Higher Order Expectations

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Iterated Expectations

- Alice's expectation of random variable f
- Bob's expectation of Alice's expectation of f
- Charlie's expectation of Bob's expectation of Alice's expectation of f
- ...etc...

Samet 1998a

Samet (1998a) "Iterated Expectations and the Common Prior Assumption" showed that

1. All such sequences of iterated expectations converge to a constant
2. The following two statements are equivalent:
   - A collection of agents' beliefs satisfy the belief-consistency (a.k.a., common prior assumption)
   - The limit of iterated expectations is independent of the sequence of agents

This Talk

1. Review Samet 1998a
2. Present two results about limit under weaker conditions:
    2.1 if we fix a external state space Θ, limit of iterated expectations of Θ-dependent random variables is order-independent if and only if beliefs satisfy (weaker) expectation-consistency condition
    2.2 limit depends only on last agent in the sequence if and only if beliefs satisfy higher-order expectation-consistency
3. One motive for extensions: resolves/clarifies paradox about iterated expectations interim foundation for common prior assumption?
4. Another (more substantive) motive later....

Setup

- Finite agents $I = \{1, .., n\}$
- Finite State Space Ω
- Agent i's information partition $\mathcal{P}_i$
  - Write $P_i(\omega)$ for the unique element of $\mathcal{P}_i$ containing ω
- Belief type function $t_i : \Omega \to \Delta(\Omega)$ with
  - t_i measurable w.r.t. $\mathcal{P}_i$
  - $t_i(\omega)$ has support within $P_i(\omega)$
- Assumptions of convenience:
  - Each $t_i(\omega)$ has full support within $P_i(\omega)$
  - Meet of $\mathcal{P}_i$ consists of whole state space (no non-trivial common knowledge events)

Random Variables and Priors

- A random variable is $f : \Omega \to \mathbb{R}$
  - For a measure $p \in \Delta(\Omega)$, write

    $$p.f = \sum_{\omega} p(\omega) f(\omega)$$

    for the expectation of f
  - Now i's expectation of f is $E_i f(\omega) := t_i(\omega) f$
  - $E_i f$ is a random variable
- A *prior* for agent i is a convex combination of his interim beliefs, i.e., for some weights $\alpha_i \in \Delta(\mathcal{P}_i)$,

  $$p_i = \sum_{P_i \in \mathcal{P}_i} \alpha_i(P_i) \, t_i(P_i)$$

Iterated Expectations

- A *sequence* of agents is a map $\sigma : \{0, 1, 2, ...\} \to I$
- An $I-$*sequence* is sequence in which each agent appears infinitely often
- For any sequence σ, write

$$S\left(k;\sigma\right) = E_{\sigma(k-1)}E_{\sigma(k-2)}....E_{\sigma(1)}E_{\sigma(0)}$$

- Observe that $S\left(k;\sigma\right) : \mathbb{R}^{\Omega} \to \mathbb{R}^{\Omega}$ (i.e., maps random variables into random variables)

Convergence

There is *convergence to a deterministic limit* along sequence σ if

$$S\left(k,\sigma\right)$$

converges to some limit operator

$$S\left(\infty,\sigma\right)$$

and $S\left(\infty,\sigma\right).f$ is constant (i.e., non-random) for all f.

Samet 1998a part 1: Convergence

THEOREM. (Samet 1998a part 1). There is convergence to a deterministic limit along every *I*-sequence.
INTUITION / IDEA OF PROOF:

- Suppose that we fixed a sequence of all players and looked at σ corresponding to repetitions of the sequence
- Each application of sequence of expectation operators "averages" across types
- Corresponds to a Markov process on Ω and has a deterministic limit (the ergodic distribution)
- More general *I*-sequences bounded by this repeated cycle

Samet 1998a part 2: Characterizing the Limit

Definition
There is full order-independence if $S(\infty, \sigma).f$ is independent of σ for every random variable f.

Definition
There is belief-consistency if there exists prior p such that p is a prior for each agent.

This is an interim expression of the common prior assumption.

THEOREM (Samet 1998a part 2). Full-order independence and belief-consistency are equivalent.

Common Prior Assumption Interim Foundation

- The common prior assumption is often assumed in settings where there is not a natural meaningful prior stage: e.g., universal type space, common knowledge foundation of correlated equilibrium.
- (1990s debate) So what is the interim meaning of the common prior assumption? (Gul, Aumann, Dekel and Gul....)
- Samet has *two* papers on this topic in 1998 GEB:
  - no trade (Feinberg 2000; Morris 1994; Samet 1998b)
  - iterated expectations (Samet 1998a)

But paradox about iterated expectations characterization?

- Two complaints about no trade characterization:
  1. self-referential?
  2. lacks natural interpretation of the common prior?
- But iterated expectations foundation suggests a paradox?
  - common prior assumption (a.k.a. belief consistency) depends on all beliefs (including 1's beliefs about how 2 and 3's beliefs are correlated)
  - iterated expectations should depend on strictly less information (shouldn't include 1's beliefs about how 2 and 3's beliefs are correlated?)
- Resolution of paradox: higher order expectations about what?

Higher Order Beliefs/Expectations about What?

- Samet (1998) concerns all random variables on a fixed finite type space
- Mertens and Zamir (1985) started from a fixed set of "external states" (parameters) Θ and use state space as a way of representing beliefs and higher orders beliefs about Θ; suppose that we analogously considered expectations about Θ-dependent random variables and higher-order expectations about such random variables.
- Formally, let Θ be an arbitrary partition of Ω
- Write $m\mathcal{Q}$ for collection of random variables measurable with respect to arbitrary partition $\mathcal{Q}$

Iterated Expectation "Universal Vector Space"

- Increasing collections of random variables corresponding to kth order iterated expectations
- First order expectations random variables:

$$V_i^1 = \{E_i f \,|f \in m\Theta\}$$

- $k+1$th order expectations

$$V_i^{k+1} = \left\{ E_i f \,\middle|\, f \in mV_j^k \text{ for some } j \right\}$$

- Write V_i^∞ for the limit of the V_i^k
- universal space of higher-order expectations is "smaller" than corresponding Mertens-Zamir space

Order-Independence

Definition

There is order-independence if $S(\infty, \sigma).f$ is independent of σ for every $f \in m\Theta$

Definition

Beliefs are expectation-consistent if there exists priors p_i for the agents such $p_i f = p_j f$ for all $f \in V_k^{\infty}$ for some k.

PROPOSITION 1. There is order-independence if and only if beliefs are expectation-consistent.

Discussion

- Expectation-consistency weakens belief-consistency because...
  - no restrictions on beliefs about "redundant" (in Mertens-Zamir higher order belief sense) states
  - even if no belief-redundant states, no restrictions on some events in space not relevant for higher order expectations
  - no restrictions on 1's beliefs about the correlation of 2 and 3's types......
- consider common knowledge of heterogeneous priors case

Weaker versions of order-independence

- Any interesting cases where order-independence fails?
- At least two interesting cases where order-independence fails but the limit depends only on the last agent in the sequence:
  - two agent case
  - common knowledge of heterogeneous priors
- We study this....

Higher-Order-Independence

Definition

There is higher-order-independence if $S(\infty, \sigma).f$ is independent of $\sigma(k)$, $k = 1, 2, ...$

Let $\widehat{V}_i^\infty \subset V_i^\infty$ be the set of vectors corresponding to i's expectations about some other agent's expectations (and in particular excluding i's first order expectations of Θ-measurable random variables...)

$$\widehat{V}_i^\infty = \left\{ E_i f \;\middle|\; f \in \cup_j V_j^\infty \right\}$$

Definition

Beliefs are higher-order expectation-consistent there exists priors p_i for the agents such $p_i f = p_j f$ for all i, j and $f \in \widehat{V}_k^\infty$ for some k.

PROPOSITION 1. There is higher-order-independence if and only if beliefs are higher-order expectation-consistent.

Proof of Proposition 2: Higher-Order Expectation-Consistency implies Higher-Order Independence

- Assume higher-order expectation-consistency with priors p_i
- Fix any I-sequence σ
- Write $I(\sigma)$ for measure corresponding to limit of iterated expectations for sequence σ
- Enough to show that $I(\sigma) f = p_{\sigma(0)} f$ for all $f \in m\Theta$

Higher Order Expectation-Consistency implies Higher-Order-Independence

- By definition of prior..

$$p_{\sigma(0)} f = p_{\sigma(0)} E_{\sigma(0)} f$$

- By higher-order expectation-consistency (since $E_{\sigma(0)} f \in mV^1_{\sigma(0)}$)

$$p_{\sigma(0)} E_{\sigma(0)} f = p_{\sigma(1)} E_{\sigma(0)} f$$

- and so

$$p_{\sigma(0)} f = p_{\sigma(1)} E_{\sigma(0)} f$$

Higher-Order Expectation-Consistency implies Higher-Order Independence

- Replacing f with $E_{\sigma(0)}f$ and moving labels, we have

$$p_{\sigma(0)}f = p_{\sigma(2)}E_{\sigma(1)}E_{\sigma(0)}f$$

- Iterating gives...

$$p_{\sigma(0)}f = p_{\sigma(k)}S\left(k,\sigma\right)f$$

for each k and

$$p_{\sigma(0)}f = I\left(\sigma\right)f$$

Higher-Order-Independence implies Higher-Order Expectation-Consistency

- Assume higher-order-independence
- We must construct p_i with $p_i g = p_j g$ for all i, j and $g \in \widehat{V}_k^\infty$
- Let $p_i = S(\infty, \sigma) E_i$ for arbitrary σ (i.e., independent of σ)

Higher-Order-Independence implies Higher-Order Expectation-Consistency

- Any $g \in \widehat{V}_i^{\infty}$ can be written as

$$E_{\sigma(k)} E_{\sigma(k-2)} .... E_{\sigma(1)} f$$

  for some finite sequence σ
- Enough to show that $p_i g = l(\sigma) E_i g$ is independent of σ and i.
- But substituting above expression

$$p_i g = l(\sigma) E_i E_{\sigma(k)} E_{\sigma(k-2)} .... E_{\sigma(1)} f$$

- Higher-Order Independence $\Rightarrow$ does not depend on σ or i except though $\sigma(1)$
- So $p_i g = p_j g$ for all i, j and $g \in \widehat{V}_k^{\infty}$ for some k

Next Talk: Linear Best Response Games on a Network

- Let $f \in m\Theta$
- Each agent i sets his action a_i equal to

$$(1-\beta) E_i f + \beta E_i \left(\sum_{j \neq i} \Gamma(i,j) E_i a_j \right)$$

  where each $a_j \in m\mathcal{P}_j$
- Γ is a (irreducible) network ($=$ Markov matrix)

Iterated Average Expectations and Rationalizable Play

- Let
$$A_i(0,\Gamma) = E_i$$
and
$$A_i(k+1,\Gamma) = \sum_{j \in I} \Gamma(i,j) A_j(k,\Gamma)$$
- Unique rationalizable action in game is
$$\sum_{k=0}^{\infty} \beta^k A_i(k,\Gamma) f$$
- As $\beta \to 1$, unique rationalizable action converges to $A_i(\infty,\Gamma) f$ (if defined) for each i

Preliminary Results

- Each $A_i(k, \Gamma)$ converges to deterministic limit $A(\infty, \Gamma)$ independent of i...
  - $A(\infty, \Gamma)$ is a implicit common prior
- Belief-consistency implies that limit is independent of Γ(immediately from Samet 98a)
  - in fact, expectation-consistency is sufficient
  - and equal to expectation under "common priors"...
- "De-Coupling": Higher-order belief-consistency implies

  $$A(\infty, \Gamma) = \sum_i e_i(\Gamma) p_i$$

  for some p_i where $e_i(\Gamma)$ are network centrality weights,

  $$e_i(\Gamma) = \sum_j e_j(\Gamma) \Gamma(j, i)$$

- Ben will tell you more results...
- I will mention subtleties in establishing these preliminary results as corollaries of properties of Samet I-sequences...

Forward versus Backward Looking Iterated Expectations

- Samet 98a looked at "backward looking" iterated expectations

$$S\left(k;\sigma\right)=E_{\sigma(k-1)}E_{\sigma(k-2)}....E_{\sigma(1)}E_{\sigma(0)}$$

- We could have looked at forward looking version....

$$F\left(k,\sigma\right)=E_{\sigma(0)}E_{\sigma(1)}....E_{\sigma(k-2)}E_{\sigma(k-1)}$$

- Obviously does not converge: in case of complete information....

$$F\left(k,\sigma\right)=E_{\sigma(k-1)}$$

- Our iterated average expectations are forward looking.....

Stochastic Interpretation of Iterated Average (Forward Looking) Expectations

- Pick sequence of players using Γ, i.e., fix $\sigma(0)$ and draw $\sigma(1)$ according to $\Gamma(i, \cdot)$ and so on...
- Now $A_i(k, \Gamma)$ equals $\mathbb{E}_\sigma F(k, \sigma)$ where $\sigma(0) = i$ and we pick (forward) sequence σ as above

Iterated Average Backward Looking Expectations

- Fix another network $\widetilde{\Gamma}$
- Let
$$R_i\left(0,\widetilde{\Gamma}\right)=E_i$$
- Let $R_i\left(k+1,\widetilde{\Gamma}\right)$ be j's expectation of $R_i\left(k,\widetilde{\Gamma}\right)$ where j is picked according to $\widetilde{\Gamma}$
- Thus $R_i\left(k,\Gamma\right)$ equal $\mathbb{E}_\sigma S\left(k,\sigma\right)$ where $\sigma\left(0\right)=i$ and we pick (backward) sequence σ according to Markov process $\widetilde{\Gamma}$

Iterated Average Expectations

- Corresponding to $\widetilde{\Gamma}$, the time reversal of Γ defined by

$$\widetilde{\Gamma}(i,j) = \frac{e_j(\Gamma)\,\Gamma(j,i)}{e_i(\Gamma)}$$

  where $e_i(\Gamma)$ are the agents' network centralities
- Limiting behavior of $A_i(k,\Gamma)$ and $R_i\left(k,\widetilde{\Gamma}\right)$ are related
- higher-order-independence (=higher-order expectation-consistency) implies

$$A(\infty,\Gamma) = \sum_i e_i(\Gamma)\, R_i\left(\infty,\widetilde{\Gamma}\right)$$

- This is de-coupling
- Other results (convergence, network independence) can also be proved using this connection

Takeaways?

- Samet (1998a) is a remarkable and important result
- Higher-order expectations should be studied with the seriousness of higher-order beliefs?
  - Syntactic approach and universal space? Interpretation of order-independence is subtle.....
  - Universal belief space of Mertens and Zamir (1985) is the "right" space for studying rationalizable behavior (Dekel, Fudenberg and Morris (2007)); universal expected space is the right space for studying linear best response games / or perturbations of smooth games...
- Network defines a natural way of generating an implicit common prior
- Markov process view of beliefs insight is invaluable
  - In studying expectations, natural (as in next talk) to use Markov process on union of types instead of state space / type profiles... removes redundancies and provides unified analysis of networks and incomplete information, see Morris (2000) "Contagion" building on Monderer and Samet (1989)....