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"Topologies on types": Correction

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We show by an example that Proposition 2 in "Topologies on types" by Dekel, Fudenberg, and Morris [Theoretical Economics 1 (2006), 275–309] is not true.

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In a recent paper, Dekel et al. (2006) (hereafter, DFM) propose the strategic topology, which is defined to be just strong enough to guarantee that the correspondence mapping types into ε-interim-correlated-rationalizable actions is continuous. That is, two types are close under the strategic topology if and only if they have similar ε-interim-correlated-rationalizable actions in every finite game. They show that the strategic topology is still weak enough that finite types are dense in the universal type space.

In contrast to the strategic topology, DFM consider also the *uniform* strategic topology, which requires the degree of similarity of strategic behavior to be uniform over all finite games. DFM use their Proposition 2 to argue that finite types are not dense under the uniform strategic topology. In this note, we present a counterexample to show that the direction of Proposition 2 that DFM use in their non-denseness argument is not correct.[1] We also fill a gap in their proof of the other direction of Proposition 2.

In order to make our discussion self-contained, we briefly define the following notation. For any topological space Y, let $\Delta(Y)$ be the space of Borel probability measures on Y endowed with the standard weak* topology. Let $Y^0 = \Theta$ be the finite set of basic uncertainty endowed with the discrete topology. For every $k \geq 1$, let $Y^k = Y^{k-1} \times \Delta(Y^{k-1})$. Let (T^*, π^*) be the resulting Mertens–Zamir universal type space, where $T^* \subset \times_{k=0}^{\infty} \Delta(Y^k)$ and π^* is the homeomorphism between T^* (endowed with the product topology) and $\Delta(\Theta \times T^*)$. For $i = 1, 2$, let $T_i^* = T^*$ and $\pi_i^* = \pi^*$. For any $y \in Y$, let δ_y denote the Dirac measure on y.

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[1]In Chen and Xiong (2008), we nonetheless confirm their conclusion by explicitly constructing a type that is not the limit of any sequence of finite types under the uniform strategic topology.

Let $G = (A_i, g_i)_{i=1,2}$ be a finite game, where A_i is a finite set of actions and $g_i : A_1 \times A_2 \times \Theta \to [-1,1]$ is the payoff function for player i. For any $\varepsilon \geq 0$, DFM define the ε-interim-correlated-rationalizable set $R(G,\varepsilon)$ to be the largest (with respect to set inclusion) set in $((2^{A_i})^{T_i^*})_{i=1,2}$ with the best reply property that for any $i = 1,2$, $j = 3-i$, and $a_i \in R_i(t_i, G, \varepsilon)$, there exists $\nu \in \Delta(A_j \times \Theta \times T_j^*)$ such that

$$\nu[\{(a_j,\theta,t_j) : a_j \in R_j(t_j,G,\varepsilon)\}] = 1$$

$$\operatorname*{marg}_{\Theta\times T_j^*} \nu = \pi_i^*[t_i]$$

$$\int_{(a_j,\theta,t_j)} [g_i(a_i,a_j,\theta) - g_i(a_i',a_j,\theta)]\, d\nu \geq -\varepsilon \text{ for all } a_i' \in A_i.$$

For each $t_i \in T_i^*$, define $h_i(t_i|a_i,G) = \min\{\varepsilon : a_i \in R_i(t_i,G,\varepsilon)\}$.

The purpose of DFM's Proposition 2 is to establish the equivalence between the two metrics d^{US} and d^{**} on T_i^*, which are defined as follows. For $t_i, t_i' \in T_i^*$,

$$d^{US}(t_i,t_i') \equiv \sup_{a_i\in A_i(G),G} \left| h_i(t_i|a_i,G) - h_i(t_i'|a_i,G) \right|$$

$$d^{**}(t_i,t_i') \equiv \sup_k \sup_{f\in F_k} \left| E(f|\pi^*[t_i]) - E(f|\pi^*[t_i']) \right|,$$

where F_k is the collection of bounded real-valued functions on $\Theta \times T^*$ that are measurable with respect to k^{th}-order beliefs. In particular, they aim to show d^{US} convergence implies d^{**} convergence, so that an argument in Morris (2002) can be invoked to show that finite types are not dense under d^{US}.

First, we present an example showing that $d^{US}(t^n,t) \to 0$ does not necessarily imply $d^{**}(t^n,t) \to 0$. Let $\Theta = \{0,1\}$. Consider a hierarchy $t = (\mu_1,\mu_2,\mu_3 \ldots)$, where it is common 1-belief that $\theta = 0$. Let $t^n = (\mu_1^n,\mu_2^n,\mu_3^n \ldots)$ be a hierarchy under which both players believe $\theta = 0$ with probability $1-1/n$ and it is common 1-belief that both players believe $\theta = 0$ with probability $1-1/n$. Hence, $\pi^*[t] = \delta_{(0,t)}$ and $\pi^*[t^n] = (1-1/n)\delta_{(0,t^n)} + (1/n)\delta_{(1,t^n)}$ (cf. Mertens and Zamir 1985). Now consider the measurable function $f : \Delta(\Theta) \to [0,1]$ such that $f(\mu_1) = 1$ if $\mu_1 = \delta_{\{\theta=0\}}$ and $f(\mu_1) = 0$ otherwise. Observe that f can be identified with a bounded function $f^* : \Theta \times T^* \to [0,1]$ by defining $f^*(\theta,\tilde{\mu}_1,\tilde{\mu}_2,\tilde{\mu}_3,\ldots) = f(\tilde{\mu}_1)$ for every $(\theta,\tilde{\mu}_1,\tilde{\mu}_2,\tilde{\mu}_3,\ldots)$ in $\Theta \times T^*$. Hence, the value of f^* depends only on $\Delta(\Theta)$ and f^* is measurable with respect to $\Delta(\Theta)$, i.e., $f^* \in F_1$. Observe that $E(f^*|\pi^*[t]) = 1$ and $E(f^*|\pi^*[t^n]) = 0$ for every n. Therefore, $\left|E(f^*|\pi^*[t]) - E(f^*|\pi^*[t^n])\right| = 1$ and hence $d^{**}(t^n,t) \geq 1$ for every n. However, it is straightforward to verify that the Prohorov metric between the k^{th}-order beliefs of t^n and t equals $1/n$ for every n and $k \geq 1$, which can be used to show that $d^{US}(t^n,t) \to 0$. (A detailed proof is provided in Chen and Xiong 2008.)

Second, DFM also show that $d^{**}(t_i,t_i') \to 0$ implies $d^{US}(t_i,t_i') \to 0$. They start with two types t_i and t_i' with $d^{**}(t_i,t_i') \leq \varepsilon$ and aim to show that $R_i(t_i,G,\gamma) \subseteq R_i(t_i',G,\gamma+4\varepsilon)$ for any $\gamma \geq 0$, which implies $d^{US}(t_i,t_i') \leq 4\varepsilon$. However, for $a_i \in R_i(t_i,G,\gamma)$, when DFM choose a conjecture ν' to $(\gamma+4\varepsilon)$-rationalize a_i for t_i', they do not explicitly check if $\nu'[\{(a_j,\theta,t_j) : a_j \in R_j(t_j,G,\gamma+4\varepsilon)\}] = 1$ is true. We propose one way to deal with this issue. Suppose that $a_i \in R_i(t_i, G, \gamma)$ and ν is a conjecture that γ-rationalizes a_i. Since $A_j \times \Theta \times T_j^*$ is a standard separable measure space, there exist conditional probabilities $\nu(\cdot|\theta, t_j) \in \Delta(A_j)$. Also, since $t_j \mapsto R_j(t_j, G, \gamma + 4\varepsilon)$ is upper hemicontinuous under the product topology on T_j^*, by the Kuratowski–Ryll–Nardzewski Theorem (see Aliprantis and Border 1999), there is a measurable function $d : T_j^* \to A_j$ with $d(t_j) \in R_j(t_j, G, \gamma + 4\varepsilon)$ for all $t_j \in T_j^*$. Let $S^* = \{(\theta, t_j) : \text{support}[\nu(\cdot|\theta, t_j)] \subseteq R_j(t_j, G, \gamma)\}$. To define ν', we first define a measurable function $b_j : \Theta \times T_j^* \to \Delta(A_j)$ by

$$b_j(\theta, t_j) = \begin{cases} \nu(\cdot|\theta, t_j), & \text{if } (\theta, t_j) \in S^* \\ \delta_{d(t_j)} & \text{if } (\theta, t_j) \notin S^*. \end{cases}$$

Then we define the conjecture $\nu' \in \Delta(A_j \times \Theta \times T_j^*)$ such that for any measurable set $E \subseteq T_j^*$ and $(a_j, \theta) \in A_j \times \Theta$, $\nu'(E \times \{(a_j, \theta)\}) \equiv \int_E b_j(a_j|\theta, t_j)\pi_i^*(t_i')[(\theta, dt_j)]$. Observe that $\text{marg}_{\Theta \times T_j^*} \nu' = \pi_i^*[t_i']$. Moreover, we have $\nu'[\{(a_j, \theta, t_j) : a_j \in R_j(t_j, G, \gamma + 4\varepsilon)\}] = 1$, because $\text{support}[b_j(\theta, t_j)] \subseteq R_j(t_j, G, \gamma + 4\varepsilon)$ for all $t_j \in T_j^*$ by the definitions of S^* and $d(\cdot)$. Then, we can use equation (8) in Dekel et al. (2006, p. 306) to verify that a_i is a $(\gamma+4\varepsilon)$-best reply to ν'. (A detailed proof is provided in Chen and Xiong 2008.) Therefore, $a_i \in R_i(t_i', G, \gamma + 4\varepsilon)$ and $d^{US}(t_i, t_i') \leq 4\varepsilon$.

References

Aliprantis, Charalambos D. and Kim C. Border (1999), *Infinite Dimensional Analysis*, second edition. Springer, Berlin. [285]

Chen, Yi-Chun and Siyang Xiong (2008), "Non-denseness of finite types under the uniform strategic topology." Unpublished paper, Department of Economics, Northwestern University. [283, 284, 285]

Dekel, Eddie, Drew Fudenberg, and Stephen Morris (2006), "Topologies on types." *Theoretical Economics*, 1, 275–309. [283, 285]

Mertens, Jean-François and Shmuel Zamir (1985), "Formulation of Bayesian analysis for games with incomplete information." *International Journal of Game Theory*, 14, 1–29. [284]

Morris, Stephen (2002), "Typical types." Unpublished paper, Department of Economics, Princeton University. [284]

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