Online Appendix to
Beyond Ricardo:
Assignment Models in International Trade

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Abstract

This addendum provides formal statements and proofs of all theoretical results mentioned in Sections 4.3 and 5.2.

A Section 4.3: Factor complementarity

The production function that we consider is a strict generalization of both the standard neoclassical production function with two factors and the linear production function of our main paper. Output of good σ in country γ is given by

$$Q(\sigma,\gamma) = F\left[K^{agg}(\sigma,\gamma), L^{agg}(\sigma,\gamma)|\sigma,\gamma\right],$$

where $K^{agg}(\sigma,\gamma)$ and $L^{agg}(\sigma,\gamma)$ denote the aggregate amounts of capital and labor, respectively, and $F(\cdot,\cdot|\sigma,\gamma)$ is a constant returns to scale production function. Like in a standard neoclassical model, we assume that capital is a homogeneous input, but like in a baseline R-R model, we allow workers of different type ω to have different productivity in different sectors,

$$L^{agg}(\sigma,\gamma) = \int_{\Omega} A(\omega,\sigma,\gamma_A)L(\omega,\sigma,\gamma)d\omega.$$

For expositional purposes, we further assume identical Cobb-Douglas production functions across sectors:

$$F(K^{agg}, L^{agg}|\sigma,\gamma) = \left(K^{agg}\right)^{\alpha}\left(L^{agg}\right)^{1-\alpha},$$

with $\alpha \in (0,1)$. The baseline R-R model presented in Section 3.1 corresponds to the special case with $\alpha = 0$. Most of our results do not hinge on this assumption, as we argue below.

Finally, in order to set aside issues related to FPE in a standard neoclassical trade model with two factors, we assume that all workers are immobile, but that both goods and capital are freely traded. We let $r > 0$ denote the world price of capital.

A.1 Competitive Equilibrium

To study competitive equilibria in this more general environment, one can separate the firm's cost minimization problem into an inner and an outer problem. In the inner problem, the firm minimizes the unit cost, $c_L(\sigma,\gamma)$, of producing the labor input by choosing $L(\omega,\sigma,\gamma)$ for all ω, exactly as in the baseline R-R model. In equilibrium, firms minimize unit labor costs,

$$c_L(\sigma,\gamma) = \min_{\omega\in\Omega}\{w(\omega,\gamma)/A(\omega,\sigma,\gamma_A)\}, \tag{1}$$

which yields the allocation of workers to sectors,

$$\Omega(\sigma,\gamma) \subset \arg\min_{\omega\in\Omega}\{w(\omega,\gamma)/A(\omega,\sigma,\gamma_A)\}. \tag{2}$$

In the outer problem, the firm minimizes the cost of producing good σ by choosing $K^{agg}(\sigma,\gamma)$ and $L^{agg}(\sigma,\gamma)$, exactly as in a standard neoclassical model with two factors. Conditional on unit labor costs, the cost minimization problem of the firm implies

$$K^{agg}(\sigma,\gamma) = \left(\frac{\alpha}{1-\alpha}\frac{c(\sigma,\gamma)}{r}\right)^{1-\alpha} Q(\sigma,\gamma) \tag{3}$$

and

$$L^{agg}(\sigma,\gamma) = \left(\frac{1-\alpha}{\alpha}\frac{r}{c(\sigma,\gamma)}\right)^{\alpha} Q(\sigma,\gamma), \tag{4}$$

Profit maximization further implies that

$$p(\sigma) \leq \left(\frac{r}{\alpha}\right)^{\alpha}\left(\frac{c(\sigma,\gamma)}{1-\alpha}\right)^{1-\alpha} \text{ with equality if } \Omega(\sigma,\gamma) \neq \emptyset. \tag{5}$$

The labor market clearing condition is

$$\int_{\Sigma} L(\omega,\sigma,\gamma)d\sigma = L(\omega,\gamma_L) \text{ for all } \omega, \gamma, \tag{6}$$

and the good market clearing condition is

$$\int_{\Gamma} D(\sigma,\gamma)d\gamma = \int_{\Gamma} Q(\sigma,\gamma)d\gamma \text{ for all } \sigma. \tag{7}$$

A.2 Cross-Sectional Predictions

Since the cost function associated with the inner problem is unchanged relative to our main paper, Assumption 1 and condition (1) imply PAM within each country for the exact same reasons as in our main paper.

PAM (I) [Complementarity]. *Suppose Assumption 1 holds. Then for any country γ, $\Omega(\sigma,\gamma)$ is increasing in σ.*

Since the previous result does not rely on any restriction on the aggregate production F, it extends to arbitrary neoclassical production functions.

To derive PAM between countries, note that conditions (1) and (5) imply

$$(1-\alpha)(p(\sigma))^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha}} \leq \frac{w(\omega,\gamma)}{A(\omega,\sigma,\gamma_A)} \text{ with equality if } \omega \in \Omega(\sigma,\gamma).$$

Thus, we have

$$w(\omega,\gamma) = \max_{\sigma\in\Sigma}\left\{(1-\alpha)(p(\sigma))^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha}} A(\omega,\sigma,\gamma_A)\right\} \tag{8}$$

and

$$\Sigma(\omega,\gamma) \subset \arg\max_{\sigma\in\Sigma}\left\{(1-\alpha)(p(\sigma))^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{r}\right)^{\frac{\alpha}{1-\alpha}} A(\omega,\sigma,\gamma_A)\right\}. \tag{9}$$

Using equation (9), we then obtain the following result.

PAM (II) [Complementarity]. *Suppose Assumption 1 holds. Then for any factor ω, $\Sigma(\omega,\gamma)$ is increasing in γ_A.*

Consider two countries with the same technology, $\gamma_A = \gamma'_A$. Equation (8) implies that wages are equalized across countries.

FPE Theorem [Complementarity]. *If there are no technological differences between countries, then factor prices are equalized under free trade and free capital mobility, $w(\omega, \gamma) = w(\omega)$ for all γ.*

Like in our main paper, we now restrict attention to economies with a continuum of factors in which the allocation of factors to sectors can be summarized by a matching function, M, such that $\Sigma(\omega, \gamma) = \{M(\omega, \gamma)\}$ is a singleton. For any particular country, PAM implies that the matching function M is increasing in ω and γ_A. Starting from equation (8) and invoking the Envelope Theorem, we get

$$\frac{d \ln w(\omega, \gamma)}{d\omega} = \frac{\partial \ln A(\omega, M(\omega, \gamma), \gamma_S)}{\partial \omega}. \tag{10}$$

Under the assumption of common technology, $\gamma_A = \gamma'_A$, we still have $\Omega(\sigma, \gamma) = \Omega(\sigma, \gamma') \equiv \Omega(\sigma)$. This implies that

$$L^{agg}(\sigma, \gamma) = \int_{\Omega(\sigma)} A(\omega, \sigma, \gamma_A) L(\omega, \gamma_L) d\omega. \tag{11}$$

Combining this expression with conditions (4) and (5), we obtain

$$Q(\sigma, \gamma) = \left(\frac{\alpha p(\sigma)}{r}\right)^{\frac{\alpha}{1-\alpha}} \int_{\Omega(\sigma)} A(\omega, \sigma, \gamma_A) L(\omega, \gamma_L) d\omega. \tag{12}$$

Starting from this expression and noting that $\left(\frac{\alpha}{r} p(\sigma)\right)^{\frac{\alpha}{1-\alpha}}$ does not vary with γ, we can use the same argument as in our main paper. PAM [Complementarity], Property 1, and Property 2 imply the following Rybczynski Theorem.

Rybczynski [Complementarity]. *Suppose Assumptions 1 and 2 hold. Then $Q(\sigma, \gamma)$ is log-supermodular in (σ, γ_L).*

Note that the crucial assumption here is that r does not vary across countries, which implies that $\Omega(\sigma, \gamma) \equiv \Omega(\sigma)$ if all countries have the same technology. The assumption that production functions are Cobb-Douglas does not play any role in our derivation of the Rybczynski Theorem. With more general production functions, the term in front of the integral in equation (12) would be different, but would remain a function of σ only, thereby allowing us to apply the same logic.

A.3 Comparative Static Predictions

Like in Section 3.4.1 of our main paper, we now restrict attention to economies with a continuum of both goods and factors. We first let prices depend on foreign characteristics, $p(\sigma, \phi)$, and focus on comparative statics for one small open economy whose characteristics γ are held fixed. Again, it is convenient to write $w(., \gamma, \phi)$ and $M(., \gamma, \phi)$. By differentiating equation (8) with respect to ϕ, we get

$$\frac{d \ln w(\omega, \gamma, \phi)}{d\phi} = \frac{1}{1-\alpha} \frac{\partial \ln p(M(\omega, \gamma, \phi), \phi)}{\partial \phi}$$

Further differentiating with respect to ω, we obtain

$$\frac{d^2 \ln w(\omega,\gamma,\phi)}{d\phi d\omega} = \frac{\partial^2 \ln p\left(M(\omega,\gamma,\phi),\phi\right)}{\partial\phi\partial\sigma}\frac{dM(\omega,\gamma,\phi)}{d\omega} \geq 0,$$

where the inequality follows from PAM (I) [Complementarity] and Assumption 3 in our main paper. This leads to the following Stolper-Samuelson Theorem.

Stolper-Samuelson [Complementarity]. *Suppose that Assumptions 1 and 3 hold. Then $w(\omega,\gamma,\phi)$ is log-supermodular in (ω,ϕ).*

Now like in Section 3.4.2, consider a closed economy. In this case, we allow the autarkic price of capital $r^a(\gamma)$ to vary endogenously so that the demand for capital is equal to its exogenous supply,

$$\int_\Sigma K^{agg}(\sigma,\gamma)d\sigma = K. \tag{13}$$

We are interested in the effects of labor endowment and taste shocks. Given PAM I [Complementarity], the factor market clearing condition (6) implies

$$\int_{\underline{\sigma}}^{M^a(\omega,\gamma)} L^{agg,a}(\sigma,\gamma)/A((M^a)^{-1}(\sigma,\gamma_A),\sigma,\gamma_A)d\sigma = \int_{\underline{\omega}}^{\omega} L(v,\gamma_L)dv \text{ for all } \omega.$$

From equations (11) and (12), we know that

$$Q^a(\sigma,\gamma) = \left(\frac{\alpha}{r^a(\gamma)}p^a(\sigma)\right)^{\frac{\alpha}{1-\alpha}} L^{agg,a}(\sigma,\gamma).$$

Combining the two previous equations, we therefore get

$$\int_{\underline{\sigma}}^{M^a(\omega,\gamma)} Q^a(\sigma,\gamma)\left(\frac{\alpha}{r^a(\gamma)}p(\sigma)\right)^{\frac{\alpha}{\alpha-1}} /A((M^a)^{-1}(\sigma,\gamma_A),\sigma,\gamma_A)d\sigma = \int_{\underline{\omega}}^{\omega} L(v,\gamma_L)dv \text{for all } \omega.$$

Differentiating this expression and using the goods market clearing condition (7) and equation (8), we obtain

$$\frac{dM^a(\omega,\gamma)}{d\omega} = \left(\frac{\alpha}{1-\alpha}\frac{w(\omega,\gamma)}{r^a(\gamma)}\right)^{\alpha}\frac{A^{1-\alpha}(\omega,M^a(\omega,\gamma),\gamma_A)L(\omega,\gamma_L)}{D(p^a,I^a(\gamma)|M^a(\omega,\gamma),\gamma^D)}.$$

In the case of a CES economy, normalizing the CES price index to one and using equation (8) again to express good prices as a function of wages, the previous expression simplifies into

$$\frac{dM^a(\omega,\gamma)}{d\omega} = \frac{\left(A(\omega,M^a(\omega,\gamma),\gamma_A)\right)^{(1-\alpha)(1-\varepsilon)}\left(w(\omega,\gamma)\right)^{\alpha(1-\varepsilon)+\varepsilon}L(\omega,\gamma_L)}{B(M^a(\omega,\gamma),\gamma_D)Z^a(\gamma)}, \tag{14}$$

where $Z^a(\gamma) \equiv I^a(\gamma)(1-\alpha)^{\alpha(1-\varepsilon)+\varepsilon}\left(r^a(\gamma)/\alpha\right)^{\alpha(1-\varepsilon)}$. For a given value of $r^a(\gamma)$, equations (10) and (14) still form a system of two differential equations in (M^a,w^a). Under PAM [Complementarity], the boundary conditions are given by $M^a(\underline{\omega},\gamma) = \underline{\sigma}$ and $M^a(\bar{\omega},\gamma) = \bar{\sigma}$. The only difference compared to our main paper is that $r^a(\gamma)$ is now jointly determined so that the capital market clearing condition

(13) holds. Nevertheless, since $Z^a(\gamma)$ enters equation (14) in the same way as $I(\gamma)$, we can still use the same argument as in Costinot & Vogel (2010) to show the two following comparative static results.

Comparative Statics (I): Factor Allocation [Complementarity]. *Suppose that Assumptions 1, 2, and 4 hold in a CES economy under autarky. Then, $M^a(\omega, \gamma)$ is decreasing in γ_D and γ_L.*

Comparative Statics (II): Factor Prices [Complementarity]. *Suppose that Assumptions 1, 2, and 4 hold in a CES economy under autarky. Then, $w^a(\omega, \gamma)$ is log-submodular in (ω, γ_L) and (ω, γ_D).*

B Section 4.3: Heterogeneous preferences

Starting with Roy (1951), labor markets have been one of the most common applications of the model presented in Section 3. This application raises a number of interesting issues. As noted by Rosen (1987), most labor market transactions involve a tied sale in which workers sell the services of their labor and buy the attributes of their jobs. Thus, in general, the assignment of workers to jobs depends both on their productivity in different economic activities and on their preferences over different working conditions. In our baseline model, we have emphasized the first source of heterogeneity. We now turn to the second.

We focus on the polar case in which individuals are endowed with one unit of homogeneous labor, $A(\omega, \sigma, \gamma_A) = 1$, for all ω, σ, and γ_A, but workers with different characteristics ω in country γ derive different utility from being employed in sector σ,

$$V(\omega, \sigma, \gamma_U) = w^c(\sigma, \gamma) U(\omega, \sigma, \gamma_U).$$

One should think of $w^c(\sigma, \gamma)$ as the difference between the country-specific wage paid to labor services and the country-and-sector specific fee received by firms from country γ in sector σ, that is the compensating differential. Under the assumption that preferences over goods are homothetic, $w^c(\sigma, \gamma)$ is also proportional to the consumption index derived from consuming goods all goods in the set Σ.

B.1 Competitive Equilibrium

In this environment, the equilibrium conditions are given by the sectoral choice of workers,

$$\Sigma(\omega, \gamma) \subset \arg\max_{\sigma \in \Sigma} \{w^c(\sigma, \gamma) U(\omega, \sigma, \gamma_U)\}, \tag{15}$$

the profit maximization condition,

$$p(\sigma) \leq w^c(\sigma, \gamma) \text{ with equality if } \Omega(\sigma, \gamma) \neq \emptyset, \tag{16}$$

the factor market clearing condition,

$$\int_{\Sigma} L(\omega, \sigma, \gamma) d\sigma = L(\omega, \gamma_L) \text{ for all } \omega, \gamma, \tag{17}$$

and the goods market clearing condition,

$$\int_\Gamma D(\sigma,\gamma)d\gamma = \int_\Gamma Q(\sigma,\gamma)d\gamma \text{ for all } \sigma. \tag{18}$$

B.2 Cross-Sectional Predictions

In line with our previous analysis, we impose the following restriction on preferences.

Assumption 1 [Preferences]. *$U(\omega,\sigma,\gamma_U)$ is strictly log-supermodular in (ω,σ) and (σ,γ_U).*

Property 3 in our main paper and condition (15) therefore imply PAM.

PAM [Preferences]. *Suppose that Assumption 1 [Preferences] holds. Then for any country γ, $\Sigma(\omega,\gamma)$ is increasing in ω.*

By condition (16), we must also have

$$V(\omega,\gamma) = \max_\sigma\{p(\sigma)U(\omega,\sigma,\gamma_U)\}, \tag{19}$$

which implies the following version of the FPE Theorem for indirect utility functions.

FPE Theorem [Preferences]. *If there are no differences in preferences between countries, then indirect utility levels are equalized under free trade, $V(\omega,\gamma) = V(\omega)$ for all γ.*

Like in our main paper, we now restrict attention to economies with a continuum of factors in which the allocation of factors to sectors can be summarized by a matching function, M, such that $\Sigma(\omega,\gamma) = \{M(\omega,\gamma)\}$ is a singleton. For any particular country, PAM then implies that the matching function M is increasing in ω. Let $V(\omega,\gamma) \equiv \max_\sigma\{w^c(\sigma,\gamma)U(\omega,\sigma,\gamma_U)\}$ denote the indirect utility function. By the Envelope Theorem, we have

$$\frac{d\ln V(\omega,\gamma)}{d\omega} = \frac{\partial \ln U(\omega, M(\omega,\gamma),\gamma_U)}{\partial\omega}. \tag{20}$$

To get predictions with respect to output, we maintain the assumption of common preferences, $\gamma_U = \gamma'_U$, which implies that $\Omega(\sigma,\gamma) = \Omega(\sigma,\gamma') \equiv \Omega(\sigma)$, as in our main paper. Thus, aggregate output can be expressed as

$$Q(\sigma,\gamma) = \int_{\Omega(\sigma)} L(\omega,\gamma_L)d\omega.$$

Starting from this expression, we can use the exact same argument as in our main paper. PAM [Preferences], Property 1, and Property 2 imply the following Rybczynski Theorem.

Rybczynski [Preferences]. *Suppose that Assumptions 1 [Preferences] and 2 hold. Then $Q(\sigma,\gamma)$ is log-supermodular in (σ,γ_L).*

B.3 Comparative Static Predictions

Like in Section 3.4.1 of our main paper, we now restrict attention to economies with a continuum of both goods and factors. We first let prices depend on foreign characteristics, $p(\sigma,\phi)$, and focus on comparative statics for one small open economy whose characteristics γ are held fixed. It is convenient to write $w^c(., \gamma, \phi)$, $V(., \gamma, \phi)$ and $M(., \gamma, \phi)$. Differentiating equation (19) with respect to ϕ and invoking the Envelope Theorem, we get

$$\frac{d \ln V(\omega, \gamma, \phi)}{d\phi} = \frac{\partial \ln p(M(\omega, \gamma, \phi), \phi)}{\partial \phi}.$$

Further differentiating with respect to ω, we obtain

$$\frac{d^2 \ln V(\omega, \gamma, \phi)}{d\omega d\phi} = \frac{\partial^2 \ln p\left(M(\omega, \gamma, \phi), \phi\right)}{\partial \phi \partial \sigma} \frac{dM(\omega, \gamma, \phi)}{d\omega} \geq 0,$$

where the inequality follows from PAM [Preferences] and Assumption 3 in our main paper. This leads to the following Stolper-Samuelson Theorem.

Stolper-Samuelson [Preferences]. *Suppose that Assumptions 1 [Preferences] and 3 hold. Then $V(\omega, \gamma, \phi)$ is log-supermodular in (ω, ϕ).*

Now like in Section 3.4.2, consider a closed economy. We are interested in the effects of endowment and taste shocks. Given PAM [Preferences], the factor market clearing condition (18) implies

$$\int_{\underline{\sigma}}^{M^a(\omega,\gamma)} Q^a(\sigma, \gamma) d\sigma = \int_{\underline{\omega}}^{\omega} L(v, \gamma_L) dv \text{ for all } \omega.$$

Differentiating this expression and using the good market clearing, we obtain

$$\frac{dM^a(\omega, \gamma)}{d\omega} = \frac{L(\omega, \gamma_L)}{D(p^a, I^a(\gamma) | M^a(\omega, \gamma), \gamma^D)}.$$

Noting that $p^a(M^a(\omega, \gamma)) = w^c(M^a(\omega, \gamma), \gamma) = V^a(\omega, \gamma)/U(\omega, M^a(\omega, \gamma), \gamma_U)$, we can rearrange this expression in the case of a CES economy as

$$\frac{dM^a(\omega, \gamma)}{d\omega} = \frac{\left(U(\omega, M^a(\omega, \gamma), \gamma_U)\right)^{-\varepsilon} \left(V^a(\omega, \gamma)\right)^{\varepsilon} L(\omega, \gamma_L)}{B(M^a(\omega, \gamma), \gamma_D) \int_{\Omega} \left(U(\omega'^a(\omega', \gamma), \gamma_U)\right)^{-\varepsilon} \left(V^a(\omega', \gamma)\right)^{\varepsilon} L(\omega', \gamma_L) d\omega'}. \tag{21}$$

where the CES price index is normalized to one.

Equations (20) and (21) form a system of two differential equations in (M^a, V^a). Under PAM [Preferences], the boundary conditions are given by $M^a(\underline{\omega}, \gamma) = \underline{\sigma}$ and $M^a(\bar{\omega}, \gamma) = \bar{\sigma}$. Using this system of differential equations, we can then use the exact same argument as in Costinot & Vogel (2010) to show the two following comparative static results about factor allocation and factor prices.

Comparative Statics (I): Factor Allocation [Preferences]. *Suppose that Assumptions 1 [Preferences], 2, and 4 hold in a CES economy under autarky. Then $M^a(\omega, \gamma)$ is decreasing in γ_D and γ_L.*

Comparative Statics (II): Factor Prices [Preferences]. *Suppose that Assumptions 1 [Preferences], 2, and 4 hold in a CES economy under autarky. Then $V^a(\omega, \gamma)$ is log-submodular in (ω, γ_L) and (ω, γ_D).*

C Section 4.3: Endogenous Skills

In the baseline R-R model, factors of production are characterized by their exogenous productivity in various economic activities. In this environment, workers that are better at a job are those that can readily produce more. In practice, workers may also differ in terms of how costly it is for them to acquire the skills required for production to take place. Here we follow Blanchard & Willmann (2013) and incorporate this type of heterogeneity. Heterogeneity in learning costs is also central to a number of other assignment models such as the knowledge-based hierarchical model of Garicano (2000).

To isolate the role of heterogeneity in learning costs, we adopt the same approach as in the previous subsection and assume that upon learning, all workers are homogeneous: $A(\omega, \sigma, \gamma_A) = 1$ for all ω, σ, and γ_A. A firm located in country γ, however, now needs to pay $S(\omega, \sigma, \gamma_S) > 0$ in order to train a worker of type ω in sector σ. Learning costs are proportional to the consumer price index, which we normalize to one.

C.1 Competitive Equilibrium

Cost functions are now given by

$$c(\sigma, \gamma) = \min_{L(\omega,\sigma,\gamma)\geq 0} \left\{ \int_\Omega (w(\omega, \gamma) + S(\omega, \sigma, \gamma_S))\, L(\omega, \sigma, \gamma)\, d\omega \,\middle|\, \int_\Omega L(\omega, \sigma, \gamma) \geq 1 \right\}.$$

While the cost function is no longer the same as in Section 3.2 of our main paper, the linearity of the production function immediately implies

$$c(\sigma, \gamma) \equiv \min_{\omega} \{w(\omega, \gamma) + S(\omega, \sigma, \gamma_S)\}.$$

Hence, in this environment, the equilibrium conditions are given by the sectoral choice of workers,

$$\Omega(\sigma, \gamma) \subset \arg\min_{\omega\in\Omega} \{w(\omega, \gamma) + S(\omega, \sigma, \gamma_S)\}, \tag{22}$$

the profit maximization condition,

$$p(\sigma) \leq c(\sigma, \gamma) \equiv \min_{\omega\in\Omega} \{w(\omega, \gamma) + S(\omega, \sigma, \gamma_S)\} \text{ with equality if } \omega \in \Omega(\sigma, \gamma), \tag{23}$$

the factor market clearing condition,

$$\int_\Sigma L(\omega, \sigma, \gamma) d\sigma = L(\omega, \gamma_L) \text{ for all } (\omega, \gamma), \tag{24}$$

and the goods market clearing condition,

$$\int_\Gamma D(\sigma, \gamma) d\gamma = \int_\Gamma Q(\sigma, \gamma) d\gamma \text{ for all } \sigma. \tag{25}$$

For future reference, note that $D(\sigma,\gamma)$ now refers to absorption of good σ, which includes both demand by consumers and firms.

C.2 Cross-Sectional Predictions

In the baseline R-R model, where unit cost is multiplicative in wages and productivity, a sufficient condition to obtain PAM is Assumption 1. In the present model, where the cost function is additive in wages and training costs, we impose the following assumption.

Assumption 1 [Skills]. *$S(\omega,\sigma,\gamma_S)$ is strictly submodular in (ω,σ) and (σ,γ_S).*

Unit costs here take the same functional form as the logarithm of unit costs in an R-R model, where the wage level here replaces the log of the wage and the training cost $S(\omega,\sigma,\gamma_S)$ replaces $-\log A(\omega,\sigma,\gamma_A)$ in the baseline model. Hence, the log-supermodularity of $A(\omega,\sigma,\gamma_A)$—which is equivalent to the submodularity of $-\log A(\omega,\sigma,\gamma_A)$—in an R-R model has equivalent implications as the submodularity of $S(\omega,\sigma,\gamma_S)$ in the present model; they both imply PAM. Specifically, the Monotonicity Theorem of Milgrom & Shannon (1994) and condition (22) yield the following result.[1]

PAM (I) [Skills]. *Suppose that Assumption 1 [Skills] holds. Then for any country γ, $\Omega(\sigma,\gamma)$ is increasing in σ.*

By condition (23), we also have

$$p(\sigma) - S(\omega,\sigma,\gamma_S) \leq w(\omega,\gamma) \text{ for all } \sigma \in \Sigma, \tag{26}$$

with equality if $\omega \in \Omega(\sigma,\gamma)$. Since $\sigma \in \Sigma(\omega,\gamma)$ if and only if $\omega \in \Omega(\sigma,\gamma)$, this further implies that

$$\Sigma(\omega,\gamma) \subset \arg\max_{\sigma\in\Sigma}\{p(\sigma) - S(\omega,\sigma,\gamma_S)\}. \tag{27}$$

Equation (27) and Assumption 1 [Skills] directly imply the following prediction.

PAM (II) [Skills]. *Suppose that Assumption 1 [Skills] holds. Then for any factor ω, $\Sigma(\omega,\gamma)$ is increasing in γ_S.*

Factor market clearing requires that $\omega \in \Omega(\sigma,\gamma)$ for some $\sigma \in \Sigma$. Thus, condition (26) implies that

$$w(\omega,\gamma) = \max_{\sigma\in\Sigma}\{p(\sigma) - S(\omega,\sigma,\gamma_S)\}, \tag{28}$$

which yields the following FPE Theorem.

[1] The previous discussion also sheds light on the relationship between R-R models and one-to-one matching models à la Becker (1973). In the latter class of models, it is assumed that if a firm of type σ matches with a worker of type ω, it produces $A(\omega,\sigma)$ units of a homogeneous final good. Hence, the profit-maximization problem of firm σ is $\max_\omega\{A(\omega,\sigma) - w(\omega)\} = -\min_\omega\{w(\omega) - A(\omega,\sigma)\}$. Omitting the country characteristic γ, this is identical to the cost-minimization problem above for $A(\omega,\sigma) \equiv -S(\omega,\sigma)$. Thus, the previous discussion explains why log-supermodularity is sufficient for PAM in an R-R model, whereas supermodularity is sufficient for PAM in a matching model à la Becker (1973). In both cases, PAM derives from the Monotonicity Theorem of Milgrom & Shannon (1994), with log-supermodularity and supermodularity the critical sufficient conditions for the single-crossing property to hold in each environment.

FPE Theorem [Skills]. *If there are no differences in training costs between countries, then factor prices are equalized under free trade, $w(\omega, \gamma) = w(\omega)$ for all γ.*

Like in our main paper, we now restrict attention to economies with a continuum of factors in which the allocation of factors to sectors can be summarized by a matching function, M, such that $\Sigma(\omega, \gamma) = \{M(\omega, \gamma)\}$ is a singleton. For any particular country, PAM implies that the matching function M is increasing in ω and γ_S. By the Envelope Theorem, we therefore have

$$\frac{dw(\omega, \gamma)}{d\omega} = -\frac{\partial S(\omega, M(\omega, \gamma), \gamma_S)}{\partial \omega}. \tag{29}$$

Under the assumption of common training costs, $\gamma_S = \gamma'_S$, we still have $\Omega(\sigma, \gamma) = \Omega(\sigma, \gamma') \equiv \Omega(\sigma)$. Thus aggregate output is given by

$$Q(\sigma, \gamma) = \int_{\Omega(\sigma)} L(\omega, \gamma_L) d\omega.$$

Starting from this expression, we can use the exact same argument as in our main paper. PAM [Skills], Property 1, and Property 2 imply the following Rybczynski Theorem.

Rybczynski [Skills]. *Suppose Assumptions 1 [Skills] and 2 [Skills] hold. Then $Q(\sigma, \gamma)$ is log-supermodular in (σ, γ_L).*

C.3 Comparative Static Predictions

We now derive variants of the comparative static predictions in our main paper.

Like in Section 3.4.1 of our main paper, we now restrict attention to economies with a continuum of both goods and factors. We first let prices depend on foreign characteristics, $p(\sigma, \phi)$, and focus on comparative statics for one small open economy whose characteristics γ are held fixed. Again, it is convenient to write $w(., \gamma, \phi)$ and $M(., \gamma, \phi)$. Because the price level here replaces the log of the price in our baseline model, we impose the following variant of Assumption 3.

Assumption 3 [Skills]. *$p(\sigma, \phi)$ is supermodular in (σ, ϕ).*

Differentiating equation (28) with respect to ϕ and invoking the Envelope Theorem, we get

$$\frac{dw(\omega, \gamma, \phi)}{d\phi} = \frac{\partial p\left(M(\omega, \gamma, \phi), \phi\right)}{\partial \phi}.$$

Further differentiating with respect to ω, we obtain

$$\frac{d^2 w(\omega, \gamma, \phi)}{d\phi d\omega} = \frac{\partial^2 p\left(M(\omega, \gamma, \phi), \phi\right)}{\partial \phi \partial \sigma} \frac{dM(\omega, \gamma, \phi)}{d\omega} \geq 0,$$

where the inequality follows from PAM (I) [Skills] and Assumption 3 [Skills]. This leads to the following Stolper-Samuelson Theorem.

Stolper-Samuelson [Skills]. *Suppose that Assumptions 1 [Skills] and 3 [Skills] hold. Then, $w(\omega, \gamma, \phi)$ is supermodular in (ω, ϕ).*

Now like in Section 3.4.2, consider a closed economy. We are interested in the effects of endowment and taste shocks. Given PAM I [Skills], the factor market clearing condition in (25) implies that the quantity supplied satisfies

$$\int_{\underline{\sigma}}^{M^a(\omega,\gamma)} Q^a(\sigma,\gamma)d\sigma = \int_{\underline{\omega}}^{\omega} L(v,\gamma)dv \text{ for all } \omega.$$

Differentiating this expression and using the goods market clearing condition (25), we obtain

$$\frac{dM^a(\omega,\gamma)}{d\omega} = \frac{L(\omega,\gamma_L)}{D(p^a, E^a(\gamma)|M^a(\omega,\gamma),\gamma^D)}, \tag{30}$$

where $E^a(\gamma)$ denotes total expenditure, which equals total consumer income plus total firm training costs,

$$E^a(\gamma) = \int_\Omega w^a(\omega,\gamma)L(\omega,\gamma_L)d\omega + \int_\Sigma \int_\Omega S(\omega,\sigma,\gamma_S)L^a(\omega,\sigma,\gamma_S)d\omega d\sigma \text{ for all } \gamma.$$

Equations (29) and (30) form a system of two differential equations in (M^a, w^a). Under PAM [Skills], the boundary conditions are given by $M^a(\underline{\omega},\gamma) = \underline{\sigma}$ and $M^a(\bar{\omega},\gamma) = \bar{\sigma}$. If preferences are Leontief, equation (30) is independent of prices. It is the same equation that would prevail in our baseline model under Leontief preferences, with $E^a(\gamma)$ substituting for $I^a(\gamma)$. Thus, the exact same argument as in Costinot & Vogel (2010) implies the following comparative static result about factor allocation.

Comparative Statics (I): Factor Allocation [Skills]. *Suppose that Assumptions 1 [Skills], 2, and 4 hold in a Leontief economy under autarky. Then $M^a(\omega,\gamma)$ is decreasing in γ_D and γ_L.*

Combining the previous result with equation (29) and Assumption 1 [Skills], we then obtain the following comparative static result about factor prices.

Comparative Statics (II): Factor Prices [Skills]. *Suppose that Assumptions 1 [Skills], 2, and 4 hold in a Leontief economy under autarky. Then, $w^a(\omega,\gamma)$ is submodular in (ω,γ_L) and (ω,γ_D).*

Given the mapping between log prices in the baseline model and prices here, similar results can be established in an environment with logit demand functions, where log demand depends on prices rather than log prices, as in a CES economy. See Khandelwal (2010) and Fajgelbaum, Grossman & Helpman (2011) for examples of such preferences in international trade.

D Section 5.2: Unobserved Productivity Shocks across Factors

We impose the same assumptions as in Section 5.2 of our main paper.

D.1 Competitive Equilibrium

A competitive equilibrium is characterized by the distribution of factors to sectors,

$$\pi(\sigma|\omega,\gamma) = \frac{(p(\sigma)T(\omega,\sigma,\gamma_A))^{\theta(\omega,\gamma_A)}}{\sum_{\sigma'}(p(\sigma')T(\omega,\sigma',\gamma_A))^{\theta(\omega,\gamma_A)}}, \tag{31}$$

the average returns of factors,

$$\bar{w}(\omega,\gamma) = \chi(\omega,\gamma_A)\left[\sum_{\sigma}\left(p(\sigma)T(\omega,\sigma,\gamma_A)\right)^{\theta(\omega,\gamma_A)}\right]^{1/\theta(\omega,\gamma_A)}, \tag{32}$$

with $\chi(\omega,\gamma_A) \equiv \Gamma\left(\frac{\theta(\omega,\gamma_A)-1}{\theta(\omega,\gamma_A)}\right)$, and the good market clearing condition,

$$\sum_{\gamma} D(p, I(\gamma)|\sigma,\gamma_D) = \sum_{\gamma'} Q(\sigma,\gamma') \text{ for all } \sigma, \tag{33}$$

where aggregate output satisfies

$$Q(\sigma,\gamma) = \sum_{\omega} \frac{1}{p(\sigma)}\bar{w}(\omega,\gamma)\pi(\sigma|\omega,\gamma)L(\omega,\gamma_L), \tag{34}$$

by zero profits. For future reference, note that equations (31), (32), and (34) imply

$$Q(\sigma,\gamma) = \sum_{\omega} \chi(\omega,\gamma_A)T(\omega,\sigma,\gamma_A)L(\omega,\gamma_L)\left(\pi(\sigma|\omega,\gamma)\right)^{\frac{\theta(\omega,\gamma_A)-1}{\theta(\omega,\gamma_A)}}. \tag{35}$$

D.2 Cross-Sectional Predictions

By Assumption 1 [Fréchet], $T(\omega,\sigma,\gamma_A)$ is strictly log-supermodular in (ω,σ) and (σ,γ_A). As noted in our main paper, combining this assumption with equation (31), we obtain the following weaker forms of PAM.

PAM (I) [Fréchet]. *Suppose that Assumption 1 [Fréchet] holds and that $\theta(\omega,\gamma_A) \equiv \theta(\gamma_A)$ for all ω. Then for any country γ, $\pi(\sigma|\omega,\gamma)$ is log-supermodular in (ω,σ).*

PAM (II) [Fréchet]. *Suppose that Assumption 1 [Fréchet] holds and that $\theta(\omega,\gamma_A) \equiv \theta(\omega)$ for all γ. Then for any factor ω, $\pi(\sigma|\omega,\gamma)$ is log-supermodular in (σ,γ_A).*

Under the assumption $\theta(\omega,\gamma_A) \equiv \theta$ for all ω, the two previous results simultaneously hold, as discussed in our main paper. Equation (32) also implies the following FPE Theorem.

FPE Theorem [Fréchet]. *If there are no technological differences between countries, then average factor prices are equalized under free trade, $\bar{w}(\omega,\gamma) = \bar{w}(\omega)$ for all γ.*

Finally, under the assumption that $\theta(\omega,\gamma_A) \equiv \theta(\gamma_A)$ *for all* ω, Assumption 1 [Fréchet], Assumption 2, PAM (I) [Fréchet], and Property 1 imply that $\chi(\omega,\gamma_A)T(\omega,\sigma,\gamma_A)L(\omega,\gamma_L)\left(\pi(\sigma|\omega,\gamma)\right)^{\frac{\theta(\omega,\gamma_A)-1}{\theta(\omega,\gamma_A)}}$ is log-supermodular in (ω,σ,γ_L). By Property 2, equation (35) therefore implies the following Rybczynski Theorem.

Rybczynski [Fréchet]. *Suppose Assumptions 1 [Fréchet] and 2 hold with $\theta(\omega,\gamma_A) \equiv \theta(\gamma_A)$ for all ω. Then $Q(\sigma,\gamma)$ is log-supermodular in (σ,γ_L).*

D.3 Comparative Statics Predictions

Like in Section 3.4.1 of our main paper, we let prices depend on foreign characteristics, $p(\sigma,\phi)$, and focus on comparative statics for one small open economy whose characteristics γ are held fixed. Here, it is convenient to write $\pi(\sigma|\omega,\gamma,\phi)$ and $\bar{w}(\omega,\gamma,\phi)$. Throughout this subsection, we assume that $\theta(\omega,\gamma_A)\equiv\theta(\gamma_A)$ for all ω.

Under the previous assumption, equation (32) implies

$$\bar{w}(\omega,\gamma,\phi)=\chi(\omega,\gamma_A)\left[\sum_{\sigma}\left(p(\sigma,\phi)T(\omega,\sigma,\gamma_A)\right)^{\theta(\gamma_A)}\right]^{1/\theta(\gamma_A)}.$$

By Property 1, if Assumption 1 [Fréchet] and Assumption 3 hold, then $\left(p(\sigma,\phi)T(\omega,\sigma,\gamma_A)\right)^{\theta(\gamma_A)}$ is log-supermodular in (ω,σ,ϕ). Thus by Property 2, $\left(\bar{w}(\omega,\gamma,\phi)\right)^{\theta(\gamma_A)}$ and hence $\bar{w}(\omega,\gamma,\phi)$ must be log-supermodular in (ω,ϕ). This leads to the following Stolper-Samuelson Theorem.

Stolper-Samuelson [Fréchet]. *Suppose that Assumptions 1 [Fréchet] and 3 hold and that $\theta(\omega,\gamma_A)\equiv\theta(\gamma_A)$ for all ω. Then $\bar{w}(\omega,\gamma,\phi)$ is log-supermodular.*

E Section 5.2: Unobserved Preference Shocks

In Section 5.2 we focused on applications featuring heterogeneous factors of production, like in our baseline R-R model. Building on the analysis of Section B in this Online Appendix, we conclude by discussing applications featuring preference heterogeneity.

The economic environment combines elements from Section B of this Online Appendix and Section 5.2 of our main paper. There is a discrete number of sectors and countries, whose characteristics are perfectly observed, $\sigma^*=\sigma$ and $\gamma^*=\gamma$, and a continuum of workers, whose characteristics are not, $\omega^*\neq\omega$. Workers are productively homogeneous, but differ in their preferences over working conditions. A worker of type ω^* from country γ employed at a wage $w^c(\sigma,\gamma)$ in a sector σ derives utility

$$V(\omega^*,\sigma,\gamma)=U(\omega^*,\sigma,\gamma_U)\,w^c(\sigma,\gamma) \tag{36}$$

Given observables (ω,σ,γ) preference shocks are independently drawn from a Fréchet distribution:

$$\Pr\left\{U(\omega^*,\sigma,\gamma_U)\le a|\omega\right\}=\exp\left(-\left[a/T(\omega,\sigma,\gamma_U)\right]^{-\theta(\omega,\gamma_U)}\right)$$

A competitive equilibrium is characterized by the same system of equations as in Section B of this Online Appendix, whereas the probability that employment in sector σ offers the maximum utility for an individual with observable characteristic ω in country γ is given by

$$\pi(\sigma|\omega,\gamma)=\frac{\left[p(\sigma)\,T(\omega,\sigma,\gamma_U)\right]^{\theta(\omega,\gamma_U)}}{\sum_{\sigma'}\left[p(\sigma')\,T(\omega,\sigma',\gamma_U)\right]^{\theta(\omega,\gamma_U)}}$$

as in Section 5.2 of our main paper. Again, if the Fréchet version of Assumption 1 holds—now expressed in terms of γ_U—and $\theta(\omega, \gamma_U) \equiv \theta$, we get the same version of PAM as in Sections 5.1 and 5.2 of our main paper. The Stolper-Samuelson result presented in Section B of our Online Appendix extends in a similar manner.

A final remark is in order. In this section and in Section 5.2 of our main paper, we have assumed multiplicative preference and productivity shocks. This particular assumption creates a tight connection between R-R models with productivity and preference heterogeneity. Structural empirical work, however, often assumes additive preference shocks. Using our notation, this corresponds to a situation in which utility satisfies

$$V(\omega^*, \sigma, \gamma) = U(\omega^*, \sigma, \gamma_U) + w^c(\sigma, \gamma) \tag{37}$$

It is equally easy to incorporate this alternative assumption in R-R models. Specifically, if one assumes that additive preference shocks are independently drawn from a Gumbel distribution, then one can show that cross-sectional and comparative static results extend to this environment under the same qualifications as in Section C of this Online Appendix, i.e. one should substitute "wages" and "prices" for "log wages" and "log prices." To see this, take the log of equation (36), compare it to equation (37), and note that if a random variable is distributed Fréchet, its logarithm is distributed Gumbel.

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