Incomplete Information Correlated Equilibrium

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- [Bergemann PSE talk spring 2010]
- Game Theoretic Predictions are very sensitive to "higher order beliefs" or "expectational coordination" or (equivalently) information structure
- Higher order beliefs are rarely observed
- What predictions can we make and analysis can we do if we do not observe higher order beliefs?

- Fix "payoff relevant environment"
  - = action sets, payoff-relevant variables ("states"), payoff functions, distribution over states
  - = incomplete information game without higher order beliefs about states

- Assume payoff relevant environment is observed by the econometrician

- Analyze what could happen for all possible higher order beliefs (maintaining common prior assumption and equilibrium assumptions)

- Make set valued predictions about joint distribution of actions and states

Incomplete Information Correlated Equilibrium

- Set valued prediction is an incomplete information version of correlated equilibrium

- "Robust Predictions" paper and Dirk's talk:
  - develop robust predictions agenda in an important class of games for applications (continuum player, linear best response, normal distributions, etc....)

- This paper and talk
  - describe "epistemic" result in general finite game setting
  - illustrate robust predictions agenda in other examples
    - first price auction
    - single person binary choice
  - describe relation to existing definitions of incomplete information correlated equilibrium (Forges 93, 06)
  - discuss comparative statics of background information (Gossner 00, Lehrer-Rosenberg-Schmaya 06, 10)
  - incentive compatibility and signed covariance (Chwe 06)

- players $i = 1, ..., I$
- (payoff relevant) states Θ

- actions $(A_i)_{i=1}^{I}$
- utility functions $(u_i)_{i=1}^{I}$, each $u_i : A \times \Theta \to \mathbb{R}$
- state distribution $\psi \in \Delta(\Theta)$
- $G = \left((A_i, u_i)_{i=1}^{I}, \psi \right)$
- ("basic game", "pre-game")

- signals (types) $(T_i)_{i=1}^{I}$
- signal distribution $\pi : \Theta \rightarrow \Delta (T_1 \times T_2 \times ... \times T_I)$
- $S = \left((T_i)_{i=1}^{I} , \pi \right)$
- ("higher order beliefs", "type space," "signal space")

- The pair (G, S) is a standard game of incomplete information
- A (behavioral) strategy for player i is a mapping
  $b_i : T_i \to \Delta(A_i)$

DEFINITION. A strategy profile b is a Bayes Nash Equilibrium (BNE) of (G, S) if, for all i, t_i and a_i with $b_i(a_i|t_i) > 0$,

$$\sum_{a_{-i}\in A_{-i}, t_{-i}\in T_{-i}, \theta\in\Theta} \left(\prod_{j\neq i} b_j(a_j|t_j) \right) u_i((a_i, a_{-i}), \theta) \, \psi(\theta) \, \pi(t|\theta)$$

$$\geq \sum_{a_{-i}\in A_{-i}, t_{-i}\in T_{-i}, \theta\in\Theta} \left(\prod_{j\neq i} b_j(a_j|t_j) \right) u_i((a_i', a_{-i}), \theta) \, \psi(\theta) \, \pi(t|\theta)$$

for all $a_i' \in A_i$.

DEFINITION. An action state distribution $\mu \in \Delta(A \times \Theta)$ is a BNE action state distribution of (G, S) if there exists a BNE strategy profile b such that

$$\mu(a, \theta) = \sum_{t \in T} \psi(\theta)\, \pi(t|\theta) \left(\prod_{i=1}^{I} b_i(a_i|t_i) \right).$$

DEFINITION. An action state distribution $\mu \in \Delta(A \times \Theta)$ is a Bayes Correlated Equilibrium (BCE) of G if is *obedient*, i.e., for each i, a_i and a_i',

$$\sum_{\theta \in \Theta} u_i\left((a_i, a_{-i}), \theta\right) \mu\left((a_i, a_{-i}), \theta\right)$$

$$\geq \sum_{\theta \in \Theta} u_i\left((a_i', a_{-i}), \theta\right) \mu\left((a_i, a_{-i}), \theta\right)$$

and *consistent*, i.e., for each θ

$$\sum_{a \in A} \mu(a, \theta) = \psi(\theta).$$

PROPOSITION 1. Action state distribution μ is a BNE action state distribution of (G, S) for some S if and only if it is a BCE of G.

c.f. Aumann 1987, Forges 1993

- We know players observe S but we dont know what additional information they observe.
- Augmented information system $\widetilde{S} = \left((Z_i)_{i=1}^I, \phi \right)$, where
  $$\phi : \Theta \times T \to \Delta(Z)$$
- Augmented information information game $\left(G, S, \widetilde{S} \right)$
- Player i's behavioral strategy $\tau_i : T_i \times Z_i \to \Delta(A_i)$
  DEFINITION. A strategy profile τ is a Bayes Nash Equilibrium (BNE) of (G, S, S') if, for all i, t_i, z_i and a_i with $b_i(a_i|t_i, z_i) > 0$,

  $$\sum_{a_{-i}, t_{-i}, z_{-i}, \theta} u_i\left((a_i, b_{-j}(t_{-j}, z_{-j})), \theta \right) \psi(\theta) \pi(t|\theta) \phi(z|t, \theta)$$

  $$\geq \sum_{a_{-i}, t_{-i}, z_{-i}, \theta} u_i\left((a_i', b_{-j}(t_{-j}, z_{-j})), \theta \right) \psi(\theta) \pi(t|\theta) \phi(z|t, \theta)$$

  for all $a_i' \in A_i$.

DEFINITION. An action type state distribution $\nu \in \Delta(A \times T \times \Theta)$ is a BNE action type state distribution of (G, S, S') if there exists a BNE strategy profile τ such that

$$\mu(a, t, \theta) = \psi(\theta)\,\pi(t|\theta) \sum_{z \in Z} \left(\prod_{i=1}^{I} b_i(a_i|t_i, z_i) \right) \phi(z|t, \theta).$$

DEFINITION. An action type state distribution $\nu \in \Delta \left(A \times T \times \Theta \right)$ is a Bayes Correlated Equilibrium (BCE) of (G, S) it is *obedient*, i.e., for each i, t_i, a_i and a'_i,

$$\sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} u_i \left((a_i, a_{-i}), \theta \right) \nu \left((a_i, a_{-i}), (t_i, t_{-i}), \theta \right)$$

$$\geq \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} u_i \left(\left(a'_i, a_{-i} \right), \theta \right) \nu \left((a_i, a_{-i}), (t_i, t_{-i}), \theta \right)$$

and *consistent*, i.e.,

$$\sum_{a \in A} \nu \left(a, t, \theta \right) = \psi \left(\theta \right) \pi \left(t | \theta \right).$$

- If S is null information system, reduces to earlier definition.

PROPOSITION 2. Action type state distribution ν is a BNE action type state distribution of (G, S, S') for some S' if and only if it is a BCE of (G, S).

c.f. Forges 1993

- If S is null information system, reduces to Proposition 1.

- Two bidders
- $\Theta = \left\{ \frac{1}{K}, \frac{2}{K}, ..., \frac{K-1}{K}, 1 \right\}^2$ with element $\theta = (\theta_1, \theta_2) \in \Theta$.

$G = \left((A_i, u_i)_{i=1,2}, \psi \right)$ where

- $A_1 = A_2 = \left\{ 0, \frac{1}{M}, \frac{2}{M}, ..., \frac{M-1}{M}, 1 \right\}$
- utility functions

$$u_i \left((a_i, a_j), (\theta_i, \theta_j) \right) = \begin{cases} \theta_i - a_i, \text{ if } \theta_i > \theta_j \\ \frac{1}{2} (\theta_i - a_i), \text{ if } \theta_i = \theta_j \\ 0, \text{ if } \theta_i < \theta_j \end{cases} .$$

- $\psi(\theta) = \frac{1}{K^2}$ for each $\theta \in \Theta$;

$S = \left((T_i)_{i=1,2}, \pi \right)$

- $T_1 = T_2 = \left\{ \frac{1}{K}, \frac{2}{K}, ..., \frac{K-1}{K}, 1 \right\}$
- signal distribution

$$\pi(t|\theta) = \left\{ \begin{array}{l} 1, \text{ if } t = \theta \\ 0, \text{ otherwise} \end{array} \right. .$$

- Distribution $\nu \in \Delta \left(A \times T \times \Theta \right)$
- Consistency: for all t and θ

$$\sum_{a \in A} \nu \left(a, t, \theta \right) = \left\{ \begin{array}{l} \frac{1}{K^2}, \text{ if } t = \theta \\ 0, \text{ otherwise} \end{array} \right.$$

- Obedience: for all i, θ_i, t_i, a_i and a_i'.

$$\sum_{a_j, t_j, \theta_j} \nu \left(\left(a_i, a_j \right), \left(t_i, t_j \right), \left(\theta_i, \theta_j \right) \right) u_i \left(\left(a_i, a_j \right), \left(\theta_i, \theta_j \right) \right)$$

$$\geq \sum_{a_j, t_j, \theta_j} \nu \left(\left(a_i, a_j \right), \left(t_i, t_j \right), \left(\theta_i, \theta_j \right) \right) u_i \left(\left(a_i', a_j \right), \left(\theta_i, \theta_j \right) \right)$$

- Let $K = 3$ and $M = 8$.
- unique BNE and revenue minimizing BCE have marginals:

$a_i \ \ \theta_i$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$
0	0	0	0
$\frac{1}{8}$	0	0	0
$\frac{2}{8}$	1	0	0
$\frac{3}{8}$	0	1	0
$\frac{4}{8}$	0	0	1

$a_i \ \ \theta_i$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$
0	0.18	0	0
$\frac{1}{8}$	0.56	0	0
$\frac{2}{8}$	0.26	0.50	0
$\frac{3}{8}$	0	0.50	0.70
$\frac{4}{8}$	0	0	0.30

- BNE revenue: 0.43; BCE revenue: 0.33

- one player
- two states $\Theta = \{\theta_0, \theta_1\}$

$G = (A, u, \psi)$ with

- $A = \{a_0, a_1\}$
- payoff function u

	θ_0	θ_1
a_0	κ	$1 - \kappa$
a_1	0	0

- $\psi(\theta_0) = \xi$ and $\psi(\theta_1) = 1 - \xi$

$S = (T, \pi)$

- arbitrary T
- write $\pi_k(t)$ for $\pi(t|\theta_k)$
- With arbitrary informations structure, close relation to Gentskow and Kaminica

- "mediator" recommends action a_1 if the player observes signal t in state θ_k with probability $\beta_k(t)$ (and thus a_0 with probability $1 - \beta_k(t)$).
- mediator's behavior is given by (β_1, β_2) with each $\beta_k : T \to [0, 1]$.
- player observing t advised to play a_1 attaches probability

$$\frac{\xi \pi_0(t) \beta_0(t)}{\xi \pi_0(t) \beta_0(t) + (1 - \xi) \pi_1(t) \beta_1(t)}$$

  to state θ_0
- follows the recommendation if

$$(1 - \xi) \pi_1(t) \beta_1(t) (1 - \kappa) \geq \xi \pi_0(t) \beta_0(t) \kappa$$

  or

$$\beta_1(t) \geq \left(\frac{\kappa}{1 - \kappa}\right) \left(\frac{\xi}{1 - \xi}\right) \left(\frac{\pi_0(t)}{\pi_1(t)}\right) \beta_0(t).$$

- player observing t advised to play a_0 attaches probability

$$\frac{\xi \pi_0 (t) (1 - \beta_0 (t))}{\xi \pi_0 (t) (1 - \beta_0 (t)) + (1 - \xi) \pi_1 (t) (1 - \beta_1 (t))}$$

  to state θ_0

- follows the recommendation if

$$(1 - \xi) \pi_1 (t) (1 - \beta_1 (t)) (1 - \kappa) \leq \xi \pi_0 (t) (1 - \beta_0 (t)) \kappa$$

  or

$$(1 - \xi) \pi_1 (t) \beta_1 (t) (1 - \kappa) \geq \xi \pi_0 (t) \beta_0 (t) \kappa + (1 - \xi) \pi_1 (t) (1 - \kappa)$$

- follows the recommendation if

$$(1 - \xi) \pi_1(t) (1 - \beta_1(t)) (1 - \kappa) \leq \xi \pi_0(t) (1 - \beta_0(t)) \kappa$$

or

$$(1 - \xi) \pi_1(t) \beta_1(t) (1 - \kappa) \geq \xi \pi_0(t) \beta_0(t) \kappa + (1 - \xi) \pi_1(t) (1 - \kappa)$$

or

$$\beta_1(t) \geq \left\{ \begin{array}{c} \left(\frac{\kappa}{1-\kappa}\right) \left(\frac{\xi}{1-\xi}\right) \left(\frac{\pi_0(t)}{\pi_1(t)}\right) \beta_0(t) \\ + \left(1 - \left(\frac{\kappa}{1-\kappa}\right) \left(\frac{\xi}{1-\xi}\right) \left(\frac{\pi_0(t)}{\pi_1(t)}\right)\right) \end{array} \right\}.$$

- obedience constraints combine to

$$\beta_1\left(t\right) \geq \left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\xi}{1-\xi}\right)\beta_0\left(t\right) + \max\left(0, 1 - \left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\xi}{1-\xi}\right)\right)$$

- Now distribution $\nu \in \Delta\left(A \times T \times \Theta\right)$ is a Bayes Correlated Equilibrium if and only if

$$\nu\left(a,t,\theta\right) = \begin{cases} \left(1-\xi\right)\pi_1\left(t\right)\beta_1\left(t\right), \text{ if } \left(a,\theta\right) = \left(a_1,\theta_1\right) \\ \left(1-\xi\right)\pi_1\left(t\right)\left(1-\beta_1\left(t\right)\right), \text{ if } \left(a,\theta\right) = \left(a_0,\theta_1\right) \\ \xi\pi_0\left(t\right)\beta_0\left(t\right), \text{ if } \left(a,\theta\right) = \left(a_1,\theta_0\right) \\ \xi\pi_0\left(t\right)\left(1-\beta_0\left(t\right)\right), \text{ if } \left(a,\theta\right) = \left(a_0,\theta_0\right) \end{cases}$$

for some $\left(\beta_1, \beta_2\right)$ satisfying these top inequality.

- Ex ante utility

$$\sum_{t \in T} \left(\xi \kappa \pi_0 \left(t \right) \left(1 - \beta_0 \left(t \right) \right) + \left(1 - \xi \right) \left(1 - \kappa \right) \pi_1 \left(t \right) \beta_1 \left(t \right) \right).$$

- Maximized setting $\beta_0 \left(t \right) = 0$ and $\beta_1 \left(t \right) = 1$ for all $t \in T$, giving maximum ex ante utility

$$\overline{U} \left(S \right) = \xi \kappa + \left(1 - \xi \right) \left(1 - \kappa \right).$$

Since we have that

$$(1 - \xi) \pi_1 (t) \beta_1 (t) (1 - \kappa) - \xi \pi_0 (t) \beta_0 (t) \kappa$$
$$\geq \max \{0, (1 - \xi) \pi_1 (t) (1 - \kappa) - \xi \pi_0 (t) \kappa\}$$

minimum utility minimizing BCE is attained by setting
$\beta_0 (t) = \beta_1 (t) = 0$ if

$$\left(\frac{\kappa}{1 - \kappa} \right) \left(\frac{\xi}{1 - \xi} \right) \left(\frac{\pi_0 (t)}{\pi_1 (t)} \right) \leq 1$$

and $\beta_0 (t) = \beta_1 (t) = 1$ if

$$\left(\frac{\kappa}{1 - \kappa} \right) \left(\frac{\xi}{1 - \xi} \right) \left(\frac{\pi_0 (t)}{\pi_1 (t)} \right) > 1$$

This gives minimum ex ante utility

$$\underline{U}(S) = \left\{ \begin{array}{c} \xi \kappa \Pr\left(\left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\xi}{1-\xi}\right)\left(\frac{\pi_0(t)}{\pi_1(t)}\right) > 1\right) \\ + (1-\xi)(1-\kappa)\Pr\left(\left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\xi}{1-\xi}\right)\left(\frac{\pi_0(t)}{\pi_1(t)}\right) \leq 1\right) \end{array} \right\}.$$

Notice that more information will increase the minimum ex ante utility and not change the maximum ex ante utility.

- Now consider the probability that action a_1 is chosen,

$$\sum_{t \in T} \left(\xi \pi_0 \left(t \right) \beta_0 \left(t \right) + \left(1 - \xi \right) \pi_1 \left(t \right) \beta_1 \left(t \right) \right).$$

- This is maximized by setting $\beta_0 \left(t \right) = \beta_1 \left(t \right) = 1$ if

$$\left(\frac{\kappa}{1 - \kappa} \right) \left(\frac{\xi}{1 - \xi} \right) \left(\frac{\pi_0 \left(t \right)}{\pi_1 \left(t \right)} \right) \leq 1$$

and $\beta_1 \left(t \right) = 1$ and $\beta_0 \left(t \right)$ solves

$$\beta_0 \left(t \right) = \left(\frac{1 - \kappa}{\kappa} \right) \left(\frac{1 - \xi}{\xi} \right) \left(\frac{\pi_1 \left(t \right)}{\pi_0 \left(t \right)} \right)$$

otherwise.

- Thus the maximum probability of action a_1 in a *BCE* is

$$\overline{\Pi}(S) = 1 - \Pr\left(\left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\xi}{1-\xi}\right)\left(\frac{\pi_0(t)}{\pi_1(t)}\right) > 1\right)\left(1 - \left(\frac{1-\kappa}{\kappa}\right)\right.$$

- This is minimized by setting $\beta_0(t) = \beta_1(t) = 0$ if

$$\left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\xi}{1-\xi}\right)\left(\frac{\pi_0(t)}{\pi_1(t)}\right) > 1$$

and $\beta_0(t) = 0$ and $\beta_1(t)$ solves

$$\beta_1(t) = 1 - \left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\xi}{1-\xi}\right)\left(\frac{\pi_0(t)}{\pi_1(t)}\right)$$

otherwise.

- Thus the minimum probability of action a_1 in a *BCE* is $\underline{\Pi}(S)$

$$\Pr\left(\left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\xi}{1-\xi}\right)\left(\frac{\pi_0(t)}{\pi_1(t)}\right) \leq 1\right)$$
$$\times\left(1-\left(\frac{\kappa}{1-\kappa}\right)\left(\frac{\xi}{1-\xi}\right)\left(\frac{\pi_0(t)}{\pi_1(t)}\right)\right)$$
$$\times\left(\frac{(1-\xi)\pi_1(t)}{\xi\pi_0(t)+(1-\xi)\pi_1(t)}\right)$$

- Forges (1993): "Five Legitimate Definitions of Correlated Equilibrium in Games with Incomplete Information"; Forges (2006) gives #6

- This definition is "illegitimate" because it fails "join feasibility"

DEFINITION. Action type state distribution ν is join feasible for (G, S) if there exists $f : T \to \Delta(A)$ such that

$$\nu(a, t, \theta) = \psi(\theta)\,\pi(t|\theta)\,f(a|t)$$

for each a, t, θ.

DEFINITION. Action type state distribution ν is a Bayesian solution of (G, S) if it is a BCE (i.e., is obedient and consistent) and also satisfies join feasibility.

- This is Forges' weakest solution concept
- If there is a dummy player who observes θ, then join feasibility is "free" and BCE = Bayesian solution

- $I = 1$
- $\Theta = \left\{ \theta, \theta' \right\}$
- $\psi\left(\theta\right) = \psi\left(\theta'\right) = \frac{1}{2}$
- Payoffs u_1

	θ	θ'
a_1	2	-1
a_1'	0	0

- unique Bayesian solution: $\mu\left(a_1, \theta\right) = \mu\left(a_1, \theta'\right) = \frac{1}{2}$
- a BCE: $\mu\left(a_1, \theta\right) = \mu\left(a_1', \theta'\right) = \frac{1}{2}$

DEFINITION. Distribution $\nu \in \Delta\left(A \times T \times \Theta\right)$ is belief invariant for (G, S) if, for all $t_i \in T_i$ and $a_i \in A_i$ such that

$$\nu\left(t_{-i}|\left(a_i, t_i\right)\right) = \frac{\displaystyle\sum_{\theta \in \Theta} \psi\left(\theta\right)\pi\left(\left(t_i, t_{-i}\right)|\theta\right)}{\displaystyle\sum_{\theta \in \Theta}\sum_{t'_{-i} \in T'_{-i}} \psi\left(\theta\right)\pi\left(\left(t_i, t'_{-i}\right)|\theta\right)}$$

for each $t_{-i} \in T_{-i}$.

- "the omniscient mediator can use his knowledge of the types to make his recommendations but the players should not be able to infer anything on the others' types from their recommendations."

DEFINITION. A probability distribution $\nu \in \Delta \left(A \times T \times \Theta \right)$ is a belief invariant Bayesian solution of (G, S) if it is a BCE that is join feasible and belief invariant.

DEFINITION. Distribution $\nu \in \Delta(A \times T \times \Theta)$ is agent normal form feasible for (G, S) if there exists $q \in \Delta(\Sigma)$ such that

$$\nu(a, t, \theta) = \psi(\theta) \pi(t|\theta) \sum_{\{\sigma \in \Sigma | \sigma(t) = a\}} q(\sigma)$$

for each $a \in A$, $t \in T$ and $\theta \in \Theta$.

DEFINITION. A probability distribution $\nu \in \Delta(A \times T \times \Theta)$ is an agent normal form correlated equilibrium of (G, S) if it is a BCE that is agent normal form feasible (and thus join feasible and belief invariant).

DEFINITION. Distribution $\nu \in \Delta (A \times T \times \Theta)$ is strategic form incentive compatible for (G, S) if there exists $q \in \Delta (\Sigma)$ such that

$$\nu \left(a, t, \theta \right) = \psi \left(\theta \right) \pi \left(t | \theta \right) \sum_{\{\sigma \in \Sigma | \sigma(t) = a\}} q \left(\sigma \right)$$

for each $a \in A$, $t \in T$ and $\theta \in \Theta$; and, for each $i = 1, .., I$, $t_i \in T_i$, $a_i \in A_i$ and $\sigma_i \in \Sigma_i$ such that $\sigma_i \left(t_i \right) = a_i$, we have

$$\sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} \psi \left(\theta \right) \pi \left(t | \theta \right) \left(\sum_{\{\sigma_{-i} \in \Sigma_{-i} | \sigma_{-i}(t_{-i}) = a_{-i}\}} q \left(\sigma_i, \sigma_{-i} \right) \right)$$

$$\geq \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} \psi \left(\theta \right) \pi \left(t | \theta \right) \left(\sum_{\{\sigma_{-i} \in \Sigma_{-i} | \sigma_{-i}(t_{-i}) = a_{-i}\}} q \left(\sigma_i, \sigma_{-i} \right) \right)$$

for all $a_i' \in A_i$.

DEFINITION. A probability distribution $\nu \in \Delta\left(A \times T \times \Theta\right)$ is a strategic form correlated equilibrium of (G, S) if it is a BCE that is strategic form incentive compatible (and thus agent normal form feasible, belief invariant and join feasible).

Write $\xi_\nu : T \times \Theta \rightarrow A$ for the mediator's recommendation strategy implied by $\nu \in \Delta(A \times T \times \Theta)$, so that, for each $t \in T$ and $\theta \in \Theta$ with $\displaystyle\sum_{a' \in A} \nu(a', t, \theta) > 0$,

$$\xi_\nu(a|t, \theta) = \frac{\nu(a, t, \theta)}{\displaystyle\sum_{a' \in A} \nu(a', t, \theta)}$$

for each $a \in A$.

DEFINITION. Distribution $\nu \in \Delta (A \times T \times \Theta)$ is truth telling for (G, S) if, for each $i = 1, .., I$ and $t_i \in T_i$, we have

$$\sum_{a \in A, t_{-i} \in T_{-i}, \theta \in \Theta} \psi (\theta) \pi ((t_i, t_{-i}) | \theta) \xi_\nu ((a_i, a_{-i}) | (t_i, t_{-i}), \theta)$$

$$\geq \sum_{a \in A, t_{-i} \in T_{-i}, \theta \in \Theta} \psi (\theta) \pi ((t_i, t_{-i}) | \theta) \xi_\nu ((\delta_i (a_i), a_{-i}) | (t'_i, t_{-i}), \theta)$$

for all $t'_i \in T_i$ and $\delta_i : A_i \rightarrow A_i$.

DEFINITION. A probability distribution $\nu \in \Delta (A \times T \times \Theta)$ is a communication equilibrium of (G, S) if it is a BCE that is join feasible and truth-telling.

- BCE > Bayesian solution > belief invariant Bayesian solution > agent normal form CE > strategic form CE
- Bayesian solution > communication equilibrium > strategic form CE

DEFINITION. Information system S' is less informed than S if there exist $\sigma : T \times \Theta \to \Delta \left(T' \right)$ and, for each i, $\phi_i : T_i \to \Delta \left(T'_i \right)$, such that

$$\pi' \left(t'|\theta \right) = \sum_{t \in T} \sigma \left(t'|t, \theta \right) \pi \left(t|\theta \right)$$

for each $t' \in T'$ and $\theta \in \Theta$, satisfying also that for each $i = 1, ..., I$, $t_i \in T_i$, $t'_i \in T'_i$,

$$\sum_{t'_{-i} \in T'_{-i}} \sigma \left(\left(t'_i, t'_{-i} \right) | \left(t_i, t_{-i} \right), \theta \right) = \phi_i \left(t'_i|t_i \right)$$

for all $t_{-i} \in T_{-i}$ and $\theta \in \Theta$.

Write $BCE(G, S)$ for the set of action state distributions induced by Bayes Correlated Equilibria of $BCE(G, S)$.

DEFINITION. Information system S' is BCE-larger than information system S if $BCE(G, S) \subseteq BCE(G, S')$ for all games G.

THEOREM. S' is BCE-larger than S if and only if S' is less informed than S.

Intuition: adding more information only adds incentive constraints

- Lehrer, Rosenberg and Shmaya 06, 10

DEFINITION. Information system S' garbling of S if there exists $\phi : T \rightarrow \Delta \left(T' \right)$ with

$$\pi' \left(t'|\theta \right) = \sum_{t \in T} \pi \left(t|\theta \right) \phi \left(t'|t \right)$$

for each $t' \in T'$ and $\theta \in \Theta$. Map ϕ is a garbling that transforms S to S'.

Garbling ϕ is *non-communicating* if, for each $i = 1, ..., I$, $t_i \in T_i$, $t_i' \in T_i$,

$$\sum_{t'_{-i} \in T'_{-i}} \phi\left(\left(t_i', t_{-i}'\right) \mid \left(t_i, t_{-i}\right)\right) = \sum_{t'_{-i} \in T'_{-i}} \phi\left(\left(t_i', t_{-i}'\right) \mid \left(t_i, \widetilde{t}_{-i}\right)\right)$$

for all $t_{-i}, \widetilde{t}_{-i} \in T_{-i}$.

DEFINITION. Information system S' is a non-communicating garbling of S if there exists a non-communicating garbling ϕ that transforms S into S'.

If garbling ϕ is a non-communicating garbling, we write $\phi_i\left(t_i'|t_i\right)$ for the (t_{-i} independent) probability of t_i' conditional on t_i, i.e.,

$$\phi_i\left(t_i'|t_i\right) \equiv \sum_{t'_{-i} \in T_i} \phi\left(\left(t_i', t_{-i}'\right) \mid \left(t_i, t_{-i}\right)\right)$$

Garbling ϕ is *coordinated* if there exist $\lambda \in \Delta\left(\{1, ..., K\}\right)$ and, for each i, $\phi_i : T_i \times \{1, ..., K\} \rightarrow \Delta\left(T_i\right)$ such that

$$\phi\left(t'|t\right) = \sum_{k=1}^{K} \lambda\left(k\right) \prod_{i=1}^{I} \phi_i\left(t_i'|t_i, k\right)$$

for each $t \in T$ and $t' \in T'$.

DEFINITION. Information system S' is a coordinated garbling of S if there exists a coordinated garbling ϕ that transforms S into S'.

A garbling is independent if it is coordinated with $K = 1$.

DEFINITION. Information system S' is an independent garbling of S if there exists a independent garbling ϕ that transforms S into S'

- if S' is a non-communicating garbling of S then S' is less informed than S
- converse is false

An information system S is larger that S' under a given equilibrium concept if, for every game G, every action state distribution induced by an equilibrium of (G, S') is also induced by an equilibrium of (G, S). Information system S is equivalent to S' under a given equilibrium concept if S is larger than S' and S' is larger than S under that equilibrium.

Lehrer, Rosenberg and Shmaya (2006) show two information systems are....

1. ...equivalent under Bayes Nash Equilibrium if and only if they are independent garblings of each other.

2. ...equivalent under Agent Normal Form Correlated Equilibrium if and only if they are coordinated garblings of each other.

3. ...equivalent under the Belief Invariant Bayesian Solution if and only if they are non-communicating garblings of each other.

Lehrer, Rosenberg and Shmaya (2010): restrict attention to common interest games: information system is S better than S' under a given solution concept if, for every common interest game G, the maximum (common) equilibrium payoff is higher in (G, S) than (G, S'). They show information system S is better than S' under

1. Bayes Nash Equilibrium if and only if S' is a coordinated garbling of S.
2. Agent Normal Form Correlated Equilibrium if and only if S' is a coordinated garbling of S.
3. Strategic Form Correlated Equilibrium if and only if S' is a coordinated garbling of S.
4. Belief Invariant Bayesian Solution if and only if S' is a non-communicating garbling of S.
5. Communication Equilibrium if and only if S' is a garbling of S.

An independent garbling ϕ is *faithful* if whenever for each i, $t_i \in T_i$ and $t_i' \in T_i'$ with $\phi_i\left(t_i'|t_i\right) > 0$, we have

$$\frac{\psi\left(\theta\right)\pi'\left(\left(t_i', t_{-i}'\right)|\theta\right)}{\displaystyle\sum_{\widetilde{t}_{-i}'\in T_{-i}', \widetilde{\theta}\in\Theta}\psi\left(\widetilde{\theta}\right)\pi'\left(\left(t_i', \widetilde{t}_{-i}'\right)|\widetilde{\theta}\right)}$$

$$= \frac{\psi\left(\theta\right)\displaystyle\sum_{t_{-i}\in T_{-i}}\pi\left(\left(t_i, t_{-i}\right)|\theta\right)\left(\displaystyle\prod_{j\neq i}\phi_j\left(t_j'|t_j\right)\right)}{\displaystyle\sum_{t_{-i}\in T_{-i}, \widetilde{\theta}\in\Theta}\psi\left(\widetilde{\theta}\right)\pi\left(\left(t_i, t_{-i}\right)|\widetilde{\theta}\right)}$$

for all $t_{-i}' \in T_{-i}'$ and $\theta \in \Theta$.

DEFINITION. Information system S is BNE-larger than information system S' if $BNE\,(G, S') \subseteq BNE\,(G, S)$ for all basic games G.

DEFINITION. Information system S' is a faithful independent garbling of S if there exists a faithful independent garbling ϕ that transforms S into S'.

THEOREM. Information System S is BNE-larger than S' if and only if S' is a faithful garbling of S.
NOTE: more information is BAD.

1. Gossner and Mertens (2001)
2. Peski (2008)

Chwe (2006): "Incentive Compatibility implies Signed Covariance"

- Fix G and BCE $\mu \in \Delta(A \times \Theta)$
- Fix player i and action $a_i^* \in A_i$.
- Consider random variables

$$
\begin{aligned}
I_{a_i^*}(a, \theta) &= \begin{cases} 1, \text{ if } a_i = a_i^* \\ 0, \text{ otherwise} \end{cases} \\
\Pi_{a_i^*, a_i'}(a, \theta) &= u_i\left(\left(a_i^*, a_{-i}\right), \theta\right) - u_i\left(\left(a_i', a_{-i}\right), \theta\right)
\end{aligned}
$$

- $I_{a_i^*}$ and $\Pi_{a_i^*, a_i'}$ have positive covariance

$$E_\mu\left(I_{a_i^*}\,\big|\,\left\{a_i^*, a_i'\right\}\right) = \frac{\displaystyle\sum_{a_{-i},\theta} \mu\left(\left(a_i^*, a_{-i}\right), \theta\right)}{\displaystyle\sum_{a_{-i},\theta} \mu\left(\left(a_i^*, a_{-i}\right), \theta\right) + \sum_{a_{-i},\theta} \mu\left(\left(a_i', a_{-i}\right), \theta\right)}$$

$$E_\mu \left(\Pi_{a_i^*, a_i'} \left| \left\{ a_i^*, a_i' \right\} \right. \right) = \frac{\left\{ \begin{array}{l} \displaystyle\sum_{a_{-i}, \theta} \mu \left((a_i^*, a_{-i}), \theta \right) \left(u_i \left((a_i^*, a_{-i}), \theta \right) - u_i \left((a' \right. \right. \\ \displaystyle + \sum_{a_{-i}, \theta} \mu \left((a_i', a_{-i}), \theta \right) \left(u_i \left((a_i^*, a_{-i}), \theta \right) - u_i \left((a \right. \right. \end{array} \right.}{\displaystyle\sum_{a_{-i}, \theta} \mu \left((a_i^*, a_{-i}), \theta \right) + \sum_{a_{-i}, \theta} \mu \left((a_i', a_{-i}) \right.}$$

$$E_\mu \left(I_{a_i^*} \Pi_{a_i^*, a_i'} \big| \left\{ a_i^*, a_i' \right\} \right) = \frac{\displaystyle\sum_{a_{-i}, \theta} \mu \left((a_i^*, a_{-i}), \theta \right) \left(u_i \left((a_i^*, a_{-i}), \theta \right) - u_i \left((a_i', \right. \right.}{\displaystyle\sum_{a_{-i}, \theta} \mu \left((a_i^*, a_{-i}), \theta \right) + \sum_{a_{-i}, \theta} \mu \left((a_i', a_{-i}), \right.}$$

- IC1: i prefers a_i^* to a_i' when advised to play a_i^* :

$$E_\mu \left(I_{a_i^*} \Pi_{a_i^*, a_i'} \left| \left\{ a_i^*, a_i' \right\} \right. \right) \geq 0$$

- IC2: i prefers a_i' to a_i^* when advised to play a_i':

$$E_\mu \left(\left(1 - I_{a_i^*} \right) \Pi_{a_i^*, a_i'} \left| \left\{ a_i^*, a_i' \right\} \right. \right) \leq 0$$

or

$$E_\mu \left(I_{a_i^*} \Pi_{a_i^*, a_i'} \left| \left\{ a_i^*, a_i' \right\} \right. \right) \geq E_\mu \left(\Pi_{a_i^*, a_i'} \left| \left\{ a_i^*, a_i' \right\} \right. \right).$$

Now the covariance of $I_{a_i^*}$ and $\Pi_{a_i^*, a_i'}$, conditional on $\{a_i^*, a_i'\}$, is

$$E_\mu \left(I_{a_i^*} \Pi_{a_i^*, a_i'} \mid \{a_i^*, a_i'\} \right) - E_\mu \left(I_{a_i^*} \mid \{a_i^*, a_i'\} \right) E_\mu \left(\Pi_{a_i^*, a_i'} \mid \{a_i^*, a_i'\} \right)$$

If $E_\mu \left(\Pi_{a_i^*, a_i'} \mid \{a_i^*, a_i'\} \right) \geq 0$, IC1 and $E_\mu \left(I_{a_i^*} \mid \{a_i^*, a_i'\} \right) \geq 0$ imply that this is non-negative. If $E_\mu \left(\Pi_{a_i^*, a_i'} \mid \{a_i^*, a_i'\} \right) \leq 0$, IC2 and $E_\mu \left(I_{a_i^*} \mid \{a_i^*, a_i'\} \right) \leq 1$ imply that this is non-negative.

1. Although there are many definitions of incomplete information correlated equilibrium, we can always create one more (but see Milchtaich 09)

2. Possible to obtain restrictions on behavior without know the information structure

3. We described an approach to doing so and related it to the incomplete information correlated equilibrium literature