Selling to Intermediaries: Optimal Auction Design in a Common Value Model

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Selling to Intermediaries

- a single unit of a good is to be sold to intermediaries

- each of N intermediaries has a client with private value (i.i.d. draws)

- winning intermediary can re-sell the good (at a take-it-or-leave-it price) to the best (highest value) client

- what is the revenue maximizing selling mechanism in this context?

Posted Prices with No Exclusion

- sometimes, no exclusion is optimal (because the worst possible client is sufficiently good)

- in this case, a posted price is optimal

- posted price is the expectation of the highest of $N - 1$ client values, so even the intermediary with the worst possible client buys

- object allocated randomly among intermediaries

What is Going On?

- if seller could sell to uninformed intermediary (or act as the intermediary himself), could extract all surplus (i.e., the expectation of the best client value)

- therefore two competing forces:
  - would like to sell to intermediary with good client, since has high unconditional willingness to pay

  - would like to sell to intermediary with bad client, since he has less information about best client value

- compromise: sell with equal probability to all intermediaries independent of their clients' values

More Generally

- sometimes, exclusion **is** optimal (to reduce information rents)

- same two competing forces:
  - would like to sell to intermediary with good client, since has high unconditional willingness to pay

  - Would like to sell to an intermediary with a bad client, since he has less information about the best client value

- now exclusion makes it possible/incentive compatible to lower winning probability of intermediary with better client below that of intermediaries with worse clients

- minimize probability of intermediaries with high value clients being allocated the object, subject to incentive constraints

- global binding constraints: intermediaries are indifferent between misreporting down to **any** level

Auction Design in a Common Value Model

- alternative interpretations:
  - buyers not intermediaries but will discover alternative uses of the good
  - re-sale not sale to intermediaries
- abstract interpretation (remainder of talk)
  - each buyer observes i.i.d. signal
  - value is maximum of signals, so (pure) common value
- we characterize the optimal auction in this common value setting

Contributions: Substantive

- we examine a setting where bidders with higher signals have more accurate information about a common value;
  - naturally arises in a market with intermediaries, and other settings
- countervailing screening incentives:
  - tension between selling to those with a higher expected value and those with more information
- it becomes optimal to screen "less" and in a more nuanced way (with no screening in the no exclusion limit)
- new foundation for posted price mechanisms
- implementation of optimal mechanism when not posted price

Contributions: Methodological

- very few results extend characterization of optimal auctions beyond private value case

- we extend our knowledge of optimal auctions in a fundamentally new direction

- new technical issues arise:
  - in private value case, "local" incentive constraints are often sufficient to pin down optimal mechanism
  - in our case, "global" constraints matter: at optimum, bidders are indifferent to reporting all signals less than their true signal!

Two Bulow and Klemperer papers

- "maximum game" introduced in Bulow and Klemperer (2002, Rand);
  - we generalize their observation that posted price mechanisms beat second price auction in this "maximum game"

- Bulow and Klemperer (1996, AER) study optimal auctions with interdependent values:
  - providing a revenue equivalence theorem (we use/adapt it)

  - show that transfers depend not just on allocation but also on sensitivity of value to private information; no general analogue for monotonicity

  - solve for special cases where local constraints are sufficient (in common value case, because most optimistic bidder is not more informed than less informed bidder....as in "wallet game"; Myerson (1981)

Plan for the talk

- model

- implementation of optimal mechanism: descending auction in quantities

- uniform example

- optimality:
  - revenue equivalence
  - upper bound
  - upper bound attained

Pure Common-Value Model

- N bidders for a single unit of object
- bidder i receives signal $s_i \in S = [\underline{s}, \overline{s}]$, $s \in S^N$
- signals are iid from cumulative distribution $F(s_i)$ with density $f(s_i)$
- pure common value is maximum signal:

$$v(s_1, \ldots, s_N) = \max\{s_1, \ldots, s_N\}.$$

- signal distribution $F(s_i)$ induces value distribution $G(v)$:

$$G(v) = (F(v))^N$$

- "maximum value game", Bulow and Klemperer (RAND, 2002)

Utility and Allocation

- bidders are expected utility maximizers with quasilinear preferences over probability q_i of receiving good and transfers t_i, represented by:

$$u_i(s, q_i, t_i) = v(s) q_i - t_i$$

- feasibility:

$$q_i(s) \geq 0, \text{ with } \sum_{i=1}^{N} q_i(s) \leq 1$$

- interim probability that bidder i receives the object:

$$Q_i(s_i) = \int_{s_{-i} \in S^{N-1}} q_i(s_i, s_{-i}) f_{-i}(s_{-i}) ds_{-i},$$

where

$$f_{-i}(s_{-i}) = \prod_{j \neq i} f(s_j)$$

Transfers and Revenues

- **ex-post transfer** $t_i(s)$ of bidder i, **interim expected transfer**:

$$T_i(s_i) = \int_{s_{-i} \in S^{N-1}} t_i(s_i, s_{-i}) f_{-i}(s_{-i}) ds_{-i},$$

- **Revenue** is expected sum of transfers:

$$R = \sum_{i=1}^{N} \int_{s_i \in S} T_i(s_i) f(s_i) ds_i$$

- bidder i's **surplus** when reporting s_i' with true signal s_i:

$$U_i(s_i, s_i') = \int_{s_{-i} \in S^{N-1}} q_i(s_i', s_{-i}) v(s_i, s_{-i}) f_{-i}(s_{-i}) ds_{-i} - T_i(s_i')$$

- direct mechanism $\{q_i, t_i\}_{i=1}^{N}$ is **incentive compatible** (IC) if

$$U_i(s_i) = U_i(s_i, s_i) \geq U_i(s_i, s_i'), \text{ for all } i \text{ and } s_i, s_i' \in S$$

and **individually rational** (IR) if $U_i(s_i) \geq 0$, for all i and $s_i \in S$

- Seller maximizes R over all IC and IR direct mechanisms

Descending Clock Auction in Probabilities

- "probability of sale" starts at $q^* < 1/N$ and declines to 0

- when an intermediary stops the clock, he gets the good with the probability q on the clock, with remaining probability $\frac{1}{N-1}(1-q)$ divided equally among other intermediaries (who thus have a higher probability of getting the object)

- if no one stops the clock before the time that the clock hits 0, the good is not allocated

Equilibrium

- symmetric equilibrium stopping function $\gamma : [\underline{s}, \bar{s}] \to [0, q^*]$ will be:

$$\gamma(s) = \begin{cases} \frac{1}{N}\left(1 - \frac{1 - Nq^*}{F(s)}\right), & \text{if} \quad s \geq F^{-1}\left(1 - Nq^*\right); \\ 0, & \text{otherwise} \end{cases}$$

- key qualitative properties:
  - highest bidder stop the clock immediately at q^*
  - bidders with signals below "reserve" $r^* = F^{-1}\left(1 - Nq^*\right)$ never stop
  - stopping probabilities decreasing (and so stopping times increase) as signals decrease
  - quantities continuously decreasing (with strictly monotonicity under we hit 0 probability)

Intuition for Key Qualitative Properties

- (generalized) war of attrition

- bidders trade off possibility of total exclusion versus lower share

- higher signal bidders stop sooner because higher opportunity cost of exclusion

- no gain to waiting if no one else is stopping

Imposing Global Indifference

- a bidder observing signal x assigns probability $F^{N-1}(x)$ to his signal being the highest
  - so his probability of getting the object is $\beta(x).F^{N-1}(x)$.

- the unconditional probability that the highest signal is between r^* and x is $F^N(x) - F^N(r^*)$

- if bidder x randomized between stopping quantities of lower bidders y according to cdf $F(x)$, then by symmetry, his probability of the getting the object would be $\frac{1}{F(x)}\left(F^N(x) - F^N(r^*)\right)$

- averaging across players, ex ante probability is

- y according to cdf $\frac{F(y)}{F(x)}$

- equating $\beta(x).F^{N-1}(x)$ and $\frac{1}{F(x)}\left(F^N(x) - F^N(r^*)\right)$ gives the equilibrium strategy

Entry Fee

- now ask bidders to pay fixed entry fee T^*equal to the expected surplus of types $s \leq r^*$:

$$\int\limits_{x=r^*}^{\overline{s}} x\left(1 - \gamma\left(x\right)\right) F^{N-2}\left(x\right) f\left(x\right) dx$$

- we will show that the optimal mechanism is equivalent to this one for some q^*, where $q^*$is chosen to trade off efficiency and information rent

- observe that as $q^* \uparrow 1/N$, we approach the posted price mechanism

Uniform Example: Second Price Auction

- two bidders, signals distributed uniformly on unit in

- terval

- it is an equilibrium to bid your signal
    - in equilibrium, bidder i is indifferent between bidding s_i and any $s_i' > s_i$
    - upward deviation leads bidder i to win sometimes when the high bid is the high signal, which is the value
    - upward deviator wins the object but pays its value

- expected revenue is the expected second-highest value, which is $1/3$ (c.f., private value case)

Uniform Example: Posted Price

- Suppose good is randomly allocated to one of the two bidders and charged posted price $1/2$

- Every type of every bidder accepts

- Revenue improves from $1/3$ to $1/2$

Uniform Example: Optimal Auction

- Downward auction strategy

$$\gamma(s) = \left\{ \begin{array}{cc} \frac{1}{2}\left(1 - \frac{r}{s}\right), & \text{if} \quad s \geq r; \\ 0, & \text{otherwise} \end{array} \right.$$

where $r = 1 - 2q^*$.

- Transfer

$$T = \int_{x=r}^{1} x\left(1 - \gamma\left(x\right)\right) dx$$
$$= \frac{1}{4}\left(1 - r^2 - 2r^2 \log\left(r\right)\right)$$

- Maximum satisfies the FOC

$$-2r - 4r^2 \log\left(r\right) - 2r = 0 \iff r = \frac{1}{e}$$

- Revenue then is

Winning Probability and Indirect Utility

- Local incentive compatibility: indirect utility depends only on probability of receiving the object

- Useful to distinguish between
  - Winning probability of i when i has highest signal realization x,
  - When somebody including i has the highest signal realization x.

- Likelihood conditional on bidder j's signal being x, that
  - (i) the highest signal is x and
  - (ii) bidder i is allocated the good

  $$\widehat{Q}_{i,j}(x) = \int_{s_{-j} \in [\underline{s}, x]^{N-1}} q_i(x, s_{-j}) f_{-j}(s_{-j}) \, ds_{-j},$$

- Probability that i is allocated good **and** that i has high signal:

  $$\widehat{Q}_i(x) \equiv \widehat{Q}_{i,i}(x),$$

Social Surplus

- Probability that bidder i gets object and highest signal is x:

$$\overline{Q}_i(x) \equiv \sum_{j=1}^{N} \widehat{Q}_{i,j}(x)$$

- Probability that **some** bidder gets the good and highest signal is x:

$$\overline{Q}(x) \equiv \sum_{i=1}^{N} \overline{Q}_i(x)$$

- Social surplus generated by the mechanism:

$$TS = \int_{x \in S} x\overline{Q}_i(x) f(x) \, dx$$

Envelope Formula

- Recall a type s_i's interim utility from reporting s_i':

$$U_i\left(s_i, s_i'\right) = \int_{s_{-i} \in S^{N-1}} q_i\left(s_i', s_{-i}\right) \max\left\{s_1, \ldots, s_n\right\} f_{-i}\left(s_{-i}\right) ds_{-i} - T_i\left(s_i'\right)$$

- If U is sufficiently differentiable, then the envelope theorem says

$$U_i\left(s_i, s_i\right) = \max_{s_i'} U_i\left(s_i, s_i'\right) \quad \Longrightarrow \quad \frac{d}{ds_i} U_i\left(s_i, s_i\right) = \left.\frac{\partial}{\partial x} U_i\left(x, s_i\right)\right|_{x=s_i}$$

- So as long as things are sufficiently well-behaved,

$$\frac{d}{ds_i} U_i\left(s_i, s_i\right) = \int_{s_{-i} \in S^{N-1}} q_i\left(s_i, s_{-i}\right) \mathbb{I}_{s_i \geq \max_{j \neq i} s_j} f_{-i}\left(s_{-i}\right) ds_{-i} = \widehat{Q}_i\left(s_i\right)$$

Proposition

In any incentive compatible mechanism indirect utility is

$$U_i\left(s_i\right) = U_i\left(\underline{s}\right) + \int_{x=\underline{s}}^{s_i} \widehat{Q}_i\left(x\right) dx.$$

Revenue equivalence

- Reduce information rent of lowest signal: $U_i\left(\underline{s}\right) = 0$, for all i
- So, if we write $\widehat{Q}\left(x\right) = \sum_i \widehat{Q}_i\left(x\right)$, total rents to bidders is
-

$$U = \int_{x \in S} \int_{y=0}^{x} \widehat{Q}\left(y\right) dy\, f\left(x\right) dx = \int_{x \in S} \widehat{Q}\left(x\right)\left(1 - F\left(x\right)\right) dx$$

Proposition

If $U_i\left(\underline{s}\right) = 0$ for all i, then the expected revenue from the direct mechanism $\{q_i, t_i\}_{i=1}^{N}$ is

$$R = TS - U = \int_{x \in S} \left(x\overline{Q}\left(x\right) f\left(x\right) - \widehat{Q}\left(x\right)\left(1 - F\left(x\right)\right)\right) dx.$$

Virtual Utilities

- Equivalent formulation:

$$R = \sum_{i=1}^{N} \int_{s \in S^N} \left(s_i - \mathbb{I}_{s_i = \max_j s_j} \frac{1 - F(s_i)}{f(s_i)} \right) q_i(s) f_N(s) \, ds$$

- **Virtual utility** of each bidder, $\pi_i(s_i, s_{-i})$:

$$\pi_i(s_i, s_{-i}) = \left\{ \begin{array}{ll} \max_j\{s_j\}, & \text{if} \quad s_i \leq \max\{s_{-i}\}; \\ \max\{s_j\} - \frac{1 - F_i(s_i)}{f_i(s_i)}, & \text{if} \quad s_i > \max\{s_{-i}\}. \end{array} \right.$$

- Downward discontinuity in virtual utility suggests that seller wishes to minimize the probability of assigning the object to the bidder with the highest signal

False Start

- Revenue equivalence formula tells us:

$$R = \int_{x \in S} \left(x \overline{Q}(x) f(x) - \widehat{Q}(x) (1 - F(x)) \right) dx$$

- Bidder receives information rents only when he is allocated the good **and** when he has highest signal

- Maximizing revenue: sell object to one of the bidders whose signal is less than maximum

- $\overline{Q}(x) = N(F(x))^{N-1}$, its maximal value, and $\widehat{Q}(x) = 0$

- Bidders receive no information rents, seller extracts full surplus

Global Incentive Constraints

- Bidders want to misreport lower signals, i.e., violates global IC

- Bidder with highest signal $\bar{s}$ would never be allocated good under this mechanism, and would receives zero rents

- Bidder with lowest signal $\underline{s}$ receives object with probability $1/(N-1)$ and pays $1/(N-1)$ share of expectation of the highest of the $N-1$ other signals:

$$\hat{s} \triangleq \int_{x=\underline{s}}^{\bar{s}} x (N-1) F^{N-2}(x) f(x) \, dx.$$

- Highest type could pretend to be lowest type and obtain $(\bar{s} - \hat{s})/(N-1)$

- Have to incorporate global constraints into optimization problem

Which Global Constraints and Deviations?

- Consider one-dimensional family of constraints: **misreporting a redrawn lower signal**

- Instead of reporting signal s_i, report a random signal s_i' that is drawn from the truncated prior

$$F\left(s_i'\right)/F\left(s_i\right)$$

  on the support $[\underline{s}, s_i]$

- We use these incentive constraints to derive an upper bound on maximum revenue

A Revenue Upper Bound

- What are gains from **misreporting a redrawn lower signal**?

- Equilibrium surplus of a bidder with type x is

$$U_i(x) = \int_{y=\underline{s}}^{x} \widehat{Q}_i(y)\, dy$$

- Surplus from misreporting the redrawn lower signal

$$\frac{1}{F(x)} \int_{y=\underline{s}}^{x} U_i(x, y)\, f(y)\, dy$$

- Gains vary depending on realized misreport, but average gains across all misreports is easy to compute

$$\frac{1}{F(x)} \int_{y=\underline{s}}^{x} \left[(x-y)\overline{Q}_i(y) + \int_{z=\underline{s}}^{y} \widehat{Q}_i(z)\, dz \right] f(y)\, dy$$

Average Gains from Misreporting

.

- Misreport is redrawn from prior, bidder i is equally likely to fall anywhere in distribution of signals, unconditional on misreport, ex-ante likelihood that i receives good and y is highest signals

$$\overline{Q}_i(y) f(y)$$

- If highest report is less than x, surplus that bidder i obtains from being allocated good is x rather than y, so $x - y$ is difference between deviator and truthtelling surplus:

$$\frac{1}{F(x)} \int_{y=\underline{s}}^{x} \left[(x - y) \overline{Q}_i(y) + \int_{z=\underline{s}}^{y} \widehat{Q}_i(z) \, dz \right] f(y) \, dy$$

- Integrating by parts and comparing with truthful reporting:

$$\int_{y=\underline{s}}^{x} (x - y) \overline{Q}_i(y) f(y) \, dy \leq \int_{y=\underline{s}}^{x} \widehat{Q}_i(y) F(y) \, dy$$

An Inequality

- Misreporting a redrawn lower signal is not attractive iff

$$\int_{y=\underline{s}}^{x} (x - y)\,\overline{Q}_i(y)\,f(y)\,dy \le \int_{y=\underline{s}}^{x} \widehat{Q}_i(y)\,F(y)\,dy$$

- Summing across i, we conclude that direct mechanism deters misreporting redrawn lower signals only if

$$\int_{y=\underline{s}}^{x} (x - y)\,\overline{Q}(y)\,f(y)\,dy \le \int_{y=\underline{s}}^{x} \widehat{Q}(y)\,F(y)\,dy$$

- and also satisfy the feasibility constraints

$$0 \le \overline{Q}(x) \le N\,(F(x))^{N-1} \text{ and } 0 \le \widehat{Q}(x) \le N\,(F(x))^{N-1}$$

Revenue after Integration by Parts

- The earlier expression for revenue

$$\int_{x \in S} \left(x \overline{Q}(x) - \int_{y=\underline{s}}^{x} \widehat{Q}(y) \, dy \right) f(x) \, dx$$

can be integrated by parts:

$$\int_{x \in S} \left(x \overline{Q}(x) f(x) - \frac{1 - F(x)}{F(x)} \widehat{Q}(x) F(x) \right) dx,$$

- And integrating the second term by parts again:

$$\int_{x \in S} \left(x \overline{Q}(x) f(x) - \frac{f(x)}{F^2(x)} \int_{y=\underline{s}}^{x} \widehat{Q}(y) F(y) \, dy \right) dx.$$

- Revenue is maximal where $\int_{y=\underline{s}}^{x} \widehat{Q}(y) F(y) \, dy$ is minimal

Generalized Virtual Utility

- Revenue is maximal where $\int_{y=\underline{s}}^{x} \widehat{Q}(y) F(y) \, dy$ is minimal subject to the random deviation constraint:

$$\int_{y=\underline{s}}^{x} (x-y) \overline{Q}(y) f(y) \, dy \leq \int_{y=\underline{s}}^{x} \widehat{Q}(y) F(y) \, dy$$

- As a result, we can solve out $\widehat{Q}$ in terms of $\overline{Q}$:

$$\widehat{Q}(x) = \frac{1}{F(x)} \int_{y=\underline{s}}^{x} \overline{Q}(y) f(y) \, dy,$$

- We obtain the following expression for revenue:

$$\int_{x \in S} \left(x \overline{Q}(x) f(x) - \frac{1 - F(x)}{F(x)} \int_{y=\underline{s}}^{x} \overline{Q}(y) f(y) \, dy \right) dx.$$

A Generalized Virtual Utility Formulation

- Integrating by parts one last time

$$\int_{x \in S} \left(x \overline{Q}(x) f(x) - \frac{1 - F(x)}{F(x)} \int_{y=\underline{s}}^{x} \overline{Q}(y) f(y) \, dy \right) dx.$$

we obtain our final formula for revenue, which is

$$R = \int_{x \in S} \psi(x) \overline{Q}(x) f(x) \, dx$$

where

$$\psi(x) = x - \int_{y=x}^{\overline{s}} \frac{1 - F(y)}{F(y)} dy,$$

Upper Bound on Revenue

- Let $r = \inf \{x | \psi(x) > 0\}$,

- Pointwise optimum of the revenue formula is:

$$\overline{Q}(x) = \begin{cases} 0 & \text{if } x < r; \\ NF^{N-1}(x) & \text{otherwise.} \end{cases}$$

Proposition

The revenue of the optimal auction is bounded above by

$$\overline{R} = \int_{x=r}^{\overline{s}} \psi(x) \, NF^{N-1}(x) \, f(x) \, dx.$$

Trade-Offs

- Bound is generated by allocation that favors low-signal bidders by making $\widehat{Q}(x)$ as small as possible

$$\widehat{Q}(x) = \frac{1}{F(x)} \int_{y=\underline{s}}^{x} \overline{Q}(y) f(y) \, dy$$

- Increasing $\overline{Q}(x)$ has two competing effects on revenue:
  1. increases total surplus generated by auction,
  2. generates additional information rents for types greater than x

- $\psi(x)$ represents net contribution to revenue of allocating object taking into account both forces

- Allocate the good if and only if $\psi(x) \geq 0$

Construct a Direct Mechanism

- Construct a direct mechanism that attains the bound

- Probability of getting the good with highest signal:

$$\gamma(x) = \frac{1}{N}\left(1 - \left(\frac{F(r)}{F(x)}\right)^{N}\right).$$

- Allocation is as follows:

- If highest signal x is at least r, then the good is allocated to bidder with highest signal with probability $\gamma(x)$....

- ...with residual probability $1 - \gamma(x)$, good is allocated to one of $N-1$ bidders who do not have highest signal

- if the highest signal is less than r, then the good is not allocated all

Allocation Rule

- Probability of bidder i receiving the object:

$$q_i(s) = \begin{cases} \gamma(\max s), & \text{if } s_i > s_j \; \forall j \neq i \text{ and } s_i \geq r; \\ \frac{1}{N-1}(1 - \gamma(\max s)), & \text{if } s_i < \max s \text{ and } \max s \geq r; \\ 0, & \text{otherwise.} \end{cases}$$

- Reverse engineered to implement the allocation corresponding to the solution to the relaxed program

- Total surplus coincides with solution to relaxed program:

$$\gamma(x) = \frac{\widehat{Q}(x)}{\overline{Q}(x)} = \frac{\widehat{Q}(x)}{N F^{N-1}(x)}$$

and correspondingly

$$\widehat{Q}(x) = \frac{F^N(x) - F^N(r)}{F(x)}$$

Main Result

- Implied interim transfer is constant in s_i:

$$T_i\left(s_i\right) = T = \int_{x=r}^{\overline{s}} x\left(1 - \gamma\left(x\right)\right) F^{N-2}\left(x\right) f\left(x\right) dx,$$

- Simply the expected surplus generated by allocating the good to any type $s_i < r$.

Theorem

The direct mechanism described above is IC and IR and attains maximum revenue. The interim transfer payment $T_i(s_i)$ and the interim probability $Q_i(s_i)$ of receiving the good are constant in s_i.

Resale Interpretation

- Seller biases the allocation towards those bidders who are likely to sell the good to a different buyer than the one they know about

- Since they have less private information about the resale value, cheaper to incentivize them to reveal their signals

- $\widehat{Q}$ cannot be too low, however, or else bidders would want to deviate by misreporting redrawn lower signals

- Constraint boils down to the requirement that $\widehat{Q}(x)$ cannot be smaller than the probability that the good is allocated conditional on the highest signal being less than x

Uniform Distribution

- $[\underline{v}, \overline{v}] = [0, 1]$, $F(x) = x^{1/N}$, so value is standard uniform

- The generalized virtual utility $\psi(x)$ takes the form:

$$\psi(x) = x - \int_{y=x}^{1} \left(x^{-\frac{1}{N}} - 1 \right) dx = \frac{1}{N-1} \left(N x^{\frac{N-1}{N}} - 1 \right)$$

- Optimal cutoff r is therefore

$$r = \left(\frac{1}{N} \right)^{\frac{N}{N-1}}, \tag{1}$$

  which is strictly decreasing in N

- Optimal revenue

$$\frac{1}{2N-1} \left(N - 1 - \frac{1}{N}^{\frac{N}{N-1}} \right)$$

- Strictly increasing in N as well, converges to $1/2$

Descending Clock Auction

- Natural indirect implementation of the optimal mechanism that uses a "descending clock"

- In Dutch auction, the value of the clock is the **price** at which the bidder who stops the clock will purchase the good

- In our indirect mechanism, the value of the clock is the **probability** with which the bidder who stops the clock gets the good

- All of the bidders must pay an entry fee of T to enter the auction as determined earlier

Descending Probability Auction

- Probability p starts at $\gamma(\bar{s}) \leq 1/N$ and descends gradually

- If bidder i is first to release his button at $p > 0$, then i is allocated the good with probability p, and each of the other bidders receive it with probability$(1 - p)/(N - 1)$

Proposition

The descending probability auction implements the optimal auction.

- Even as p gets arbitrarily close to zero, bidders are willing to wait and see if someone else stops the auction,

- Probability of being allocated the good as bidder who stops the auction is sufficiently small compared to the corresponding probability when someone else stops the auction

Posted Prices

- Optimal selling mechanism is achieved with constant interim transfer $T = T_i(s_i)$ and constant interim winning probability $Q = Q_i(s_i)$.

- But it distorts the ex post allocation $q_i(s)$ as a function of the threshold value r which the highest signal has to exceed before the object is allocated.

- Posted price becomes optimal if the threshold value r were to coincide with lowest signal in the support of S, or $r = \underline{s}$.

Optimality of Posted Price

Proposition

A posted price mechanism is optimal if and only if $\psi\left(\underline{s}\right) \geq 0$. If a posted price p is optimal, then

$$p = T \cdot N = \int_{\underline{s}}^{\overline{s}} x(N-2)F(x)^{N-1}f(x)dx.$$

- p is the expected value of the lowest type

- If posted price is optimal, it will not exclude any type

- Posted price is limit of descending probability auction as $r \to 0$

- Initial value of the clock $\gamma\left(\overline{s}\right)$ converges to $1/N$, and in equilibrium, all types stop the clock immediately

Uniform Distribution

- Family of translated uniform distributions on $[a, a+1]$, $a > 0$.

- Marginal revenue function for these distributions is

$$\psi_a(x) = x - \int_{y=x}^{a+1} \left((x-a)^{-\frac{1}{N}} - 1 \right) dx,$$

- Lowest marginal revenue is

$$\psi_a(a) = a - \int_{y=a}^{a+1} \left((x-a)^{-\frac{1}{N}} - 1 \right)$$

$$= a - \frac{1}{N-1}.$$

- Thus, a posted price is optimal for $a > 1/(N-1)$

Posted Prices and Large N

- Generally, posted prices are approximately optimal for N large

- Suppose seller sets a posted price of

$$t = \int_v vP(dv) - \epsilon$$

  for some $\epsilon > 0$,

- As $N \to \infty$, the probability that at least one of the bidders assigns a value of at least t to the good goes to 1,

- For N large, when a bidder has a low signal, the expectation of the highest of the others' signals is converging to the unconditional expectation of the value

- Probability that at least one bidder has low signal $\to 1$

Maximum Value Game

- Bulow and Klemperer (2002) show that in the "maximum value game" with a second price auction in equilibrium each bidder bids his signal

$$v_i = max_j\{x_1, ..., x_j, ..., x_N\}$$

- In equilibrium, the bidder with the highest signal wins the auction and pays the second-highest signal

- In fact, it is optimal to bid any amount which is at least your signal, and, in particular, it is optimal to bid your signal

- By contrast, in optimal auction, each bidder is indifferent between reporting his signal and reporting any **lower** signal

Comparison with IPV

- Suppose now that the signals are the values, thus independent private value environment:

$$v_i = x_i \leq max_j\{x_1, ..., x_N\}$$

- In second price auction bidding your value remains optimal

- Thus, we find that in the second price auction of the pure common value environment, each bidder behaves as if his signal is his true private value rather than a signal and in particular a lower bound on the pure common value.

- Observation can be generalized

Strategic Equivalence

- Consider independent private value (IPV) model: $v_i(s_1, \ldots, s_N) = s_i$

- Denote the set of bidders with high signals

$$H(s) = \left\{ i \,|\, s_i = \max_j s_j \right\}$$

- An allocation q_i is **conditionally efficient** if (i) $q_i(s) > 0$ if and only if $s_i \in H(s)$ and (ii) there exists a cutoff r such that the good is allocated whenever $\max_i s_i > r$.

Proposition

Suppose a direct mechanism $\{q_i, t_i\}$ is IC and IR for the IPV model in which $v_i(s) = s_i$ and that the allocation is conditionally efficient. Then $\{q_i, t_i\}$ is also IC and IR for the maximum common value model in which $v_i(s) = \max_j \{s_j\}$.

Implications for First Price Auction

- We say that an auction is **standard** if it induces a conditionally efficient allocation on IPV type spaces

- e.g., first-price, second-price, all-pay, with reserves

Corollary

Suppose there is a pure common value v with fixed distribution $P(v)$. Then there exists a reserve price r such that the first-price auction with minimum bid r generates greater minimum revenue than any standard mechanism, where the minimum is taken across all Bayes Nash equilibria and across all common-value common-prior type spaces where the distribution of the common value is P.

- While many standard mechanisms are optimal under IPV, standard mechanisms other than the FPA are more susceptible to low revenue in other informational environments (BBM, Ecta 2017)

Auctions vs Optimal Mechanism

- Bulow and Klemperer (1996) establish the limited power of optimal mechanisms as opposed to standard auction formats

- Revenue of optimal auction with N bidders is strictly dominated by standard absolute auction with $N + 1$ bidders

- Pure common value environment is an instance of their more general interdependent value environment with one exception

- Virtual utility function—or marginal revenue function—is not monotone due to maximum operator in common value model

A Closer Look at the Virtual Utility

- Non-monotonicity leads to an optimal mechanism with features distinct from standard first or second price auction.

- It elicits information from bidder with highest signal but minimizes probability of assigning him the object subject to incentive constraint

- **Virtual utility** of each bidder, $\pi_i(s_i, s_{-i})$:

$$\pi_i(s_i, s_{-i}) = \left\{ \begin{array}{ll} \max_j\{s_j\}, & \text{if} \quad s_i \leq \max\{s_{-i}\}; \\ \max\{s_j\} - \frac{1 - F_i(s_i)}{f_i(s_i)}, & \text{if} \quad s_i > \max\{s_{-i}\}. \end{array} \right.$$

- Downward discontinuity in virtual utility indicates why seller wishes to minimize the probability of assigning the object to the bidder with the high signal

Revenue Comparison

- Virtual utility of bidder i fails the monotonicity assumption even when the hazard rate of the distribution function is increasing everywhere

- Bulow and Klemperer (1996) require the monotonicity of the virtual utility when establishing their main result that an absolute English auction with $N + 1$ bidders is more profitable than any optimal mechanism with N bidders

- Consider class of power distribution functions:

$$G(v) = v^{\alpha}, \ v \in [0, 1], \ \alpha \in \mathbb{R}_+.$$

Proposition

For every $N \geq 2$, there exists $\overline{\alpha}$, with $1 < \overline{\alpha} < \infty$, such that an optimal auction with N bidders is more profitable than a second price auction with $N + 1$ bidders if and only if $\alpha < \overline{\alpha}$.

Revenue with Uniform Distribution

- In special case of the uniform distribution:

$$\alpha = 1$$

- Optimal mechanism with N bidders is therefore more profitable than a second price auction with $N + 1$ bidders irrespective of the number N of bidders

Auctions with Resale

- Characterization of the optimal mechanism remains valid if we interpret the model as one where the object is initially sold optimally among N bidders with independent private values

- Then offered for resale under complete information

- In contrast to previous work on auctions with resale, such as Gupta and le Brun (1999) and Haile (2003) we analyze optimal mechanism in primary market

- Carroll and Segal (2016) study robust resale mechanisms

Conclusion

- Characterized novel revenue maximizing auctions for a class of common value models

- Naturally arise when demand is derived from reselling downstream

- Existing characterizations of optimal auctions depend on information rents being smaller for bidders who are more optimistic about the value

- Qualitative impact was that optimal auctions discriminate in favor of more optimistic bidders

- Today: optimal auctions discriminate in favor of less optimistic bidders since they obtain less information rents from being allocated the good

- In some cases, leads to optimality of posted prices