Cost Based Nonlinear Pricing

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- optimal nonlinear pricing,
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  - depends heavily on information about demand distribution
  - e.g., optimal mark-up is equal to reciprocal of demand elasticity

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- we will devise informationally minimal pricing policy that:
    1. is independent of specific distribution of willingness-to-pay
    2. exhibits profit guarantee across all distributions
- without any restrictions on demand, such as moment restrictions, support restrictions, etc., profit guarantee must be relative (or proportional) rather than absolute.

Profit Guarantee as Competitive Ratio

- a mechanism will be evaluated by the ratio of the realized profit to feasible surplus (=complete information / first-degree price discrimination profit)
- the profit guarantee / "competitive ratio" is the infinum of this ratio across all demand distributions
- term originated in analysis of online vs. offline algorithms to express related informational constraints

Two Classes of Pricing Problems

1. quality differentiated pricing problems
   Mussa and Rosen (1978)
   - linear willingness-to-pay for quality
   - cost is increasing, convex function of quality
2. quantity differentiated pricing
   Maskin and Riley (1984)
   - concave willingness-to-pay for quantities
   - constant marginal cost of producing additional units

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1. first pass: iso-elastic cost function
2. then, general cost functions

First Result: Positive Profit Guarantee

- profit guarantee / competitive ratio is strictly positive and bounded away from zero
- profit guarantee / competitive ratio is a simple monotone function of cost elasticity
- profit guarantee / competitive ratio is sharp:
    - identify demand distribution under which robust policy coincides with Bayes optimal mechanism

Second Result: Constant Mark-Up

- derive indirect mechanism – quality tariff–that attains profit guarantee
- optimal tariff is a constant mark-up policy
- simple and transparent pricing policy that attains profit guarantee
- mark-up is determined by cost elasticity alone without reference to demand data, thus:
- cost based nonlinear pricing

One More Result: Consumer Surplus

- profit guarantee is solution of profit optimization problem
- solution is agnostic about consumer surplus
- yet, robust pricing rule generates large consumer surplus
- how large? for every cost elasticity, find the largest share of consumer surplus across all distributions (and Bayes optimal mechanisms)
- the maximum is attained by robust pricing rule for all demand distributions
- thus robust pricing policy succeeds by creating consumer surplus

Literature: Pricing and Competitive Ratio

- optimal monopoly pricing for single unit demand
- Neeman (2003), Maglaras (2009), Hartline and Roughgarden (2014), etc...
- with support $[1, h]$, competitive ratio: $1/(1 + \ln h)$
- competitive ratio vanishes as support restriction weakens
- single unit pricing requires randomized reserve price, i.e., many prices and assignment probabilities
- menu already arises for efficient allocation, now finds second use to hedge against demand uncertainty

Model

Model

- buyer has value $v \in \mathbb{R}_+$ (willingness-to-pay) for quality $q \in \mathbb{R}_+$

$$u(v, q, t) = v \cdot q - t$$

- value v is private information
- seller offers quality differentiated products q at cost

$$c(q) = q^\eta / \eta, \qquad \eta \in (1, \infty)$$

- cost elasticity η:

$$\frac{\frac{dc(q)}{c(q)}}{\frac{dq}{q}} = \frac{\frac{dc(q)}{dq}}{\frac{c(q)}{q}} = \frac{c'(q)\,q}{c(q)} = \eta$$

Payoffs and Menu

- seller chooses menu M (or direct mechanism) with qualities $Q(v)$ at prices $T(v)$:

$$M \triangleq \{(Q(v), T(v))\}_{v \in \mathbb{R}_+}$$

- incentive compatibility and participation constraints,

$$\begin{aligned} vQ(v) - T(v) &\geq vQ(v') - T(v'); \\ vQ(v) - T(v) &\geq 0; \qquad \forall v, v' \in \mathbb{R}_+ \end{aligned}$$

- profit and consumer surplus with menu M and value v:

$$\Pi_M(v) \triangleq T(v) - c(Q(v)),$$

and

$$U_M(v) \triangleq Q(v)v - T(v).$$

First Degree Price Discrimination

- profit with complete information is profit with perfect or first-degree price discrimination

$$\overline{\Pi}(v) \triangleq \max_q \{vq - c(q)\} =$$

- supported by socially efficient allocation:

$$\overline{Q}(v) \triangleq \arg\max_q \{vq - c(q)\} = v^{\frac{1}{\eta - 1}}$$

- first degree price discrimination captures social surplus

$$\overline{\Pi}(v) \triangleq \max_q \{vq - c(q)\} \triangleq S(v)$$

Second Degree Price Discrimination

- given distribution F:

$$F \in \Delta([\underline{v}, \bar{v}]), \qquad 0 \leq \underline{v} < \bar{v} \leq \infty$$

- expected profit and surplus with M :

$$\Pi_{F,M} \triangleq \mathbb{E}[T(v) - c(Q(v))],$$

and

$$U_{F,M} \triangleq \mathbb{E}[Q(v)v - T(v)].$$

- Bayes optimal menu with distribution F:

$$M_F \triangleq \arg\max_{M} \Pi_{F,M}.$$

- with some abuse of notation

$$\Pi_F = \Pi_{F,M_F} \text{ and } U_F = U_{F,M_F}$$

Competitive Ratio

- we are interested in ratio of profit under unknown distribution to profit under known distribution
- competitive ratio of mechanism M :

$$\inf_F \frac{\Pi_{F,M}}{\overline{\overline{\Pi}}_F}$$

- find optimal profit-guarantee menu M^* defined as:

$$M^* = \arg\max_M \inf_F \frac{\Pi_{F,M}}{\overline{\overline{\Pi}}_F}$$

- as by-product find distribution of values that minimizes seller's normalized profit:

$$\inf_F \max_M \frac{\Pi_{F,M}}{\overline{\overline{\Pi}}_F}$$

Analysis

Profit Guarantee

- consider a given mechanism $M = \{Q(v), T(v)\}$
- how well is mechanism M performing across different demand distributions F?
- how well is the mechanism M performing against a most challenging distribution F^*?
- referred to as competitive ratio of M :

$$\inf_F \frac{\Pi_{F,M}}{\overline{\Pi}_F} < 1$$

- profit-guarantee menu M^* maximizes competitive ratio

$$M^* = \arg\max_M \inf_F \frac{\Pi_{F,M}}{\overline{\Pi}_F}$$

Competitive Ratio and Adversarial Nature

- profit-guarantee menu M^* maximizes competitive ratio

$$M^* = \arg\max_M \inf_F \frac{\Pi_{F,M}}{\overline{\overline{\Pi}}_F}$$

- minmax theorem suggest distribution F^* that minimizes

$$\max_M \frac{\Pi_{F,M}}{\overline{\overline{\Pi}}_F}$$

- and thus

$$F^* = \arg\min_F \max_M \frac{\Pi_{F,M}}{\overline{\overline{\Pi}}_F}$$

- and indeed there is a saddle-point:

$$\max_M \inf_F \frac{\Pi_{F,M}}{\overline{\overline{\Pi}}_F} = \min_F \sup_M \frac{\Pi_{F,M}}{\overline{\overline{\Pi}}_F}$$

First Step Toward Solution: Local

- competitive ratio is stated in terms of expectations:

$$\inf_F \frac{\Pi_{F,M}}{\overline{\Pi}_F} = \inf \left\{ \frac{\int \left(T(v) - c(Q(v))\right) dF(v)}{\int \left(\overline{T}(v) - c\left(\overline{Q}(v)\right)\right) dF(v)} \right\}$$

- given menu $M = \{T(v), Q(v)\}$, nature chooses demand F that lowers the profit guarantee
- nature puts weight on values v where guarantee is weak:

$$\inf_v \left\{ \frac{T(v) - c(Q(v))}{\overline{T}(v) - c\left(\overline{Q}(v)\right)} \right\}$$

- to defend against such attacks find menu M where pointwise (local) guarantee is as high as possible, uniformly across all v :

$$\frac{T(v) - c(Q(v))}{\overline{T}(v) - c\left(\overline{Q}(v)\right)} = k, \quad \forall v.$$

Second Step Toward Solution: Proportional

- social surplus is generated by efficient choice $\overline{Q}(v)$
- maintain profit guarantee by staying with a constant proportion s of $\overline{Q}(v)$:

$$s \cdot \overline{Q}(v), \quad s \in (0,1)$$

- gross revenue grows at rate s, cost increases at rate s^{η}
- find optimal trade-off

$$\max_{s} \{s - s^{\eta}\} \Leftrightarrow s^{*} = \left(\frac{1}{\eta}\right)^{\frac{1}{\eta-1}}$$

A Profit Guarantee Menu

- construct a menu with a profit guarantee

Theorem (Profit Guarantee Mechanism)

The menu M^*:

$$Q^*(v) = s^* \cdot \overline{Q}(v) = \left(\frac{1}{\eta}\right)^{\frac{1}{\eta-1}} \cdot v^{\frac{1}{\eta-1}},$$

generates a profit guarantee

$$\frac{\Pi^*(v)}{\overline{\Pi}(v)} = \left(\frac{1}{\eta}\right)^{\frac{\eta}{\eta-1}},$$

for every value v and a fortiori every distribution F.

- thus profit guarantee is share s^* powered by elasticity η

Return to Minmax

- profit-guarantee menu M^* must be Bayes-optimal
- given F, M^* solves

$$\arg\max_M \frac{\Pi_{F,M}}{\overline{\Pi}_F} \Leftrightarrow \arg\max_M \Pi_{F,M}$$

- candidate optimal quality Q^* is constant share s^* of socially efficient quality $\overline{Q}(v)$
- candidate optimal quality Q^* is obtained by virtual value proportional to value
- Pareto distribution uniquely generates virtual value that is linear in value

Pareto Distribution

- Pareto distribution with shape parameter $\alpha \in [1, \infty)$:

$$F_\alpha(v) \triangleq \begin{cases} 0, & \text{if } v < 1; \\ 1 - \frac{1}{v^\alpha}, & \text{if } v \geq 1; \end{cases}$$

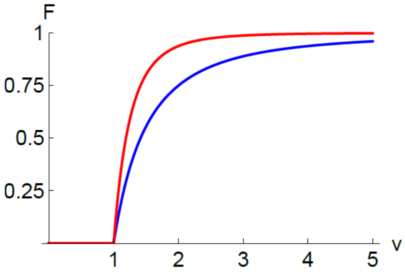
- virtual value with Pareto distribution

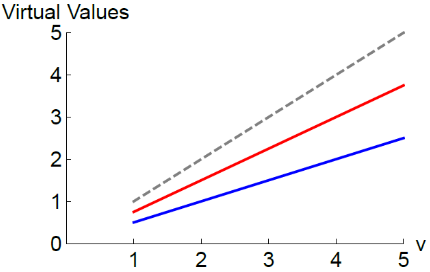
$$\phi(v) \triangleq v - \frac{1 - F_\alpha(v)}{f_\alpha(v)} = v - \frac{v^\alpha}{\alpha v^{\alpha - 1}} = \frac{\alpha - 1}{\alpha} v$$

- $\alpha = 1$ is equal revenue distribution prominent in unit demand pricing analysis

Pareto Distribution and Virtual Values

- Pareto distribution and virtual values





Profit Guarantee Menu is Optimal

- profit guarantee gave us a specific lower bound, can we do better?

Theorem (Minmax Distribution)

Menu M^* *is Bayes optimal for Pareto distribution* α*:*

$$\alpha = \frac{\eta}{\eta - 1},$$

and attains infimum:

$$\inf_F \frac{\Pi_F}{\overline{\overline{\Pi}}_F} = \frac{\Pi_{F_\alpha}}{\overline{\overline{\Pi}}_{F_\alpha}}\bigg|_{\alpha=\frac{\eta}{\eta-1}} = \left(\frac{1}{\eta}\right)^{\frac{\eta}{\eta-1}}.$$

- Pareto distribution $\alpha = \eta/(\eta - 1)$: least normalized profit
- profit guarantee is a sharp bound

Consumer Surplus

- minmax solution generates particular pair of surplus sharing among seller and buyers

Corollary (Consumer Surplus with M^*)

Menu M^* generates constant consumer surplus:

$$\frac{U_{M^*}(v)}{\overline{\Pi}(v)} = \frac{U_{M^*}(v)}{S(v)} = \left(\frac{1}{\eta}\right)^{\frac{1}{\eta-1}}$$

for every value v and a fortiori every distribution F.

- in profit-guarantee menu, each consumer receives the same share of the efficient social surplus.
- how does consumer surplus guarantee compare to consumer surplus attained across all Bayes optimal menus?

Maximum Consumer Surplus

- recall consumer surplus with known demand:

$$U_F = U_{F,M_F}$$

Corollary (Maximum Consumer Suplus)

The consumer surplus is bounded above as follows,

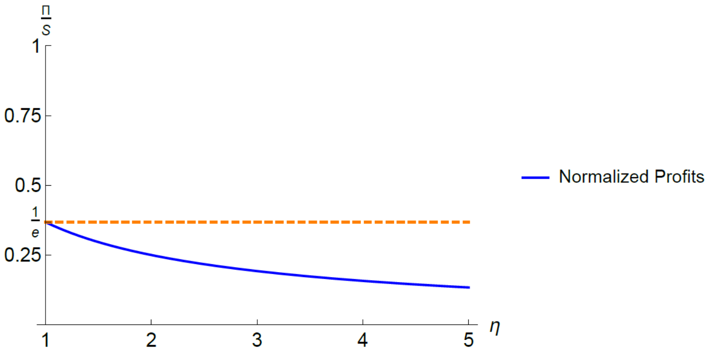
$$\sup_F \frac{U_F}{S_F} = \left(\frac{1}{\eta}\right)^{\frac{1}{\eta-1}}$$

and is attained by the Pareto distribution with shape parameter

$$\alpha = \frac{\eta}{\eta - 1}.$$

- profit guarantee concedes consumer surplus to stay near efficient allocation

Profit Share and Cost Elasticity

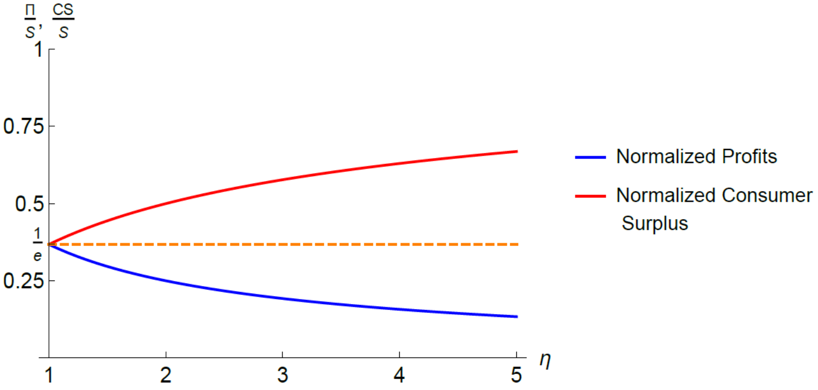


- profit (share) guarantee

$$(1/\eta)^{\frac{\eta}{\eta-1}}$$

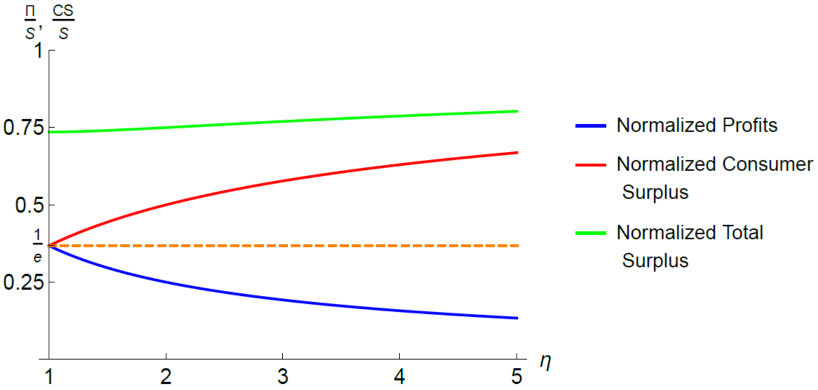
- limit $\eta \to 1$ corresponds to nearly constant marginal cost
- limit $\eta \to \infty$ corresponds to selling an indivisible good.

Profit and Consumer Surplus



- profit and consumer surplus share move in opposite direction as cost elasticity increases

Social Surplus



- profit and cs move in opposite direction as η increases
- realized social surplus increases with cost elasticity η
- uniform lower bound $2/e$

Indirect Mechanism

- indirect mechanism (tariff) asks price $P(q)$ for quality q
- marginal price for quality, the price-per-quality increment:

$$P'(q) \triangleq p(q)$$

- for quality q the total payment is:

$$P(q) = \int_0^q p(s)ds$$

- incentive compatibility will imply that

$$p(q(v)) = Q^{-1}(q(v)) = v.$$

Mark-Up Pricing

Corollary (Constant Mark-Up)

The menu M^* is implemented by offering quality increments $q \in \mathbb{R}$ at a price $p(q)$ satisfying:

$$\frac{p(q) - c'(q)}{c'(q)} = \eta - 1 \quad \Leftrightarrow p(q) = \eta c'(q).$$

- constant mark-up of cost:

$$\eta > 1$$

- price depend on cost information only, demand information is entirely absent
- alternatively, expressing pricing in terms of *Lerner's index:*

$$\frac{p(q) - c'(q)}{p(q)} = \frac{\eta - 1}{\eta}$$

- again, a constant measure of market power

Contrast to Bayesian Optimal Menu

- for a given prior distribution F optimal quantity is:

$$q(v) \in \arg\max_{q} \left\{ \left(v - \frac{1 - F(v)}{f(v)} \right) q - c(q) \right\}.$$

- first-order condition is given by:

$$v - \frac{1 - F(v)}{f(v)} - c'(q(v)) = 0,$$

- incentive compatible transfers:

$$T'(v) = q'(v)v.$$

- price per marginal unit of quality is given by:

$$p(q(v)) = \frac{T'(v)}{q'(v)} = v.$$

Demand Elasticity

- demand for quality q at incremental quality price $p(q(v))$:

$$D(p(q(v))) = 1 - F(v).$$

- resulting markup:

$$\frac{p(q) - c'(q)}{p(q)} = \frac{v - (v - \frac{1-F(v)}{f(v)})}{v} = \frac{1 - F(v)}{f(v)v}.$$

- rhs is negative of reciprocal of demand elasticity:

$$\frac{1 - F(v)}{f(v)v} = -\frac{\frac{D(p(v))}{p(v)}}{D'(p(v))}$$

- classic formula for Lerner's index

Lerner's Index

- classic formula for the Lerner's index:

$$\frac{p(q) - c'(q)}{p(q)} = \frac{1 - F(v)}{f(v)v} = -\frac{\frac{D(p(v))}{p(v)}}{D'\left(p\left(v\right)\right)}$$

- Bayes-optimal mechanism determined by demand elasticity– expressed in terms of product of value v and hazard rate $f\left(v\right) / \left(1 - F\left(v\right)\right)$
- profit-guarantee menu is determined only by cost elasticity

$$\frac{p(q) - c'(q)}{p(q)} = \frac{\eta - 1}{\eta}$$

- profit-guarantee is accomplished across all possible distribution of values, no reference to specific distribution

Constant Mark-Up and Mirrlees

- we did not impose any restrictions on distribution of willingness-to-pay
  $\longrightarrow$ no monotonicity or regularity restrictions on F
  $\longrightarrow$ no support restrictions on F
- critical demand is Pareto with unbounded support
- thus "no distortion at the top" fails to hold, instead constant mark-up
- related insights in optimal taxation literature

Beyond Constant Elasticity

Non-Constant Cost Elasticity

- pointwise cost elasticity:

$$\eta(q) = \frac{dc(q)}{dq} \frac{q}{c(q)}.$$

- pointwise marginal cost elasticity

$$\gamma(q) = \frac{dc'(q)}{dq} \frac{q}{c'(q)}.$$

- cost with constant elasticity has simple relation:

$$\gamma(q) = \eta(q) - 1,$$

Approximation

- use constant elasticity informed pricing
- obtain profit guarantees for non-constant elasticity
- weaker guarantees, transparent approximation

Proportional Mark Up Pricing

- tariff $P(q)$ and price $p(q)$ for quality increment:

$$p(q) = P'(q)$$

- price per quality proportional to marginal cost elasticity

$$\widehat{p}(q) \triangleq (1 + \gamma(q))c'(q)$$

- tariff in terms of markup:

$$\frac{\widehat{p}(q) - c'(q)}{c'(q)} = \gamma(q)$$

Lower Bound

- establish a relationship between profit and social surplus
- profit with v is equal surplus with w, where

$$w(v) < v$$

Lemma (Profit as Downward Shifted Social Surplus)

The tariff attains profit as downward shifted social surplus:

$$w(v) = \frac{v}{1 + \gamma(q(v))},$$

and

$$\Pi(v) = S(w(v)).$$

- monotone relationship between profit and social surplus

Bounded Cost Elasticity

- consider bounds on marginal cost elasticity:

$$\gamma(q) \in [\underline{\gamma}, \overline{\gamma}], \quad \forall q$$

Proposition

Suppose the elasticity is bounded $\gamma(q) \in [\underline{\gamma}, \overline{\gamma}]$, $\forall q$, *then:*

$$\frac{\Pi(v)}{S(v)} \geq \left(\frac{1}{\overline{\gamma}+1}\right)^{\frac{\underline{\gamma}+1}{\underline{\gamma}}}.$$

- relative to constant elasticity, bound is weaker as base and exponent are formed by lower and upper bound of marginal cost elasticity
- coincides with constant elasticity result if $\underline{\gamma} = \overline{\gamma}$

Sharp Bound

- consider class of increasing cost elasiticity:

$$\gamma'(q) \geq 0 \text{ and } \lim_{q \to \infty} \gamma(q) = \overline{\gamma} < \infty.$$

Proposition

If marginal cost elasticity $\gamma(q)$ is increasing with limit $\overline{\gamma}$, then proportional pricing generates decreasing ratio:

$$\frac{\Pi(v)}{S(v)} \geq \left(\frac{1}{\overline{\gamma}+1}\right)^{\frac{\overline{\gamma}+1}{\overline{\gamma}}}$$

and the bound is attained in the limit $v \to \infty$.

- a generalization of Pareto distribution with variable shape parameter delivers a Bayesian optimal mechanism

Additional Demand Information

Additional Demand Information

- we have worked without any information about demand
- with additional information about demand, we may increase the profit guarantee
- suppose we know lower and upper bounds on the support of the value distribution, thus

$$0 \leq \underline{v} < \bar{v} < \infty.$$

New Results

- main insights of Theorem 1 and 2 remain in the presence of additional support information:

1. there exists a minmax solution
2. competitive ratio between realized profit and social surplus are constant at every point in support of demand

- some changes with finite support:

1. optimal menu does not display constant mark-up anymore
2. "no-distortion at the top" result re-emerges

Minmax Solution

- find allocation $q(v)$ and distribution $F(v)$ such that:

$$\max_{\{q:[\underline{v},\bar{v}]\to\mathbb{R}\}} \inf_{F\in\Delta[\underline{v},\bar{v}]} \frac{\int \Pi_q(v)dF(v)}{\int S(v)dF(v)},$$

- denote a solution by (q^*, F^*).

Proposition (Existence)

There exists (q^*, F^*) such that:

$$\max_{\{q:[\underline{v},\bar{v}]\to\mathbb{R}\}} \inf_{F\in\Delta[\underline{v},\bar{v}]} \frac{\int \Pi_q(v)dF(v)}{\int S(v)dF(v)} = \min_{F\in\Delta[\underline{v},\bar{v}]} \sup_{\{q:[\underline{v},\bar{v}]\to\mathbb{R}\}} \frac{\int \Pi_q(v)dF(v)}{\int S(v)dF(v)}.$$

And q^* is the optimal Bayesian mechanism when the distribution is F^*.

Construction of Constant Competitive Ratio

- define a family of allocations, parameterized by $\beta \in [0, 1]$, denoted by $q_\beta : [\underline{v}, v_\beta] \to \mathbb{R}$
- defined implicitly by:

$$\frac{\Pi_{q_\beta}(v)}{S(v)} = \beta$$

- upper bound of domain v_β is upper bound on which condition can be maintained

Proposition (Constant Profit-Surplus Ratio)

1. *The profit-guarantee mechanism q^*, is given by $q^*(v) = q_\beta(v)$, with β such that $v_\beta = \bar{v}$.*
2. *The allocation rule q_β is increasing in β and v_β is decreasing in β.*

Upper Bound

- with finite support there is no explicit solution for competitive ratio even with constant elasticity
- provide an upper bound by means of a Bayes optimal mechanism
- converges to exact solution as upper boud of support diverges to ∞.

Proposition (Bounded Support)

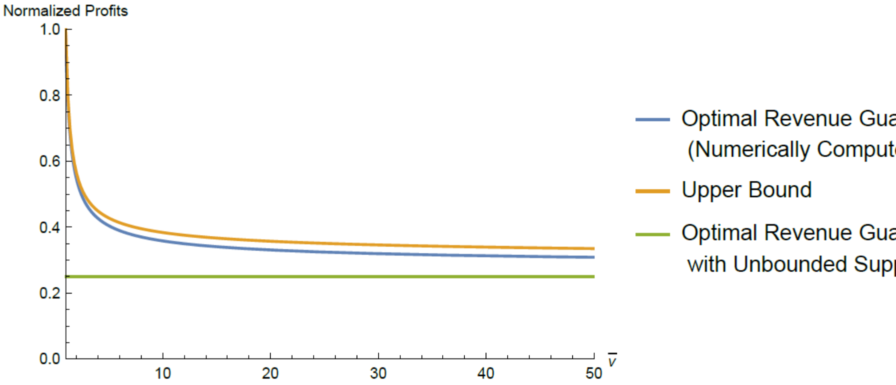
There exists a distribution F with support in $[\underline{v}, \bar{v}]$ such that the Bayesian optimal mechanism generates normalized profits:

$$\frac{\Pi}{S} = \frac{1}{\eta^{\frac{\eta}{\eta-1}}} + (1 - \frac{1}{\eta^{\frac{\eta}{\eta-1}}})\frac{1}{1 + \frac{\eta}{\eta-1}\log(\bar{v})}.$$

- constitutes an upper bound on profit guarantee

Approximation and Finite Support

- how does competitive ratio degrade with size of support?



- here $\eta = 2, \underline{v} = 1$; result are invariant for $\overline{v}/\underline{v}$

Boundaries of Surplus Sharing

Constrained Efficient Surplus Sharing

- profit guarantee is attained as Bayes optimal outcome for specific Pareto distribution
- upper frontier of the feasible consumer surplus and profit share across all distributions and Bayes optimal solutions:

$$\sup_F \left\{ \frac{U_F}{S_F} : \quad \frac{\Pi_F}{S_F} = \beta \right\}$$

- identify maximum consumer surplus given profit is greater than or equal to some fraction $\beta \in [0, 1]$ of the social surplus

Surplus Frontier

- consider all possible distributions F

Proposition (Surplus Frontier)

The surplus frontier is given by:

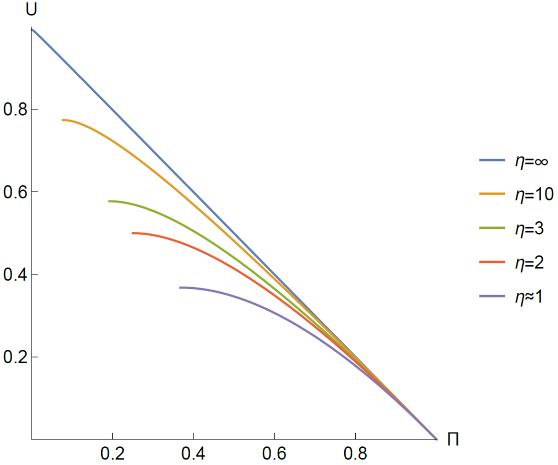
$$\sup_F \left\{ \frac{U_F}{S_F} : \quad \frac{\Pi_F}{S_F} = \beta \right\} = \frac{\eta}{\eta - 1} \left(\beta^{\frac{1}{\eta}} - \beta \right).$$

The constraint is feasible if and only if $\beta \in \left[1/\eta^{\frac{\eta}{\eta-1}}, 1 \right]$.

- lower bound is given by profit guarantee

Surplus Frontier

- surplus frontier and elasticity η



- all equilibrium points on surplus frontier by Pareto distributions with different shape parameters $\alpha \geq \eta / (\eta - 1)$

Lower Bound on Social Surplus

- we note that:

$$\left.\frac{U_{P_\alpha}}{S_{P_\alpha}}\right|_{\alpha=1} = 0 \quad \text{and} \quad \left.\frac{\Pi_{P_\alpha}}{S_{P_\alpha}}\right|_{\alpha=1} = \frac{1}{\eta}$$

- when distribution of values is the Pareto distribution with shape parameter $\alpha = 1$ the consumer's surplus is 0

Proposition (Lower Bound on Social Surplus)

When $\eta \geq 2$, social surplus is bounded below by:

$$\inf_F \frac{U_F + \Pi_F}{S_F} = \left.\frac{U_{P_\alpha} + \Pi_{P_\alpha}}{S_{P_\alpha}}\right|_{\alpha=1} = \frac{1}{\eta}.$$

Entire Surplus Set I

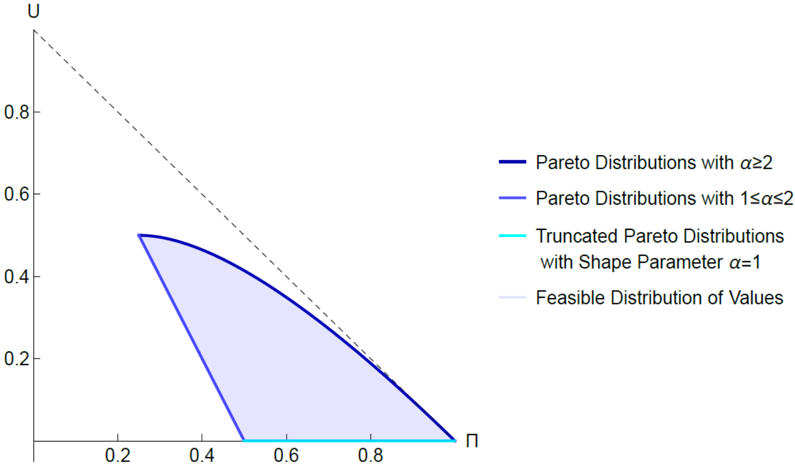


Figure: Equilibrium feasible normalized profits and consumer surplus for quadratic cost, $\eta = 2$

Entire Surplus Set II

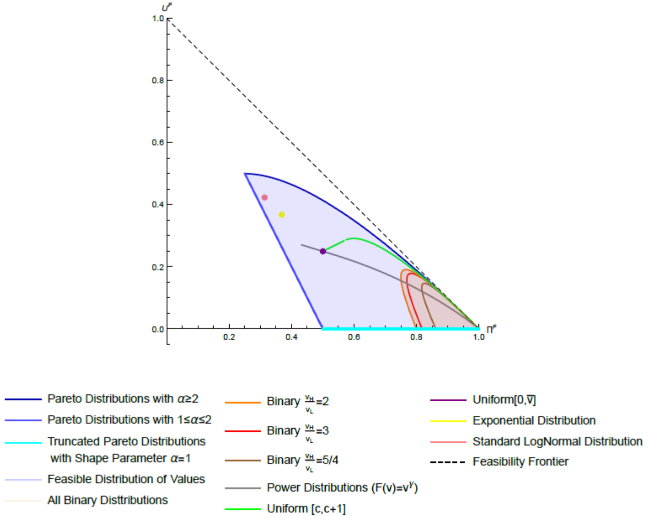


Figure: Illustration of Consumer Surplus and Profits for Different Distributions with Quadratic Costs

Conclusion

- cost-based rather than demand-based pricing can attain positive profit guarantee, and even higher social surplus guarantee
- menu in nonlinear pricing acts as a hedge against demand uncertainty
- menu provides stronger profit guarantee than could be anticipated from single-unit analysis
- robust menu attains guarantee through simple, transparent mark-up pricing

Variations

Quantity Discrimination

- provide a profit guarantee for the case of multiplicatively separable utility functions:

$$u(v,q) = v\frac{\eta}{\eta+1}q^{\frac{\eta+1}{\eta}},$$

for some

$$\eta \in (-\infty, -1)$$

- utility function is increasing and concave
- cost of production is linear $c(q) = cq$, wlog $c = 1$
- demand is inverse of marginal utility:

$$D(v,p) \triangleq u_q^{-1}(v,p),$$

- demand elasticity is

$$\frac{\partial D(v,p)}{\partial p}\frac{p}{D(v,p)} \triangleq \eta$$

Profit Guarantee with Quantity Discrimination

- Pareto distribution with shape parameter $\alpha \in (1, \infty)$

Theorem (Profit Guarantee with Quantities)

The uniform-price menu $t = p^* q$ *with*

$$p^* = \eta / (\eta + 1) > 1,$$

guarantees profits:

$$\Pi^* (v) = (\eta / (\eta + 1))^\eta S (v) ,$$

for every v *and every* F*.*

- profit-guarantee menu is Bayes optimal with Pareto distribution and $\alpha = |\eta|$:

$$\lim_{\alpha \to |\eta|} \frac{\Pi_{P_\alpha}}{S_{P_\alpha}} = \left(\frac{\eta}{\eta + 1} \right)^\eta$$

Nonlinear Utility

- nonlinearity in utility function:

$$u(v, q, t) = h(v, q) - t,$$

  where h is concave in q given v distributed with F
- cost of production remains linear $c(q) = cq$ wlog $c = 1$
- demand function is inverse of marginal utility:

$$D(v, p) \triangleq h_q^{-1}(v, p),$$

- demand elasticity

$$\eta(v, p) \triangleq \frac{\partial D(v, p)}{\partial p} \frac{p}{D(v, p)}, \qquad \eta(v, p) < 0, \forall v, p$$

- all $p \in [1, \infty]$,

$$\eta(v, p) \text{ is non-increasing in } p \text{ and } \eta(v, p) \in [\bar{\eta} - 1, \bar{\eta}],$$

  for some $\bar{\eta} \in (-\infty, -1)$

Robust Profit Guarantee

- for given $D(v, p)$, optimal uniform price $\hat{p}$:

$$\hat{p} = \arg\max_{p} D(v, p)(p - c).$$

- first-order condition:

$$\hat{p} = c\frac{\eta(v, \hat{p})}{\eta(v, \hat{p}) + 1}.$$

- and thus

$$\hat{p} \leq c\frac{\bar{\eta}}{\bar{\eta} + 1}.$$

Robust Profit Guarantee

Theorem (Robust Profit-Guarantee Mechanism)

The uniform-price menu $t = p^* q$*, where*

$$p^* = \frac{\bar{\eta}}{\bar{\eta} + 1}$$

guarantees a profit share of the social surplus:

$$\Pi^* \geq \left(\frac{\bar{\eta}}{\bar{\eta} + 1} \right)^{\bar{\eta}} S.$$

Procurement

Procurement

- single buyer procures from sellers with private information about their cost
- robust procurement policies by competitive ratio
- seller has cost $\theta \cdot c(q)$ to provide a good of quality q
- θ is private information for seller with distribution F
- costs have constant elasticity

$$\theta c(q) = \theta \frac{q^\eta}{\eta}, \quad \eta > 1$$

Socially Efficient Procurement

- efficient social surplus:

$$S(\theta) = \max_q \{q - \theta c(q)\}.$$

- efficient quality is inversely related to cost parameter θ :

$$q^* = \left(\frac{1}{\theta}\right)^{\frac{1}{\eta-1}}$$

- generates a social surplus of:

$$S(\theta) = \frac{\eta - 1}{\eta}(\frac{1}{\theta})^{\frac{1}{\eta-1}}.$$

Surplus Guarantee

- constant p for every marginal unit of quality:

$$\theta c'(q) = p \Longleftrightarrow q = (\frac{p}{\theta})^{\frac{1}{\eta-1}}.$$

Corollary (Surplus Guarantee Mechanism)

The surplus guarantee menu has constant unit price

$$p = 1/\eta$$

for incremental quality and the buyer is guaranteed a share:

$$\left(\frac{1}{\eta}\right)^{\frac{1}{\eta-1}}$$

of the efficient social surplus.

Procurement with Concave Cost

- buyer has a utility function $u(q)$

$$u(q) = \eta q^{(\eta+1)/\eta}/(\eta + 1)$$

with a demand elasticity:

$$\eta \in (-\infty, -1) \, .$$

- seller has cost

$$c\,(q) = \theta \cdot q,$$

- marginal cost θ is private information for the seller and given by a common prior distribution

Socially Efficient Procurement

- first-best surplus is:

$$S(\theta) = \max_q \{u(q) - \theta q\}.$$

- efficient quantity is:

$$q^* = \theta^\eta$$

- social surplus is

$$S(\theta) = -\frac{\theta^{\eta+1}}{\eta + 1}.$$

Surplus Guarantee

Corollary (Surplus Guarantee Mechanism)

The surplus guarantee menu has constant mark-up

$$p(q) = \frac{\eta + 1}{\eta} q^{1/\eta}$$

for quantity and the buyer is guaranteed a share:

$$\left(\frac{\eta}{\eta + 1} \right)^{\eta + 1}$$

of the optimal social surplus.