to aggregate volatility? The most transparent and tractable way of doing this is to consider a sequence of supply networks $\{W_n\}_{n\in\mathbb{N}}$ corresponding to different levels of disaggregation of the economy.[3] In this context, we can ask two questions. The first is whether as $n \to \infty$, aggregate output becomes increasingly concentrated around a constant value (i.e., whether the law of large numbers holds). The second and the more interesting question concerns the rate at which such concentration takes place, if at all. For many types of supply networks, in particular those that feature no input-output linkages and those in which each sector relies equally on all other sectors, the "too interconnected to fail" phenomenon does not take place: the law of large numbers holds and convergence of aggregate output takes place at the rate $\sqrt{n}$.[4] This has two important implications. First, in such cases, interconnections and network effects would have little impact on aggregate volatility. Second and relatedly, for reasonably large values of n, without sizable aggregate shocks, aggregate output would have a very small standard deviation. For example, if the logarithm of the output of each sector has a standard deviation of 10% and there are 100 sectors in the economy, log per capita output would have a standard deviation of 1%, whereas with 1000 sectors, this standard deviation would be 0.3%, and with 10,000 sectors (corresponding to very detailed products), it would only be 0.01%. Even if there is additional correlation between the fluctuations of the output of sectors, the standard deviation of aggregate output would tend to be very small. This last observation is also the reason why many macroeconomic analyses build on aggregate productivity, credit, demand or monetary shocks in order to generate sizable output fluctuations.

As already noted by Jovanovic (1987), Durlauf (1993) and Bak, Chen, Scheinkman, and Woodford (1993), when there are strategic (or economic) connections among sectors and firms, or as shown by Gabaix (2010), when some firms play a disproportionately important role (due to their much larger size relative to others), the standard central limit theorems need not hold, and aggregate fluctuations may result from firm-level (idiosyncratic) shocks. Our approach highlights that more complex supply networks (rather than the simple networks mentioned in the previous paragraph) also play a similar role. In particular, our main results characterize conditions on the underlying supply network of the economy under which relatively small shocks can create *cascade effects* and as a result, convergence at the rate $\sqrt{n}$ no longer applies.

More specifically, we first show that if some sectors have a disproportionately large role in the supply network (the extreme case being a star-like network, where one sector is an input supplier to a very large number of other sectors), the law of large numbers does not hold. In this

[2]We justify the focus on this measure of fluctuations in aggregate output in the text. We also discuss other measures, and in particular, tail risks, in Section 6.

[3]In our model economy, the total supply of labor is considered to be fixed. Therefore, an increase in the number of sectors is equivalent to an increase in the level of disaggregation of the economy.

[4]More precisely, as is well-known from the central limit theorem, the distribution of the realization of output minus its limiting value scaled by $1/\sqrt{n}$ converges to a normal distribution as n becomes large.

case, as $n \to \infty$, aggregate volatility does not disappear (even though there are no aggregate shocks). The presence of such highly interconnected sectors can create cascade effects in that negative shocks to those sectors are transmitted to a sufficiently large fraction of other sectors in the economy. Notably, such cascade effects can arise even in cases in which the fraction of sectors supplied by the most connected sector in the economy goes to zero relatively fast.

However, even in cases in which the law of large numbers holds, the structure of the supply network may still affect the behavior of aggregate output by making the convergence rate much slower than $\sqrt{n}$. First, this can happen because, again, some sectors are suppliers to many other sectors; that is, because of first-order interconnections, as measured by the degree of the sectors in the supply network. Second and more interestingly, this can happen due to higher-order interconnections—i.e., the possibility that high degree sectors are themselves being supplied by common sources; a notion more closely related to cascade effects. If the supply network exhibits such higher-order interconnections, low productivity in one sector can potentially create cascade effects through the entire economy, as its high-degree downstream sectors will suffer and this will in turn affect a large number of further downstream sectors. These types of higher-order interconnections appear to capture concerns similar to that of Ford Motor Co. regarding the failure of common suppliers because of failure of General Motors or Chrysler.[5] Note that the intuition for why such interconnections matter is related to Gabaix (2010). Whereas in his work idiosyncratic shocks translate into aggregate shocks because the firm size distribution is sufficiently heavy-tailed and the largest firms contribute a sufficiently large fraction to aggregate output, in our economy, such a role is played by the supply network, leading to some sectors to have a disproportional effect on the aggregate output.

We provide several lower bounds on the rate of convergence as functions of structural properties of the supply network.[6] Before explaining these lower bounds, it is useful to illustrate the implications of slower convergence. For example, if convergence takes place at the rate $n^{1/4}$ instead of $\sqrt{n}$, in an economy with 100 sectors each with a standard deviation of 10%, aggregate volatility would be about 3%, whereas it would be 1.7% and 1% with 1000 and 10,000 sectors, respectively.[7]

We prove two key theorems characterizing lower bounds on the rate of convergence (and thus scaling factors for aggregate volatility). In terms of first-order interconnections, the lower bound we provide depends on *the coefficient of variation* of the degree sequence, defined as the sample standard deviation of the degrees of the sectors in the economy divided by average degree. When the coefficient of variation is high, a few sectors are highly interconnected relative to the rest. A corollary of this result is that if the degree distribution of the sequence of supply networks can be approximated by a *power law* (Pareto distribution) with shape parameter β be-

[5]In this paper, for simplicity, we assume that the markets are competitive and all firms have Cobb-Douglas technologies, which imply that shocks to a sector only affects its downstream sectors not its upstream suppliers. This issue is further discussed in the Conclusion.

[6]Structural properties here refer to properties that can be determined without full knowledge of the entire supply network, such as certain moments of the degree distribution.

[7]And of course, it can also be much slower. For example, if convergence is at the rate $\log(\log n)$, then aggregate volatility would remain around 1.7% even if the economy contained trillions of sectors/firms.

tween 1 and 2, then aggregate volatility decays at a rate slower than $n^{(\beta-1)/\beta}$. More importantly, we provide lower bounds in terms of second and higher-order interconnections. We also show that if the *second-order degree* sequence (appropriately defined so as to incorporate cascading effects) has a power law tail with shape parameter ζ between 1 and 2, then aggregate volatility decays at a rate slower than $n^{(\zeta-1)/\zeta}$. Notably, two economies with identical degree sequences (first-order connections) can vary to an arbitrary degree with respect to their coefficients of higher-order interconnections and to their distribution of second-order degree.

Finally, we study the relationship between the likelihood of "tail events"—e.g., very large drops in output—and the structure of the supply network. Such tail events are relevant for understanding large recessions or crashes. Some have claimed that events like the 2008–09 recession and the Long-Term Capital Management collapse in 1998 were results of one in 10,000 likelihood events and thus should not have much bearing for our understanding of risk and fluctuations. But if these "rare events" are one in 100 rather than one in 10,000, they should clearly be incorporated into our models and practices. Whether this is so or not requires an analysis of tail events. We first remark that two economies exhibiting the same level of aggregate volatility (as defined by our standard deviation-based measure) can have widely different likelihoods of tail events. We then characterize the asymptotic distribution of aggregate output and the limiting probabilities of tail events using large deviation bounds.

Our paper is most closely related to Gabaix (2010) and to the independent but clearly prior work by Carvalho (2010). Gabaix's pioneering work shows that when the firm size distribution has sufficiently heavy tails (in particular, when it is Pareto with a shape coefficient of one), idiosyncratic shocks will not wash out at the rate $\sqrt{n}$ and provides evidence to support that this source of aggregate volatility is indeed important in practice. While Gabaix takes the firm size distribution as given, our approach can be used to endogenize the firm size distribution as a function of network structure (input-output linkages) and relate the aggregate effects of idiosyncratic shocks to structural properties of the network structure (though in what follows, we interpret our micro units as sectors rather than firms). Carvalho (2010) which is even more closely related to our paper considers a multi-sector model with network effects. He notes that the law of large numbers does not apply when the economy has a star network structure and provides results on power law degree distributions that parallel the results we also present. Carvalho uses a different approach (based on random graphs) and more specific and restrictive assumptions on the distribution of shocks, whereas our analysis applies to any sequence of supply networks and to any distribution of sectoral shocks (with finite variance). As a result, we are able to develop a general framework and more powerful results. First, our more general approach confirms and extends Carvalho's results (his main results can be derived as a corollary of Theorem 3). Second, Carvalho's analysis does not contain any analog of our results on higher-order interconnection and cascade effects, which are our main focus. Finally, he does not study tail events and asymptotic distributions, instead focusing on dynamic implications of network effects in a multi-sector growth model (a direction we do not pursue).

Like Carvalho (2010), we also build on the literature on the role of sectoral shocks in macro fluctuations, such as Long and Plosser (1983), Horvath (1998, 2000), Dupor (1999), Conley and

Dupor (2003) and Shea (2002). In particular, the debate between Horvath (1998, 2000) and Dupor (1999) centered around whether sectoral shocks would translate into aggregate shocks. Our results provide fairly complete answers to the questions raised by these pioneering papers. This literature presents a variety of empirical evidence on the role of sectoral shocks, but does not provide a general mathematical framework similar to the one developed here.[8]

Our work also naturally builds on the related literature prior to Gabaix (2010), in particular, Jovanovic (1987), Durlauf (1993) and Bak et al. (1993). Jovanovic and Durlauf construct models in which there are strong strategic complementarities across firms, so that shocks to some firms can create cascade effects. Our approach can therefore be viewed as providing a general framework for a rigorous analysis of this class of models (of which Jovanovic and Durlauf's models would be special cases). Bak et al. (1993) construct an economic version of a "sandpile" type (self-organized criticality) model along the lines of work in physics and statistical mechanics by Bak and co-authors (e.g., Bak (1996), Bak, Tang, and Wiesenfeld (1987)), as well as Jensen (1998). Though a very interesting direction of work, their paper only offers a specific example, and one in which not only weak law of large numbers fails, but in which also mean average output is infinite. We view our line of research as complementary to these alternatives.

Finally, our work builds on the literature on non-classical central limit theorems in statistics and the literature on laws of large numbers and limit theorems for non-identically distributed random variables. It is also related to the *PageRank* vector as well as the concept of *Bonacich centrality* in the network science literature, which capture the importance of vertices of a given graph (see, e.g., Chung and Zhao (2008), Jackson (2008) and Bonacich (1987)). To the best of our knowledge, the literature does not provide lower bounds on the asymptotic behavior of these measures as functions of structural network parameters, which constitutes our main contribution.

The rest of the paper is organized as follows. In Section 2, we present our model and define the concept of the supply network. Section 3 characterizes the *influence vector* (which summarizes the relevant features of the supply network), provides conditions under which aggregate output, with the appropriate scaling, converges to an asymptotic distribution, and finally provides conditions under which it is asymptotically normally distributed. Section 4 contains our main results which characterize how structural properties of the supply network, such as its degree sequence and higher-order interconnectivities, determine the rate at which aggregate volatility vanishes. Section 5 illustrates some of the results using information from the US input-output matrix (and shows that second-order interconnections indeed appear to be quite important). Section 6 studies asymptotic distributions and tail events. Section 7 concludes. All proofs and some additional mathematical details are presented in the Appendix.

[8]Our model is also related to the smaller literature on the implications of input-output linkages on economic growth and cross-country income differences, see, for example, Ciccone (2002) and Jones (2009).

Notation

Throughout the paper, unless otherwise noted, all vectors are assumed to be column vectors. We denote the transpose of a matrix X by X'. We write $x \geq y$, if vector x is element-wise greater than or equal to vector y. Similarly, we write $x > y$, if every element of x is strictly greater than the corresponding element in y. We use $\mathbf{1}$ to denote the vector of all ones, the size of which is adjusted to and clear from the context.

Given two sequences of positive real numbers $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$, we write $a_n = \mathcal{O}(b_n)$, if they satisfy $\limsup_{n\to\infty} a_n/b_n < \infty$, whereas $a_n = \Omega(b_n)$ if $\liminf_{n\to\infty} a_n/b_n > 0$. On the other hand, $a_n = o(b_n)$ means that $\lim_{n\to\infty} a_n/b_n = 0$, and $a_n = \omega(b_n)$ means that $\lim_{n\to\infty} a_n/b_n = \infty$. Finally, we write $a_n = \Theta(b_n)$, if $a_n = \mathcal{O}(b_n)$ and $a_n = \Omega(b_n)$ hold simultaneously.

2 Model

In this section, we present our economic model. We start by describing the interactions between different sectors in the economy and define the notion of the supply network. Our model is a static variant of the multi-sector model of Long and Plosser (1983). We choose to focus on a static model, since the economics (and in fact even the mathematics) is very similar, except that less notation is necessary to set up and analyze the model. A related dynamic model is presented and analytically and quantitatively investigated in Carvalho (2010). We omit the dynamic extension to save space.

2.1 The Environment

Consider a static economy consisting of n sectors, $\mathcal{I}_n \equiv \{1, 2, \ldots, n\}$, each producing a distinct product. The output of any given sector can be either consumed or used by other sectors as inputs (intermediate goods) for production. Each sector consists of a unit-mass continuum of identical firms with Cobb-Douglas production technologies that use labor and intermediate goods purchased from other sectors. The production technologies of all firms exhibit constant returns to scale. More specifically, the output of sector i, denoted by x_i, is given by

$$x_i = z_i^\alpha l_i^\alpha \prod_{j\in\mathcal{N}_i} x_{ij}^{(1-\alpha)w_{ij}}, \tag{1}$$

where $\mathcal{N}_i \subseteq \mathcal{I}_n$ is the set of sectors that supply sector i with inputs, $\alpha \in (0, 1]$ is the labor share in the production technologies,[9] x_{ij} is the amount of commodity j used in production of good i, and z_i is the idiosyncratic productivity shock to firms in sector i. We assume that productivity shocks z_i are independent across sectors, and denote the distribution of $\epsilon_i \equiv \log(z_i)$ by F_i. l_i denotes the amount of labor employed by sector i.

For any given pair of sectors i and j with $j \in \mathcal{N}_i$, the exponent $w_{ij} > 0$ captures the share of good j in the total intermediate input use of firms in sector i. The fact that firms in a given sector

[9]Our results are largely unaffected if α is sector-specific (provided that it remains bounded away from 0 and 1 as $n \to \infty$).

use the output of firms in other sectors as inputs for production is the source of interconnectivity in the economy. We capture such inter-sectoral supply relations more concisely by defining the *input-output* matrix W_n with a generic entry $(W_n)_{ij} = w_{ij}$.[10] We also adopt the convention that $w_{ij} = 0$ if sector j is not an input supplier to sector i. By definition, W_n is a non-negative $n \times n$ matrix. The following assumption guarantees that the sectoral production functions exhibit constant returns to scale to their labor inputs and the intermediate goods provided by their suppliers.

Assumption 1. The input shares of any firm $i \in \mathcal{I}_n$ in the economy add up to one; that is, $\sum_{j=1}^{n} w_{ij} = 1$.

Note that the above assumption also implies that input-output matrix W_n is always a *stochastic matrix*, meaning that its row sums are one (and thus has an eigenvalue equal to one with the corresponding right eigenvector consisting of all ones). Also note that we do not assume $w_{ii} = 0$.

We assume perfectly competitive markets, where firms in each sector take the prices and wage as given, and maximize profits. In particular, the representative firm in sector i chooses the amount of labor and intermediate goods in order to maximize its profits

$$p_i x_i - h l_i - \sum_{j=1}^{n} p_j x_{ij}, \tag{2}$$

subject to its production possibility (1), where h denotes the hourly wage in the market and p_j is the market price of commodity j.

In addition to the firms, there is a continuum of identical consumers in the economy, whose mass is normalized to one. The representative household is endowed with one unit of labor that can be hired by firms at wage h. We assume that she has symmetric Cobb-Douglas preferences over all goods in the economy and supplies labor inelastically:

$$u(c_1, c_2, \ldots, c_n) = A_n \prod_{i=1}^{n} (c_i)^{1/n},$$

where c_i is the consumption of good i, and A_n is a normalization constant which depends on the inter-sectoral supply structure of the economy (see (23) in the Appendix).[11] The symmetry assumption enables us to focus on the interconnections created through the input-output relations in the economy, since all products have the same consumption demand. The representative household maximizes her utility by choosing a consumption bundle $(c_1, \ldots, c_n)$ subject to her budget constraint

$$\sum_{i=1}^{n} p_i c_i = h.$$

[10]Throughout, for notational simplicity, we do not use the index n for the entries of the matrix W_n (though we do typically write the subscript n for other objects).

[11]A_n only affects the expected value of log GDP in the economy and is introduced to guarantee that the mean output in economies with different number of sectors remains constant. Its value does not affect aggregate volatility or other distributional properties of (log) aggregate output, the main quantities of interest in this paper.

Due to the competitive markets assumption, h is equal to the aggregate nominal value added in the economy. If the ideal price index is chosen as the numeraire, then h also reflects the real output in the economy, as well as real GDP per capita.

Given the setup presented above, an economy consisting of n sectors is completely specified by the tuple $\mathcal{E}_n = (\mathcal{I}_n, W_n, \{F_i\}_{i\in\mathcal{I}_n})$, capturing the set of sectors, the input-output matrix, and the distributions of the idiosyncratic productivity shocks. Since we are interested in the asymptotic behavior of aggregate output upon disaggregation, throughout the paper, we focus on a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ where the number of sectors increases. Note that in our model, the total supply of labor is normalized to one for all values of n. Therefore, the increase in the number of sectors in the sequence $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ corresponds to disaggregating the structure of the economy. For any such sequence, we impose the following assumption on the sectoral productivity shocks.

Assumption 2. Given a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ and for any sector $i \in \mathcal{I}_n$, F_i is such that

(a) $\mathbb{E}\epsilon_i = 0$, and

(b) $\mathrm{var}(\epsilon_i) = \sigma_i^2 \in (\underline{\sigma}^2, \bar{\sigma}^2)$, where $0 < \underline{\sigma} < \bar{\sigma}$ are independent of n.

The first part of Assumption 2 is a normalization which entails no loss of generality. The second part requires that in any given sequence of economies, the variances of ϵ_i's are uniformly bounded from below and above.

As our description highlights, there are no externalities or strategic interactions in this economy. The only non-trivial interactions are through the input-output relations. In particular, as in Long and Plosser (1983), a negative shock to a sector will reduce its output, and in equilibrium, raise the price of its product, reducing the amount demanded by each of its downstream sectors as input (sector i is a downstream sector for sector j if $w_{ij} > 0$, or alternatively put, if $j \in \mathcal{N}_i$). As a consequence, a negative shock to a sector will reduce the output of all of its downstream sectors. This is its first-order effect (we can think of the direct impact of the shock on the productivity level of the sector as its the zeroth-order effect). There will also be higher-order effects as the reduction in the output of the downstream sectors will in turn reduce the output of their own downstream sectors and so on. We will quantify the extent of these effects in the next section. The characterization of the competitive equilibrium is presented in Appendix B.

2.2 The Supply Network

Pairwise interactions between different sectors through their input supply relations can also be represented by a weighted, directed graph. In particular, the *supply network* of economy $\mathcal{E}_n = (\mathcal{I}_n, W_n, \{F_i\}_{i\in\mathcal{I}_n})$ can be represented by the graph $G_n = (V_n, E_n, W_n)$ with vertex set $V_n = \mathcal{I}_n$, edge set E_n, and edge weight matrix W_n, where each vertex in V_n corresponds to a sector in the economy, $(j, i) \in E_n$ if sector j supplies firms in sector i with intermediate goods for production, and the weight of edge $(j, i) \in E_n$ is equal to w_{ij}, the share of sector j's output in sector i's production technology. Note that by construction, the supply network corresponding to an economy is a directed graph. Also, note that our definition of the supply network allows for self-loops $(i, i) \in E_n$ whenever $w_{ii} > 0$.

We also define the *weighted outdegree*, or simply the *degree*, of sector i as the share of sector i's output in the input supply of the entire economy normalized by constant $1 - \alpha$; that is,

$$d_i \equiv \sum_{j=1}^{n} w_{ji}. \tag{3}$$

Clearly, when all supply linkage weights are identical, the outdegree of vertex i is proportional to the number of sectors it is a supplier of. We refer to the sequence $d^{(n)} = (d_1, d_2, \ldots, d_n)$ as the *degree sequence* of economy $\mathcal{E}_n$.[12]

Throughout the paper, due to the one-to-one correspondence between the supply network of an economy and its input-output matrix, we use the two concepts interchangeably.

3 The Influence Vector and Aggregate Volatility

In this section, we provide a simple representation of the relationship between the supply network and aggregate output in terms of an *influence vector*, which summarizes the influence of each sector on aggregate output. We then use this representation to discuss the conditions under which the law of large numbers holds and then derive key results about convergence to an asymptotic distribution, which will be the basis of the rest of our analysis. We also briefly discuss alternative interpretations of the model.

3.1 The Influence Vector

Our main aggregate statistic throughout the paper will be *aggregate output*, defined as the logarithm of the real value added in the economy, i.e.,

$$y_n \equiv \log(h).$$

Given the environment described in the previous section, it is straightforward to characterize the competitive equilibrium of the economy $\mathcal{E}_n = (\mathcal{I}_n, W_n, \{F_i\}_{i \in \mathcal{I}_n})$.[13] By choosing an appropriate constant A_n and setting the ideal price index to one, we obtain

$$y_n = \log(h) = v_n' \epsilon, \tag{4}$$

where $\epsilon \equiv [\epsilon_1 \ldots \epsilon_n]'$, and the n-dimensional vector v_n is the *influence vector*, defined as

$$v_n \equiv \frac{\alpha}{n} \left[I - (1 - \alpha) W_n' \right]^{-1} \mathbf{1}. \tag{5}$$

[12] There is no need to define indegree sequences, since in view of Assumption 1, the (weighted) indegrees of all sectors are equal to one.

[13] The details are provided in Appendix B. Clearly, one could also obtain a competitive equilibrium by computing the Pareto efficient allocation (maximizing the utility of the representative household) and invoking the Second Welfare Theorem.

As we show in Appendix B, the i-th element of v_n is also equal to the equilibrium share of sales of sector i in the economy, so that v_n is also the "sales vector" of the economy. This is of course not surprising in view of the results in Hulten (1978) and Gabaix (2010), relating changes in aggregate TFP (total factor productivity) to changes in firm-level TFP weighted by sales.

Thus, aggregate output of economy $\mathcal{E}_n$ can be written in terms of the supply network, as represented by the matrix W_n, the labor share in the technology α, and the idiosyncratic productivity shocks. Assumption 1 guarantees that $[I - (1-\alpha)W_n]$ is non-singular, and thus, its inverse is well-defined. Moreover, since none of W_n's eigenvalues lie outside of the unit circle, it is possible to express v_n in terms of a convergent power series:

$$v_n' = \frac{\alpha}{n}\mathbf{1}'\sum_{k=0}^{\infty}(1-\alpha)^k W_n^k. \tag{6}$$

Alternatively, from (5), one could also write the influence vector v_n as the unique solution to the following linear system

$$v_n' = \frac{\alpha}{n}\mathbf{1}' + (1-\alpha)v_n' W_n. \tag{7}$$

The presence of higher-order interconnections can already be seen from equation (7). Indeed, the second term on the right-hand side implies that the effect of a sector's idiosyncratic shock on the aggregate output is greater if it supplies inputs to sectors that are themselves central in determining the aggregate output, rather than to sectors with marginal effects.

The influence vector in our model coincides with the definition of the *PageRank* vector of a graph as well as the concept of *Bonacich centrality* in the network science literature. Certain properties of the PageRank vector (so-called because it corresponds to the algorithm used by Google's search engine in ranking webpages) and some generalizations of it are discussed in Chung and Zhao (2008), while Jackson (2008) shows how the PageRank vector is related to Bonacich centrality, another measure for evaluating the significance of a vertex in a given graph.

Until Section 6, we primarily focus on the behavior of the standard deviation of aggregate output, to which we refer as *aggregate volatility*.[14] The key observation is that since productivity shocks to all sectors are mutually independent, aggregate volatility is simply equal to

$$(\text{var } y_n)^{1/2} = \sqrt{\sum_{i=1}^{n}\sigma_i^2 v_{n,i}^2}, \tag{8}$$

where $v_{n,i}$ denotes the i-th element of the influence vector v_n. Since, in view of Assumption 2, σ_i is uniformly bounded, (8) implies that aggregate volatility scales with $\|v_n\|_2$, where $\|\cdot\|_2$ is the Euclidean (vector) norm. Equivalently, we can write this relation as

$$(\text{var } y_n)^{1/2} = \Theta(\|v_n\|_2). \tag{9}$$

Though simple, this relationship is central for the rest of our analysis as it will enable us both to understand whether and under which conditions the appropriate law of large numbers holds, and when it does, at what rate aggregate output concentrates around a constant.

[14] Note that given our normalizations, the mean of $y_n = v_n'\epsilon$ is zero.

Throughout the rest of the paper, we adopt the convention that whenever we refer to a sequence of influence vectors $\{v_n\}_{n \in \mathbb{N}}$, this is associated with a sequence of economies $\{\mathcal{E}_n\}_{n \in \mathbb{N}}$, and vice versa, and that all limiting statements are for $n \to \infty$.

3.2 Alternative Interpretations

We have so far derived equations (4) and (5) from the model economy described in the previous section. In the rest of the analysis, we will only use these equations. These or similar relations could have been derived from alternative models of economic behavior.

Consider, for example, a reduced-form model, involving n units (agents, firms, sectors, etc.)

$$y = \tilde{W}_n y + \tilde{\epsilon}, \tag{10}$$

where y is the vector consisting of the output levels, value added or other actions (or the logarithms thereof) of these units, $\tilde{W}_n$ is an $n \times n$ matrix capturing the interactions between them, and $\tilde{\epsilon}$ is a vector of independent shocks to each unit. Clearly, equations (4) and (5) can be derived exactly or approximately from (10). Therefore, one could study the relationship between the structure of interactions and aggregate volatility in different economic environments that can be cast in this way. We briefly mention some examples.

First, we should note that while we have modeled the ϵ_i's as productivity shocks, nothing in our analysis depends on this interpretation. Any other shock affecting sectoral outputs would lead to identical results.

Second, we could have (10) apply to a set of firms (rather than sectors), in which case the matrix $\tilde{W}_n$ would represent the input-output linkages between firms. Even though each firm could, in general, buy inputs from several suppliers producing closely substitutable inputs, the failure of a particular supplier could still reduce the outputs of all its downstream customers if long-term relation-specific investments are involved in the supplier-producer interactions. If so, (10) could be a reduced-form representation of these firm-level interlinkages, where the failure of a firm will create downward pressure on the output of its downstream customers. These firm-level interactions can be further enriched by noting that the failure of a downstream customer will also depress the output of a supplier, or even potentially cause its failure, because given the long-term firm-specific investments that the supplier has made, the demand for its output from other firms will be limited, at least in the short run. This richer environment would be the one necessary for interpreting the discussion in the Introduction, related to the concerns of Ford Motor Co. about the failure of their competitors, General Motors and Chrysler, which could then induce the failure of upstream suppliers that they all share.

Third, equation (10) could also be a reduced-form representations of the counterparty relationships between financial institutions. In this case, $w_{ij} > 0$ would correspond to firm i being a counterparty to firm j (i.e., holding some of firm j's debt or other liabilities on its balance sheet). Such interlinkages, which have become increasingly common over the last decade, have been argued to be at the root of the liquidity and insolvency problems faced by many financial institutions during the 2008–2009 crisis.

Finally, equation (10) could also be derived from various strategic complementarities, for example as in Jovanovic (1987) or Durlauf (1993), which would link the input or output choices of different firms or sectors.

In the rest of this section, we study the behavior of aggregate output as a function of the structure of the economy when the number of sectors is large. More specifically, we consider a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$, and study the distribution of y_n as $n \to \infty$. First, we provide conditions, in terms of the Euclidean norm of the influence vector, $\|v_n\|_2$, as highlighted by equation (9), under which sectoral shocks in a well-behaved sequence of economies fail to average out, implying that the standard shock diversification argument (and the law of large numbers) does not hold. We then show that even for sequences of economies for which the law of large numbers holds, the convergence rate need not be $\sqrt{n}$ as is generally assumed, but is instead given by the same object that determines whether the law of large numbers applies; i.e., by $\|v_n\|_2$.

3.3 Dominant Sectors and the Law of Large Numbers

We first introduce the following simple definition.

Definition 1. A sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ has dominant sectors if $\|v_n\|_\infty = \Theta(1)$, where $\|\cdot\|_\infty$ is the sup vector norm; i.e., the largest element of the vector in absolute value.

Recall that v_n is the vector of sectoral sales. Thus $\|v_n\|_\infty = \Theta(1)$ implies that the share of sales of the largest sector remains bounded away from zero as the number of sectors increases. This also makes it clear why it is natural to refer to such sectors as "dominant". If all sectors were "symmetric", equation (5), together with the fact that sectoral shocks are independent, would imply that the influence of each sector gradually declines and $\|v_n\|_2 \to 0$. This also implies $\|v_n\|_\infty \to 0$, so that there are no dominant sectors in this case (see the proof of Theorem 1). However, if the influence of some sector does not die down even as the economy becomes highly disaggregated, then we would have $\|v_n\|_2 \not\to 0$ and $\|v_n\|_\infty \not\to 0$ as captured by the above definition. The relationship between the presence of dominant sectors and $\|v_n\|_2$ is further clarified by considering star-like supply network structures introduced in the next definition.

Definition 2. A sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ has a *star-like structure* if $\max_{i\in\mathcal{I}_n} d_i = \Theta(n)$, where d_i denotes the degree of sector i.

In other words, a sequence of economies has a star-like structure if there is a sequence of *central sectors* $i_n \in \mathcal{I}_n$ supplying inputs to a non-vanishing fraction of the sectors in $\mathcal{E}_n$. For example, the supply network depicted in Figure 1 belongs to this class with the first sector in each economy taking the role of the central sector. The next proposition shows that star-like structures do indeed have dominant sectors as defined in Definition 1.

Proposition 1. *A sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ with a star-like structure has dominant sectors.*

The main result of this subsection is the following theorem, which follows from a simple application of Chebychev's inequality:

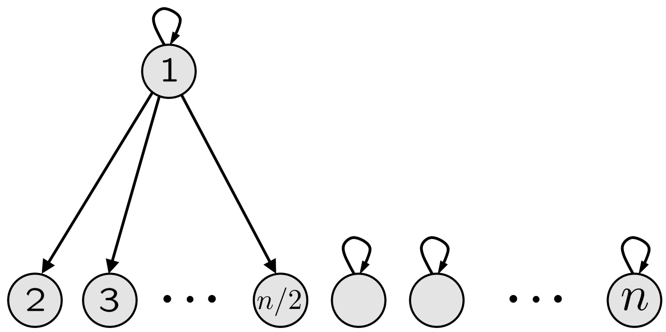


Figure 1: The supply network corresponding to a star-like sequence of economies, where all supply links are identical. Sector 1 plays the role of the central sector, whose outdegree is of order $\Theta(n)$.

Theorem 1. *Aggregate output in a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ converges to zero in probability (as $n \to \infty$) if and only if $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ does not have dominant sectors.*

The simple theorem shows that, even though each sector has independent productivity shocks given by ϵ_i, the (weak) law of large numbers need not apply because of the interconnections created by the supply network. Star-like structures provide one concrete example.[15] Another example is provided next. The role of second and higher-order interconnections in creating dominant firms will be discussed further and clarified in the next section.

Example 1. Consider a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$, where the supply network of the economy consisting of n sectors is depicted in Figure 2. As the figure suggests, the number of sectors who depend on sector 1 as their sole input supplier scales at rate $\log n$, whereas the outdegree of sectors who are supplied by sector 1 scale at rate $\Theta(n/\log n)$. On the other hand, using equation (7) it is easy to verify that $\|v_n\|_2$, and hence $\mathrm{var}(y_n)$, vary with n at rate $\Theta(1)$; that is, aggregate volatility is bounded away from zero as the number of sectors grows. Thus, existence of a star-like structure is sufficient, but not necessary for the failure of the law of large numbers.

It is also worth noting that, loosely speaking, economies with dominant sectors will be neither the most heavily connected nor the least connected economies. In particular, it is straightforward to see that either if $w_{ij} = 0$ for all $i \neq j$ (i.e., a disconnected sequence of economies) or if $w_{ij} = 1/n$ for all i and j (i.e., a sequence of complete symmetric networks), we have $\|v_n\|_2 \to 0$. It is only "in the middle" that $\|v_n\|_2$ is uniformly bounded away from zero (see Proposition 2 in the next section).

Having noted the possible failure of the law of large numbers, in the rest of the analysis we will focus on the rate at which convergence to zero happens (though our results will be stated without assuming such convergence). We next present the main result of this section.

[15]Carvalho (2010) also notes that the law of large numbers fails when the economy has a star network. Theorem 1 generalizes this result and provides an "if and only if" condition for the failure of the law of large numbers.

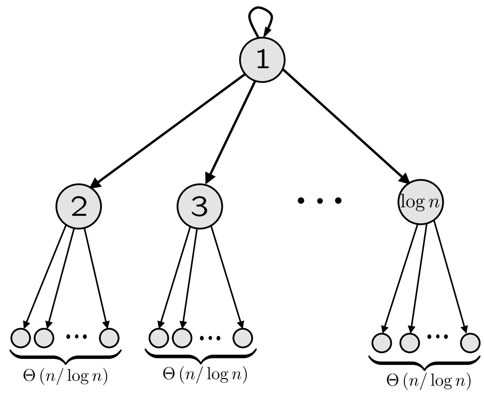


Figure 2: The supply network corresponding to a sequence of economies for which the risk diversification argument predicted by the law of large numbers does not apply, despite the fact the sequence does not have a star-like structure.

3.4 Asymptotic Distribution and Aggregate Volatility

Even though Theorem 1 shows that in the presence of interconnections (particularly, in the presence of dominant sectors), the law of large numbers need not hold in general, we would expect that it would apply in most interesting and realistic situations. However, even then, inter-sector linkages in the economy might still have a first-order effect on volatility. In this subsection, we address this question and show that even in sequences of economies in which aggregate output converges to zero, the convergence rate need not be equal to $\sqrt{n}$. Instead aggregate output y_n has a non-degenerate asymptotic distribution, when scaled by $\|v_n\|_2$—a value distinct from $1/\sqrt{n}$ in general. The result, which is based on generalizations of the central limit theorem to independent, but not identically distributed random variables, also provides conditions under which aggregate output is asymptotically normally distributed, and will be the basis of all of our results in the next section.

Theorem 2. *Consider a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ and assume that $\mathbb{E}\epsilon_i^2 = \bar{\sigma}^2$ for all $i \in \mathcal{I}_n$.*

(a) Suppose that $\{\epsilon_i\}$ are normally distributed for all $i \in \mathcal{I}_n$ and all n , then $\frac{1}{\|v_n\|_2} y_n \xrightarrow{d} \mathcal{N}(0, \bar{\sigma}^2)$.

(b) If $\frac{\|v_n\|_\infty}{\|v_n\|_2} \longrightarrow 0$ (with F_i's arbitrary), then $\frac{1}{\|v_n\|_2} y_n \xrightarrow{d} \mathcal{N}(0, \bar{\sigma}^2)$.

(c) Suppose that $\{\epsilon_i\}$ are identically distributed for all $i \in \mathcal{I}_n$ and all n, and are not normally distributed. If $\frac{\|v_n\|_\infty}{\|v_n\|_2} \not\to 0$, then the asymptotic distribution of $\frac{1}{\|v_n\|_2} y_n$, when it exists, is non-normal and has finite variance $\bar{\sigma}^2$.

This theorem establishes that aggregate output, when normalized by the Euclidean norm of the influence vector, $\|v_n\|_2$, converges to a non-degenerate distribution. It is thus a natural complement to, and strengthens, equation (9), which showed that aggregate volatility scales with $\|v_n\|_2$. Notably, $1/\|v_n\|_2$ is in general distinct from (slower than) the $\sqrt{n}$ scaling implied by the standard central limit theorem (see Lemma 2 in the next section). The most important result implied by Theorem 2 is that the rate of decay is determined by the same factor that captures aggregate volatility, namely $\|v_n\|_2$, which, as we have already seen and will discuss in greater detail in the next section, is itself shaped by the structural properties of the supply network.

Theorem 2 also shows that the supply structure of the economy not only affects the convergence rate, but also determines the asymptotic distribution of aggregate output: depending on $\|v_n\|_\infty$, which captures the influence level of the most influential sector, aggregate output (properly normalized) can have a non-normal distribution. In fact, if the conditions in part (c) of the theorem hold, then the asymptotic distribution of aggregate output necessarily depends on the specific distribution of the sectoral-level productivity shocks. In either case, however, the limiting variance of $y_n/\|v_n\|_2$ is finite and equal to $\bar{\sigma}^2$.

Note that the last part of the theorem is stated conditional on such an asymptotic distribution existing. This is necessary, since we have not put any restriction on the sequence of economies (and in fact we have not even imposed the law of large numbers), and thus $\|v_n\|_\infty$ and $\|v_n\|_2$ need not have well-behaved limits. This does not create a problem for part (b) of the theorem, which shows that *any* sequence of economies satisfying $\|v_n\|_\infty / \|v_n\|_2 \to 0$ will necessarily have a well-behaved distribution when scaled by $\|v_n\|_2$. Also note that, the assumption that $\|v_n\|_\infty / \|v_n\|_2 \to 0$ ensures that the sequence has no dominant firms, and as a consequence, y_n necessarily converges to its mean, by Theorem 1.

4 Characterizing Aggregate Volatility

In this section, we build on Theorem 2 and further characterize the behavior of aggregate volatility in terms of the structural properties of the supply network. In particular, we provide lower bounds on aggregate volatility depending on simple first-order and higher-order interconnectivity properties of the supply network.

4.1 Preliminary Results

Recall throughout that aggregate volatility refers to the standard deviation of aggregate output, $(\text{var } y_n)^{1/2}$.

Lemma 1. *All of the elements of the influence vector v_n are positive, and $\|v_n\|_1 = 1$.*

Recalling that the ℓ_1-norm of a non-negative vector, $\|\cdot\|_1$, is simply the sum of the elements of that vector, this lemma combined with equation (4) which states that $y_n = v_n'\epsilon$, implies that aggregate output is simply a convex combination of the logarithm of the idiosyncratic shocks to each sector. This result holds regardless of the structure of the economy. The supply network, on the other hand, determines how the unit mass of vector v_n is distributed between its n elements, and hence, the rate at which aggregate volatility decays to zero. The next lemma provides global bounds on this rate.

Lemma 2. *Given a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$, aggregate volatility satisfies $(\text{var } y_n)^{1/2} = \mathcal{O}(1)$ and $(\text{var } y_n)^{1/2} = \Omega(1/\sqrt{n})$.*

This result is quite intuitive. It states that aggregate volatility cannot grow unbounded as $n \to \infty$, since this corresponds to the economy becoming further disaggregated. In other words, as the familiar logic of the law of large numbers implies, having more units (further disaggregation) can only lead to greater "diversification" of the sector-specific shocks. The lemma also shows that aggregate volatility in the economy does not decay to zero faster than the rate predicted by the regular law of large numbers for independent variables. In other words, correlations between sectors' outputs due to interconnections can only decrease the degree of diversification in the economy.

4.2 First-Order Interconnections and Aggregate Volatility

We now focus on the effects of first-order interconnections, as measured by the degree distribution, on aggregate volatility. As already hinted at by our discussion in the previous subsection, this analysis will reveal the importance of the distribution of influences across different sectors. In particular, our main result in this subsection, Theorem 3, will show that if a "large" fraction of the sectors rely on a "small" number of sectors for their inputs, then aggregate output exhibits higher volatility.

Definition 3. Given an economy $\mathcal{E}_n = (\mathcal{I}_n, W_n, \{F_i\}_{i \in \mathcal{I}_n})$ with degree sequence $d^{(n)} = (d_1, d_2, \ldots, d_n)$, the *coefficient of variation* is

$$\mathrm{CV}(d^{(n)}) \equiv \frac{\mathrm{STD}(d^{(n)})}{\bar{d}}$$

where $\bar{d} \equiv \frac{1}{n} \sum_{i=1}^{n} d_i$ is the average degree, and $\mathrm{STD}(d^{(n)}) \equiv \left[\sum_{i=1}^{n} (d_i - \bar{d})^2/(n-1)\right]^{1/2}$ is the standard deviation of the degree sequence.

Theorem 3. *Consider a sequence of economies $\{\mathcal{E}_n\}_{n \in \mathbb{N}}$. Then, aggregate volatility satisfies*

$$(\mathrm{var}\, y_n)^{1/2} = \Omega\left(\frac{1}{n}\sqrt{\sum_{i=1}^{n} d_i^2}\right) \tag{11}$$

and

$$(\mathrm{var}\, y_n)^{1/2} = \Omega\left(\frac{1 + \mathrm{CV}(d^{(n)})}{\sqrt{n}}\right). \tag{12}$$

Theorem 3 states that if the degree sequence of the supply network exhibits high variability as measured by the coefficient of variation, then there is also high variability in the effect of different sector-specific shocks on the aggregate output. Such heterogeneity then implies that aggregate volatility decays at a rate slower than $\sqrt{n}$. The supply network can thus have a considerable effect on aggregate volatility—even when the law of large numbers holds. Intuitively, when the coefficient of variation is high, only a small fraction of sectors are responsible for the majority of the input supplies in the economy. Shocks to these sectors then propagate through the entire economy as their low productivity leads to lower production for all of their downstream sectors.

It is worth noting that Theorem 3 is stated without assuming that the law of large numbers holds (or that $\|v_n\|_\infty / \|v_n\|_2 \to 0$); in fact, the failure of the law of large numbers for star-like supply network structures (e.g., Proposition 1 and Theorem 1) can be obtained as a corollary of Theorem 3. In particular, for a star-like supply network $\mathrm{CV}(d^{(n)}) = \Theta(\sqrt{n})$, so that $(\mathrm{var}\; y_n)^{1/2} = \Omega(1)$; i.e., aggregate volatility stays bounded away from zero as $n \to \infty$.

A complementary intuition for the results in Theorem 3 can be obtained from equation (11), which can also be interpreted as a condition on the tail of the degree sequence of the network: aggregate volatility is higher in economies whose corresponding degree sequences have *heavy-tailed* distributions. This observation leads to an immediate corollary to Theorem 3 for supply networks where the degree sequence satisfies a power law (Pareto) distribution. A similar corollary appears in Carvalho's work for random graphs.

Definition 4. A sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ has a *power law tail structure*, if there exist a constant $\beta > 1$, a function $L(\cdot)$, and a sequence of positive numbers $c_n = \Theta(1)$ such that for all $n \in \mathbb{N}$ and all $k < d^{(n)}_{\max} = \Theta(n^{1/\beta})$, we have

$$P_n(k) = c_n k^{-\beta} L(k)$$

where $P_n(k) \equiv \frac{1}{n} |\{i \in \mathcal{I}_n : d_i > k\}|$ is the empirical counter-cumulative distribution function of the outdegrees, $d^{(n)}_{\max}$ is the maximum outdegree of $\mathcal{E}_n$, and $L(\cdot)$ is a slowly-varying function satisfying $\lim_{t\to\infty} L(t)t^{\epsilon} = \infty$ and $\lim_{t\to\infty} L(t)t^{-\epsilon} = 0$ for all $\epsilon > 0$.

In the above definition, $\beta > 1$ is called the *scaling index* or the *shape parameter* and captures the scaling behavior of the tail of the degree sequence of the economy. Lower values of β correspond to heavier tails and thus, larger variations in the degree sequence. We now apply the result derived in Theorem 3 to a sequence of economies with power law tail structures to obtain the following corollary.

Corollary 1. *Consider a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ with a power law tail structure and scaling index $\beta \in (1, 2)$. Then, aggregate volatility satisfies*

$$(\mathrm{var}\; y_n)^{1/2} = \Omega\left(n^{-\frac{\beta-1}{\beta}-\epsilon}\right),$$

where $\epsilon > 0$ is arbitrary.

The corollary establishes that if the degree sequence of the supply network exhibits relatively heavy tails, then aggregate volatility decreases at a much slower rate than the one predicted by the regular law of large numbers. We emphasize that Theorem 3, and hence Corollary 1, provide a lower bound on the rate at which aggregate volatility vanishes. Therefore, even if the scaling index of the power law structure is in the interval $(2, \infty)$, other, higher-order structural properties of the supply network, besides the degree sequence, can still prevent the output volatility from decaying at rate $\sqrt{n}$. We discuss such effects in the following subsections.

4.3 Second-Order Interconnections and Aggregate Volatility

First-order interconnections give only limited information about the structure of the supply network. In particular, as the next example demonstrates, two sequences of economies with identical degree distributions can have arbitrarily large differences in their structural properties and influence vectors. Intuitively, first-order interconnections, and hence degree distributions, provide no information on the true extent of "cascades," where low productivity in one sector affects not only its downstream sectors, but the downstream customers of these sectors and so on.

Example 2. Consider two sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ and $\{\widehat{\mathcal{E}}_n\}_{n\in\mathbb{N}}$, with corresponding supply networks depicted in Figures 3(a) and 3(b), respectively. Each edge corresponds to a supply link of weight one, and all other supply links have weight zero. As the figures suggest, the two network structures have identical degree sequences for all $n \in \mathbb{N}$. More precisely, the economy indexed n in each sequence contains a sector of degree d (labeled sector 1), $d-1$ sectors of degree d' (labeled 2 through d), with the rest of sectors having degrees zero.[16] However, the two supply networks can exhibit considerably different behaviors as far as aggregate volatility is concerned.

The influence vector corresponding to the sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ depicted in Figure 3(a) is equal to

$$v_{n,i} = \begin{cases} 1/n + v_{n,2}(1-\alpha)(d-1)/\alpha & \text{if} \quad i = 1 \\ \alpha/n + \alpha(1-\alpha)d'/n & \text{if} \quad 2 \le i \le d \\ \alpha/n & \text{otherwise,} \end{cases}$$

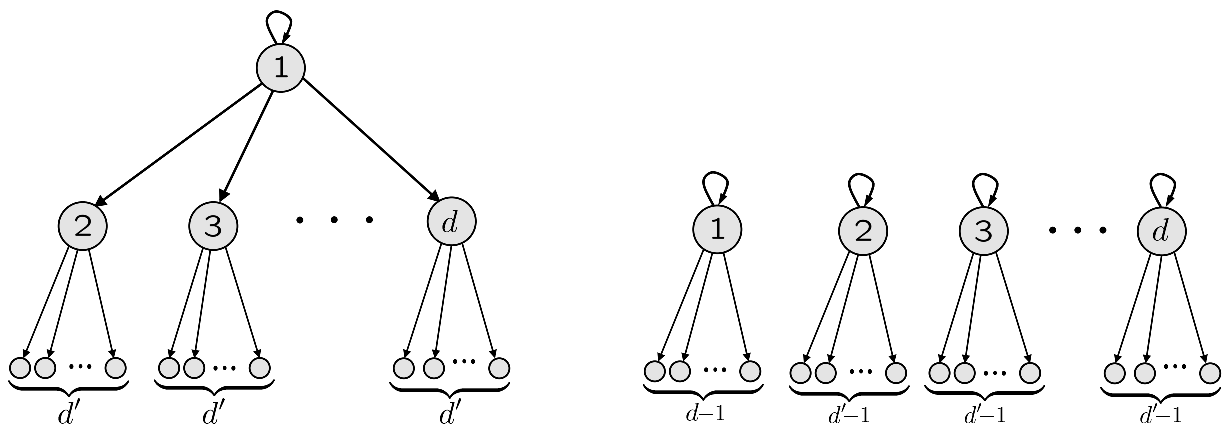
implying that $\|v_n\|_2 = \Theta(1)$; i.e., aggregate volatility of $\mathcal{E}_n$ does not converge to zero as $n \to \infty$, regardless of the values of d and d'.

On the other hand, elements of the influence vector corresponding to n-th element in the sequence of economies $\{\widehat{\mathcal{E}}_n\}_{n\in\mathbb{N}}$ depicted in 3(b) are given by

$$\widehat{v}_{n,i} = \begin{cases} 1/n + (1-\alpha)(d-1)/n & \text{if} \quad i = 1 \\ 1/n + (1-\alpha)(d'-1)/n & \text{if} \quad 2 \le i \le d \\ \alpha/n & \text{otherwise,} \end{cases}$$

implying that $\|\widehat{v}_n\|_2 = \Theta(d/n + 1/\sqrt{d})$. Therefore, in general, the rate of decay of aggregate volatility in $\{\widehat{\mathcal{E}}_n\}_{n\in\mathbb{N}}$ is significantly different from that of $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$, despite the fact that both have identical degree sequences for all n. For example, if d is of order $\Theta(\sqrt{n})$, then $\|\widehat{v}_n\|_2 = \Theta(1/\sqrt[4]{n})$, whereas $\|v_n\|_2 = \Theta(1)$. The source of this difference is of course the fact that in $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$, unlike in $\{\widehat{\mathcal{E}}_n\}_{n\in\mathbb{N}}$, the high-degree sectors labeled 2 through d share a common supplier with one another, creating the possibility of sizable "cascade" effects.

[16]Note that d and d' are not independent, as the total number of sectors in the economy is equal to n. More specifically, it must be the case that $(d-1)d' + d = n$. Also note that there are values of $n \in \mathbb{N}$ for which no such decomposition in terms of positive integers d and d' exists. For such values of n, one needs to sacrifice symmetry in the degrees of sectors labeled 2 through d. However, this does not affect the issue highlighted by the example, as only the growth rates of d and d', and not their actual values, matter.



(a) High second-order interconnectivity

(b) Low second-order interconnectivity

Figure 3: The two structures have identical outdegree sequences. However, depending on the values of d and d', the aggregate output volatility can exhibit considerably different behaviors for large values of n.

Motivated by this example and the general intuition that it conveys, we now provide a lower bound on the decay rate of aggregate volatility in terms of second-order interconnections. The key concept is the following new statistic, which we refer to as the *second-order interconnectivity coefficient*.

Definition 5. Given an economy $\mathcal{E}_n = (\mathcal{I}_n, W_n, \{F_i\}_{i\in\mathcal{I}_n})$, the *second-order interconnectivity coefficient* is defined as

$$\tau_2(W_n) \equiv \sum_{i=1}^{n}\sum_{j\neq i}\sum_{k\neq i,j} w_{ji}w_{ki}d_jd_k, \tag{13}$$

where d_j is the degree of sector $j \in \mathcal{I}_n$, and w_{ji} is the (j,i) element of input-output matrix W_n.

The second-order interconnectivity coefficient measures the extent to which sectors with high degrees (those that are major suppliers to other sectors) are interconnected to one another through common suppliers. For example, in terms of the discussion in the Introduction, the situation with Ford, General Motors and Chrysler would correspond to a high second-order interconnectivity coefficient because all three companies have high degrees (are major suppliers and important for the economy) and have common suppliers. More specifically, τ_2 takes higher values when high-degree sectors share the same suppliers with other high-degree sectors, as opposed to sharing suppliers with low-degree ones. This observation is a consequence of the *Rearrangement Inequality*, which states that if $a_1 \geq a_2 \geq \cdots \geq a_r$ and $b_1 \geq b_2 \geq \cdots \geq b_r$, then for any permutation $(\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_r)$ of $(a_1, a_2, \ldots, a_r)$, we have (e.g., Wu and Liu (1995) or Steele (2004)):

$$a_1b_1 + a_2b_2 + \cdots + a_rb_r \geq \hat{a}_1b_1 + \hat{a}_2b_2 + \cdots + \hat{a}_rb_r.$$

Note that the information captured by τ_2 is fundamentally different from the information encoded in the degree sequence of a network. For example, the second-order interconnectivity coefficient of networks depicted in Figures 3(a) and 3(b) are significantly different even though they both have identical degree sequences.

It is also worth noting that the second-order interconnectivity coefficient τ_2 is different from the related notion of *s-metric* introduced by Li, Alderson, Doyle, and Willinger (2005):

$$s(W_n) = \sum_{i \neq j} w_{ji} d_i d_j.$$

Whereas τ_2 measures the extent to which high-degree vertices share common ancestors (high-degree sectors having common suppliers), the s-metric measures whether ancestors of high-degree vertices have high degrees themselves. Our next result shows that as far as aggregate volatility in a highly disaggregated economy is concerned, the effect captured by the s-metric is dominated by the effects captured by τ_2 (and the degree sequence), and thus the second-order interconnectivity coefficient appears to be the right concept for our purposes.

Theorem 4. *Consider a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$. Then, aggregate volatility satisfies*

$$(\text{var } y_n)^{1/2} = \Omega\left(\frac{1}{\sqrt{n}} + \frac{\text{CV}(d^{(n)})}{\sqrt{n}} + \frac{\sqrt{\tau_2(W_n)}}{n}\right), \tag{14}$$

where $\text{CV}(d^{(n)})$ is the coefficient of variation of the degree sequence of $\mathcal{E}_n$ and τ_2 is the corresponding second-order interconnectivity coefficient.

Theorem 4 above shows how second-order interconnections, captured by coefficient τ_2, affect aggregate volatility. It also shows that even if two sequence of economies have identical degree sequences for all values n, it is possible for the aggregate volatility to exhibit considerably different behavior. In this sense, Theorem 4 is a refinement of Theorem 3, taking both first and second-order relations between different sectors into account. It can also be considered to be the economically more interesting result, as it captures not only the fact that some sectors are "large" (dominant or quasi-dominant) in the economy, but also the more subtle notion that there is a clustering of significant sectors caused by the fact that they have common suppliers. In this way, we believe that Theorem 4 captures the essence of "cascade effects". We will see in the next section that Theorem 4 will turn out to be more relevant in the context of the US input-output linkages than Theorem 3.

Example 2 (continued). Recall the sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ and $\{\widehat{\mathcal{E}}_n\}_{n\in\mathbb{N}}$ depicted in Figure 3. As mentioned earlier, the two supply networks have identical degree sequences for all $n \in \mathbb{N}$. On the other hand, it is straightforward to verify that the second-order interconnectivity coefficients are very different; in particular, $\tau_2(W_n) = \Theta(n^2)$ and $\tau_2(\widehat{W}_n) = 0$. In fact, this is the reason why there is a stark difference in the decay rate of aggregate volatility of these two sequences of economies. Moreover, note that despite the fact that $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ does not have a star-like structure, Theorem 4 and the fact that $\tau_2(W_n) = \Theta(n^2)$ guarantee that aggregate volatility does not vanish as $n \to \infty$, implying the presence of a dominant sector (which is sector 1) in the sequence. The fact that sector 1 plays the role of a dominant sector is exactly due to the presence of second-order cascade effects.

Similar to the case of the first-order connections as in (11), one can also think of the effect of second-order interconnection in terms of the tail distribution of the *second-order degree* distribution. We define the second-order degree of a sector i as

$$q_i \equiv \sum_{j=1}^{n} d_j w_{ji}. \tag{15}$$

We then have the following counterpart to Corollary 1.

Corollary 2. *Suppose $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ is a sequence of economies whose second-order degree sequences have power law tails; that is, there is constant $\zeta > 1$, a slowly-varying function $L(\cdot)$, and a sequence of positive numbers $c_n = \Theta(1)$ such that for all $n \in \mathbb{N}$ and all $k < q_{\max}^{(n)} = \Theta(n^{1/\zeta})$, the empirical counter-cumulative distribution function of the second-order degrees, $Q_n(k)$, satisfies*

$$Q_n(k) = c_n k^{-\zeta} L(k).$$

Then, if $\zeta \in (1,2)$, aggregate volatility satisfies

$$(\text{var } y_n)^{1/2} = \Omega\left(n^{-\frac{\zeta-1}{\zeta}-\epsilon}\right),$$

for any $\epsilon > 0$.

The above corollary establishes that if the distribution of second-order degrees also exhibits relatively heavy tails, then aggregate volatility decreases at a much slower rate than the one predicted by the regular law of large numbers. Note that as Example 2 shows, the second-order effects can dominate the first-order effect of the degree distribution in determining the decay rate of aggregate volatility of the economy. In particular, in a sequence of economies with both first-order and second-order degree distributions exhibiting power law tails, it might be the case that the shape parameters satisfy $\zeta < \beta$, and if so, the bound provided in Corollary 2 (which incorporates cascade effects) will be tighter than the one in Corollary 1.

4.4 Higher-Order Interconnections

The results of the previous subsection can be extended even further in order to capture higher-order interconnections and more complex patterns of cascades. Mathematically, this will correspond to tighter lower bounds than the one we provided in Theorem 4. Let us define the *higher-order interconnectivity coefficients* as follows.

Definition 6. Given an economy $\mathcal{E}_n = (\mathcal{I}_n, W_n, \{F_i\}_{i\in\mathcal{I}_n})$, the $(m+1)^{th}$*-order interconnectivity coefficient* is defined as

$$\tau_{m+1}(W_n) \equiv \sum_{i=1}^{n} \sum_{\substack{j_1,\dots,j_m \\ k_1,\dots,k_m \\ \text{all distinct}}} (d_{j_1} d_{k_1}) (w_{j_m i} w_{k_m i}) \prod_{s=1}^{m-1} w_{j_s j_{s+1}} \prod_{r=1}^{m-1} w_{j_r j_{r+1}}$$

This coefficient captures supply relations between different sectors of order $m+1$. In particular, the third-degree coefficient will be high when the suppliers of high-degree sectors have common suppliers and so on. Once again the Rearrangement Inequality implies that higher levels of τ_m correspond to higher interconnectivities among different sectors.

The main result here is straightforward generalization of Theorem 4.

Theorem 5. *Consider a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$. Then for any $m \in \mathbb{N}$, aggregate volatility satisfies*

$$(\mathrm{var}\, y_n)^{1/2} = \Omega\left(\frac{1}{\sqrt{n}} + \frac{\mathrm{CV}(d^{(n)})}{\sqrt{n}} + \frac{\sqrt{\tau_2(W_n)}}{n} + \cdots + \frac{\sqrt{\tau_m(W_n)}}{n}\right).$$

4.5 Balanced Structures

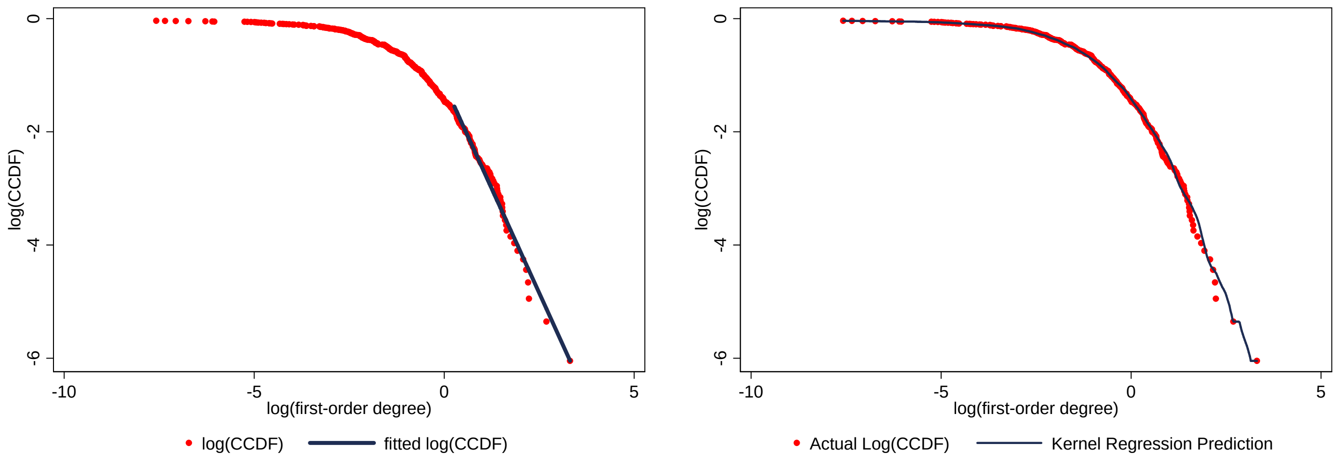
In this subsection, we show that when all sectors are "balanced" in terms of their first-order interconnections, the decay rate of aggregate volatility is always $\sqrt{n}$ and other properties of the supply network, such as higher-order interconnectivity coefficients, do not influence the nature of volatility.

Definition 7. A sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ is *balanced* if $\max_{i\in\mathcal{I}_n} d_i = \Theta(1)$.

In balanced structures, each sector's output is used approximately to the same extent that the sector itself depends on outside suppliers. In this sense, balanced structures are the polar opposite of star-like structures studied in the previous section. The complete graph with equal supply weights and the fully disconnected graph are examples of balanced structures. We have the following simple result:

Proposition 2. *Consider a sequence of balanced economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$. Then there exists $\bar{\alpha} \in (0,1)$ such that for $\alpha \geq \bar{\alpha}$, $(\mathrm{var}\, y_n)^{1/2} = \Theta(1/\sqrt{n})$.*

Thus when the supply network has a balanced structure and when the role of the intermediate inputs in production is sufficiently small, volatility decays at the rate $\sqrt{n}$, implying that other structural properties of the supply network cannot contribute further to aggregate volatility. This proposition is a generalization of Theorem 2 of Dupor (1999) and Carvalho (2010)'s results for complete networks. As noted by Dupor as well as Horvath (1998) in a related context, Proposition 2 is both an aggregation and irrelevance result for economies with balanced structures. As an aggregation result it suggests an observational equivalence between the single aggregate sector economy and the multi-sector economy where the variance of sector-specific shocks scales by the level of disaggregation. On the other hand, as an irrelevance result, it shows that within the class of balanced structures, different input-output matrices generate roughly the same amount of volatility. However, note that, differently from what is suggested in Lucas (1977) and Dupor (1999), our earlier results clearly establish that neither the aggregation nor the irrelevance interpretations hold for more general supply networks.



(a) Linear regression

(b) Non-parametric estimation

Figure 4: Empirical counter-cumulative distribution function (CCDF) of first-order degrees.

5 Application: First and Second-Order Interconnections in the US Supply Network

In this section, we briefly look at the structure of interconnections implied by the 2002 US input-output matrix (Commodity-by-Commodity direct requirements table) published by the Bureau of Economic Analysis (see, e.g., Streitwieser (2009)).[17] Even though we think of our model as applying at a finer level than the most detailed level at which the US input-output matrix is available, this exercise is useful to obtain a rough empirical grounding for the results we have derived so far and to perform a back-of-the-envelope calculation of how the interconnections implied by the US input-output structure affect the relationship between sectoral variability and aggregate volatility.

The US input-output matrix we use is the Commodity-by-Commodity direct requirement matrix, which comprises 423 commodities.[18] These are based on five-digit NAICS (North American Industry Classification System) sectors but are further aggregated and reclassified by the Bureau of Economic Analysis. Typical sectors are fairly broad and include, for example, Semiconductor and related device manufacturing, Wholesale trade, Retail trade, Truck transportation, Advertising and related services, etc.

The direct requirements table gives the equivalent of our W matrix.[19] From this, we computed first-order and second-order degrees, d_i and q_i, according to equations (3) and (15). Figure 4 plots the logarithm of the empirical counter-cumulative distribution function for first-order

[17]The results are very similar if we instead use the Commodity-by-Industry or the Industry-by-Industry tables.

[18]There are 425 industries in the Commodity-by-Industry table.

[19]Except that its row sums are not necessarily equal to one. This is because the direct requirement tables report input needs as a fraction of total output. But once we exclude from the matrix the value added, tax components, etc., rows no longer sum to one. We solve this by re-normalizing all the row entries by the total input requirement of the industry. We checked that all the results still apply when we do not perform this normalization.

degrees (i.e., one minus the empirical cumulative distribution function) against the logarithm of first-order degrees, and Figure 5 plots the logarithm of the empirical counter-cumulative distribution function of second-order degrees against the logarithm of second-order degrees (the graphs only include 407 sectors that have non-zero first-order and second-order degrees). In both cases, and particularly for second-order degrees, the tail is well-approximated by a power law distribution as shown by the approximate linear relationship (recall that for distributions with a Pareto tail, we have, $\log P(x) \simeq \gamma_0 - \gamma_1 \log x$, where $P(x)$ is the empirical counter-cumulative distribution function for $x = d$ or q).

The first panels in both figures also plot the estimated slopes at the tails of the two distributions. The tail is taken to corresponds to the 82 industries (20% of the sample) with the highest d and q, respectively). The shape parameter estimates (which are equal to the negative of the slopes at the tails) are $\beta = 1.47$ for the first-order degree distribution and $\zeta = 1.29$ for the second-order degree distribution.[20] The second panels in both figures show non-parametric estimates of the empirical counter-cumulative distributions using the Nadaraya-Watson kernel regression with a bandwidth selected using least squares cross-validation (Nadaraya (1964) and Watson (1964)).[21] Using again the 20% of the sample at the tail of the distribution, the average slopes are, respectively, -1.54 and -1.26, thus fairly close to the estimates from the linear regression.[22] For the remainder of the discussion, we take the shape parameters for the tail of the distribution as $\beta = 1.5$ and $\zeta = 1.3$ for the first-order and second-order degree distributions, respectively.

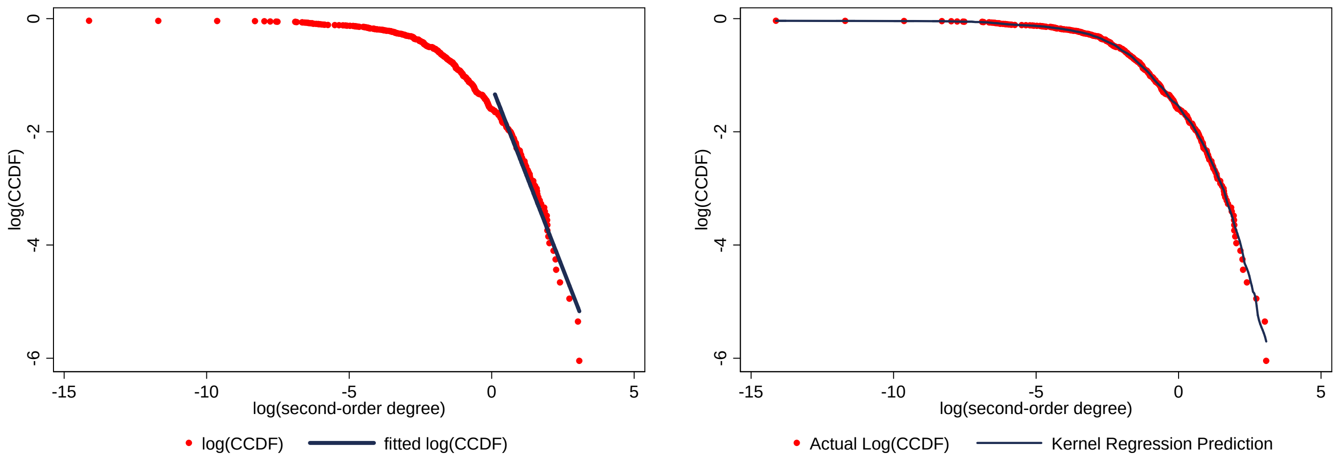
These numbers imply that, as suggested by the discussion following Corollary 2, the lower bound on the scaling of standard deviation obtained by looking at second-order degrees is considerably tighter than that obtained from the first-order degree distribution.

Clearly, we only observe the input-output matrix, which is the equivalent of our W, for a single economy, rather than a sequence of economies. Thus any extrapolation from the distribution of first-order or second-order degrees in this economy to their potential limiting distributions is bound to be speculative. Nevertheless, we can appeal to the scale free nature of the power law distribution and presume that the power law tail will capture the behavior of a sequence of

[20]These coefficients are estimated very precisely, though, for reasons explained in Gabaix and Ibragimov (2009), ordinary least squares standard errors are biased downwards. Gabaix (2009) and Gabaix and Ibragimov (2009) suggest a different method of estimation using a modified regression of the logarithm of the rank of a sector on its first- or second-order degree. These regressions, with or without Gabaix and Ibragimov's correction, also lead to a similar pattern (in particular, using Gabaix and Ibragimov's estimation procedure, the β coefficient increases from 1.47 to 1.57, while the ζ coefficient increases from 1.30 to 1.38).

[21]The optimal bandwidth is estimated as 0.5 for the first-order degree and as 0.35 for the second-order degree. The cross-validation criterion puts zero weight on observations near the support of x (first-order or second-order degree). In general, optimal bandwidths are sensitive to the observations near the boundary. We excluded the smallest number of observations close to the boundary, after which excluding further observations led to little change in the estimate of optimal bandwidth. (Naturally, the observations near the boundary are always included in the estimation once the optimal bandwidth is obtained). As discussed in greater detail in the next footnote, the bandwidth choice has little effect on the average slopes at the tail, which are our main focus.

[22]Different choices of bandwidth change the goodness of fit of the kernel regression. Nevertheless, the average slope at the tail is not significantly affected by different choices of bandwidth. For example, with various values for bandwidth between 0.1 and 1, the average slope at the tail for the first-order degree distribution is between -1.49 and -1.58 (except at 0.2, where it is -1.21) and for the second-order degree distribution, it is only between -1.22 and -1.3.



(a) Linear regression

(b) Non-parametric estimation

Figure 5: Empirical counter-cumulative distribution function (CCDF) of second-order degrees.

economies (as n grows large and thus the level of disaggregation increases). Even though there is no guarantee that such extrapolation is reliable, it is still useful to give an impression of the possible quantitative magnitudes of the types of input-output interactions observed in the US economy.

Let us focus on the second-order degree distribution, which clearly gives a better lower bound on the rate at which aggregate volatility decays. The coefficient $\zeta = 1.3$ implies that aggregate volatility will decay no faster than $n^{(\zeta-1)/\zeta} \simeq n^{0.23}$, which is obviously much slower than $\sqrt{n}$ (the lower bound implied by the first-order degree distribution is $n^{(\beta-1)/\beta} \simeq n^{0.33}$, which corresponds to a significantly faster convergence rate). To get a sense of the implications of this rate of decay, we computed the average (over-time) standard deviation of the logarithm of value added across 459 four-digit (SIC) manufacturing industries from the NBER productivity database between 1958 and 2005 (after controlling for a linear time trend to account for the secular decline in several manufacturing industries). This average standard deviation is estimated as 0.219.[23] Between 1958 and 2005, the average of US GDP accounted for by manufacturing is about 20%,[24] so that we can think of the 459 four-digit manufacturing sectors as corresponding to 1/5th of the GDP. Hence, for the purposes of our back-of-the-envelope calculation, we suppose that the economy comprises $5 \times 459 = 2295$ sectors at the same level of disaggregation as four-digit manufacturing industries.[25] With a sectoral volatility of 0.219, if aggregate volatility decayed at the rate $\sqrt{n}$ (as would be the case with a balanced structure),

[23]If we instead weight different industries by the logarithm of value added so that small industries do not receive disproportionate weight, the average is very similar, 0.214.

[24]Data from http://www.bea.gov/industry/gdpbyind_data.htm.

[25]One might be concerned that manufacturing is more volatile than non-manufacturing. This does not appear to be the case, however, at the three-digit level, where we can compare manufacturing and non-manufacturing industries. If anything, manufacturing industries appear to be somewhat less volatile with or without controlling for industry size (though this difference is not statistically significant in either case).

we would expect aggregate volatility to be approximately about $0.219/\sqrt{2295} \simeq 0.005$, which is clearly a very small amount of variability, underscoring the result that with a reasonably large number of sectors, sector-specific variability will average out and will not translate into a sizable amount of aggregate volatility.

If, instead, as suggested by our lower bound from the second-order degree distribution, aggregate volatility decays at the rate $n^{0.23}$, the same number would be $0.219/\left(2295\right)^{0.23} \simeq 0.037$, which is a sizable number. This back-of-the-envelope calculation thus suggests that the types of interconnections implied by the US input-output structure can generate significant aggregate fluctuations from sectoral shocks. We should add, however, that as we have already emphasized, these calculations are merely suggestive and are no substitute for a systematic econometric and quantitative investigation of the implications of the input-output linkages in the US economy, which we leave for future work.

6 Asymptotic Distributions and Tail Events

We have so far focused on the standard deviation (or variance) of the logarithm of aggregate output as the measure of aggregate fluctuations in the economy. This is a good measure of macro fluctuations "near the mean" as it follows from a second-order approximation to the moment generating function. However, the standard deviation is not the proper measure for capturing significant deviations from the mean (and in fact, we will see that this is true even when the asymptotic distribution is normal). In this section, we first provide sufficient conditions on structural properties of the supply network to ensure that the asymptotic distribution is normal. Then, more importantly, we relate the probability of large deviations of output from the mean, i.e., the probability of so-called *tail events*, to the supply network.

6.1 Asymptotic Normal Distributions

We first define the concept of k_n-decomposability for a sequence of economies.

Definition 8. Consider a sequence of positive integers $\{k_n\}_{n\in\mathbb{N}}$ and a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$. The sequence $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ is said to be k_n*-decomposable*, if there exist a sequence $a_n = o(k_n)$, a constant $c > 0$, and a partition of $\mathcal{I}_n$ denoted by $\mathcal{P}^{(n)} = \{P_{n1}, P_{n2}, \ldots, P_{nm}\}$, such that for all $n \in \mathbb{N}$:

(a) the size of any given element of partition $\mathcal{P}^{(n)}$ is at most a_n.

(b) $\sum_{j\in P_{ni}, s\notin P_{ni}} w_{sj} < c$ for all $P_{ni} \in \mathcal{P}^{(n)}$.

In other words, the sequence of economies is k_n-decomposable, whenever there are cuts partitioning the corresponding networks to components of size growing slower than k_n, and the cut values do not grow with n. Given the above definition, it is straightforward to show that if a sequence of economies is k_n-decomposable, then it is also $\hat{k}_n$-decomposable as long as $\hat{k}_n = \omega(k_n)$. The next theorem provides a sufficient condition for asymptotic normality of aggregate output in terms of decomposability of the supply structure.

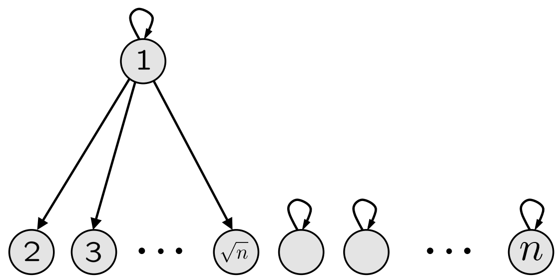


Figure 6: The supply network structure of a sequence of economies in which aggregate volatility decays at rate $\sqrt{n}$, but aggregate output is not normally distributed.

Theorem 6. *Suppose that $\mathbb{E}\epsilon_i^2 = \bar{\sigma}^2$ for all $i \in \mathcal{I}_n$ and all n and that the sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ is $\sqrt{n}$-decomposable. Then there exists $\bar{\alpha} \in (0,1)$ such that for $\alpha \geq \bar{\alpha}$, $\frac{1}{\|v_n\|_2} y_n \xrightarrow{d} \mathcal{N}(0, \bar{\sigma}^2)$.*

The theorem establishes that if by removing a reasonably small number of supply links an economy consisting of n sectors can be divided into sub-economies each of size at most $\sqrt{n}$, then aggregate output is asymptotically normally distributed, regardless of the distribution of sector-specific shocks. Note that balanced structures are clearly $\sqrt{n}$-decomposable.

The next two examples show that in general, the rate at which the law of large numbers applies and the asymptotic distribution of aggregate output are independent.

Example 3. Consider the sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ with the supply network as depicted in Figure 6, where each edge in the supply network corresponds to a supply weight of value one, and all other supply weights are zero. Then the elements of the influence vector are given by:

$$v_{n,i} = \begin{cases} 1/n + (1-\alpha)(d_1 - 1)/n & \text{if } \ i = 1 \\ \alpha/n & \text{if } \ 2 \leq i \leq n, \end{cases}$$

where d_1 is the out–degree of sector 1, and satisfies $d_1 = \Theta(\sqrt{n})$. Therefore,

$$\begin{aligned} \|v_n\|_\infty &= \Theta\left(1/\sqrt{n}\right) \\ \|v_n\|_2 &= \Theta\left(1/\sqrt{n}\right), \end{aligned}$$

implying that the condition for asymptotic normality in Theorem 2 does not apply, despite the fact that $\|v_n\|_2$ and thus, aggregate volatility decay at rate $\sqrt{n}$, the convergence rate predicted by the regular law of large numbers.

Our next example shows that it is possible to have asymptotic normal distribution for the properly normalized aggregate output despite the fact that aggregate volatility decays to zero at an arbitrarily slow rate.

Example 4. Consider the supply network structure depicted in Figure 7, where the first d sectors are connected to all sectors labeled $d+1$ through n. Assume that $d = o(n)$ and $d = \omega(1)$. Each edge connecting the supplier sectors to the downstream sectors have a weight equal to $1/d$, and

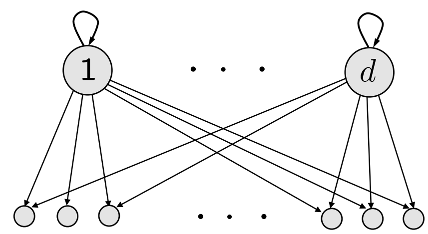


Figure 7: The supply network structure of a sequence of economies in which aggregate output converges to a normal distribution. Aggregate output volatility converges to zero at rate $\sqrt{d}$ which can be arbitrarily slow.

the self-loops have value 1. This guarantees that the indegree of each sector is exactly equal to one. Simple calculations show that

$$v_{n,i} = \begin{cases} 1/n + (1-\alpha)(n-d)/(nd) & \text{if } 1 \le i \le d \\ \alpha/n & \text{otherwise.} \end{cases}$$

Therefore, we have the following relations:

$$\begin{aligned} \|v_n\|_\infty &= \Theta\left(1/d\right) \\ \|v_n\|_2 &= \Theta\left(1/\sqrt{d}\right), \end{aligned}$$

implying that $\|v_n\|_\infty = o(\|v_n\|_2)$. At the same time, one can pick the rate of growth of d such that $\|v_n\|_2$, and thus aggregate volatility, decays to zero arbitrarily slowly.

6.2 Tail Events

It might first appear that when the asymptotic distribution of aggregate output is normal, for example, when the supply network is $\sqrt{n}$-decomposable, our measure of aggregate volatility (the standard deviation of aggregate output) should give us all the information we need about tail events and the probabilities of large deviations (based on the intuition that a normal distribution is fully described by its mean and variance). In this subsection, we will see that this is *not* true, and the probability of large deviations can be significantly different for two economies converging to a normal distribution with identical variances. We will then relate the probability of tail events to the structural properties of the supply network.

We first define the probability of *tail events* more precisely as the probability that aggregate output is smaller than a certain level; that is, as $\mathbb{P}\left(y_n < -c\right)$ where c is a positive constant.[26] The behavior of this probability can be characterized by using large deviations analysis. We next present an example illustrating that two sequences of economies converging to an identical asymptotic (normal) distribution can exhibit significantly different probabilities of tail events.

[26]Symmetrically, we can also define the tail probabilities as $\mathbb{P}\left(y_n > c\right)$ for large positive deviations (recall that the mean of y_n is equal to zero). When the asymptotic distribution is symmetric, these two notions coincide.

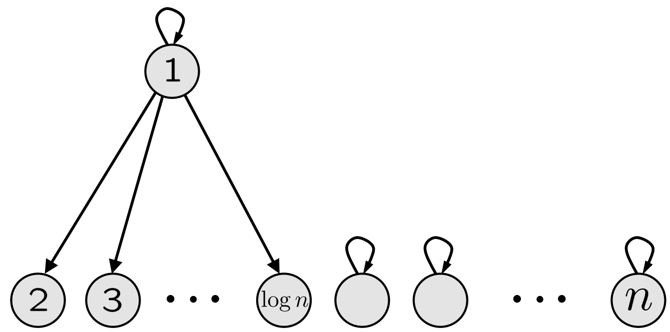


Figure 8: The supply network corresponding to a sequence of economies, for which aggregate volatility decays at rate $\sqrt{n}$.

Example 5. Consider a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$, where the input-output matrix of the economy with n sectors has the block diagonal form

$$W_n = \begin{bmatrix} \mathbf{1}e_1' & 0 \\ 0 & I \end{bmatrix},$$

where e_1 is the first unit vector, and the upper-left block component is a matrix of dimension $\lfloor \log n \rfloor$. In other words, the economy consists of one hub-like sector (indexed 1), acting as the input supplier to a collection of sectors whose size grows at rate $\Theta(\log n)$. On the other hand, the outputs of the rest of the sectors are not used as intermediate goods for production at all (except maybe by the sectors themselves). Figure 8 depicts the structure of the corresponding supply network.

It is easy to verify that the element of the influence vector corresponding to sector 1 is $v_{n,1} = \Theta(\log n/n)$, whereas $v_{n,i} = \Theta(1/n)$ when $i \neq 1$. Therefore, $\|v_n\|_2 = \Theta(1/\sqrt{n})$, implying that aggregate volatility vanishes at rate $\sqrt{n}$ regardless of the distribution of sectoral shocks. Moreover, Theorem 2 implies that $y_n/\|v_n\|_2$ is asymptotically normally distributed for all distributions that satisfy the conditions of the theorem. However, large deviation probabilities depend on the distribution of sectoral shocks. In particular, if shocks are normally distributed, then $-\log \mathbb{P}(y_n < -c)$ grows at rate $\Theta(n)$, a result similar to standard large deviation theorems for sums of independent and identically distributed random variables (see for example, Durrett (2005)). On the other hand, if the shocks have an exponential tail, then it can be shown (and it follows from Theorem 8 below) that

$$-\log \mathbb{P}\left(y_n < -c\right) = \Theta\left(n/\log(n)\right),$$

establishing a much slower decay rate for the tail event probabilities than the standard case.

This example motivates the need to characterize the tail event probabilities more closely. In general, these probabilities depend both on the distribution of sector-specific productivity shocks and on the supply network. We next present two results illustrating these relationships.

Theorem 7. *Consider a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ in which the distribution of idiosyncratic productivity shocks $\{F_i\}_{i\in\mathcal{I}_n}$ are normal with variance σ^2, for all $n \in \mathbb{N}$. Then, for all $c > 0$,*

$$\mathbb{P}\left(y_n < -c\right) = \Phi\left(\frac{c}{\sigma\|v_n\|_2}\right),$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal. Consequently,

$$-\log\mathbb{P}\left(y_n < -c\right) = \Theta\left(1/\left\|v_n\right\|_2^2\right). \tag{16}$$

This theorem establishes that the the supply network of the economy not only affects the level of deviations from the mean, but also has a significant influence on the probabilities of large deviations. The result suggests that in the specific case of normally distributed productivity shocks, this dependence is captured by the Euclidean norm of the influence vector—the same quantity that measures aggregate volatility. It is then straightforward to provide lower bounds on tail event probabilities using the same kind of analysis, and in terms of the same structural properties such as first-order and higher-order interconnectivity coefficients, as in Section 4. This nice parallel between our baseline measure of aggregate volatility and tail events is, unfortunately, not true in general (with other distributions). We next establish the interesting result that with non-normal distributions, other properties of the supply network can become crucial for the rate of decay of large deviation probabilities.

Definition 9. A random variable with distribution function $F(\cdot)$ has an *exponential tail* if $0 < \liminf_{t\to\infty} -(1/t)\log\bar{F}(t) \leq \limsup_{t\to\infty} -(1/t)\log\bar{F}(t) < \infty$, where $\bar{F}(t) \equiv 1 - F(t)$ denotes the counter-cumulative distribution function.

In other words, a random variable has an exponential tail if the tail probabilities decay exponentially. For example, if $\bar{F}(t) = L(t)e^{-\gamma t}$ for some constant $\gamma > 0$ and some polynomial function $L(t)$, then the corresponding random variable has an exponential tail. Naturally, random variables with exponential tails have unbounded support. Our next result establishes the relationship between large deviation probabilities and the supply structure of the economy in presence of exponential tails.

Theorem 8. *Consider a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ in which the distribution of idiosyncratic productivity shocks $\{F_i\}_{i\in\mathcal{I}_n}$ are identical for all $n \in \mathbb{N}$, and are symmetric around their mean $\mathbb{E}\epsilon_i = 0$. If the shock distributions have an exponential tail, then for all $c > 0$,*

$$-\log\mathbb{P}\left(y_n < -c\right) = \Theta\left(1/\|v_n\|_\infty\right). \tag{17}$$

The significance of Theorem 8 is highlighted by comparing it to Theorem 7 for normally distributed shocks. Whereas with normally distributed shocks it is $\|v_n\|_2$ that captures the rate of decay of the tail events, this role is played by $\|v_n\|_\infty$ for shocks with exponential tails. The fact that the influence vector satisfies $\|v_n\|_2^2 \leq \|v_n\|_\infty$, a consequence of Hölder's inequality (Steele (2004)), implies that tail events are always more likely with exponential tails than with normally distributed sector specific shocks. Example 5 above illustrates the contrast of these two theorems. [27]

Our final example shows that economies with non-trivial input-output interconnections and thin-tailed shocks can exhibit properties similar to economies consisting of isolated sectors but with heavy-tailed productivity shocks.

Example 6. Consider a sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ similar to the one depicted in Figure 8, but instead assume that the degree of sector 1 scales as $n/\log(n)$ rather than $\log(n)$. By Theorem 8, the large deviation probability in such a sequence and in the presence of productivity shocks with exponential tails satisfies

$$-\log \mathbb{P}\left(y_n < -c\right) = \Theta(\log n).$$

Consider next another sequence of economies $\{\widehat{\mathcal{E}}_n\}_{n\in\mathbb{N}}$, where no sector is an input supplier to any other sector; i.e., $\widehat{W}_n$ is the identity matrix for all $n \in \mathbb{N}$, in which case the corresponding influence vector is equal to $\widehat{v}_n = \frac{1}{n}\mathbf{1}$. Also suppose that the sector-specific productivity shocks have a symmetric distribution, with a power law tail with exponent $\xi > 2$, that is $\bar{F}_i(t) = L(t)t^{-\xi}$, where $L(\cdot)$ is a slowly-varying function. Then, by the large deviation results for heavy-tailed random variables (see, for example, Mikosch and Nagaev (1998)), we have

$$\lim_{n\to\infty} \frac{\mathbb{P}(y_n > c)}{n\bar{F}(nc)} = 1,$$

implying that

$$-\log \mathbb{P}(y_n < -c) = \Theta\left(\log n\right).$$

That is, the probability of extreme events in $\{\widehat{\mathcal{E}}_n\}_{n\in\mathbb{N}}$ decays at a rate similar to that in $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$, despite the fact the productivity shocks in the latter are thin-tailed, as opposed to the former.

7 Conclusion

Recent events have highlighted that interconnections across firms or sectors linked through supply networks or financial linkages might create cascades as shocks to some firms/sectors spread to the rest of the economy. In this paper, we provide a general framework for the analysis of the relationship between the network structure of an economy and its aggregate volatility. We study a sequence of economies consisting of a set of sectors linked through a supply (input-output) network, specifying the extent to which each sector needs to use the output of other sectors as intermediate goods for production. We define aggregate volatility as the standard deviation of the logarithm of GDP per capita in the economy and study the relationship between sector-specific volatility and aggregate volatility as a function of the structural properties of the supply network.

[27] Stanley et al. (1996) find that the distribution of annual growth rate of Compustat manufacturing firms is well approximated by an exponential distribution. We find the same pattern when we look at the annual growth rate of four-digit manufacturing sectors in the NBER productivity database.

More formally, we consider a sequence of economies (in which the number of sectors grows) and investigate whether and at what rate aggregate volatility disappears upon disaggregation. First, we show that in the presence of *dominant sectors* (for example when the sequence has a star-like structure with certain "central" sectors supplying to a non-vanishing fraction of the sectors in the economy), the weak law of large numbers fails and aggregate volatility does not disappear. This case starkly illustrates how aggregate fluctuations may result from purely idiosyncratic (sectoral) variability.

Our main focus, however, has been on sequences of economies for which the weak law of large numbers holds. For such economies, when the network structure is such that there exists a uniform upper bound on the extent to which each firm supplies the rest of the economy, aggregate volatility declines asymptotically at the rate $\sqrt{n}$, consistent with the standard central limit theorem. In this case, in reasonably-sized economies, sectoral-level (or when units are interpreted as firms, firm-level) shocks wash out rapidly. On the other hand, we show that in the presence of richer network interactions, the rate of convergence can be significantly lower (and convergence is not necessarily to a normal distribution). Our analysis also shows that in the debate between Dupor (1999) and Horvath (1998, 2000), the former's critique applies to economies with balanced structures (such as the ones he studies), whereas as emphasized by the latter, more generally, the sectoral structure of the economy is important for understanding business cycles (see also the discussion in Carvalho (2010)).

We start with a general characterization result linking aggregate volatility to the Euclidean norm of an *influence vector* measuring the importance of each sector in the supply network. Other properties of this influence vector also determine whether aggregate output is asymptotically normally distributed or not.

Our main results provide various lower bounds on the rate of convergence of aggregate volatility as the size of the economy grows. A first bound is provided by analyzing the effects of first-order connections, represented by the degree sequence induced by the supply network. This lower bound is in terms of the coefficients of variation of the degree sequence. An immediate corollary of this result is that for degree sequences that exhibit a power law (meaning that the tail of the degree sequence is Pareto), the lower bound can be characterized in terms of the shape parameter of the Pareto distribution.

Our most important results provide a more useful (and economically more subtle) bound by analyzing the effects of second and higher-order connections, which more closely correspond to the idea of "cascades". For example, just taking into account second-order interconnections, the lower bound is in terms of two-hop ahead connections, capturing whether two highly connected sectors also share common suppliers. This is important for aggregate volatility because when these suppliers are hit by negative shocks, there is a cascade in the supply network—their low output translates into low inputs for their downstream sectors, which are themselves important and supply many other downstream sectors. We also provide an analog of this result for higher-order interconnections.

Interestingly, even though it is at a fairly aggregated level, the US input-output matrix suggests that second-order interconnections are more important for aggregate volatility than first-order interconnections, highlighting the importance of the types of cascades discussed in the previous paragraph.

Finally, we also provide an analysis of tail events—the likelihood of large deviations of aggregate output from its mean. Even for two sequences of economies converging to identical asymptotically normal distributions, the probabilities of tail events can be significantly different. This suggests that even when the central limit theorem holds, the supply network can have significant implications for tail events. We show that when sector-specific shocks are normally distributed, tail events are determined by the same structural properties of the supply network that shapes our baseline measure of aggregate volatility. But for other distributions of sector specific shocks, yet other structural properties influence the events.

Several areas of future research are opened up by this analysis. First, it would be interesting to relate the probabilities of tail events to structural properties of supply networks for distributions other than the ones we studied here.

Second, the same analysis can be carried out at the firm level (rather than at the sectoral level), but in this case substitution between different suppliers needs to be modeled more carefully. Interestingly, substitution possibilities might be different in the short run than in the long run. For example, many firms might be beholden to their suppliers in the short run, but can switch suppliers with sufficient advance planning.

Third, our analysis focused on a competitive economy with Cobb-Douglas technologies, which implied that only "outdegrees", that is, supply linkages, mattered. The discussion of the US auto industry during the crisis in the Introduction suggests that both supply and demand linkages are important in general; Ford's concern during the crisis was that the failure of either GM or Chrysler would lead to failures of auto suppliers (their upstream suppliers) and auto dealers (their downstream costumers) that would then cascade to Ford. Extending the current model to a monopolistically competitive environment would allow us to have a flexible framework in which both supply and demand linkages matter, and is thus a natural next step.

Fourth and more importantly, we have throughout taken the supply network as given. In practice, however, the supply network is endogenously determined both at the firm and the sectoral levels. For example, at the firm level, firms can decide how many suppliers (and downstream customers) to form long-term relationships with. They can also choose which ones to form such relations with. The more substitutable suppliers a firm has, the less subject to supplier-level productivity shocks it will be. But making duplicate investments for building relationships with several suppliers (or customers) is costly, creating a trade-off. Part of this trade-off will be shaped by how risky different suppliers are perceived to be and how risk is evaluated and priced in the economy. Another source of trade-off arises from the fact that complex supply networks are likely to increase productivity by enabling specialization, and yet through the cascade effects quantified in this paper, may also increase volatility.

Fifth, both the analysis of scale effects and the trade-offs between profitability and risks created through interconnections are particularly important in the context of financial networks (e.g., Allen, Babus, and Carletti (2010), Brunnermeier, Gorton, and Krishnamurthy (2010) and Caballero and Simsek (2010)). A first step might be to derive a theoretically-grounded version of our influence vector that might summarize several important dimensions of interactions in financial networks. However, in this context it might be the tail risks that are transmitted through the financial network, rather than all shocks to output or profitability as in our model. This set of issues requires a different type of analysis.

Finally, an analysis of dynamics in extended model including potential switches between suppliers and endogenous relationships is also a potential area for future research. We think that all of these areas are very promising for enriching our understanding of aggregate volatility and risk in the economy.

Appendix

A Central Limit Theorems

A.1 The Lindeberg-Feller Theorem

The Lindeberg-Feller Theorem provides sufficient conditions under which distribution of sums of independent, but not necessarily identically distributed random variables converge to the normal law, and thus, can be considered as a generalization of the central limit theorem. The statement and proof of the theorem can be found in many probability textbooks, e.g., Durrett (2005, p. 114).

Theorem 9 (Lindeberg-Feller)**.** *Consider the triangular array of independent random variables* ζ_{in}*,* $1 \leq i \leq n$*, with zero expectations and finite variances such that*

$$\sum_{i=1}^{n} \mathbb{E}\zeta_{in}^2 = 1.$$

Also suppose that Lindeberg's condition holds:

$$\lim_{n\to\infty} \sum_{i=1}^{n} \mathbb{E}\left(\zeta_{in}^2 \mathbb{I}_{\{|\zeta_{in}|>\delta\}}\right) = 0 \qquad \text{for all } \delta > 0, \tag{18}$$

where $\mathbb{I}$ *denotes the indicator function. Then,*

$$\zeta_{1n} + \zeta_{2n} + \cdots + \zeta_{nn} \xrightarrow{d} \mathcal{N}(0,1).$$

It is easy to verify that for Lindeberg's condition (18) to hold, the triangular array of random variables $\{\zeta_{in}\}$ must satisfy the *asymptotic negligibility* property, which requires that

$$\max_{1\leq i\leq n} \mathbb{P}\left(|\zeta_{in}| > \delta\right) \longrightarrow 0 \qquad \forall\, \delta > 0, \tag{19}$$

as $n \to \infty$. The asymptotic negligibility property guarantees that the limit distribution of $\zeta_{1n} + \zeta_{2n} + \cdots + \zeta_{nn}$ is insensitive to the behavior of finitely many components in the sequence (Linnik and Ostrovskiĭ (1977)).

A.2 Non-Classical Central Limit Theorems

To establish asymptotic normality for triangular arrays of random variables $\{\zeta_{in}\}$ that violate asymptotic negligibility property, one needs to apply "non-classical" generalizations of the central limit theorem. The following theorem is from Rotar (1975). A detailed treatment of the subject can be found in Chapter 9 of Linnik and Ostrovskiĭ (1977).

Theorem 10. *Consider a triangular array of independent random variables* ζ_{in}*,* $1 \leq i \leq n$*, with distributions* G_{in}*, zero expectations, and finite variances* σ_{in}^2*, such that* $\sum_{i=1}^{n} \sigma_{in}^2 = 1$*. Then* $\zeta_{1n} + \zeta_{2n} + \cdots + \zeta_{nn} \to \mathcal{N}(0,1)$ *in distribution, only if*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \int_{|t|>\delta} |t| \left| G_{in}(t) - \Phi_{in}(t) \right| dt = 0 \qquad \text{for all } \delta > 0, \tag{20}$$

where $\Phi_{in}(t) = \Phi(t/\sigma_{in})$ and Φ denotes the standard normal distribution.

B Competitive Markets Equilibrium

In this section, we define and compute the competitive markets equilibrium of the static economy studied in the paper. Consider an economy $\mathcal{E}_n = (\mathcal{I}_n, W_n, \{F_i\}_{i\in\mathcal{I}_n})$ consisting of n different sectors with input-output matrix W_n. The representative consumer's problem can be stated as

$$\begin{aligned} \max_{l,\{c_i\}_{i\in\mathcal{I}_n}} \quad & \frac{1}{n}\sum_{i=1}^{n}\log(c_i) + \log(A_n) \\ \text{subject to} \quad & \sum_{i=1}^{n} p_i c_i = hl \\ & 0 \le l \le 1, \end{aligned} \tag{21}$$

where l is total amount of labor supplied by the consumer. The representative firm in sector i maximizes profits subject to its production possibilities, captured by the sector's production technology; that is,

$$\begin{aligned} \max_{l_i, x_i, \{x_{ij}\}_{j\in\mathcal{I}_n}} \quad & p_i x_i - h l_i - \sum_{j=1}^{n} p_j x_{ij} \\ \text{subject to} \quad & x_i = z_i^{\alpha} l_i^{\alpha} \prod_{j\in\mathcal{N}_i} x_{ij}^{(1-\alpha)w_{ij}}. \end{aligned} \tag{22}$$

Finally, market clearing conditions for the economy $\mathcal{E}_n$ are given by

$$\begin{aligned} c_i + \sum_{j=1}^{n} x_{ji} &= x_i \qquad \forall i \in \mathcal{I}_n \\ \sum_{i=1}^{n} l_i &= l. \end{aligned}$$

Given the above, we can now define the equilibrium of $\mathcal{E}_n$.

Definition 10. A *competitive markets equilibrium* of economy $\mathcal{E}_n$ consists of prices $(p_1, p_2, \ldots, p_n)$, wage h, consumption bundle $(c_1, c_2, \ldots, c_n)$, labor supply l, and quantities $(l_i, x_i, (x_{ij}))$ such that

(a) the representative consumer maximizes her utility,

(b) the representative firms in each sector maximize profits,

(c) labor and commodity markets clear.

To characterize the competitive markets equilibrium of the above economy, note that since the representative consumer does not value leisure, she supplies one unit of labor inelastically. It is also easy to verify that given the commodity prices and wage, the optimal consumption bundle of the consumer is given by $c_i = h/(np_i)$. On the other hand, taking first-order conditions with respect to l_i and x_{ij} in firm i's problem implies that

$$\begin{aligned} l_i &= \frac{\alpha p_i x_i}{h} \\ x_{ij} &= \frac{(1-\alpha) p_i w_{ij} x_i}{p_j}, \end{aligned}$$

the substitution of which in the firm's production technology leads to

$$\alpha \log(h) = \alpha \epsilon_i - H_{(\alpha)} + \log(p_i) - (1-\alpha) \sum_{j=1}^{n} w_{ij} \log(p_j) - (1-\alpha) H_i$$

where $H_{(\alpha)} = -\alpha \log(\alpha) - (1-\alpha)\log(1-\alpha)$ and H_i is the input weight entropy of sector i, defined as $H_i \equiv -\sum_{j=1}^{n} w_{ij} \log(w_{ij})$. Writing the above equality in vector form, and premultiplying both sides by the influence vector $v_n' = \frac{\alpha}{n} \mathbf{1}'[I - (1-\alpha) W_n]^{-1}$ yields

$$\log(h) = v_n' \epsilon + \frac{1}{n} \sum_{i=1}^{n} \log(p_i) - \frac{H_{(\alpha)}}{\alpha} - \frac{1-\alpha}{\alpha} v_n' H,$$

where $H = [H_1 \dots H_n]'$ is the vector of input entropies of all sectors. Finally, by setting

$$A_n = n \exp\left(\frac{1}{\alpha} \left[H_\alpha + (1-\alpha) v_n' H \right] \right) \tag{23}$$

and normalizing the ideal price index to 1, i.e.,

$$\frac{n}{A_n} (p_1 p_2 \dots p_n)^{1/n} = 1,$$

we obtain

$$y_n = \log(h) = v_n' \epsilon.$$

That is, logarithm of real value added in a given economy (which we also refer to as its aggregate output) is a weighted sum of sector-specific productivity shocks, where the weights are determined by the corresponding influence vector.

We now show that the influence vector also captures the equilibrium share of sales of different sectors. By plugging the optimal values of labor and goods purchased by the firms and the optimal consumption of the consumers in the market clearing condition for commodity i we have:

$$h/n + (1-\alpha) \sum_{j=1}^{n} w_{ji} p_j x_j = p_i x_i,$$

which implies that

$$s_i = h/n + (1-\alpha) \sum_{j=1}^{n} s_j w_{ji},$$

where $s_i = p_i x_i$ is the equilibrium value of sales of sector i. Thus, the vector of equilibrium sales is related to the influence vector through

$$s' = (h/n)\mathbf{1}' \left[I - (1-\alpha)W\right]^{-1} = (h/\alpha)v_n',$$

implying that

$$v_{n,i} = \frac{p_i x_i}{\sum_{j=1}^{n} p_j x_j},$$

where we have used the fact that $v_n'\mathbf{1} = 1$, proved in Lemma 1.

C Proofs

C.1 Proofs of Section 3

Proof of Proposition 1: By definition, the presence of central sectors in the economy implies that $\|W_n\|_1 = \Theta(n)$, where $\|W_n\|_1$ is the induced ℓ_1-norm of matrix W_n and is equal to the maximum outdegree in the economy. On the other hand, equation (6) implies that

$$v_n' \geq \frac{\alpha(1-\alpha)}{n}\mathbf{1}'W_n$$

and as a result,

$$\|v_n\|_\infty \geq \frac{\alpha(1-\alpha)}{n}\|W_n\|_1.$$

which guarantees that $\|v_n\|_\infty = \Omega(1)$. Therefore, $\|v_n\|_\infty$ is uniformly bounded away from zero for all n, completing the proof. ■

Proof of Theorem 1: Suppose that the sequence of economies $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ has a dominant sector; i.e., $\|v_n\|_\infty = \Theta(1)$, by definition. Therefore, inequality $\|v_n\|_\infty \leq \|v_n\|_2$ implies that $\|v_n\|_2 = \Omega(1)$. This means that the standard deviation (variance) of aggregate output y_n remains bounded away from zero as $n \to \infty$, ruling out the possibility of convergence of y_n to zero in probability.

Now suppose that the sequence has no dominant sectors; that is $\|v_n\|_\infty = o(1)$. Then, Hölder's inequality $\|v_n\|_2^2 \leq \|v_n\|_\infty \|v_n\|_1$ (see, e.g., Steele (2004)) and the fact that $\|v_n\|_1 = 1$ (proved in Lemma 1 below) imply that $\|v_n\|_2$ converges to zero as $n \to \infty$. Thus, by Chebychev's inequality, aggregate output y_n converges to its mean in probability. ■

Proof of Theorem 2: The proof of part (a) is trivial and is omitted.

In order to prove part (b) of the theorem, we define the triangular array of real numbers $\{\zeta_{in}\}_{1\leq i\leq n}$ whose elements are given by $\zeta_{in} = v_{n,i}\epsilon_i/\bar{\sigma}\|v_n\|_2$ for all $i \in \mathcal{I}_n$. Thus, by definition, $\frac{1}{\bar{\sigma}\|v_n\|_2}y_n = \zeta_{1n} + \zeta_{2n} + \cdots + \zeta_{nn}$. It is also straightforward to verify that the following relations hold:

$$\begin{aligned}\mathbb{E}\zeta_{in} &= 0. \\ \sum_{i=1}^{n}\mathbb{E}\zeta_{in}^2 &= 1.\end{aligned}$$

Therefore, by the Lindeberg-Feller Theorem, which we have stated in Appendix A, $y_n/\bar{\sigma}\|v_n\|_2$ converges in distribution to the standard normal law, as long as Lindeberg's condition (18) is satisfied. In order to verify that Lindeberg's condition indeed holds, notice that we have,

$$\begin{aligned}\sum_{i=1}^{n}\mathbb{E}\left(\zeta_{in}^2\mathbb{I}_{\{|\zeta_{in}|>\delta\}}\right) &= \frac{1}{\bar{\sigma}^2\|v_n\|_2^2}\sum_{i=1}^{n}v_{n,i}^2\,\mathbb{E}\left[\epsilon_i^2\mathbb{I}_{\left\{|\epsilon_i|>\frac{\delta\bar{\sigma}\|v_n\|_2}{|v_{n,i}|}\right\}}\right] \\ &\leq \frac{1}{\bar{\sigma}^2\|v_n\|_2^2}\sum_{i=1}^{n}v_{n,i}^2\,\mathbb{E}\left[\epsilon_i^2\mathbb{I}_{\left\{|\epsilon_i|>\frac{\delta\bar{\sigma}\|v_n\|_2}{\|v_n\|_\infty}\right\}}\right] \\ &= \frac{1}{\bar{\sigma}^2}\mathbb{E}\left[\epsilon_i^2\mathbb{I}_{\left\{|\epsilon_i|>\frac{\delta\bar{\sigma}\|v_n\|_2}{\|v_n\|_\infty}\right\}}\right].\end{aligned}$$

By the dominated convergence theorem and the assumption that $\|v_n\|_\infty = o(\|v_n\|_2)$, the right-hand side of the above equality converges to zero as $n \to \infty$, and therefore,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\mathbb{E}\left(\zeta_{in}^2\mathbb{I}_{\{|\zeta_{in}|>\delta\}}\right) = 0 \qquad \text{for all } \delta > 0,$$

i.e., Lindeberg's condition is satisfied. As a result, the Lindeberg-Feller Theorem guarantees that $\frac{1}{\|v_n\|_2}y_n \to \mathcal{N}(0,\bar{\sigma}^2)$ in distribution. This completes the proof of part (b).

In order to prove part (c), note that assumption $\frac{\|v_n\|_\infty}{\|v_n\|_2} \not\to 0$ implies that the triangular array of random variables $\zeta_{in} = v_{n,i}\epsilon_i/\bar{\sigma}\|v_n\|_2$ does not satisfy the asymptotic negligibility property (19) stated in Appendix A, and hence, the Lindeberg-Feller Theorem is not applicable. Instead, one needs to apply the non-classical variant of the central limit theorem stated as Theorem 10 in Section A.2.[28]

To prove that asymptotic normality does not hold for non-normal ϵ_i's, we need to show that condition (20) is not satisfied. By the definition of ζ_{in}, we know that its distribution function is given in terms of the common distribution of ϵ_i's as $G_{in}(t) = F(t\bar{\sigma}\|v_n\|_2/v_{n,i})$. Therefore, we have

$$\begin{aligned}\sum_{i=1}^{n}\int_{|t|>\delta}|t|\,|G_{in}(t)-\Phi_{in}(t))\,|dt &= \frac{1}{\bar{\sigma}^2\|v_n\|_2^2}\sum_{i=1}^{n}v_{n,i}^2\int_{-\infty}^{\infty}|s|\,|F(s)-\Phi(s)|\,\mathbb{I}_{\left\{|s|>\frac{\delta\bar{\sigma}\|v_n\|_2}{|v_{n,i}|}\right\}}ds \\ &\geq \left(\frac{\|v_n\|_\infty}{\bar{\sigma}\|v_n\|_2}\right)^2\int_{-\infty}^{\infty}|s||F(s)-\Phi(s)|\mathbb{I}_{\left\{|s|>\frac{\delta\bar{\sigma}\|v_n\|_2}{\|v_n\|_\infty}\right\}}ds.\end{aligned}$$

Therefore, unless $F = \Phi$, for small enough $\delta > 0$, the right-hand side of the above relation is bounded away from zero for infinitely many n. Hence, Theorem 10 implies that $\frac{1}{\|v_n\|_2}y_n$ is not normally distributed as $n \to \infty$. ■

[28]For a similar argument, see Christopeit and Werner (2001).

C.2 Proofs of Section 4

Proof of Lemma 1: By definition, $y_n = v_n'\epsilon$, where $v_n = \alpha/n\,[I-(1-\alpha)W_n']^{-1}\,\mathbf{1}$. Also note that by equation (6), it is possible to express v_n in terms of a convergent power series of W_n:

$$v_n' = \frac{\alpha}{n}\mathbf{1}'\sum_{k=0}^{\infty}(1-\alpha)^k W_n^k$$

which implies that v_n is the sum of infinitely many non-negative vectors. Moreover, since the first term in the summation is equal to $\frac{\alpha}{n}\mathbf{1}'$, it must be the case that v_n is entry-wise positive. To prove the second part, we multiply both sides of the above equation by the vector of all ones and observe that $v_n'\mathbf{1} = \alpha\sum_{k=0}^{\infty}(1-\alpha)^k = 1$. Thus, $\|v_n\|_1 = \sum_{i=1}^{n} v_{n,i} = 1$. ■

Proof of Lemma 2: The proof immediately follows from the inequalities $\|v_n\|_2 \le \|v_n\|_1$ and $1/\sqrt{n}\|v_n\|_1 \le \|v_n\|_2$, and the fact that $\mathrm{var}(y_n) = \Theta(\|v_n\|_2^2)$. ■

Proof of Theorem 3: Recall that aggregate volatility is of order $\|v_n\|_2$. Moreover, equation (6) and the fact that W_n is element-wise non-negative imply that

$$v_n' \ge \frac{\alpha}{n}\mathbf{1}' + \frac{\alpha(1-\alpha)}{n}\mathbf{1}'W_n.$$

Therefore,

$$\begin{aligned}\|v_n\|_2^2 &\ge \frac{\alpha^2}{n^2}\mathbf{1}'\mathbf{1} + \frac{2\alpha^2(1-\alpha)}{n^2}\mathbf{1}'W_n\mathbf{1} + \frac{\alpha^2(1-\alpha)^2}{n^2}\|W_n'\mathbf{1}\|_2^2 \\ &= \frac{\alpha^2(3-2\alpha)}{n} + \frac{\alpha^2(1-\alpha)^2}{n^2}\|W_n'\mathbf{1}\|_2^2 \\ &= \Theta(1/n) + \Theta\left(\frac{1}{n^2}\sum_{i=1}^{n} d_i^2\right) \qquad (24)\end{aligned}$$

where we have used the fact that the i-th column sum of W_n is the outdegree of sector i, and that the sum of all its elements is equal to n. Given that inequality $\sqrt{n}\|z\|_2 \ge \|z\|_1$ holds for any n-dimensional vector z, we conclude that

$$\sum_{i=1}^{n} d_i^2 \ge \frac{1}{n}\left(\sum_{i=1}^{n} d_i\right)^2 = n.$$

Thus, the first term in (24) is always dominated by the second term. This establishes the first relation.

To prove the second part of the theorem, note that the average outdegree $\bar{d}$ is equal to one, and therefore, we can write the sum of degree squares in terms of the standard deviation of the degree sequence as:

$$\frac{1}{n^2}\sum_{i=1}^{n} d_i^2 = \frac{n-1}{n^2}\left[\mathrm{CV}(d^{(n)})\right]^2 + \frac{1}{n},$$

establishing that $\mathrm{var}(y_n) = \Omega\left(\frac{1+\mathrm{CV}^2(d^{(n)})}{n}\right)$. This completes the proof. ■

Proof of Corollary 1: We define

$$\hat{P}_n(k) \equiv \frac{1}{n}\left|\{i \in \mathcal{I}_n : d_i^2 > k\}\right|$$

as the empirical counter-cumulative distribution function of the outdegrees-squared. Clearly, by definition, we have $\hat{P}_n(k) = P_n(\sqrt{k})$ for all k. We also define $B = \{b_1, \ldots, b_m\}$ to be the set of values that the outdegrees-squared of $\mathcal{E}_n$ take, where $b_{k+1} > b_k$ for all k. Thus, we have

$$\begin{aligned}\sum_{i=1}^{n} d_i^2 &= n\sum_{k=1}^{m} b_k\left[\hat{P}_n(b_{k-1}) - \hat{P}_n(b_k)\right] \\ &= n\sum_{k=0}^{m-1}\left(b_{k+1} - b_k\right)\hat{P}_n(b_k)\end{aligned}$$

with the convention that $b_0 = 0$. Therefore,

$$\sum_{i=1}^{n} d_i^2 = n\int_0^{b_m}\hat{P}_n(t)dt = 2n\int_0^{d_{\max}^{(n)}} tP_n(t)dt$$

where the last equality is due to a simple change of variables. The fact that $L(\cdot)$ is a slowly-varying function, satisfying $\lim_{t\to\infty} L(t)t^{\epsilon} = \infty$ for any positive $\epsilon > 0$, implies that

$$\sum_{i=1}^{n} d_i^2 \geq n\hat{c}_n\int_0^{d_{\max}^{(n)}} t^{(1-\beta-2\epsilon)}dt,$$

where $\hat{c}_n = \Theta(1)$ is a sequence of positive numbers. Thus, from (11) in Theorem 3 and since $\beta \in (1,2)$, we have

$$(\mathrm{var}\ y_n)^{1/2} = \Omega\left(n^{\frac{1-\beta}{\beta}-\epsilon}\right),$$

proving the result. ■

Proof of Theorem 4: Once again, recall that the influence vector corresponding to economy $\mathcal{E}_n$ can be written as a power series of W_n specified by equation (6). Given the fact that all terms in this infinite sum are non-negative vectors, we have

$$v_n' \geq \frac{\alpha}{n}\mathbf{1}'\left[I + (1-\alpha)W_n + (1-\alpha)^2W_n^2\right].$$

Therefore,

$$\begin{aligned}\|v_n\|_2^2 &\geq \frac{\alpha^2}{n^2}\mathbf{1}'\left[I + (1-\alpha)W_n + (1-\alpha)^2W_n^2\right]\left[I + (1-\alpha)W_n + (1-\alpha)^2W_n^2\right]'\mathbf{1} \\ &= \Theta\left(\frac{1}{n^2}\|\mathbf{1}'W_n\|_2^2\right) + \Theta\left(\frac{1}{n^2}\mathbf{1}'W_n^2W_n'\mathbf{1}\right) + \Theta\left(\frac{1}{n^2}\left\|\mathbf{1}'W_n^2\right\|_2^2\right),\end{aligned} \quad (25)$$

where we have used the fact that $\frac{1}{n^2}\|\mathbf{1}'W_n\|_2^2 = \frac{1}{n^2}\sum_{i=1}^n d_i^2$ dominates $1/n$ for large values of n. For the second term on the right-hand side of (25), we have

$$
\begin{aligned}
\mathbf{1}'W_n^2 W_n'\mathbf{1} &= \sum_{i=1}^n \sum_{j=1}^n w_{ji} d_i d_j \\
&= \sum_{i=1}^n \sum_{j\neq i} w_{ji} d_i d_j + \sum_{i=1}^n w_{ii} d_i^2 \\
&= s(W_n) + \mathcal{O}\left(\sum_{i=1}^n d_i^2\right),
\end{aligned}
$$

where s denotes the s-metric corresponding to the economy, defined in Section 4. On the other hand, for the third term on the right-hand side of (25), we have

$$
\begin{aligned}
\|\mathbf{1}'W_n^2\|_2^2 &= \sum_{i=1}^n \left[\sum_{j=1}^n w_{ji} d_j\right]^2 = \sum_{i=1}^n \left[w_{ii} d_i + \sum_{j\neq i} w_{ji} d_j\right]^2 \\
&= \sum_{i=1}^n w_{ii}^2 d_i^2 + 2\sum_{i=1}^n \sum_{j\neq i} w_{ii} w_{ji} d_i d_j + \sum_{i=1}^n \left[\sum_{j\neq i} w_{ji} d_j\right]^2 \\
&= \mathcal{O}\left(\sum_{i=1}^n d_i^2\right) + \mathcal{O}\left(s(W_n)\right) + \sum_{i=1}^n \sum_{j\neq i} d_j^2 w_{ji}^2 + \sum_{i=1}^n \sum_{j\neq i} \sum_{k\neq i,j} w_{ji} w_{ki} d_j d_k \\
&= \mathcal{O}\left(\sum_{i=1}^n d_i^2\right) + \mathcal{O}\left(s(W_n)\right) + \Theta\left(\tau_2(W_n)\right)
\end{aligned}
$$

where in the next to last equality we have used the fact that $w_{ii} \leq 1$ for all i. The last equality holds because of the fact that $\sum_{i=1}^n w_{ji}^2 \leq 1$ for all j. Thus, combining all the above leads to

$$\|v_n\|_2^2 = \Omega\left(\frac{1}{n^2}\left[\sum_{i=1}^n d_i^2 + s(W_n) + \tau_2(W_n)\right]\right).$$

Now, inequality

$$\sum_{i=1}^n \left[d_i - \sum_{j\neq i} w_{ji} d_j\right]^2 \geq 0$$

guarantees that

$$\sum_{i=1}^n d_i^2 + \sum_{i=1}^n \sum_{j\neq i} d_j^2 w_{ji}^2 + \tau_2(W_n) \geq 2s(W_n)$$

implying that $s(W_n) = \mathcal{O}\left(\sum_{i=1}^n d_i^2 + \tau_2(W_n)\right)$. Therefore, in highly disaggregated economies, the effect captured by the s-metric is dominated by the sum of the other two terms, and as a result

$$\|v_n\|_2 = \Omega\left(\frac{1}{n}\sqrt{\sum_{i=1}^n d_i^2} + \frac{\sqrt{\tau_2(W_n)}}{n}\right),$$

completing the proof. ■

Proof of Corollary 2: By equation (25), we have $(\mathrm{var}\ y_n)^{1/2} = \Omega\left(\frac{1}{n}\|\mathbf{1}'W_n^2\|_2\right)$, which implies that

$$(\mathrm{var}\ y_n)^{1/2} = \Omega\left(\frac{1}{n}\sqrt{\sum_{i=1}^{n} q_i^2}\right).$$

The rest of the proof then simply follows from a similar argument as in proof of Corollary 1. ■

Proof of Theorem 5: Following the same logic as in the proof of Theorem 4, it is easy to verify that for any positive integer m, the influence vector satisfies the following inequality; a consequence of equation (6):

$$v_n \geq \frac{\alpha}{n}\sum_{k=0}^{m}(1-\alpha)^k \mathbf{1}'W_n^k.$$

Therefore, one can obtain the following lower bound for the Euclidean norm of the influence vector:

$$\|v_n\|_2^2 \geq \frac{\alpha^2}{n^2}\sum_{k=1}^{m}(1-\alpha)^{2k}\mathbf{1}'W_n^k W_n'^k \mathbf{1}.$$

Writing the matrix powers in terms of the input-output weights, and upon some simplification and rearrangement of terms, we get the result for any positive constant integer m. ■

Proof of Proposition 2: By definition, for a balanced sequence of economies, we have $\|W_n\|_1 = \max_{i\in\mathcal{I}_n} d_i = \Theta(1)$. Moreover, equation (7) implies that

$$\|v_n\|_\infty \leq \frac{\alpha}{n} + (1-\alpha)\|W_n\|_1\|v_n\|_\infty \leq \frac{\alpha}{n} + C(1-\alpha)\|v_n\|_\infty.$$

where C is a constant not depending on n. Therefore, for $\alpha > (C-1)/C$, we have $\|v_n\|_\infty \leq \alpha[1-(1-\alpha)C]^{-1}/n$, implying that $\|v_n\|_\infty = \Theta(1/n)$. Finally, Hölder's inequality $\|v_n\|_2 \leq \sqrt{\|v_n\|_1\|v_n\|_\infty}$ and Lemma 1 guarantee that $\|v_n\|_2 = \Theta(1/\sqrt{n})$, which completes the proof. ■

C.3 Proofs of Section 6

Proof of Theorem 6: In order to prove the theorem, we first show that the largest element of the influence vectors corresponding to the sequence $\{\mathcal{E}_n\}_{n\in\mathbb{N}}$ converges to zero at a rate slower than $\sqrt{n}$.

By equation (7) and for any sector $j \in P_{ni}$, we have

$$v_{n,j} = \frac{\alpha}{n} + (1-\alpha)\sum_{k\in P_{ni}} v_{n,k}w_{kj} + (1-\alpha)\sum_{k\notin P_{ni}} v_{n,k}w_{kj}$$

which implies that

$$\begin{aligned}\sum_{j\in P_{ni}} v_{n,j} &= \frac{\alpha\,|P_{ni}|}{n} + (1-\alpha)\sum_{k\in P_{ni}}\sum_{j\in P_{ni}} v_{n,k}w_{kj} + (1-\alpha)\sum_{j\in P_{ni}}\sum_{k\notin P_{ni}} v_{n,j}w_{kj}\\ &\leq \frac{\alpha|P_{ni}|}{n} + (1-\alpha)\sum_{k\in P_{ni}} v_{n,k} + (1-\alpha)\|v_n\|_\infty \sum_{j\in P_{ni}}\sum_{k\notin P_{ni}} w_{kj}\\ &\leq \frac{\alpha a_n}{n} + (1-\alpha)\sum_{k\in P_{ni}} v_{n,k} + (1-\alpha)c\|v_n\|_\infty\end{aligned}$$

where a_n is the n-th element of a sequence $\{a_n\}_{n\in\mathbb{N}}$ satisfying $a_n = o(\sqrt{n})$, and c is a positive constant not depending on n, the existence of which is guaranteed by the assumption of $\sqrt{n}$-decomposability. Hence,

$$\max_{j\in P_{ni}} v_{n,j} \leq \sum_{j\in P_{ni}} v_{n,j} \leq \frac{a_n}{n} + \frac{1-\alpha}{\alpha}c\|v_n\|_\infty.$$

On the other hand, note that $\|v_n\|_\infty = \max_i \max_{j\in P_{ni}} v_{n,j}$, and therefore,

$$\|v_n\|_\infty \leq \frac{a_n}{n} + \frac{1-\alpha}{\alpha}c\|v_n\|_\infty.$$

Thus, for $\alpha > \frac{c}{c+1}$, we have

$$\|v_n\|_\infty \leq \frac{\alpha a_n}{(\alpha+\alpha c - c)n} = o\left(\frac{1}{\sqrt{n}}\right).$$

So far, we have established that $\|v_n\|_\infty = o(1/\sqrt{n})$ for any $\sqrt{n}$-decomposable sequence of economies. On the other hand, the inequality $1/\sqrt{n}\|v_n\|_1 \leq \|v_n\|_2$ and Lemma 1 guarantee that $\|v_n\|_2 \geq 1/\sqrt{n}$, implying that $\frac{\|v_n\|_\infty}{\|v_n\|_2} \to 0$ as $n\to\infty$. Thus, by Theorem 2, $y_n/\|v_n\|_2$ is asymptotically normally distributed. ■

Proof of Theorem 7: The proof of the first part of the theorem is straightforward. Note that the sum of independent normal random variables is normally distributed with the variance equal to the sum of variances. We now prove (16). The statement holds trivially if $\|v_n\|_2 = \Theta(1)$. Thus, we only consider the case that $\|v_n\|_2 = o(1)$. By the first part of the proposition, we have

$$\mathbb{P}\left(y_n < -c\right) = 1 - \Phi\left(\frac{c}{\sigma\|v_n\|_2}\right) = \Theta\left(\frac{\phi\left(c/\sigma\|v_n\|_2\right)}{c/\sigma\|v_n\|_2}\right),$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ denote the cumulative distribution and the probability density functions of the standard normal, respectively, and we have used the fact that $\lim_{t\to\infty}\frac{t[1-\Phi(t)]}{\phi(t)} = 1$.[29] Taking logarithms from both sides establishes the result. ■

Proof of Theorem 8: If the sequence of economies is such that $\|v_n\|_\infty = \Theta(1)$, e.g. as in star-like structures, the law of large numbers does not apply and therefore, $\mathbb{P}(y_n < -c)$ does not

[29]See for example, Grimmett and Stirzaker (2001, p. 98).

decay to zero as $n \to \infty$. Thus, (17) holds trivially. For the rest of the proof, we assume that $\|v_n\|_\infty = o(1)$.

We first show that when all $\{\epsilon_i\}_{i \in \mathcal{I}_n}$ have a common symmetric distribution $F(\cdot)$ with an exponential tail, then $\limsup_{n\to\infty} -\|v_n\|_\infty \log \mathbb{P}(y_n < -c) < \infty$. Note that if $v_{n,i}\epsilon_i < -c$ and $\sum_{j\neq i} v_{n,j}\epsilon_j < 0$ hold for some i, then $y_n < -c$. Therefore, by independence and symmetry assumptions, we have

$$\mathbb{P}(y_n < -c) \geq \frac{1}{2}\mathbb{P}\left(\epsilon_i \|v_n\|_\infty < -c\right) = \frac{1}{2}\bar{F}\left(c/\|v_n\|_\infty\right),$$

which implies that

$$\limsup_{n\to\infty} -\|v_n\|_\infty \log \mathbb{P}(y_n < -c) \leq \limsup_{n\to\infty} -\|v_n\|_\infty \log \bar{F}\left(c/\|v_n\|_\infty\right).$$

Now, since $\|v_n\|_\infty = o(1)$ and $F(\cdot)$ has an exponential tail, the right-hand side of the above inequality is finite, which implies that $-\log \mathbb{P}(y_n < -c) = \mathcal{O}\left(1/\|v_n\|_\infty\right)$.

We now prove $\liminf_{n\to\infty} -\|v_n\|_\infty \log \mathbb{P}(y_n < -c) > 0$. To establish this, we compute an upper bound for the generating function of ϵ_i, and use Chernoff's inequality to bound the tail event probability $\mathbb{P}(y_n < -c)$.[30] However, we first remark that if $F(\cdot)$ has an exponential tail, then there exists a positive constant γ such that

$$\bar{F}(t) < e^{-\gamma t} \tag{26}$$

for all $t > 0$. This is due to the fact that the function $-(1/t)\log \bar{F}(t)$ is always positive for $t > 0$ and has a strictly positive limit inferior.

We now proceed with the proof. Note that by symmetry of the distributions, and for $k \geq 2$ we have

$$\frac{1}{2}\mathbb{E}\left|\epsilon_i\right|^k = \int_0^\infty t^k dF(t) = \int_0^\infty k t^{k-1}\left(1 - F(t)\right) dt$$

where we have used integration by parts and the fact that

$$0 \leq \lim_{t\to\infty} t^k \left(1 - F(t)\right) = \lim_{t\to\infty} \exp\left[k\log(t) + \log(\bar{F}(t))\right] = 0;$$

a consequence of the exponential tail assumption. Thus, by (26), there exists a positive constant $r = 1/\gamma$ such that

$$\frac{1}{2}\mathbb{E}|\epsilon_i|^k \leq \int_0^\infty k t^{k-1} e^{-t/r} dt = r^k k!$$

for all $k \geq 2$. Therefore, for all $i \in \mathcal{I}_n$ and for $\delta < \frac{1}{r\|v_n\|_\infty}$, we have

$$\begin{aligned}
\mathbb{E}\left(e^{\delta v_{n,i}\epsilon_i}\right) &= 1 + \sum_{k=2}^\infty \frac{\delta^k v_{n,i}^k}{k!}\mathbb{E}\left(\epsilon_i^k\right) \\
&\leq 1 + 2\sum_{k=2}^\infty \left(\delta r v_{n,i}\right)^k.
\end{aligned}$$

[30]For a similar argument, see, e.g., Teicher (1984).

The above inequality implies that

$$\mathbb{E}\left(e^{\delta v_{n,i}\epsilon_i}\right) \leq 1 + \frac{2(\delta r v_{n,i})^2}{1-\delta r v_{n,i}} \leq \exp\left(\frac{2(\delta r v_{n,i})^2}{1-\delta r\|v_n\|_\infty}\right). \tag{27}$$

Using (27), we now compute an upper bound for the large deviation probability. From Chernoff's inequality, we have

$$\mathbb{P}(y_n < -c) = \mathbb{P}(y_n > c) \leq e^{-\delta c}\mathbb{E}\left(e^{\delta y_n}\right) = e^{-\delta c}\prod_{i=1}^{n}\mathbb{E}\left(e^{\delta v_{n,i}\epsilon_i}\right)$$

implying that

$$\log \mathbb{P}(y_n < -c) \leq -\delta c + \sum_{i=1}^{n}\frac{2(\delta r v_{n,i})^2}{1-\delta r\|v_n\|_\infty} = -\delta c + \frac{2(\delta r\|v_n\|_2)^2}{1-\delta r\|v_n\|_\infty}.$$

Now, setting $\delta = c/(4r^2\|v_n\|_2^2 + rc\|v_n\|_\infty)$ leads to[31]

$$\log \mathbb{P}(y_n < -c) \leq \frac{-c^2}{8r^2\|v_n\|_2^2 + 2rc\|v_n\|_\infty} \leq \frac{-c^2}{2r\|v_n\|_\infty(4r+c)}$$

where we have used the fact that $\|v_n\|_\infty \geq \|v_n\|_2^2$. Therefore,

$$\liminf_{n\to\infty} -\|v_n\|_\infty \log \mathbb{P}(y_n < -c) \geq \frac{c^2}{8r^2+2rc} > 0,$$

completing the proof. ■

[31]Note that this choice of δ satisfies $\delta r\|v_n\|_\infty < 1$, the condition required for deriving (27).

References

Allen, Franklin, Ana Babus, and Elena Carletti (2010), "Financial connections and systemic risk." Working Paper 16177, National Bureau of Economic Research.

Bak, Per (1996), *How Nature Works: The Science of Self–Organized Criticality*. Copernicus Press, New York, NY.

Bak, Per, Kan Chen, José Scheinkman, and Michael Woodford (1993), "Aggregate fluctuations from independent sectoral shocks: self-organized criticality in a model of production and inventory dynamics." *Ricerche Economiche*, 47, 3–30.

Bak, Per, Chao Tang, and Kurt Wiesenfeld (1987), "Self–organized criticality: An explanation of the $1/f$ noise." *Physical Review Letters*, 59, 381–384.

Bonacich, Phil (1987), "Power and centrality: A family of measures." *American Journal of Sociology*, 92, 1170–1182.

Brunnermeier, Markus K., Gary Gorton, and Arvind Krishnamurthy (2010), "Risk topography." Working paper.

Caballero, Ricardo J. and Alp Simsek (2010), "Fire sales in a model of complexity." Working Paper 15479, National Bureau of Economic Research.

Carvalho, Vasco M. (2010), "Aggregate fluctuations and the network structure of intersectoral trade." Working paper.

Christopeit, Norbert and Hans Joachim Werner (2001), "A necessary and sufficient condition for the convergence of a sum of weighting random variables to a normal distribution." *Econometric Theory*, 17, 857–858.

Chung, Fan and Wenbo Zhao (2008), "Pagerank and random walks on graphs." In *Proceedings of Fete of Combinatorics and Computer Science Conference in honor of Laci Lovász*, Keszthely, Hungry.

Ciccone, Antonio (2002), "Input chains and industrialization." *Review of Economic Studies*, 63, 565–587.

Conley, Timothy G. and Bill Dupor (2003), "A spatial analysis of sectoral complementarity." *Journal of Political Economy*, 111, 311–352.

Dupor, Bill (1999), "Aggregation and irrelevance in multi–sector models." *Journal of Monetary Economics*, 43, 391–409.

Durlauf, Steven N. (1993), "Nonergodic economic growth." *Review of Economic Studies*, 60, 349–366.

Durrett, Rick (2005), *Probability: Theory and Examples*, third edition. Duxbury Press, Belmont, CA.

Gabaix, Xavier (2009), "Power laws in economics and finance." *Annual Review of Economics*, 1, 255–293.

Gabaix, Xavier (2010), "The granular origins of aggregate fluctuations." Forthcoming, *Econometrica*.

Gabaix, Xavier and Rustam Ibragimov (2009), "Rank–1/2: A simple way to improve the OLS estimation of tail exponents." Forthcoming, *Journal of Business and Economic Statistics*.

Grimmett, Geoffrey and David Stirzaker (2001), *Probability and Random Processes*, third edition. Oxford University Press, New York, NY.

Horvath, Michael (1998), "Cyclicality and sectoral linkages: Aggregate fluctuations from sectoral shocks." *Review of Economic Dynamics*, 1, 781–808.

Horvath, Michael (2000), "Sectoral shocks and aggregate fluctuations." *Journal of Monetary Economics*, 45, 69–106.

Hulten, Charles R. (1978), "Growth accounting with intermediate inputs." *The Review of Economic Studies*, 45, 511–518.

Jackson, Matthew O. (2008), *Social and Economic Networks*. Princeton University Press, Princeton, NJ.

Jensen, Henrik Jeldtoft (1998), *Self-Organized Criticality: Emergent Complex Behavior in Physical and Biological System*. Cambridge University Press, New York, NY.

Jones, Charles I. (2009), "Intermediate goods and weak links: A theory of economic development." Working paper.

Jovanovic, Boyan (1987), "Micro shocks and aggregate risk." *The Quarterly Journal of Economics*, 102, 395–409.

Li, Lun, David Alderson, John C. Doyle, and Walter Willinger (2005), "Towards a theory of scale–free graphs: Definition, properties, and implications." *Internet Mathematics*, 2, 431–523.

Linnik, Yuri V. and Iosif V. Ostrovskiĭ (1977), *Decomposition of random variables and vectors*, volume 48 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI. (Translated from Russian).

Long, John B. and Charles I. Plosser (1983), "Real business cycles." *Journal of Political Economy*, 91, 39–69.

Lucas, Robert E. (1977), "Understanding business cycles." *Carnegie-Rochester Conference Series on Public Policy*, 5, 7 – 29.

Mikosch, T. and A. V. Nagaev (1998), "Large deviations of heavy-tailed sums with applications in insurance." *Extremes*, 1, 81–110.

Mulally, Alan R. (2008a), "Examining the state of the domestic automobile industry." Hearing, United States Senate Committee on Banking, Housing, and Urban Affairs.

Mulally, Alan R. (2008b), "Review of industry plans to stabilize the financial condition of the american automobile industry." Hearing, United States House of Representatives Committee on Financial Services.

Nadaraya, E. A. (1964), "On estimating regression." *Theory of Probability and its Applications*, 9, 141–142.

Rotar, Vlademir I. (1975), "An extension of the Lindeberg-Feller theorem." *Mathematical Notes*, 18, 660–663.

Shea, John (2002), "Complementarities and comovements." *Journal of Money, Credit and Banking*, 34, 412–433.

Stanley, Michael H. R., Luís Amaral, Sergey V. Buldyrev, Shlomo Havlin, Heiko Leschhorn, Philipp Maass, Michael A. Salinger, and H. Eugene Stanley (1996), "Scaling behaviour in the growth of companies." *Nature*, 379, 804–806.

Steele, J. Michael (2004), *The Cauchy–Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*. Cambridge University Press, New York, NY.

Streitwieser, Mary L. (2009), "BEA Briefing: A primer on BEA's industry accounts." *Survey of Current Business*, 89, 40–52.

Teicher, Henry (1984), "Exponential bounds for large deviations of sums of unbounded random variables." *Sankhyā: The Indian Journal of Statistics, Series A*, 46, 41–53.

Watson, Geoffrey S. (1964), "Smooth regression analysis." *Sankhyā: The Indian Journal of Statistics, Series A*, 26, 359–372.

Wu, K. and Andy Liu (1995), "Rearrangement inequality." *Mathematics Competitions*, 8, 53–60.