

Potential Methods in Interaction Games*

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Abstract

Incomplete Information Games, Local Interaction Games and Random Matching Games are all examples of a class of *Interaction Games*. In this paper, I describe how potential methods can generalize and unify results in the different literatures.

1. Introduction

In Morris (1997b), I documented a close connection between incomplete information games and local interaction games. This connection in turn implied a relationship between robustness and infection arguments for incomplete information games and uninvadability and contagion results for best response type dynamics in local interaction games. In this paper, I summarize some ongoing work exploring the connection.

An action profile is *robust to incomplete information* in a complete information game if every incomplete information game where payoffs are almost always given by that complete information game has an equilibrium where that action profile is almost always played (see Kajii and Morris (1997a)). We showed that an

*A May 1998 version of these notes were circulated under the title “Notes on Local Interaction, Generalized Potential Games and Strategic Complementarities.” This material builds on earlier work and long discussions with Atsushi Kajii. Some of the ideas here will appear in Frankel, Morris and Pauzner (1999).

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equilibrium is robust if it is \mathbf{p} -dominant for some \mathbf{p} with $\sum_{i=1}^I p_i < 1$. Ui (1998) showed that if an equilibrium is a potential maximizer, i.e., maximizes a potential function as in Monderer and Shapley (1996), then it is robust. I provide some new sufficient conditions for robustness. An action profile a^* is a *characteristic potential maximizer* if there exists a function assigning a value to every coalition of players such that whenever player i has a conjecture over his opponents' actions such that his expected contribution to the coalition of players choosing according to a^* is positive, his best response is to player a_i^* . This condition generalizes Kajii and Morris' \mathbf{p} -dominance condition. A characteristic potential maximizer is always robust. An action profile a^* is a *local potential maximizer* if there exists a potential function assigning a value to every action profile such that whenever player i has a conjecture over his opponents' actions such that moving his action towards a_i^* increases the expected value of the potential function, he has a best response to do so. Local potential maximizers are always both characteristic potential maximizers and potential maximizers. A local potential maximizer is always robust in games with strategic complementarities. Two player, three action games with symmetric payoffs and strategic complementarities generically have a unique local potential maximizer, and these are characterized below. These results are presented in sections 2 through 4.

Any incomplete information game can be interpreted as a local interaction game as follows. Interpret the players as "roles". The types of player i (in the incomplete information interpretation) can then be interpreted as players in role i (in the local interaction interpretation). A type profile then corresponds to a collection of players, one in each role, called upon to play the game. The probability of the type profile then corresponds to the weight assigned to that interaction group. The notion of equilibrium is unchanged in this alternative interpretation. (This relation is discussed in much more detail in Morris (1997b)). Now there is an alternative interpretation of robustness: an action profile is robust in a complete information game if every interaction game where payoffs are given by that complete information game in most interactions has an equilibrium where that action profile is played by most players. The above results thus immediately translate. Much of the local interaction literature studies the dynamic evolution of play in environments where all interactions have the same payoff function. In this setting, I say that an action profile is uninvadable if it continues to be played if it is initially played by most players and players only shift to actions that give higher payoffs against the current configuration of play. All the sufficient conditions

for robustness are sufficient for uninvasibility. Although I do not present such results, action profiles that are uninvasible are also the ones that will tend to spread under the dynamics studied in the literature. These results are presented in sections 5 through 6.

Many games have no robust equilibrium. Even some games with strategic complementarities have no robust equilibrium (and thus no local potential maximizer). An example is given in section 7. It is shown how there exists an incomplete information game with payoffs almost always given by that complete information game where the unique equilibrium has one action profile always played. But there exists another incomplete information game, again with payoffs almost always given by that complete information game, where the unique equilibrium has a different action profile always played. The argument clearly holds for all games in a neighborhood of the example, i.e., for an open set of games. This result has an interesting local interaction translation. Authors in the local interaction literature (e.g., Blume (1993) and Ellison (1993)) have emphasized how local interaction will lead to fast convergence in evolutionary environments. This example shows that which action we converge to depends on some fine details of the interaction structure.

Some remaining issues are discussed in section 8:

- All the analysis of this paper takes place in a non-standard framework, where there is an unbounded weight assigned to different type profiles / interactions. In the local interaction interpretation, this is the standard infinite population assumption. In the incomplete information interpretation, this corresponds to an infinite probability mass, or “improper prior,” on the state space. This is a very convenient mathematical trick. In section 8.1, I discuss how results could be translated to the more standard setting.
- Much of the local interaction literature deals with the case where there are no “roles” and an interaction, or matching, just consists of any I (usually 2) players. In this case, we can dispense with certain restrictions on the interaction structure inherited from the incomplete information interpretation. However, this extra structure does not effect any results. This is discussed in section 8.2.
- In games with strategic complementarities, local potential maximizers are both characteristic potential maximizers and potential maximizers. Thus the local potential maximizer condition unifies existing p -dominance and

potential sufficient conditions in the incomplete information and local interaction literatures. It would be nice to have such a unifying result for games without the strategic complementarities. A conjecture along this lines is suggested in section 8.3.

- All the analysis in this paper relies on a finite action assumption. In section 8.4, I discuss the continuum analogues of the potential maximizing conditions developed in this paper. These are relevant in Frankel, Morris and Pauzner (1999).

2. Incomplete Information Games and Robustness

2.1. Incomplete Information Games with Improper Priors

There are I players. Write T_i for the infinite set of possible types of player i , and $T \equiv T_1 \times T_2 \times \dots \times T_I$. Player i has a set of possible actions, A_i ; $A = A_1 \times \dots \times A_I$. For some results, a complete order on actions is assumed; in this case, write \bar{a}_i for the largest action in A_i , \underline{a}_i for the smallest action in A_i , a_i^+ for the smallest action larger than $a_i \neq \bar{a}_i$ and a_i^- for the largest action smaller than $a_i \neq \underline{a}_i$. A strategy for player i is a function $s_i : T_i \rightarrow A_i$. A payoff function for player i is a function $u_i : A \times T \rightarrow \mathbf{R}$ with the interpretation that $u_i(a, t)$ is the payoff of type t_i if nature chooses type profile t and players choose action profile a . Let $P : T \rightarrow \mathbf{R}_+$ be the prior probability on the space of type profiles. It is assumed that this is an improper prior with infinite mass, i.e., that $\sum_{t \in T} P(t)$ is non-convergent. However, it is assumed that

$$0 < \sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) < \infty.$$

so that the conditional probability

$$P(t_{-i} | t_i) = \frac{P(t_i, t_{-i})}{\sum_{t'_{-i} \in T_{-i}} P(t_i, t'_{-i})}$$

is well-defined. An event E is said to be *finite* if $\sum_{t \in E} P(t)$ is finite; it is *co-finite* if $\sum_{t \notin E} P(t)$ is finite. The non-standard assumption of an improper prior should be thought of as a convenient mathematical way of ensuring the existence of events with non-zero but negligible probability (i.e., with finite probability). In

section 8.1, I discuss how results could be extended to standard (mass 1) priors. A strategy profile $(s_i)_{i=1}^I$ is a (pure strategy) *equilibrium* if

$$\sum_{t_i \in T_{-i}} P(t_{-i}|t_i) u_i((s_i(t_i), s_{-i}(t_{-i})), (t_i, t_{-i})) \geq \sum_{t_i \in T_{-i}} P(t_{-i}|t_i) u_i((a_i, s_{-i}(t_{-i})), (t_i, t_{-i}))$$

for all $i = 1, \dots, I$, $t_i \in T_i$ and $a_i \in A_i$. A mixed strategy for players in role i is a function $\sigma_i : T_i \rightarrow \Delta(A_i)$; utilities may be extended to mixed strategies and (mixed strategy) equilibrium defined in the usual way.

Thus holding fixed I and the action sets $(A_i)_{i=1}^I$, an information system is the pair $\mathbf{IS} = ((T_i)_{i=1}^I, P)$ and an incomplete information game is the triple $\mathbf{IG} = ((T_i)_{i=1}^I, P, (u_i)_{i=1}^I)$.

2.2. Complete Information Games

A complete information game consists of a payoff function for each player, $\mathbf{g} = (g_1, \dots, g_I)$, where each $g_i : A \rightarrow \mathbf{R}$. When a complete information game \mathbf{g} is given, I write $\beta_i : \Delta(A_{-i}) \rightarrow 2^{A_i}$ for player i 's best response correspondence, i.e.,

$$\beta_i(\lambda_i) = \arg \max_{a_i \in A_i} \sum_{a_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i}).$$

2.3. Strategic Complementarities

For some results, I will be concerned with complete information games with some monotonic structure.

Definition 2.1. Complete information game \mathbf{g} satisfies *strategic complementarities (SC)* if $g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i}) > g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i})$ if $a_i > a'_i$ and $a_{-i} > a'_{-i}$.

Lemma 2.2. If \mathbf{g} satisfies strategic complementarities and $\lambda_i \in \Delta(A_{-i})$ strictly first order stochastically dominates $\lambda'_i \in \Delta(A_{-i})$, then $\min[\beta_i(\lambda_i)] \geq \max[\beta_i(\lambda'_i)]$.

See Krishna [1992] for a proof.

Definition 2.3. Complete information game \mathbf{g} satisfies *diminishing marginal returns (DMR)* if $g_i(a_i^+, a_{-i}) - g_i(a_i, a_{-i}) \leq g_i(a_i, a_{-i}) - g_i(a_i^-, a_{-i})$, for all $a_i \in A_i \setminus \{\underline{a}_i, \bar{a}_i\}$ and $a_{-i} \in A_{-i}$.

Lemma 2.4. *If \mathbf{g} satisfies **DMR**, $a_i \in \beta_i(\lambda_i)$, and either $a_i > a'_i > a''_i$ or $a_i < a'_i < a''_i$, then*

$$\sum_{a_{-i}} \lambda_i(a_{-i}) g_i(a'_i, a_{-i}) \geq \sum_{a_{-i}} \lambda_i(a_{-i}) g_i(a''_i, a_{-i}).$$

See Krishna [1992] for a proof.

2.4. Robustness

For any fixed incomplete information game $\mathbf{IG} = ((T_i)_{i=1}^I, P, (u_i)_{i=1}^I)$, write

$$|g_i| = \left\{ t \in T : u_i(a, (t_i, t'_{-i})) = g_i(a) \text{ for each } t'_{-i} \text{ such that } P(t_i, t'_{-i}) > 0 \right\}$$

$$\text{and } |\mathbf{g}| = \bigcap_{i=1}^I |g_i|$$

Definition 2.5. *Action profile a^* is robust in \mathbf{g} if every incomplete information game where $|\mathbf{g}|$ is co-finite has an equilibrium where $\{t : \sigma_i(a_i^* | t_i) = 1 \text{ for all } i\}$ is co-finite.*

3. Generalized Potential Games

A game is a potential game [Monderer and Shapley (1996)] if each player's change in utility from switching actions could have been derived from a common payoff function.

Definition 3.1. *Action profile a^* is a potential maximizer (P -maximizer) of \mathbf{g} if there exists a potential function $v : A \rightarrow \mathbf{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$, such that for all $i = 1, \dots, I$, $a_i, a'_i \in A_i$ and $a_{-i} \in A_{-i}$,*

$$v(a_i, a_{-i}) - v(a'_i, a_{-i}) = g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i}).$$

I will also be interested in some variations on the notion of a potential maximizer. The first was studied by Monderer and Shapley (1996). The latter two are new.

Definition 3.2. *Action profile a^* is a weighted potential maximizer (WP -maximizer) of \mathbf{g} if there exists a weighted potential function $v : A \rightarrow \mathbf{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ and $\mu \in \mathbf{R}_+^I$, such that for all $i = 1, \dots, I$, $a_i, a'_i \in A_i$ and $a_{-i} \in A_{-i}$,*

$$v(a_i, a_{-i}) - v(a'_i, a_{-i}) = \mu_i [g_i(a_i, a_{-i}) - g_i(a'_i, a_{-i})].$$

Definition 3.3. Action profile a^* is a local potential maximizer (LP-maximizer) of \mathbf{g} if there exists a local potential function $v : A \rightarrow \mathbf{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ and, for each i , $\mu_i : A_i \rightarrow \mathbf{R}_+$, such that for all $i = 1, \dots, I$ and $a_{-i} \in A_{-i}$,

$$v(a_i, a_{-i}) - v(a_i^-, a_{-i}) \geq \mu_i(a_i) [g_i(a_i, a_{-i}) - g_i(a_i^-, a_{-i})] \text{ if } a_i > a_i^* \text{ (3.1)}$$

and $v(a_i, a_{-i}) - v(a_i^+, a_{-i}) \geq \mu_i(a_i) [g_i(a_i, a_{-i}) - g_i(a_i^+, a_{-i})] \text{ if } a_i < a_i^*$

Definition 3.4. Action profile a^* is characteristic potential maximizer (CP-maximizer) of \mathbf{g} if there exists a characteristic potential function $\psi : 2^{\{1, \dots, I\}} \rightarrow \mathbf{R}$ with $v(\{1, \dots, I\}) > v(S)$ for all $S \neq \{1, \dots, I\}$ and, for each i , $\mu_i : A_i \rightarrow \mathbf{R}_+$ such that for all $a_i \in A_i$ and $a_{-i} \in A_{-i}$

$$\psi(\{j \neq i : a_j = a_j^*\}) - \psi(\{j \neq i : a_j = a_j^*\} \cup \{i\}) \geq \mu_i(a_i) (g_i(a_i, a_{-i}) - g_i(a_i^*, a_{-i})).$$

A weighted potential function requires that a player's change in utility from switching actions is proportional (for that player) to the change in a common payoff function. A local potential function requires that this condition holds only locally (when a player switches from one action to a neighboring action). In addition, the equalities in the condition are replaced with inequalities: it is required only that the gain from switching away the maximizing action is no greater than the gain under the local potential. Finally, an additional action dependent condition is allowed in the inequality. Clearly a P-maximizer is always a WP-maximizer and a WP-maximizer is always an LP-maximizer.

The characteristic potential maximizer conditions both weaken and strengthen the standard potential maximizer conditions. There is a strengthening, because the potential function is allowed to depend only on the set of players choosing according to a^* (and not the exact actions of those not choosing according to a^*). But it is a weakening, since equalities are replaced by inequalities. Thus there is no general relationship between CP-maximizers and P-maximizers. However, a CP-maximizer must be an LP-maximizer. To see why, let a^* be a CP-maximizer with characteristic potential function ψ , and $\psi(\{1, \dots, I\}) > \psi(S)$ for all $S \neq \{1, \dots, I\}$. Now consider the following potential function

$$v(a) = \psi(\{i : a_i = a_i^*\}).$$

by construction, $v(a^*) > v(a)$ for all $a \neq a^*$. Thus to show that a^* is an LP-maximizer, it is enough to show that there exists, for each i , $\mu_i : A_i \rightarrow \mathbf{R}_+$, such

that for all $i = 1, \dots, I$ and $a_{-i} \in A_{-i}$,

$$\begin{aligned} v(a_i, a_{-i}) - v(a_i^-, a_{-i}) &\geq \mu_i(a_i) [g_i(a_i, a_{-i}) - g_i(a_i^-, a_{-i})] \text{ if } a_i > a_i^* \\ \text{and } v(a_i, a_{-i}) - v(a_i^+, a_{-i}) &\geq \mu_i(a_i) [g_i(a_i, a_{-i}) - g_i(a_i^+, a_{-i})] \text{ if } a_i < a_i^*. \end{aligned}$$

If $a_i > a_i^- > a_i^*$, then the left hand side of the top inequality is always equal to zero. Thus we can set $\mu_i(a_i) = 0$. If $a_i^- = a_i^*$, then the above inequality becomes:

$$\psi(\{j \neq i : a_j = a_j^*\}) - \psi(\{j \neq i : a_j = a_j^*\} \cup \{i\}) \geq \mu_i(a_i) [g_i(a_i, a_{-i}) - g_i(a_i^*, a_{-i})].$$

But by the definition of a characteristic potential, such a $\mu_i(a_i)$ exists. A symmetric argument applies for $a_i < a_i^*$.

3.1. Dual Characterizations

The following lemmas provides some alternative characterizations of the potential maximizing conditions.

Lemma 3.5. *Action profile a^* is a LP-maximizer if and only if there exists $v : A \rightarrow \mathbf{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ and, for each i and $a_i > a_i^*$,*

$$\begin{aligned} \sum_{a_{-i}} \lambda_i(a_{-i}) g_i(a_i^-, a_{-i}) &> \sum_{a_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i}) \\ \text{for all } \lambda_i &\in \Delta(A_{-i}) \text{ such that } \sum_{a_{-i}} [v(a_i^-, a_{-i}) - v(a_i, a_{-i})] \lambda_i(a_{-i}) > 0. \end{aligned}$$

Symmetrically, for each i and $a_i < a_i^*$,

$$\begin{aligned} \sum_{a_{-i}} \lambda_i(a_{-i}) g_i(a_i^+, a_{-i}) &> \sum_{a_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i}) \\ \text{for all } \lambda_i &\in \Delta(A_{-i}) \text{ such that } \sum_{a_{-i}} [v(a_i^+, a_{-i}) - v(a_i, a_{-i})] \lambda_i(a_{-i}) > 0. \end{aligned}$$

PROOF. For simplicity, consider the case where $a^* = \underline{a}$. Fix function v with $v(a^*) > v(a)$ for all $a \neq a^*$. Now a^* is LP-maximizer if for all i and $a_i > a_i^*$, there exists $\tilde{\mu} \in \mathbf{R}_+$ such that

$$\tilde{\mu} [g_i(a_i, a_{-i}) - g_i(a_i^-, a_{-i})] \leq v(a_i, a_{-i}) - v(a_i^-, a_{-i})$$

for all $a_{-i} \in A_{-i}$. By the theorem of the alternative (see, e.g., Gale (1960), page 47), this is true if and only if, for all i and $a_i \in A_i$, there does not exist $\lambda_i : A_{-i} \rightarrow \mathbf{R}_+$ such that

$$\begin{aligned} \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) [g_i(a_i, a_{-i}) - g_i(a_i^-, a_{-i})] &\geq 0 \\ \text{and } \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) [v(a_i, a_{-i}) - v(a_i^-, a_{-i})] &< 0 \end{aligned}$$

Thus if

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) [v(a_i, a_{-i}) - v(a_i^-, a_{-i})] < 0$$

for any $\lambda_i \in \Delta(A_{-i})$, we must have

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) [g_i(a_i, a_{-i}) - g_i(a_i^-, a_{-i})] < 0. \quad \blacksquare$$

Lemma 3.6. *Action profile a^* is a CP-maximizer if and only if there exists $\psi : 2^{\{1, \dots, I\}} \rightarrow \mathbf{R}$ with $\psi(\{1, \dots, I\}) > \psi(S)$ for all $S \neq \{1, \dots, I\}$ such that*

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i^*, a_{-i}) > \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i})$$

for all i , $a_i \in A_i$, and $\lambda_i \in \Delta(A_{-i})$ such that

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) [\psi(\{j : a_j = a_j^*\} \cup \{i\}) - \psi(\{j : a_j = a_j^*\})] > 0.$$

The interpretation is that a_i^* is a best response whenever the expected contribution of i to the coalition of players choosing according to a^* is non-negative. Monderer and Shapley (1996) and Ui (1997a) discuss some relationships between potential functions of non-cooperative games and Shapley values of cooperative games.

PROOF. Fix characteristic potential function ψ with $\psi(\{1, \dots, I\}) > \psi(S)$ for all $S \neq \{1, \dots, I\}$. Now a^* is a CP-maximizer if for all i and $a_i \in A_i$, there exists $\tilde{\mu} \in \mathbf{R}_+$ such that

$$\tilde{\mu} [g_i(a_i, a_{-i}) - g_i(a_i^*, a_{-i})] \leq \psi(\{j : a_j = a_j^*\}) - \psi(\{j : a_j = a_j^*\} \cup \{i\})$$

for all $a_{-i} \in A_{-i}$. By the theorem of the alternative (see, e.g., Gale (1960), page 47), this is true if and only if, for all i and $a_i \in A_i$, there does not exist $\lambda_i : A_{-i} \rightarrow \mathbf{R}_+$ such that

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) [g_i(a_i, a_{-i}) - g_i(a_i^*, a_{-i})] \geq 0$$

and

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left[\psi(\{j : a_j = a_j^*\}) - \psi(\{j : a_j = a_j^*\} \cup \{i\}) \right] < 0$$

Thus if

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left[\psi(\{j : a_j = a_j^*\}) - \psi(\{j : a_j = a_j^*\} \cup \{i\}) \right] < 0$$

for any $\lambda_i \in \Delta(A_{-i})$, we must have

$$\sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) [g_i(a_i, a_{-i}) - g_i(a_i^*, a_{-i})] < 0. \quad \blacksquare$$

This characterization has a useful corollary. Write $\mathbf{p} = (p_1, \dots, p_I)$ for a vector of probabilities, $\mathbf{p} \in [0, 1]^I$. If an action profile is \mathbf{p} -dominant, in the sense of Kajii and Morris [1997], for some \mathbf{p} with $\sum_{i=1}^I p_i < 1$, then a^* is a CP-maximizer.

Definition 3.7. *Action profile a^* is \mathbf{p} -dominant in \mathbf{g} if*

$$\sum_{a_{-i}} \lambda_i(a_{-i}) g_i(a_i^*, a_{-i}) \geq \sum_{a_{-i}} \lambda_i(a_{-i}) g_i(a_i, a_{-i}),$$

for all i , $a_i \in A_i$ and $\lambda_i \in \Delta(A_{-i})$ with $\lambda_i(a_{-i}^*) \geq p_i$.

Lemma 3.8. *If action profile a^* is \mathbf{p} -dominant for some \mathbf{p} with $\sum_{i=1}^I p_i < 1$, then a^* is a CP-maximizer.*

PROOF. Let

$$\psi(S) = \begin{cases} 1 - \sum_{i=1}^I p_i, & \text{if } S = \{1, 2, \dots, I\} \\ - \sum_{i \in S} p_i, & \text{otherwise} \end{cases}.$$

If $\sum_{i=1}^I p_i < 1$, then $\psi(\{1, \dots, I\}) > \psi(S)$ for all $S \neq \{1, \dots, I\}$. Now observe that

$$\begin{aligned} 0 &\leq \sum_{a_{-i} \in A_{-i}} \lambda_i(a_{-i}) \left[\psi(\{j : a_j = a_j^*\} \cup \{i\}) - \psi(\{j : a_j = a_j^*\}) \right] \\ &\Leftrightarrow 0 \leq \lambda_i(a_{-i}^*) - p_i \\ &\Leftrightarrow p_i \leq \lambda_i(a_{-i}^*) \quad \blacksquare \end{aligned}$$

3.2. Examples

3.2.1. Two Player Two Action Symmetric Games with Strategic Complementarities

Let $I = 2$, $A_1 = A_2 = \{0, 1\}$, $g_1(x, y) = g_2(y, x) = w_{xy}$. Let $w_{00} > w_{10}$ and $w_{11} > w_{01}$, so $(0, 0)$ and $(1, 1)$ are both strict Nash equilibria. Now a potential function v is given by the following matrix.

| | | |
|---|-------------------|-------------------------------------|
| | 0 | 1 |
| 0 | 0 | $w_{10} - w_{00}$ |
| 1 | $w_{10} - w_{00}$ | $w_{10} + w_{11} - w_{00} - w_{01}$ |

$(1, 1)$ is a P-maximizer if $w_{10} + w_{11} - w_{00} - w_{01} > 0$, and $(0, 0)$ is a P-maximizer if $w_{10} + w_{11} - w_{00} - w_{01} < 0$. Thus, generically, there is a P-maximizer. The P-maximizer is the risk dominant equilibrium in the sense of Harsanyi and Selten (1988).

3.2.2. Two Player Two Action Asymmetric Games with Strategic Complementarities

Let $I = 2$ and $A_1 = A_2 = \{0, 1\}$. Let $g_1(0, 0) > g_1(1, 0)$, $g_1(1, 1) > g_1(0, 1)$, $g_2(0, 0) > g_2(0, 1)$ and $g_2(1, 1) > g_2(1, 0)$, so $(0, 0)$ and $(1, 1)$ are both strict Nash equilibria. Now let

$$\begin{aligned} q_1^* &= \frac{g_1(0, 0) - g_1(1, 0)}{g_1(0, 0) - g_1(1, 0) + g_1(1, 1) - g_1(0, 1)} \\ q_2^* &= \frac{g_2(0, 0) - g_2(0, 1)}{g_2(0, 0) - g_2(0, 1) + g_2(1, 1) - g_2(1, 0)} \end{aligned}$$

A weighted potential function v is given by the following matrix:

| | | |
|---|-----------------|---------|
| | 0 | 1 |
| 0 | $q_1^* + q_2^*$ | q_1^* |
| 1 | q_2^* | 1 |

$(0, 0)$ is a WP-maximizer if $q_1^* + q_2^* > 1$ and $(1, 1)$ is a WP-maximizer if $q_1^* + q_2^* < 1$. Thus, generically, there is a WP-maximizer. The WP-maximizer is the risk dominant equilibrium in the sense of Harsanyi and Selten (1988).

3.2.3. Many Player Two Action Symmetric Games with Strategic Complementarities.

Let $A_i = \{0, 1\}$ and $g_i(a_i, a_{-i}) = w(a_i, \#\{j \neq i : a_j = 1\})$, where $\xi(n) = w(1, n) - w(0, n)$ is increasing in n . A potential function for this game is

$$v(a) = \begin{cases} \sum_{k=0}^{m-1} \xi(k), & \text{if } \#\{i : a_i = 1\} = m \\ 0, & \text{if } a = \mathbf{0} \end{cases}$$

Now $\mathbf{1} = (1, \dots, 1)$ is the P-maximizer if $\sum_{k=0}^{I-1} \xi(k) > 0$, $\mathbf{0}$ is the P-maximizer if

$\sum_{k=0}^{I-1} \xi(k) < 0$. Thus generically in this class of games, there exists a P-maximizer.

An equivalent characterization of the P-maximizer is the following. Suppose that a player had a uniform prior over the number of his opponents choosing action 1. If 1 is a best response to that conjecture, then $\mathbf{1}$ is the P-maximizer; if 0 is a best response to that conjecture, then $\mathbf{0}$ is the P-maximizer.

3.2.4. Many Player Two Action Asymmetric Games with Strategic Complementarities

Let $A_i = \{0, 1\}$ and $g_i(a_i, a_{-i}) = w_i(a_i, \{j \neq i : a_j = 1\})$ and $\xi_i(S) = w_i(1, S) - w_i(0, S)$ is increasing in S (where S is ordered by set inclusion). If action profile $\mathbf{1}$ is the CP-potential maximizer, then there exists $\psi : 2^{\{1, \dots, I\}} \rightarrow \mathbf{R}$ and $\mu \in \mathbf{R}_+^I$ such that $\psi(\{1, 2, \dots, I\}) > \psi(S)$ for all $S \neq \{1, 2, \dots, I\}$ and, for all i and $S \subseteq \{1, \dots, i-1, i+1, I\}$, $\psi(S) - \psi(S \cup \{i\}) \geq -\mu_i \xi_i(S)$. If action profile $\mathbf{0}$ is the CP-potential maximizer, then there exists $\psi : 2^{\{1, \dots, I\}} \rightarrow \mathbf{R}$ and $\mu \in \mathbf{R}_+^I$ such that $\psi(\{1, 2, \dots, I\}) > \psi(S)$ for all $S \neq \{1, 2, \dots, I\}$ and, for all i and $S \subseteq \{1, \dots, i-1, i+1, I\}$, $\psi(S) - \psi(S \cup \{i\}) \geq \mu_i \xi_i(S)$.

To illustrate these conditions, consider unanimity games where for some $y, z \in \mathbf{R}_{++}^I$

$$g_i(a) = \begin{cases} y_i, & \text{if } a = \mathbf{1} \\ z_i, & \text{if } a = \mathbf{0} \\ 0, & \text{otherwise} \end{cases}$$

Now $\mathbf{1}$ is a CP-maximizer if there exists $\mu \in \mathbf{R}_+^I$ such that $\mu_i y_i > \mu_j z_j$ for all $j \neq i$, $\mathbf{0}$ is a CP-maximizer if there exists $\mu \in \mathbf{R}_+^I$ such that $\mu_i y_i < \mu_j z_j$ for all $j \neq i$. Thus a necessary condition for $\mathbf{1}$ to be a CP-maximizer is that $y_i y_j > z_i z_j$ for all $j \neq i$, and a necessary condition for $\mathbf{0}$ to be a CP-maximizer is that $y_i y_j < z_i z_j$ for all $j \neq i$. Clearly, then, an open set of games do not have any CP-maximizer. For example, all games in the neighborhood of the following game do not have a CP-maximizer. There are three players, and payoffs if player 3 chooses action 0 are:

| | | | |
|-------------------|---|---------|---------|
| Player 2's Action | | | |
| | | 0 | 1 |
| Player 1's Action | 0 | 6, 1, 1 | 0, 0, 0 |
| 1 | | 0, 0, 0 | 0, 0, 0 |

while payoffs if player 3 chooses action 1 are:

| | | | |
|-------------------|---|---------|---------|
| Player 2's Action | | | |
| | | 0 | 1 |
| Player 1's Action | 0 | 0, 0, 0 | 0, 0, 0 |
| 1 | | 0, 0, 0 | 2, 2, 2 |

Note that in two action games, a LP-maximizer must be a CP-maximizer. So there is no LP-maximizer for an open set of games.

3.2.5. Two Player Three Action Symmetric Games with Strategic Complementarities

Let $I = 2$, $A_1 = A_2 = \{0, 1, 2\}$; $g_1(a_1, a_2) = g_2(a_2, a_1) = w_{a_1} w_{a_2}$, where $w_{xx} > w_{yx}$ for all $y \neq x$ and $w_{xy} - w_{x'y} > w_{xy'} - w_{x'y'}$ if $x > x'$ and $y > y'$. Write $\Delta_{x'y'}^{xy}$ for the net expected gain of choosing action x rather than y against a 50/50 conjecture on whether the opponent will choose action x' or y' . Thus

$$\Delta_{x'y'}^{xy} = w_{x'x} + w_{x'y} - w_{y'x} - w_{y'y}.$$

Note that $\Delta_{x'y'}^{xy} = \Delta_{x'y'}^{yx}$ and $\Delta_{x'y'}^{xy} = -\Delta_{y'x'}^{xy}$. Note that $\Delta_{xy}^{xy} > 0$ implies that action profile (x, x) pairwise risk dominates action profile (y, y) . Now we have the following complete (for generic games) characterization of the LP-maximizers.

- $(0, 0)$ is the LP-maximizer if $\Delta_{01}^{01} > 0$ and either (1) $\Delta_{12}^{12} > 0$ or (2) $\Delta_{21}^{21} > 0$ and $\frac{\Delta_{10}^{02}}{\Delta_{01}^{01}} < \frac{\Delta_{12}^{02}}{\Delta_{21}^{02}}$.
- $(1, 1)$ is the LP-maximizer if $\Delta_{10}^{10} > 0$ and $\Delta_{12}^{12} > 0$.
- $(2, 2)$ is the LP-maximizer if $\Delta_{21}^{21} > 0$ and either (1) $\Delta_{10}^{10} > 0$ or (2) $\Delta_{01}^{01} > 0$ and $\frac{\Delta_{10}^{02}}{\Delta_{01}^{01}} > \frac{\Delta_{12}^{02}}{\Delta_{21}^{02}}$.

The following example illustrates these conditions:

| (g_1, g_2) | 0 | 1 | 2 |
|--------------|--------|------|--------|
| 0 | 4, 4 | 0, 0 | -6, -3 |
| 1 | 0, 0 | 1, 1 | 0, 0 |
| 2 | -3, -6 | 0, 0 | 2, 2 |

$(0, 0)$ is the LP-maximizer, since $\Delta_{01}^{01} = 3$, $\Delta_{21}^{21} = 1$, $\Delta_{10}^{02} = 2$ and $\Delta_{12}^{02} = 1$. Note that $(2, 2)$ pairwise risk dominates both $(1, 1)$ and $(0, 0)$, but nonetheless is not the LP-maximizer.

Proving the above claims (i.e., constructing the local potential functions) involves tedious algebra. Here, I will just note two cases to illustrate the issues.

Case 1: $\Delta_{10}^{10} > 0$ and $\Delta_{12}^{12} > 0$. Consider the following local potential function:

| v | 0 | 1 | 2 |
|-----|---------------------|-------------------|---------------------|
| 0 | $-\Delta_{10}^{10}$ | $w_{01} - w_{11}$ | $-\varepsilon$ |
| 1 | $w_{01} - w_{11}$ | 0 | $w_{21} - w_{11}$ |
| 2 | $-\varepsilon$ | $w_{21} - w_{11}$ | $-\Delta_{12}^{12}$ |

for some small but strictly positive ε . Setting $a^* = (1, 1)$ and $\mu_1(0) = \mu_1(2) = \mu_2(0) = \mu_2(2) = 1$, one can verify that the inequalities of equation (3.1) are satisfied.

Case 2: $\Delta_{01}^{01} > 0$, $\Delta_{21}^{21} > 0$, $\Delta_{10}^{02} > 0$, $\Delta_{12}^{02} > 0$ and $\frac{\Delta_{10}^{02}}{\Delta_{01}^{01}} < \frac{\Delta_{12}^{02}}{\Delta_{21}^{02}}$. Consider the following local potential function:

| v | 0 | 1 | 2 |
|-----|---|---|---|
| 0 | ε | $\varepsilon + \lambda_1 [w(1, 0) - w(0, 0)]$ | $\lambda_1 [w_{02} - w_{12}] + \lambda_2 [w_{12} - w_{22}]$ |
| 1 | $\varepsilon + \lambda_1 [w_{10} - w_{00}]$ | $-\lambda_2 \Delta_{21}^{21}$ | $\lambda_2 [w_{12} - w_{22}]$ |
| 2 | $\lambda_1 [w_{02} - w_{12}] + \lambda_2 [w_{12} - w_{22}]$ | $\lambda_2 [w_{12} - w_{22}]$ | 0 |

for some small but strictly positive ε and positive λ_1 and λ_2 such that

$$\frac{\Delta_{21}^{21}}{\Delta_{01}^{01}} < \frac{\lambda_1}{\lambda_2} < \frac{\Delta_{12}^{02}}{\Delta_{10}^{02}}.$$

Setting $a^* = (0, 0)$, $\mu_1(1) = \mu_2(1) = \lambda_1$, $\mu_1(2) = \mu_2(2) = \lambda_2$, one can verify that the inequalities of equation (3.1) are satisfied.

3.2.6. Joint Output Games

Let $g_i(a_i, a_{-i}) = w(a) - h_i(a_i)$. Then a potential function is $v(a) = w(a) - \sum_{i=1}^I h_i(a_i)$ and a^* is a P-maximizer if $a^* \in \arg \max_a \left[w(a) - \sum_{i=1}^I h_i(a_i) \right]$.

4. Results on Robustness and Generalized Potential Games

Proposition 4.1. *If a^* is a WP-maximizer of \mathbf{g} , then a^* is robust in \mathbf{g} .*

This result is due to Ui [1998]. The current version of that paper shows that if a^* is a P-maximizer, then a^* satisfies a slightly weaker version of robustness (“robustness to canonical elaborations”). Ui has pointed out to me that the result extends to full robustness with almost the same proof. The extension to WP-maximizing action profiles is straightforward.

Proposition 4.2. *If a^* is CP-maximizer of \mathbf{g} , then a^* is robust in \mathbf{g} .*

The proof is close to that of Kajii and Morris (1997a), which in turn draws on the logic of Monderer and Samet (1989). We exploit the following lemma that has some independent interest. An event $E \subseteq T$ is said to be simple if it is the product of events in $(T_i)_{i=1}^I$.

Lemma 4.3. *Fix any information system $((T_i)_{i=1}^I, P)$. Let $\psi : 2^{\{1, \dots, I\}} \rightarrow \mathbf{R}$ with $\psi(\{1, \dots, I\}) > \psi(S)$ for all $S \neq \{1, \dots, I\}$. If simple event E is co-finite, then it has a simple, co-finite, subset $\hat{E} = \times_{i=1}^I \hat{E}_i$ with*

$$\sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) \left[\psi(\{j : t_j \in \hat{E}_j\} \cup \{i\}) - \psi(\{j : t_j \in \hat{E}_j\}) \right] \geq 0 \quad (4.1)$$

for all $i = 1, \dots, I$ and $t_i \in \hat{E}_i$.

PROOF OF LEMMA 4.3. Write $\tilde{T} = T_1 \cup \dots \cup T_I$ for the *union* of types of the players. We will construct a *labelling* of types. Each type is labelled with a pair of numbers, (τ_1, τ_2) , where $\tau_1 \in \{1, 2, 3\}$ and τ_2 is a positive integer. Thus writing $\tilde{\tau}(t_i) \equiv (\tilde{\tau}_1(t_i), \tilde{\tau}_2(t_i))$ for the label of type t_i , $\tilde{\tau}$ is an *one-to-one* function, $\tilde{\tau} : \tilde{T} \rightarrow \{1, 2, 3\} \times \mathbf{Z}_{++}$. We require a complete order on $\{1, 2, 3\} \times \mathbf{Z}_+$ defined by $(\tau_1, \tau_2) \succ (\tau'_1, \tau'_2)$ if $\tau_1 > \tau'_1$ or $\tau_1 = \tau'_1$ and $\tau_2 > \tau'_2$.

First assign labels of the form $(1, k)$ to all types not in \hat{E}_i . Now for each $k \in \mathbf{Z}_{++}$, assign label $(2, k)$ to a type t_i with

$$\sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) [\psi(\{j : \tilde{\tau}(t_j) > \tau\} \cup \{i\}) - \psi(\{j : \tilde{\tau}(t_j) > \tau\})] < 0.$$

If there is no such type, halt the labelling of such types at integer $k-1$. Otherwise, label all such types. Now assign labels of the form $(3, k)$ to all the remaining types.

Now let

$$V(k) = \sum_{t \in T} P(t) (\psi(\{j : \tilde{\tau}(t_j) > (2, k)\}) - \psi(\{1, \dots, I\}))$$

By construction, $V(1)$ is finite. But if $\tilde{\tau}(t_i) = k$,

$$V(k) - V(k-1) = - \sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) \left[\begin{array}{c} \psi(\{j : \tilde{\tau}(t_j) > (2, k)\} \cup \{i\}) \\ - \psi(\{j : \tilde{\tau}(t_j) > (2, k)\}) \end{array} \right] > 0.$$

Thus $V(k)$ is increasing.

Now let $\gamma = \min_{S \neq \{1, \dots, I\}} (\psi(\{1, \dots, I\}) - \psi(S))$. Observe that

$$V(k) \leq -\gamma \sum_{\{t : \tilde{\tau}(t_i) \leq (2, k) \text{ for some } i\}} P(t).$$

Thus if $\sum_{\{t : \tilde{\tau}(t_i) \leq (2, \infty) \text{ for some } i\}} P(t)$ were infinite, we would have $V(k) \rightarrow -\infty$ as

$k \rightarrow \infty$, a contradiction since $V(0)$ is finite and $V(k)$ is increasing in k . Now letting $\hat{E}_i = \{t_i : \tilde{\tau}(t_i) \geq (3, 1)\}$, we have that $\hat{E} = \prod_{i=1}^I \hat{E}_i$ is co-finite and satisfies the required property. ■

PROOF OF PROPOSITION 4.2. Fix a complete information game \mathbf{g} and an interaction game $\mathbf{IG} = ((T_i)_{i=1}^I, P, (u_i)_{i=1}^I)$ where $|\mathbf{g}|$ is co-finite. Applying lemma 4.3, let \hat{E} be the co-finite simple subset of $|\mathbf{g}|$ satisfying 4.1. Now consider

a modified version of incomplete information game $((T_i)_{i=1}^I, P, (u_i)_{i=1}^I)$, where each type $\hat{t}_i \in \hat{E}_i$ is required to play action a_i^* . Find an equilibrium of this game. Condition (4.1) and lemma 3.6 imply that this is also an equilibrium of the original game. ■

Proposition 4.4. *If \mathbf{g} satisfies **SC** and **DMR**, and action profile a^* is an LP-maximizer of \mathbf{g} , then a^* is robust in \mathbf{g} .*

Lemma 4.5. *Fix any information system $((T_i)_{i=1}^I, P)$. Let $v : A \rightarrow \mathbf{R}$ with $v(\{1, \dots, I\}) > v(S)$ for all $S \neq \{1, \dots, I\}$. If simple event $E = \times_{i=1}^I E_i$ is co-finite, then there exist strategy profiles \bar{s} and \underline{s} , such that (1) $\bar{s}(t) \geq a^* \geq \underline{s}(t)$ for all $t \in T$; (2) $\{t : \bar{s}(t) \neq \bar{a} \text{ or } \underline{s}(t) \neq \underline{a}\} \subseteq E$; (3) $\{t : \bar{s}(t) = a^* = \underline{s}(t)\}$ is co-finite; and (4)*

$$\sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) \left[v\left(\left(\bar{s}_i(t_i)\right)^+, \bar{s}_{-i}(t_{-i})\right) - v\left(\bar{s}(t_i, t_{-i})\right) \right] < 0 \text{ if } a_i^* \leq \bar{s}_i(t_i) \leq \bar{a}_i$$

and $\sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) \left[v\left(\left(\underline{s}_i(t_i)\right)^-, \underline{s}_{-i}(t_{-i})\right) - v\left(\underline{s}(t_i, t_{-i})\right) \right] < 0 \text{ if } a_i^* \geq \underline{s}_i(t_i) \geq \underline{a}_i$

PROOF OF LEMMA 4.5. Write $\tilde{T} = T_1 \cup \dots \cup T_I$ for the union of types of the players. Now construct a strategy sequence as follows. Let

$$\bar{s}_i^0(t_i) = \begin{cases} \bar{a}_i, & \text{if } t_i \notin E_i \\ a_i^*, & \text{if } t_i \in E_i \end{cases}.$$

For each $\tau \geq 1$, choose a player k and a type $\hat{t}_k \in E_k$ such that

$$\sum_{t_{-k} \in T_{-k}} P(\hat{t}_k, t_{-k}) \left[v\left(\left(\bar{s}_k^{\tau-1}(\hat{t}_k)\right)^+, \bar{s}_{-k}^{\tau-1}(t_{-k})\right) - v\left(\bar{s}^{\tau-1}(\hat{t}_k, t_{-k})\right) \right] \geq 0,$$

and let

$$\bar{s}_i^\tau(t_i) = \begin{cases} \left(\bar{s}_k^{\tau-1}(\hat{t}_k)\right)^+, & \text{if } i = k \text{ and } t_i = \hat{t}_k \\ \bar{s}_i^{\tau-1}(t_i), & \text{otherwise} \end{cases}$$

If for some τ , there is no such k and \hat{t}_k , let

$$\bar{s}_i^\tau(t_i) = \bar{s}_i^{\tau-1}(t_i)$$

for all i and t_i . Write \bar{s} for the limit of this sequence. Now let

$$V(\tau) = \sum_{t \in T} P(t) (v(s^\tau(t)) - v(a^*)).$$

Observe first that $V(0)$ is finite. Now suppose that $t_i \in E_i$ and that at date τ , type t_i has just switched from action $a_i = \bar{s}_i^{\tau-1}(t_i) \geq a_i^*$ to action $a_i^+ = \bar{s}_i^\tau(t_i)$. Then

$$\begin{aligned} V(\tau) - V(\tau-1) &= \sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) \left(v(a_i^+, s_{-i}^\tau(t_{-i})) - v(a_i, s_{-i}^\tau(t_{-i})) \right) \\ &\geq 0. \end{aligned}$$

Thus $V(\tau)$ is initially finite and is increasing. But letting $\gamma = \min_{a \neq a^*} (v(a^*) - v(a))$, $V(\tau) \leq -\gamma \sum_{\{t: \bar{s}^\tau(t) \neq a^*\}} P(t)$. Thus if $\sum_{\{t: \bar{s}(t) \neq a^*\}} P(t)$ were infinite, we would have $V(\tau) \rightarrow -\infty$ as $\tau \rightarrow \infty$, a contradiction. Thus $\sum_{\{t: \bar{s}(t) \neq a^*\}} P(t)$ is finite.

We can symmetrically construct a limiting strategy profile \underline{s} . Now the required properties of \bar{s} and \underline{s} follow from the construction. ■

PROOF OF PROPOSITION 4.4. Fix a complete information game \mathbf{g} and an interaction game $\mathbf{IG} = ((T_i)_{i=1}^I, P, (u_i)_{i=1}^I)$ where $|\mathbf{g}|$ is co-finite. Construct \bar{s} and \underline{s} satisfying the properties of lemma 4.5, setting $E = |\mathbf{g}|$. Now consider the modified game where each type t_i of player i must choose an action a_i with $\underline{s}_i(t_i) \leq a_i \leq \bar{s}_i(t_i)$. Find an equilibrium of this modified game. By lemma 3.5, this equilibrium remains an equilibrium of the original game. ■

Kajii and Morris (1997a) showed that if a^* was a \mathbf{p} -dominant equilibrium for some \mathbf{p} with $\sum_{i=1}^I p_i < 1$, then a^* was robust. This was sufficient to show the robustness of risk dominant equilibria in two player two action games (symmetric or asymmetric) but becomes a very stringent requirement as the number of players becomes large. Nonetheless, an open set of games (in payoff space) have such \mathbf{p} -dominant equilibria. Proposition 4.2 generalizes the Kajii and Morris result.

As noted above, Ui (1998) showed that if a^* were a WP-maximizer, then a^* is robust. While many interesting games (including many player games) have WP-maximizers, a weakness of this result is that (with many players and many actions) only a non-generic set of games have a WP-maximizer.

It would be nice to have a condition that generalized both the CP-maximizer sufficient condition and the WP-maximizer sufficient condition (section 8.3 contains a conjecture along these lines).

There is a close relation between robustness and the limit equilibria of noisy games introduced by Carlsson and van Damme (1993a). Robustness is a sufficient condition for an action profile to be played in the limit of some noisy version of the game (see Kajii and Morris (1997b) for more on the relationship). Frankel, Morris and Pauzner (1999) demonstrate how LP-maximizers must be selected in noisy games with strategic complementarities. Carlsson and van Damme (1993a) first showed that the risk dominant equilibrium of two player two action games must be selected. In a number of different settings, Carlsson and van Damme (1993b), Kim (1996) and Morris and Shin (1999) showed that the WP-maximizing action of many player two action symmetric games must be selected. Carlsson and Ganslandt (1998) showed that a certain equilibrium is selected in a minimum effort game and noted that it was a P-maximizer. Carlsson (1989) gave a three player two action asymmetric payoff example where the noisy selection depended on the noise structure. Corsetti, Morris and Shin (1999) give a many player two action asymmetric payoff application.

5. Interaction Games and Uninvadability

5.1. The Local Interaction / Matching Interpretation of the Incomplete Information Game and Robustness

An alternative local interaction interpretation of the information system $\mathbf{IS} = ((T_i)_{i=1}^I, P)$ and the incomplete information game $\mathbf{IG} = ((T_i)_{i=1}^I, P, (u_i)_{i=1}^I)$ is the following. There is an infinite population of players. Each player belongs to one of I possible roles. Write T_i for the set of players in role i , and $T \equiv T_1 \times T_2 \times \dots \times T_I$. An *interaction* is a vector of players (one in each role), $t \in T$. Each interaction was a weight $P(t)$, where $P : T \rightarrow \mathbf{R}_+$. Players in role i have a set of possible actions, A_i ; $A = A_1 \times \dots \times A_I$. A strategy for players in role i is a function $s_i : T_i \rightarrow A_i$. A payoff function for players in role i is a function $u_i : A \times T \rightarrow \mathbf{R}$ with the interpretation that $u_i(a, t)$ is the payoff of player t_i if interaction t is chosen and players choose action profile a . A strategy profile $(s_i)_{i=1}^I$ is a (pure strategy) *equilibrium* if each player's action maximizes the weighted sum of the his payoff in the interactions he is involved in, i.e.,

$$\sum_{t_i \in T_{-i}} P(t_{-i} | t_i) u_i((s_i(t_i), s_{-i}(t_{-i})), (t_i, t_{-i})) \geq \sum_{t_i \in T_{-i}} P(t_{-i} | t_i) u_i((a_i, s_{-i}(t_{-i})), (t_i, t_{-i}))$$

for all $i = 1, \dots, I$, $t_i \in T_i$ and $a_i \in A_i$.

In the random matching interpretation, only one interaction is played, but it is drawn according to (improper) probability P , and each player maximizes his expected payoff (without knowing which partners he is interacting with).

Thus we will call $\mathbf{IS} = ((T_i)_{i=1}^I, P)$ an *interaction system* and $\mathbf{IG} = ((T_i)_{i=1}^I, P, (u_i)_{i=1}^I)$ an *interaction game*.

Now there is a natural local interaction / random matching interpretation of robustness. Suppose that interaction game where almost all players have payoff functions given by \mathbf{g} , has an equilibrium where almost all players chooses according to a^* . Then a^* is robust.

5.2. Uninvadability

The classic question in the local interaction literature concerns a population that interacts according to interaction system $\mathbf{IS} = ((T_i)_{i=1}^I, P)$, and has payoffs always given by complete information game \mathbf{g} . What happens if players shift toward actions that give higher payoffs? How do such dynamics behave?

Consider a sequence of pure strategy profiles, $\{s^\tau\}_{\tau=0}^\infty$. For any fixed s^τ , write $\pi_i^\tau [t_i] \in \Delta(A_{-i})$ for the probability distribution over neighbors' action, i.e.,

$$\pi_i^\tau [t_i](a_{-i}) = \sum_{\{t_{-i}: s_{-i}^\tau(t_{-i})=a_{-i}\}} P(t_{-i}|t_i).$$

We will be concerned with two properties of such of strategy sequences:

- **Single Player Revision [S]:** for all $\tau \geq 1$, there is at most one $i \in \{1, \dots, I\}$ and $t_i \in T_i$ such that $s_i^\tau(t_i) \neq s_i^{\tau-1}(t_i)$
- **Better Response [B]:** if $s_i^{\tau-1}(t_i) = a_i$ and $s_i^\tau(t_i) = a'_i$, then a'_i gives at least as high a payoff against $\pi_i^{\tau-1}[t_i]$ than a_i

Property **S** requires that at most only player switches behavior in each period. Property **B** requires that players only switch to action that give at least as high a payoff.

Definition 5.1. Action profile a^* is invaded under strategy sequence $\{s^\tau\}_{\tau=0}^\infty$ if a^* is initially played by almost all players but is eventually not played by an infinite mass of players; i.e., writing $q^\tau = \sum_{\{t: s^\tau(t) \neq a^*\}} P(t)$, q^0 is finite, but $q^\tau \rightarrow \infty$ as $\tau \rightarrow \infty$.

Definition 5.2. Action profile a^* is *uninvadable* in \mathbf{g} if for every local interaction system and sequence $\{s^\tau\}_{\tau=0}^\infty$ satisfying property **S** and **B**, action profile a^* is not invaded.

6. Results on Uninvadability and Generalized Potential Games

Proposition 6.1. If action profile a^* is a WP-maximizer of \mathbf{g} , then a^* is uninvadable in \mathbf{g} .

PROOF. Let a^* uniquely maximize weighted potential function v with weights μ but nonetheless be invadable. Thus there exists configuration sequence $\{s^\tau\}_{\tau=0}^\infty$ with $\sum_{\{t:s^0(t) \neq a^*\}} P(t)$ co-finite, but $\sum_{\{t:s^\tau(t) \neq a^*\}} P(t)$ tends to ∞ as $\tau \rightarrow \infty$. Let

$$V(\tau) = \sum_{t \in T} P(t) (v(s^\tau(t)) - v(a^*))$$

Observe first that $V(0)$ is finite and non-positive and non-positive. Now suppose that at date τ , type t_i has just switched from action a_i to action a'_i . Then

$$\begin{aligned} V(\tau) - V(\tau - 1) &= \sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) \left(v(a'_i, s_{-i}^\tau(t_{-i})) - v(a_i, s_{-i}^\tau(t_{-i})) \right) \\ &= \mu_i \sum_{t_{-i} \in T_{-i}} P(t_i, t_{-i}) \left(g_i(a'_i, s_{-i}^\tau(t_{-i})) - g_i(a_i, s_{-i}^\tau(t_{-i})) \right) \\ &\geq 0, \text{ by best response property.} \end{aligned}$$

So $V(\tau)$ is non-decreasing; thus it remains in the interval $[-V(0), 0]$ forever. But letting $\gamma = \min_{a \neq a^*} (v(a^*) - v(a))$,

$$V(\tau) \leq -\gamma \sum_{\{t:s^\tau(t) \neq a^*\}} P(t).$$

As $\tau \rightarrow \infty$, the right hand side tends to $-\infty$, a contradiction. ■

Proposition 6.2. If action profile a^* is CP-maximizing in \mathbf{g} , then a^* is uninvadable in \mathbf{g} .

Proof. Let $\{s^\tau\}_{\tau=0}^\infty$ be a strategy sequence satisfying **S** and **B**, with $\{t : s^0(t) = a^*\}$ co-finite. By lemmas 4.3 and 3.6, there exists a simple, co-finite subset $E \subseteq \{t : s^0(t) = a^*\}$ such that $s^\tau(t) = a^*$ for all $t \in E$ and $\tau \geq 0$. ■

Proposition 6.3. *If g satisfies **SC** and **DMR** and action profile a^* is a LP-maximizer of \mathbf{g} , then a^* is uninvadable in \mathbf{g} .*

PROOF. Let $\{s^\tau\}_{\tau=0}^\infty$ be a strategy sequence satisfying **S** and **B**, with $\{t : s^0(t) = a^*\}$ co-finite. By lemmas 4.5 and 3.5, there exist strategy profiles \bar{s} and \underline{s} , such that (1) $\bar{s}(t) \geq s^\tau(t) \geq \underline{s}(t)$ for all $t \in T$; (2) $\{t : \bar{s}(t) \neq \bar{a} \text{ or } \underline{s}(t) \neq \underline{a}\} \subseteq \{t : s^0(t) = a^*\}$; and (3) $\{t : \bar{s}(t) = a^* = \underline{s}(t)\}$ is co-finite. ■

Under the dynamic revision process in this paper, a player either sticks to his current action, or switches to one giving a better payoff. Requiring the player to switch to a *best* response obviously makes it easier to be uninvadable. A number of papers have considered such best/better response dynamics in the literature, including Blume [1995], Anderlini and Ianni [1996], Ianni [1997, 1998] and Morris [1997a]. Blume [1995] and Ianni [1998] demonstrated the tendency for the potential maximizing action to spread. The uninvadability condition identified here is a key step in showing the tendency to spread, and it can be shown that CP-maximizers and LP-maximizers (in games with strategic complementarities) will spread in the dynamics that they consider. Blume [1995] also analyzed games with strategic complementarities and identified one sufficient condition for an action to spread. It can be shown that symmetric action profiles he identifies are LP-maximizers.

Another strand of the literature on local interaction work has looked at stochastic best response dynamics, where players tend to choose actions with higher payoffs, but sometimes make mistakes, e.g., Blume (1993), Ellison (1993), and Young (1998). These processes are ergodic, so no action is uninvadable. But it seems likely that CP-maximizers and LP-maximizers would represent the long run equilibria under those more general dynamics.

7. Contagion: An Example

A converse to the uninvadability question is the following. When can we construct a local interaction system and a finite “trigger group” of players in that local interaction system, such that if that trigger group of players initially plays according to a^* , and players revise their behavior towards better responses, then eventually the whole population must play according to a^* . If this is the case, we say that action profile a^* spreads contagiously. One can show that action profiles that satisfy the sufficient conditions above for robustness and uninvadability, may spread contagiously (given the right interaction structure).

But can one find games where different action profiles may spread contagiously, depending on the local interaction structure? We noted in section 3.2.4 the existence of an open set of three player, two action with asymmetric payoffs with no LP-maximizer. In this case, the two equilibria will be contagious, for different interaction structures. However, the local interaction literature mostly deals with games with two players and symmetric payoffs. So to maximize the relevance for local interaction, we will describe a two player, four action game with symmetric payoffs (and also strategic complementarities) where distinct equilibria are both contagious.

Consider the following example. There are two players, 1 and 2. The types of player i are the set of integers. The types of the players are correlated. In particular, there is $w : \mathbf{Z} \rightarrow \mathbf{R}_+$ and $P(t_1, t_2) = w(|t_1 - t_2|)$. Let $A_1 = A_2 = \{0, 1, 2, 3\}$. Consider the following complete information symmetric payoffs.

| \mathbf{g} | 0 | 1 | 2 | 3 |
|--------------|--------|--------|--------|--------|
| 0 | 50, 50 | 46, 41 | 32, 23 | 8, 3 |
| 1 | 41, 46 | 50, 50 | 42, 47 | 27, 29 |
| 2 | 23, 32 | 47, 42 | 50, 50 | 41, 52 |
| 3 | 3, 8 | 29, 27 | 52, 41 | 50, 50 |

This game has **SC** and **DMR**. Consider two different interaction weights:

$$w_1(z) = \begin{cases} 0, & \text{if } z = 0 \\ 1, & \text{for all } z \in \{1, 2, \dots, 480\} \\ 0, & \text{for all } z > 480 \end{cases}$$

$$w_2(z) = \begin{cases} 0, & \text{if } z = 0 \\ 2, & \text{for all } z \in \{1, 2, \dots, 160\} \\ 1, & \text{for all } z \in \{161, 162, \dots, 480\} \\ 0, & \text{for all } z > 480 \end{cases}$$

Claim 1 Consider the incomplete information game where the prior probability is generated by w_1 , types $\{1, \dots, 960\}$ of each player have a dominant strategy to play action 0, and all other types have payoffs given by \mathbf{g} . This game has a unique strategy profile surviving iterated deletion of dominated strategies for each player: always play action 0.

PROOF OF CLAIM 1. Consider the strategy profile s^k defined as follows:

$$s_i^k(t_i) = \begin{cases} 0, & \text{if } |t_i| \leq k + 240 \\ 1, & \text{if } k + 241 \leq |t_i| \leq k + 450 \\ 2, & \text{if } k + 451 \leq |t_i| \leq k + 660 \\ 3, & \text{if } k + 661 \leq |t_i| \end{cases}$$

This strategy profile has the property that types at the edge of two action regions always have a strict incentive to choose the lower of the two actions (and thus every player's best response is no higher than their current action). More precisely, observe that types 240 and -241 (of either player) attach probability $\frac{480}{960}$ to their opponent choosing 0, $\frac{209}{960}$ to their opponent choosing 1, $\frac{210}{960}$ to their opponent choosing 2 and $\frac{61}{960}$ to their opponent choosing 3. Thus the expected payoff to action 0 is

$$\frac{480}{960}(50) + \frac{209}{960}(46) + \frac{210}{960}(32) + \frac{61}{960}(8) = \frac{40822}{960}$$

and the expected payoff to action 1 is

$$\frac{480}{960}(41) + \frac{209}{960}(50) + \frac{210}{960}(42) + \frac{61}{960}(27) = \frac{40597}{960}$$

Thus action 0 is the best response.

The corresponding probabilities for types 451 and -451 are $(\frac{270}{960}, \frac{210}{960}, \frac{209}{960}, \frac{271}{960})$. Thus the expected payoff to action 1 is:

$$\frac{270}{960}(41) + \frac{210}{960}(50) + \frac{209}{960}(42) + \frac{271}{960}(27) = \frac{12555}{320}$$

and the expected payoff to action 2 is:

$$\frac{270}{960}(23) + \frac{210}{960}(47) + \frac{209}{960}(50) + \frac{271}{960}(41) = \frac{12547}{320}$$

Thus action 1 is the best response.

The corresponding probabilities for types 451 and -451 are $(\frac{60}{960}, \frac{210}{960}, \frac{210}{960}, \frac{480}{960})$. Thus the expected payoff to action 1 is:

$$\frac{60}{960}(23) + \frac{210}{960}(47) + \frac{210}{960}(50) + \frac{480}{960}(41) = \frac{1381}{32}$$

and the expected payoff to action 3 is:

$$\frac{60}{960}(3) + \frac{210}{960}(29) + \frac{210}{960}(52) + \frac{480}{960}(50) = \frac{1373}{32}$$

Thus action 2 is the best response.

Thus the best response to strategy profile s^k is at most s^{k+1} . Now consider what happens when we iteratively delete strictly dominated strategies. One may verify that if strategy s survives $k + 1$ rounds of iterated deletion, then $s \leq s^k$. Thus if s survives iterated deletion of strictly dominated strategies, $s_i(t_i) = 0$ for all t_i .

Claim 2 Consider the incomplete information game where the prior probability is generated by w_2 , types $\{1, \dots, 960\}$ of each player have a dominant strategy to play action 3, and all other types have payoffs given by \mathbf{g} . This game has a unique strategy profile surviving iterated deletion of dominated strategies for each player: always play action 3.

The proof of claim 2 parallels the proof of claim 1, noting that if we define strategy profile s^k by:

$$s_i^k(t_i) = \begin{cases} 3, & \text{if } |t_i| \leq k + 240 \\ 2, & \text{if } k + 241 \leq |t_i| \leq k + 320 \\ 1, & \text{if } k + 321 \leq |t_i| \leq k + 480 \\ 0, & \text{if } k + 481 \leq |t_i| \end{cases} ,$$

one can verify that the best response to s^k is at least to s^{k+1} .

Thus we have an equilibrium argument that, depending on the interaction structure, different actions must spread. To obtain a best response dynamics result, let us consider sequences that satisfy our earlier requirements that one player switches behavior at a time (**S**), and players only switch to better responses (**B**); and let us add the requirements that someone's behavior keeps switching as long as we are not in equilibrium:

- **Revision [R]**: if $s_i^{\tau-1}(t_i) \notin \beta_i(\pi_i^{\tau-1}[t_i])$ for any i and t_i , then $s^\tau \neq s^{\tau-1}$

Now suppose that payoffs are given by \mathbf{g} always. then we have:

Claim 3. If $\{s^\tau\}_{\tau=0}^\infty$ is a strategy sequence satisfying properties **B** and **R**, given w_1 and u^* , and, for some k , $s_i^0(t_i) = 0$ for all $t_i \in \{k, k + 1, \dots, k + 1920\}$, then $s_i^\tau(t_i) \rightarrow 0$ for all i and t_i .

Claim 4. If $\{s^\tau\}_{\tau=0}^\infty$ is a strategy sequence satisfying properties **B** and **R**, given w_2 and u^* , and, for some k , $s_i^0(t_i) = 3$ for all $t_i \in \{k, k + 1, \dots, k + 1920\}$, then $s_i^\tau(t_i) \rightarrow 3$ for all i and t_i .

The proofs are very similar to those for claims 1 and 2.

Blume [1993] and Ellison [1993] have emphasized how local interaction can lead to very fast convergence to a long run equilibrium. Ellison gave a non-constructive argument why the limiting behavior must depend on whether interaction is local or global. The above example provides a constructive demonstration why different limiting behavior must result from different local interaction structures.

8. Extensions

8.1. Bounded Interactions

We assumed that interaction games had unbounded probability mass, i.e., that $\sum_{t \in T} P(t)$ is non-convergent. This was a convenient mathematical device. If instead we let $\sum_{t \in T} P(t) = 1$, the above results would continue to hold with alternative (slightly more complicated) definitions of uninvasibility along the following lines.

Definition 8.1. *Action profile a^* is robust in \mathbf{g} if for every $\delta > 0$, there exists $\varepsilon > 0$ such that every interaction game where $\sum_{t \in |\mathbf{g}|} P(t) \geq 1 - \varepsilon$ has an equilibrium where $\sum_{\{t: \sigma_i(a_i^* | t_i) = 1\}} P(t) \geq 1 - \delta$.*

Definition 8.2. *Action profile a^* is ε -invaded under strategy sequence $\{s^\tau\}_{\tau=0}^\infty$ if a^* is initially played by almost all players but is eventually not played by an infinite mass of players; i.e., writing $q^\tau = \sum_{\{t: s^\tau(t) \neq a^*\}} P(t)$, $q^0 \leq \varepsilon$, but $q^\tau \rightarrow 1$ as $\tau \rightarrow \infty$.*

Definition 8.3. *Action profile a^* is uninvasible in \mathbf{g} if there exists $\varepsilon > 0$ such that for every local interaction system and sequence $\{s^\tau\}_{\tau=0}^\infty$ satisfying property **S** and **B**, action profile a^* is not ε -invaded.*

8.2. The One Population Case

[To be completed]. We considered the case where players had different roles. The analysis goes through essentially unchanged in the case with a single role. Note that almost all our examples dealt with the symmetric payoff complete information games that arise in one population local interaction models.

8.3. A Unifying Sufficient Condition

Definition 8.4. Action profile a^* is a *monotonic potential maximizer* (MP-maximizer) of \mathbf{g} if there exists a potential function $v : A \rightarrow \mathbf{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$, and for each i , $\mu_i : A_i^2 \rightarrow \mathbf{R}_+$, such that for all $i = 1, \dots, I$, $a_i \neq a_i^*$, $a_i' \in A_i$ and $a_{-i} \in A_{-i}$,

$$v(a_i, a_{-i}) - v(a_i', a_{-i}) \geq \mu_i(a_i, a_i') [g_i(a_i, a_{-i}) - g_i(a_i', a_{-i})].$$

In the special case where $\mu_i(a_i, a_i')$ is independent of a_i and a_i' and the inequalities hold with equality, this reduces to the WP-maximizer conditions. If $v(a) = v(a')$ whenever $\{i : a_i = a_i^*\} = \{i : a_i' = a_i^*\}$, and $\mu_i(a_i, a_i') = 0$ if $a_i' \neq a_i^*$, then this condition reduces to the CP-maximizer conditions.

8.4. Continuum Actions

The arguments in these notes are discrete action arguments. However, we note the continuum action analogues of the potential conditions, as these will be important in Frankel, Morris and Pauzner [1999]. Thus suppose that each $A_i = [0, 1]$.

The P-maximizer and WP-maximizer conditions are unchanged. An interesting example of a continuum potential game is a *Two Player Expected Action Game*, where player i 's best response is to choose $\beta(E_i a_j)$, where $E_i a_j$ is his expectation of his opponent's action and β is a continuous and strictly increasing function with $\beta(0) = 0$ and $\beta(1) = 1$. This best response behavior can be derived from the following symmetric payoff functions:

$$\begin{aligned} g_1(a_1, a_2) &= a_1 a_2 - \int_{x=0}^{a_1} \beta^{-1}(x) dx \\ g_2(a_1, a_2) &= a_1 a_2 - \int_{x=0}^{a_2} \beta^{-1}(x) dx \end{aligned}$$

This is an example of a joint output games (see section 3.2.6). A potential function for this game is

$$v(a_1, a_2) = a_1 a_2 - \int_{x=0}^{a_1} \beta^{-1}(x) dx - \int_{x=0}^{a_2} \beta^{-1}(x) dx;$$

(a, a) is the potential maximizing action profile if $a \in \arg \max_{\tilde{a}} \int_{x=0}^{\tilde{a}} [\beta(x) - x] dx$, i.e., if a maximizes the area under the pure strategy best response function.

The characteristic potential condition is not interesting in the case of continuum actions: it will not be satisfied in any interesting cases.

The local potential maximizer condition generalizes in a very natural way.

Definition 8.5. *Action profile a^* is a local potential maximizer (LP-maximizer) if there exists $v : A \rightarrow \mathbf{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ and, for each i , $\mu_i : A_i \rightarrow \mathbf{R}_+$, such that for all $i = 1, \dots, I$ and $a_{-i} \in A_{-i}$,*

$$\begin{aligned} \frac{dv}{da_i}(a_i, a_{-i}) &\geq \mu_i(a_i) \frac{dg_i}{da_i}(a_i, a_{-i}) \text{ if } a_i > a_i^* \\ \text{and } \frac{dv}{da_i}(a_i, a_{-i}) &\leq \mu_i(a_i) \frac{dg_i}{da_i}(a_i, a_{-i}) \text{ if } a_i < a_i^* \end{aligned}$$

The alternative characterization of the LP-maximizer condition becomes:

- Action profile a^* is a LP-maximizer if and only if there exists $v : A \rightarrow \mathbf{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ and, for each i and $a_i > a_i^*$,

$$\int_{a_{-i}} \lambda_i(a_{-i}) \frac{dg_i}{da_i}(a_i, a_{-i}) < 0$$

$$\text{for all } \lambda_i \in \Delta(A_{-i}) \text{ such that } \int_{a_{-i}} \frac{dv}{da_i}(a_i, a_{-i}) \lambda_i(a_{-i}) < 0.$$

Symmetrically, for each i and $a_i < a_i^*$,

$$\int_{a_{-i}} \lambda_i(a_{-i}) \frac{dg_i}{da_i}(a_i, a_{-i}) > 0$$

$$\text{for all } \lambda_i \in \Delta(A_{-i}) \text{ such that } \int_{a_{-i}} \frac{dv}{da_i}(a_i, a_{-i}) \lambda_i(a_{-i}) > 0.$$

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