The Folk Theorem in Repeated Games with Anonymous Random Matching*

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Abstract

We prove the folk theorem for discounted repeated games with anonymous random matching. We allow non-uniform matching, include asymmetric payoffs, and place no restrictions on the stage game other than full dimensionality. No record-keeping or communication devices—including cheap talk communication and public randomization—are necessary.

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1 Introduction

In a repeated game with anonymous random matching, a finite population of players repeatedly breaks into pairs to play 2-player games. Each period, a player observes only her partner's action—not his identity, and not any other player's action. We prove the folk theorem in this environment. In particular, when the players are sufficiently patient, they can sustain the same payoffs as if everyone's identity and actions were publicly observed at the end of each period.

Because players receive so little information under anonymous random matching, this environment has long been used as a benchmark against which to measure the value of various record-keeping devices and institutions, such as fiat money, merchant coalitions and guilds, credit bureaus, online rating systems, "standing" and "image scoring" in evolutionary biology, and in-group monitoring within ethnic groups.[1] The main implication of our result is that, even in this information-poor benchmark environment, patient players can obtain any feasible and individually rational payoffs without any record-keeping devices or institutions beyond their individual memories and the ability to count periods. Thus, any role for such institutions must result from impatience of the players, or perhaps from the possibility of constructing "simpler," "more robust," or "more realistic" equilibria when more information is available.[2]

Our folk theorem thus admits both positive and negative interpretations. The positive interpretation is that a very wide range of cooperative behaviors are possible despite minimal information. The negative interpretation is that, in a finite population of patient long-run players, it is difficult to justify the value of information-sharing institutions on efficiency grounds alone. In particular, in these environments the assumptions that monitoring is decentralized and players are anonymous—which might have been expected to restrict the set of attainable payoffs in some games—turn out to be completely payoff-irrelevant.

[1] On money, see Kiyotaki and Wright (1989, 1993), Kocherlakota (1998), Wallace (2001), Araujo (2004), Aliprantis, Camera, and Puzzello (2007). On merchants, see Greif (1993), Greif, Milgrom, and Weingast (1994), Milgrom, North, and Weingast (1990). On credit bureaus, see Klein (1992), Padilla and Pagano (2000). On online rating systems, see Friedman and Resnick (2001). On standing and image scoring, see Sugden (1986), Nowak and Sigmund, (1998). On ethnic conflict, see Fearon and Laitin (1996).

[2] Of course, our result first fixes the population size N and then takes $\delta \to 1$. If the population is very large, the required discount factor is very close to 1.

Our approach is to view the repeated random matching game as a single repeated game with imperfect private monitoring and apply techniques from the literature on the folk theorem with private monitoring. The main obstacle to this approach is that, when viewed as a single repeated game, the random matching game fails standard statistical identifiability conditions (e.g., Fudenberg, Levine, and Maskin's (1994) pairwise identifiability) and full support conditions. To overcome this obstacle, we show that players can be given incentives to truthfully share their observations—despite communicating only via payoff-relevant actions—and that the aggregated observations of a player's opponents always identify her action. Our paper thus connects three literatures: repeated games with random matching, repeated games with private monitoring, and secure communication in repeated games.

Random matching Kandori (1992), Ellison (1994), and Harrington (1995) show that cooperation can be sustained in the repeated prisoners' dilemma with anonymous random matching via "contagion strategies," where a single defection triggers the breakdown of cooperation throughout the population. This approach does not generalize beyond the prisoners' dilemma. Even within the prisoners' dilemma, it cannot be used to support asymmetric equilibria, where for example a subset of players are allowed to defect while others must cooperate. In contrast, our theorem covers all games (subject to a mild full dimensionality condition) and all feasible and individually rational payoffs.

Deb (2017) proves the folk theorem for asymmetric games where players from distinct communities fill different player-roles, cheap talk communication between partners is allowed, and all players from the same community receive the same payoff. We instead consider random matching within a single population (though our approach readily generalizes to multiple communities), allow asymmetric payoffs, and—most importantly—disallow cheap talk.[3] Deb and González-Díaz (2017) also disallow cheap talk in the 2-community model, but they impose some conditions on the stage game, restrict attention to symmetric payoffs that Pareto dominate a Nash equilibrium (thus obtaining a "Nash threat" folk theorem), and require the population to be sufficiently large. Their proof is completely different from ours, as they generalize the contagion strategy approach, while we build on the block belief-free

[3] Ruling out cheap talk seems essential, as the point of our analysis is to see what outcomes are possible in the absence of record-keeping and communication devices.

approach introduced by Hörner and Olszewski (2006) to study repeated games with almost-perfect monitoring—we compare these two approaches below. Deb, González-Díaz, and Renault (2017) prove a general folk theorem for N-community games without discounting. Another difference from all of these papers is that we allow matching to be non-uniform, and even non-i.i.d..

Other random matching models assume players directly observe some information about their partners' past play. Okuno-Fujiwara and Postlewaite (1995) and Dal Bó (2007) consider finite population models; notably, the latter paper allows asymmetric payoffs. Rosenthal (1979), Takahashi (2010), Heller and Mohlin (2017), and Bhaskar and Thomas (2018) consider continuum models.

Private monitoring The literature on repeated games with imperfect private monitoring is too large to survey here. The folk theorem with public cheap talk communication is proved by Compte (1998) and Kandori and Matsushima (1998). Piccione (2002), Ely and Välimäki (2002), Matsushima (2004), Ely, Hörner and Olszewski (2005), Hörner and Olszewski (2006), and Yamamoto (2012) develop belief-free techniques that we build on. Sugaya (2017) proves a general folk theorem under identifiability and full support conditions. These conditions are violated with anonymous random matching, but some ideas from Sugaya's proof are nonetheless useful. We explain the connection to this literature in Section 3.5.

Secure communication The most challenging part of our proof is providing incentives for secure communication with anonymous random matching, when communication can be executed only through payoff-relevant actions. As far as we know, ours is the first paper to address this problem. Incentives for secure communication have however been studied in the related setting of repeated games played on fixed networks (Ben-Porath and Kahneman, 1996; Renault and Tomala, 1998; Lippert and Spagnolo, 2011; Laclau, 2012, 2014; Nava and Piccione, 2014; Wolitzky, 2015). While the technical overlap with this literature is slight, our non-uniform matching model can approximate a fixed network, as we allow the case where a player "almost always" interacts with the same partners. As will be seen, in this setting we construct general-purpose communication protocols that are "fast," "accurate," "secure," and "error-proof."

2 Model and Folk Theorem

There is a finite set of players $I = \{1, ..., N\}$, with $N \geq 4$ even. In every period $t = 1, 2, \ldots$, players match in pairs to play a finite, symmetric 2-player game with action set A and payoff function $u : A \times A \to \mathbb{R}$, with $|A| \geq 2$. Let $a^0, a^1 \in A$ denote two arbitrary, distinct actions.

Pairs are formed as follows: (i) a *matching* μ is a partition of the population into pairs, (ii) there is an exogenous distribution p over matchings, and (iii) the period-t matching μ_t is drawn from p i.i.d. across periods.[4] We assume p has full support and let $\bar{\varepsilon} > 0$ denote the minimum of $p(\mu)$ over all matchings. As there are at least 3 possible matchings when $N \geq 4$, we have $\bar{\varepsilon} \leq \frac{1}{3}$. Let $\mu(i)$ denote player i's partner in matching μ. Let $p_{i,j} = \sum_{\mu:\mu(i)=j} p(\mu)$ denote the probability that players i and j are matched.

Players are anonymous—each player observes only the actions she faces and not her opponents' identities. Formally, letting $a_{i,t} \in A$ denote player i's period-t action, player i's observation in period t is the pair $(a_{i,t}, \omega_{i,t})$, where $\omega_{i,t} = a_{\mu_t(i),t}$. Say that a profile of observations $(a_i, \omega_i)_{i \in I}$ is *feasible* if there exists an action profile $\mathbf{a} = (a_1, \ldots a_N) \in \prod_{i \in I} A = A^N$ and a matching μ such that $\omega_i = a_{\mu(i)}$ for all $i \in I$. Player i's history at the beginning of period t is denoted $h_i^{t-1} = (a_{i,\tau}, \omega_{i,\tau})_{\tau=1}^{t-1}$, with $h_i^0 = \emptyset$. Players maximize expected discounted payoffs with common discount factor $\delta < 1$. Let $E(\delta)$ denote the sequential equilibrium payoff set with discount factor δ.[5]

For any action profile $\mathbf{a} \in A^N$, player i's expected payoff at action profile $\mathbf{a}$ is given by

$$\hat{u}_i(\mathbf{a}) = \sum_{j \neq i} p_{i,j} u(a_i, a_j).$$

Thus, the (convex hull of the) feasible payoff set in the N-player game is $F = \text{co}\left(\{\hat{\mathbf{u}}(\mathbf{a})\}_{\mathbf{a} \in A^N}\right)$, where $\hat{\mathbf{u}}(\mathbf{a}) = (\hat{u}_1(\mathbf{a}), \ldots, \hat{u}_n(\mathbf{a}))$. Let $\bar{u} = \max_{(a,a') \in A^2} |u(a, a')|$ be the greatest magnitude of any feasible payoff, and let $\underline{u} = \min_{\alpha \in \Delta(A)} \max_{a \in A} u(a, \alpha)$ be the minmax payoff. Let $\alpha^{\min} \in \text{argmin}_{\alpha \in \Delta(A)} \max_{a \in A} u(a, \alpha)$ be a minmax strategy in the 2-player game; to minmax

[4] The extension to non-i.i.d. matching is considered in Section 4.

[5] In defining sequential equilibrium, the choice of topology on the sets of beliefs and strategies does not matter for us—for concreteness, take it to be the product topology. This is another point of contrast with the approaches in Deb (2017) and Deb and González-Díaz (2017), where choosing the product topology is essential.

player i in the N-player game, every player but i plays $\alpha^{\min}$. Denote the set of feasible and individually rational payoffs by $F^* = \{\mathbf{v} \in F : v_i \geq \underline{u} \ \forall i \in I\}$.

We assume F^* has dimension N. This condition is generic: letting

$$e^i = \left(u\left(a^0, a^1\right), \left(\left(1 - p_{j,i}\right) u\left(a^1, a^1\right) + p_{j,i} u\left(a^1, a^0\right) \right)_{j \neq i} \right) \in \mathbb{R}^N$$

be the payoff vector when player i plays a^0 and all other players play a^1, the vectors $(e^i)_{i \in I}$ are linearly independent for generic values of $u\left(a^0, a^1\right)$, $u\left(a^1, a^0\right)$, and $u\left(a^1, a^1\right)$.[6]

In this setting, we establish the folk theorem:

Theorem 1 *For all* $\mathbf{v} \in \text{int}\left(F^*\right)$, *there exists* $\bar{\delta} < 1$ *such that* $\mathbf{v} \in E\left(\delta\right)$ *for all* $\delta > \bar{\delta}$.

3 Key Ideas of the Equilibrium Construction

We provide a constructive proof of the folk theorem. Most of the proof is deferred to the appendix. Here we begin the proof and introduce the key ideas underlying the construction.

We view the repeated game as an infinite sequence of finite blocks of periods. Deviations from equilibrium play are detected as a result of communication among the players (described below) and are then punished in two ways. First, within the block where the deviation occurs, players switch to mutual minmaxing. Second, the deviator's continuation payoff at the start of the next block is reduced, while other players' continuation payoffs are adjusted to compensate them for any cost of punishing the deviator.[7]

Thus, within a block, all players' payoffs are tied together, as in a contagion equilibrium of the form introduced by Kandori (1992), Ellison (1994), and Harrington (1995). Across blocks, however, each player's continuation value is independent of her opponents'. Thus, while the

[6] Full-dimensionality of F^* and full-dimensionality of the underlying 2-player game are logically independent. If the 2-player game is a pure coordination game (with payoff dimension 1) then F^* has full dimension. Conversely, with $N = 4$ and uniform matching, the 2-player game

	a^0	a^1
a^0	$4, 4$	$1, 3$
a^1	$3, 1$	$0, 0$

has full dimension, while F^* has dimension 1.

[7] This basic of idea of "rewarding the punishers" dates back to Fudenberg and Maskin (1986).

key challenge in constructing a contagion equilibrium is providing incentives to carry out punishments, in our construction the challenges are instead providing incentives for correct play within each block and (especially) providing incentives for truthful communication.

We now describe the structure of our equilibrium. Players follow automaton strategies. In each block, each player $i \in I$ has two possible states—denoted $x_i \in \{G, B\}$, for "good" and "bad." A player's state in the current block and her history in the current block jointly determine her state in the next block. We specify each player i's block strategy in state x_i—denoted $\sigma_i(x_i)$—and the state transition rules so that (i) for every realization of the other players' states $x_{-i} \in \{G, B\}^{N-1}$, both $\sigma_i(G)$ and $\sigma_i(B)$ are optimal strategies for player i (that is, as in Hörner and Olszewski (2006), the equilibrium is *block belief-free*), and (ii) player i's equilibrium continuation payoff is completely determined by the state of player $(i-1) \pmod N$. In particular, player i's continuation payoff is high (low) if $x_{i-1} = G$ (B). Player i's state transition rule can thus be used to control player $i+1$'s continuation payoff.

Play within a block proceeds as follows. First, there is an "initial talk phase," where players communicate to coordinate on the state profile $x \in \{G, B\}^N$. Then, there is a "play and talk" phase, during which players repeat the following "sub-block" multiple times: they play actions that attain the target payoffs at state profile x for many periods, and then communicate to see if anyone deviated. If players detect a deviation, they switch to the minmaxing strategy starting in the next sub-block. This is followed by a "final talk phase," where players communicate a summary of the entire block history.

Since all communication is via payoff-relevant actions, to attain the target payoffs the players must spend most of their time in the "play" phases. In particular, they cannot take the time to communicate about every play period. Instead, when players communicate to identify deviations, player $i-1$ chooses one period at random from the preceding play phase and communicates this choice to the other players, who then share their information about that period only. Since player i does not know in advance which period player $i-1$ will choose, this scheme can provide incentives for the entire play phase. However, for this to work, we need to show that (i) players $-i$ can communicate in a manner such that player i cannot profitably deviate by attempting to manipulate the outcome of communication, and (ii) once players $-i$ successfully share their information, they can identify player i's action.

In sum, the four key ideas that underlie the construction are as follows:

1. **Identifiability** We first show that it is possible to perfectly identify any player's action by aggregating the observations of all of her opponents.

2. **Communication modules** Given identifiability, the next question—and the key challenge in proving the theorem—is how to elicit players' information about their past actions and observations. We accomplish this by introducing several *communication modules*: finite repetitions of the stage game in which players communicate via actions, along with terminal payoffs (*reward functions*) that make such communication incentive compatible. As we will discuss, we must construct protocols for sharing information via actions that are "fast," "accurate," "secure," and "error-proof."

3. **Block structure** If players truthfully share information—and thus actions are identified—we can apply relatively standard techniques to sustain any feasible and individually rational payoff in a block belief-free equilibrium.

4. **Reward functions** The careful construction of reward functions (i.e., continuation payoffs from the next block, controlled by other players' state transitions) provides incentives for correct play and truthful communication within each block. This ties together the communication modules and the block structure.

We describe these four aspects of the proof in turn.

3.1 Identifiability

Suppose in some period players $-i$ play a_{-i} and observe ω_{-i}. Assume for now that players $-i$ can perfectly aggregate their observations. Then the profile (a_{-i}, ω_{-i}) of i's opponents' actions and observations perfectly identifies player i's action and observation, (a_i, ω_i).

Lemma 1 *There exists a function* $\varphi : A_{-i} \times A_{-i} \to A_i \times A_i$ *such that, if* $(a_i, \omega_i)_{i \in I}$ *is feasible, then* $\varphi(a_{-i}, \omega_{-i}) = (a_i, \omega_i)$.

Proof. Since matching occurs in pairs, the total number of players who observe the same action they play (i.e., observe $\omega_n = a_n$) is always even. Therefore, if there exists $a \in A$ such that the number of i's opponents for whom $\omega_n = a_n = a$ is odd, then $\omega_i = a_i = a$. If instead this number is even for every $a \in A$, then $a_i \neq \omega_i$. (Otherwise, the total number of players with $\omega_n = a_n = a_i$ would be odd.) In this case, there is one action a such that more of i's opponents observe $\omega_n = a$ than play $a_n = a$, and there is another action ω such that more of i's opponents play $a_n = \omega$ than observe $\omega_n = \omega$. This pair (a, ω) must then equal (a_i, ω_i). ■

Thus, if players $-i$ can aggregate their observations, they can perfectly monitor player i. While convenient, this perfect monitoring property is not necessary for our proof approach: in Section 4 and the Supplementary Appendix, we show that our proof extends to almost-perfect monitoring within matches. Nonetheless, perfect monitoring simplifies the proof while letting us focus on its most novel element: incentivizing truthful communication. We therefore maintain this assumption in the text.

3.2 Communication Protocols

The heart of the proof is the construction of *communication modules* that give players incentives to share information. It is helpful to start by explaining what properties we will need the modules to satisfy. To do so, we provide a more detailed description of each "talk" phase within a block, starting from the end of the block and moving backwards.

The final talk phase at the end of a block comprises three phases. In the last phase, player $i-1$ chooses one period t at random from all the previous periods in the block and communicates it to the other players, who then communicate their period t information to player $i-1$. Player $i-1$ then slightly adjusts her state transition probability such that the effect of discounting in player i's payoff is cancelled out: when player $i-1$ chooses period t, she increases player i's continuation payoff by $\frac{1}{\Pr(t \text{ is chosen})}\left(1-\delta^{t-1}\right)\hat{u}_i(\mathbf{a}_t)$, where $\mathbf{a}_t$ is identified from communication. Recall that player $i-1$'s state affects player i's payoff only. Hence, in this communication phase, players $-i$ are indifferent to the outcome of communication. Moreover, even player i has only a very small potential gain from manipulating communication when δ is large (once we fix the length of the block). Since it is always possible to provide small incentives without sacrificing much efficiency, for this communication phase we simply need a protocol that lets players $-i$ communicate their histories quickly and accurately. The *basic communication module* introduced below is sufficient for this.

In the penultimate talk phase, players $-i$ aggregate their histories from all previous talk phases in the block. Player $i-1$ uses this information to adjust her state transition. As will be seen, the impact of this adjustment on player i's payoff can be large, so player i may have a strong incentive to manipulate the outcome of communication if possible. Hence, for this communication phase we need a communication module with the property that (i) players $-i$ communicate their histories quickly and accurately, and (ii) there is no history at which player i believes she can manipulate the outcome of communication to her benefit. We will show that the *secure communication module* constructed below has this property. The same module will also suffice for the third-to-last talk phase.

In the remaining talk phases (i.e., the talk phase after each play phase, and the initial talk phase), there is an additional difficulty: since these phases affect not only continuation payoffs at the end of the block but also continuation play within the block, *all* players (not only the one "about whom the others are talking") have a strong incentive to manipulate communication if possible. For these communication phases, we thus need a communication module that no player can profitably manipulate. We construct the *verified communication module* to have this property.

We also introduce another communication module of secondary importance—the *jamming coordination module*. This will be described later.

A basic building block of any communication module is a *communication protocol*: a procedure for players to communicate via actions (formally, a strategy profile in a finitely repeated game). The description of a communication protocol does not include payoff functions and thus entails no claims about incentive compatibility. After constructing the communication protocols, we augment each of them with a *reward function* to construct the communication modules, and then verify sequential rationality.

There are thus four communication protocols: the *basic protocol*, the *secure protocol*, the *verified protocol*, and the *jamming coordination protocol*. In this subsection, we present the first three and derive the key statistical properties for the first two. The remaining details—as well as the reward functions used to provide incentives—are described in the appendix. The reward functions are previewed in Section 3.4.

We will repeatedly use the following exponential bound on the probability that a pair of players fails to match even once during a set of T periods:

Lemma 2 *For any set of T periods $\mathbb{T} \in \mathbb{N}^T$ and any pair of distinct players $i, j \in I$,*

$$\Pr\left(\mu_t(i) \neq j \ \forall t \in \mathbb{T}\right) \leq \exp\left(-\bar{\varepsilon}T\right).$$

Proof. $\Pr\left(\mu_t(i) \neq j \ \forall t \in \mathbb{T}\right) \leq \left(1-\bar{\varepsilon}\right)^T = \exp\left(T \log\left(1-\bar{\varepsilon}\right)\right) \leq \exp\left(-\bar{\varepsilon}T\right)$. ■

3.2.1 Basic Communication Protocol

The basic protocol lets a player $i \in I$ broadcast a message m_i from a set $M_i = \{1, \dots, |M_i|\}$. We call player i the *sender* and call the other players *receivers*. Let T be a constant to be determined, and let $\lceil x \rceil$ denote the least integer greater than x. We require the following properties:

1. Communication is *fast*: The protocol takes $2T \lceil \log_2 |M_i| \rceil$ periods.

2. Communication is *accurate*: At the end of the protocol, each receiver $j \neq i$ creates an *inference* $m_i(j) \in M_i \cup \{0\}$ (if $m_i(j) = 0$, we say j *fails to infer a message*). With probability of order $1-\exp(-T)$, j's inference is correct: $m_i(j) = m_i$.[8] Moreover, either j's inference is correct or j fails to infer a message: if $m_i(j) \neq m_i$ then $m_i(j) = 0$.

We show the following protocol has the desired properties:[9]

Basic Communication Protocol for Player i to Send Message m_i with Repetition T:

- Divide the $2T \lceil \log_2 |M_i| \rceil$ periods into $\lceil \log_2 |M_i| \rceil$ *intervals* of $2T$ periods each.

- For $t \in \{1, \dots, \lceil \log_2 |M_i| \rceil\}$,

[8] This phrasing is slightly loose: more precisely, $\Pr\left(m_i(j) = m_i\right) \geq 1 - \lceil \log_2 |M_i| \rceil \exp\left(-\bar{\varepsilon}T\right)$. Similar caveats apply whenever we summarize a protocol as having the property that a certain probability is "of order $1-\exp(-T)$."

[9] In what follows, instructions of the form "play action a in period t" are to be read as unconditional on a player's past actions and observations. Thus, a communication protocol is formally a strategy profile, not only a description of on-path play.

- If the t^{th} digit of the binary expansion of $m_i - 1$ is 0, player i plays a^0 for the first half of the t^{th} interval (i.e., the first T periods in the interval) and plays a^1 for the second half of the t^{th} interval (i.e., the last T periods in the interval).
- If the t^{th} digit of the binary expansion of $m_i - 1$ is 1, player i plays a^1 for the first half of the t^{th} interval and plays a^0 for the second half of the t^{th} interval.

We call a set of T periods where player i takes a constant action a *half-interval.*

- Each player $j \neq i$ plays a^0 throughout the protocol.
- At the end of the protocol, each player $j \neq i$ creates an inference $m_i(j) \in M_i \cup \{0\}$ as follows (as a function of her history $(a_{j,t}, \omega_{j,t})_{t=1}^{2T\lceil \log_2 |M_i| \rceil}$):
  - If, for some $t \in \{1, \ldots, \lceil \log_2 |M_i| \rceil\}$, $\omega_{j,\tau} \notin \{a^0, a^1\}$ for some period τ in the t^{th} interval, player j sets $m_i(j) = 0$.
  - If, for some $t \in \{1, \ldots, \lceil \log_2 |M_i| \rceil\}$, $\omega_{j,\tau} \neq a^1$ for every period τ in the t^{th} interval, player j sets $m_i(j) = 0$.
  - If, for some $t \in \{1, \ldots, \lceil \log_2 |M_i| \rceil\}$, $\omega_{j,\tau} = \omega_{j,\tau'} = a^1$ for some period τ in the first half of the t^{th} interval and some period τ' in the second half of the t^{th} interval, player j sets $m_i(j) = 0$.
  - Otherwise, player j constructs a number $\hat{m} \in \{0, \ldots, \lceil \log_2 |M_i| \rceil - 1\}$ as follows:
    * If $\omega_{j,\tau} = a^1$ for some period τ in the first half of the t^{th} interval and $\omega_{j,\tau} = a^0$ for every period τ in the second half of the t^{th} interval, player j sets the t^{th} digit of the binary expansion of $\hat{m}$ equal to 1.
    * If $\omega_{j,\tau} = a^1$ for some period τ in the second half of the t^{th} interval and $\omega_{j,\tau} = a^0$ for every period τ in the first half of the t^{th} interval, player j sets the t^{th} digit of the binary expansion of $\hat{m}$ equal to 0.
  - If $\hat{m} \leq |M_i| - 1$, player j sets $m_i(j) = \hat{m} + 1$. If $\hat{m} \geq |M_i|$ (which is possible if $\log_2 |M_i|$ is not an integer), player j sets $m_i(j) = 0$.

With this protocol, communication is fast by construction. Let us check that it is also accurate. When all players follow the protocol, $m_i(j) = m_i$ if and only if player j matches with player i at least once in every T-period half-interval where player i plays a^1. Hence, by Lemma 2,

$$\Pr(m_i(j) = m_i) \geq 1 - \lceil \log_2 |M_i| \rceil \exp(-\bar{\varepsilon} T) \ \ \forall j \neq i. \tag{1}$$

Moreover, when all players follow the protocol, if $m_i(j) \neq m_i$ then $m_i(j) = 0$.

In particular, $m_i(j) = m_i$ unless the realized matching process is *erroneous*, in that, for some T-period half-interval, some pair of players do not match with each other even once. Erroneous match realizations occur with low probability, but pose an important complication.

3.2.2 Secure Communication Protocol

We now define a generalization of the basic protocol, which lets player i send a message in a way that is harder for any receiver to manipulate. We do this by designating a certain set of players as *jamming players*, denoted $I_{\text{jam}} \subset I \setminus \{i\}$, and with small probability having them play in a way that "jams" attempts to manipulate communication. When $I_{\text{jam}} = \emptyset$, the secure protocol reduces to the basic protocol. More specifically, we construct a protocol with the following properties:

1. Conditional on the event that no player jams communication, communication is *fast* and *accurate* (as in the basic communication protocol).

2. Communication is *receiver-secure*: for each player $j \notin I_{\text{jam}} \cup \{i\}$, the distribution of her observations $(\omega_{j,t})_t$ is independent of her strategy, and for each observation sequence $(\omega_{j,t})_t$ one of the following two conditions is satisfied:

   (a) For each action sequence $(a_{j,t})_t$, conditional on $(a_{j,t}, \omega_{j,t})_t$, the probability that some player jammed communication is of order $1 - \exp(-T)$.

   (b) For each action sequence $(a_{j,t})_t$ and each player $j' \neq i, j$, conditional on $(a_{j,t}, \omega_{j,t})_t$ and the event that no player jams communication, the probability that $m_i(j') \in \{m_i, 0\}$ is of order $1 - \exp(-T)$; moreover, if player j follows the protocol then the probability that $m_i(j') = m_i$ is of order $1 - \exp(-T)$.

The use of jamming players to make communication secure is a key innovation in our proof. Jamming players are prescribed to mix between a^0 and a^1 on path, where playing a^1 jams communication. This guarantees that either other players attribute their observations to the on-path play of jamming players (Condition 2(a); this occurs if a^1 is observed "frequently") or player i's message m_i is transmitted successfully (Condition 2(b); this occurs if a^1 is observed "infrequently"). It is therefore impossible for any player $j \neq i$ to manipulate the protocol and successfully transmit an incorrect message $m_i' \notin \{m_i, 0\}$.

Secure Communication Protocol for Player i to Send Message m_i with Repetition T and Jamming Players I_{jam}:

- Divide the $2T \lceil \log_2 |M_i| \rceil$ periods of the protocol into $\lceil \log_2 |M_i| \rceil$ intervals of $2T$ periods each.
- Player i behaves as in the basic communication protocol.
- Each player $j \notin I_{\text{jam}} \cup \{i\}$ behaves as in the basic communication protocol (i.e., plays a^0 throughout the protocol).
- For each player $j \in I_{\text{jam}}$, in the first period of each T-period half-interval (i.e., in periods $t = kT + 1$ for $k \in \{0, 1, \ldots, 2\lceil \log_2 |M_i| \rceil - 1\}$), player j plays a^0 with probability $1 - T^{-9}$ and plays a^1 with probability T^{-9}. She then repeats the chosen action for the remainder of the half-interval (i.e., plays $a_{j,t} = a_{j,kT+1}$ for $t \in \{kT + 2, ..., (k + 1)T\}$).
- At the end of the protocol, each player $j \neq i$ infers a message $m_i(j) \in M_i \cup \{0\}$ as in the basic communication protocol.

For $j \in I_{\text{jam}}$ and $k \in \{0, 1, \ldots, 2\lceil \log_2 |M_i| \rceil - 1\}$, if $a_{j,kT+1} = a^0$ we say player j *plays REG* ("regular") in the k^{th} half-interval, and if $a_{j,(k-1)T+1} \neq a^0$ we say player j *plays JAM* ("jamming") in the k^{th} half-interval. Thus, player j plays REG and JAM with probabilities $1 - T^{-9}$ and T^{-9} in each half-interval, independently across each half-interval.

Let us check that communication is accurate and receiver-secure. Denote the event that all jamming players play REG throughout the protocol by ALLREG. Conditional on

ALLREG, all players behave identically in the secure protocol and the basic protocol. In particular, conditional on ALLREG, inequality (1) holds and $m_i(j) \neq 0$ implies $m_i(j) = m_i$ $\forall j \neq i$. Moreover,

$$\Pr(m_i(j) = m_i \ \forall j \neq i \cap ALLREG) \geq 1 - N \lceil \log_2 |M_i| \rceil \left(\exp(-\bar{\varepsilon} T) + 2T^{-9} \right). \tag{2}$$

In turn, receiver-security is captured by the following lemma:

Lemma 3 *For any player $j \neq i$ with $I_{\text{jam}} \backslash \{j\} \neq \emptyset$ and any sequence of observations $(\omega_{j,t})_{t=1}^{2T\lceil \log_2 |M_i| \rceil}$ that arises with positive probability when players $-j$ follow the secure protocol, at least one of the following two conditions holds:*

1. *For all $(a_{j,t})_{t=1}^{2T\lceil \log_2 |M_i| \rceil}$, we have*

$$\Pr\left(ALLREG | (a_{j,t}, \omega_{j,t})_{t=1}^{2T\lceil \log_2 |M_i| \rceil} \right) \leq T^9 \exp\left(-\frac{1}{4} \bar{\varepsilon} T \right). \tag{3}$$

2. *The following two conditions hold:*

   (a) *For all $(a_{j,t})_{t=1}^{2T\lceil \log_2 |M_i| \rceil}$, we have*

$$\begin{aligned} & \Pr\left(m_i(j') \in \{m_i, 0\} \ \ \forall j' \notin \{i, j\} \,|\, (a_{j,t}, \omega_{j,t})_{t=1}^{2T\lceil \log_2 |M_i| \rceil}, ALLREG \right) \\ \geq \ & 1 - N \lceil \log_2 |M_i| \rceil \exp\left(-\bar{\varepsilon}^4 T \right). \end{aligned} \tag{4}$$

   (b) *If $a_{j,t} = a^0$ for all $t \in \{1, \ldots, 2T \lceil \log_2 |M_i| \rceil\}$, then*

$$\begin{aligned} & \Pr\left(m_i(j') = m_i \ \forall j' \notin \{i, j\} \,|\, (a_{j,t}, \omega_{j,t})_{t=1}^{2T\lceil \log_2 |M_i| \rceil}, ALLREG \right) \\ \geq \ & 1 - N \lceil \log_2 |M_i| \rceil \exp\left(-\bar{\varepsilon}^4 T \right). \end{aligned} \tag{5}$$

Note that Conditions 1 and 2 are not mutually exclusive. The proof (in the appendix) shows that Condition 1 holds if $\omega_{j,t} = a^1$ for at least $(1 - \bar{\varepsilon}^3) T$ periods in some half-interval, while Condition 2 holds if $\omega_{j,t} = a^1$ for at most $(1 - \bar{\varepsilon}^3) T$ periods in every half-interval. Intuitively, in the former case, player j observes a^1 frequently, so from her perspective the probability that a jamming player played JAM is not too low. In the latter case, player j observes a^1 less frequently. In this case, assuming no jamming player played JAM, the message is likely to have transmitted successfully.[10]

In the module we will construct to support this protocol, player j's payoff is independent of her opponents' inferences whenever ALLREG does not hold, and her payoff is minimized when $m_i(j') = 0$ for some $j' \neq j$. Hence, when Condition 1 holds, player i believes that the gain from manipulating communication is very small; while when Condition 2 holds, deviations only decrease her payoff.

3.2.3 Verified Communication Protocol

In the verified communication protocol, player i first broadcasts a message $m_i \in M_i$ in $2\lceil \log_2 |M_i| \rceil$ periods using the basic communication protocol (with $T = 1$). Then, each player (including player i herself) sequentially broadcasts her actions and observations from these $2\lceil \log_2 |M_i| \rceil$ periods using the secure communication protocol with repetition T. The verified protocol thus takes a total of $\mathcal{T}(|M_i|, T)$ periods, where

$$\mathcal{T}(|M_i|, T) := 2\lceil \log_2 |M_i| \rceil + 2\left\lceil \log_2 |A|^{4\lceil \log_2 |M_i| \rceil} \right\rceil NT. \tag{6}$$

Roughly speaking, in this protocol, "cross-checking" observations ensures security against attempted manipulations by any player.

Verified Communication Protocol for Player i to Send Message m_i with Repetition T:

At the beginning of the verified protocol, each player j has two possible types, denoted $\zeta_j \in \{\text{reg}, \text{jam}\}$. A strategy in the protocol is thus a mapping from $\{\text{reg}, \text{jam}\}$ and protocol histories to actions. Let $\mathcal{I}_{\text{jam}} = \{j : \zeta_j = \text{jam}\}$. The protocol consists of $N + 1$ *rounds*.

- *Message round*

[10] Some upper bound on the frequency of $\omega_{j,t} = a^1$ is clearly needed here: for example, if player j observes a^1 in *every* period in some half-interval and no one plays JAM, then player j must have met player i in every period, so the message cannot have transmitted to the other players.

– Player i sends message $m_i \in M_i$ as in the basic communication protocol with $T = 1$.[11]

– Each player $j \neq i$ plays a^0 throughout the round.

Let $\mathbb{T}(\text{msg})$ denote the set of $2\lceil \log_2 |M_i| \rceil$ periods comprising the message round.

- *j-checking round, for each $j \in I$.* Each checking round consists of $\left\lceil \log_2 |A|^{4\lceil \log_2 |M_i| \rceil} \right\rceil$ intervals. Each interval consists of $2T$ periods. Let $\mathbb{T}(j)$ denote the set of $2T \left\lceil \log_2 |A|^{4\lceil \log_2 |M_i| \rceil} \right\rceil$ periods comprising the j-checking round.

  – Player j sends message $(a_{j,t}, \omega_{j,t})_{t \in \mathbb{T}(\text{msg})} \in A^{4\lceil \log_2 |M_i| \rceil}$ as in the basic communication protocol.

  – Each player $n \notin \mathcal{I}_{\text{jam}} \cup \{j\}$ plays a^0 throughout the round.

  – In each half-interval, each player $n \in \mathcal{I}_{\text{jam}} \setminus \{j\}$ mixes between REG and JAM with probabilities $1 - T^{-9}$ and T^{-9}, as in the secure communication protocol.

  – Each player $n \neq j$ infers a message $(a_{j,t}(n), \omega_{j,t}(n))_{t \in \mathbb{T}(\text{msg})} \in A^{4\lceil \log_2 |M_i| \rceil} \cup \{0\}$ as in the basic communication protocol.

- At the end of the protocol, each player $n \in I$ creates a *final inference* $m_i(n) \in M_i \cup \{0\}$ as follows:

  – If $(a_{j,t}(n), \omega_{j,t}(n))_{t \in \mathbb{T}(\text{msg})} = 0$ for some $j \neq n$, then $m_i(n) = 0$.

  – Otherwise, if the vector $(a_{j,t}(n), \omega_{j,t}(n))_{t \in \mathbb{T}(\text{msg}), j \in I}$ is not feasible—that is, for some $j' \in I$ and $t \in \mathbb{T}(\text{msg})$, $(a_{j',t}(n), \omega_{j',t}(n)) \neq \varphi((a_{j,t}(n), \omega_{j,t}(n))_{j \neq j'})$ (see Lemma 1 for the definition of φ)—then $m_i(n) = 0$.

  – If $(a_{j,t}(n), \omega_{j,t}(n))_{t \in \mathbb{T}(\text{msg}), j \in I}$ is feasible and $(a_{i,t}(n))_{t \in \mathbb{T}(\text{msg})}$ corresponds to the binary expansion of some $\hat{m}_i \in M_i$, then $m_i(n) = \hat{m}_i$.

  – If $(a_{j,t}(n), \omega_{j,t}(n))_{t \in \mathbb{T}(\text{msg}), j \in I}$ is feasible but $(a_{i,t}(n))_{t \in \mathbb{T}(\text{msg})}$ does not correspond to the binary expansion of some $\hat{m}_i \in M_i$, then $m_i(n)$ is set equal to an arbitrary, pre-determined element of M_i—for concreteness, let $m_i(n) = 1$.

[11] To make following the verified communication protocol sequentially rational, we will subsequently slightly modify player i's prescribed behavior after she herself deviates from the protocol. See Section C.3.

In the verified protocol, we call player i the *initial sender*, and we say player $j \in I$ is *a sender in period* t if $t \in \mathbb{T}(j)$ or $[j = i$ and $t \in \mathbb{T}(\text{msg})]$. We say *players coordinate on* m_i if $m_i(n) = m_i$ for all $n \in I$.

The core of the proof involves the interaction between the verified protocol and two other key concepts: *suspicious histories* and *erroneous opponents' histories*. Roughly speaking, a history h_j for player j is "suspicious" if it arises only after some player deviates, some jamming player plays JAM, or the realized matching process is erroneous. Similarly, a profile of player j's opponents' histories h_{-j} is "erroneous" if it arises whenever some jamming player plays JAM or the realized matching process is erroneous.

In Lemma 6 (in the appendix), we establish a key property of the verified protocol: For any player $j \neq i$, if all players follow the protocol then either (i) all players successfully receive message m_i or (ii) player j's opponents' histories h_{-j} are erroneous. If players $-j$ follow the protocol but player j deviates then either (i') all players successfully receive message m_i, (ii') player j's opponents' histories h_{-j} are erroneous, or (iii') some player $n \neq j$ becomes suspicious. Given this lemma, to provide incentives for truthful communication, we punish player j if some player $n \neq j$ becomes suspicious while h_{-j} is not erroneous, and we give player j a payoff that is greater than the punishment payoff and independent of her opponents' inferences of m_i if h_{-j} is erroneous. With this scheme, if player j attempts to manipulate her opponents' inferences, either (in case i') she fails, (in case ii') her opponents' histories are erroneous, or (in case iii') she makes someone suspicious and is punished. Moreover, if the realized matching process is such that (i) occurs on path, then player j's deviation results in either (i') or (iii'). Since player j cannot influence whether jamming players play JAM or whether the matching process is erroneous, she has an incentive to follow the protocol.

Similarly, for the initial sender i, if all players follow the protocol then either (i) all players successfully receive message m_i or (ii) h_{-i} is erroneous. If players $-i$ follow the protocol but player i deviates then either (i') all players receive some common message $\tilde{m}_i \in M_i$, (ii') player i's opponents' histories h_{-i} are erroneous, or (iii') some player $j \neq i$ becomes suspicious (intuitively, due to cross-checking in the checking rounds). A similar construction of continuation payoffs as for player $j \neq i$, together with the assumption that player i weakly prefers every player inferring m_i to every player inferring any $\tilde{m}_i \neq m_i$, establishes player i's incentive to follow the protocol.

As a player's opponents' histories are erroneous whenever a jamming player plays JAM or the realized matching process is erroneous, and a player's payoff is constant whenever her opponents' histories are erroneous, a player can condition on the event that all jamming players play REG and the realized matching process is non-erroneous when choosing her continuation strategy. This property is the key to handling erroneous histories.

In sum, we establish that the verified protocol satisfies the following properties:

1. Communication is *fast*, *accurate*, and *receiver-secure*, as in the secure communication protocol.

2. Communication is *sender-secure*: if a deviation by a player sending a message (either the initial sender or the sender in a checking round) affects another player's inference, either this is inconsequential (because the deviator's opponents' histories are already erroneous) or it induces a suspicious history.

3. Communication is *error-proof*: if the players miscoordinate, then each player's continuation payoff is independent of her own strategy and her opponents' inferences.

3.2.4 Jamming Coordination Protocol

In the jamming coordination protocol, the players jointly determine who among them will serve as jamming players in the subsequent communication protocols. The protocol takes 2 periods, and we describe it in the appendix (Section B.3). The idea is that each player mixes over all actions, playing a^1 with small probability, and players who observe a^1 become jamming players. This protocol allows each player to believe that her opponents are jamming players with positive probability at any history.

3.3 Block Belief-Free Structure

In this section, we first describe the equilibrium conditions for the block belief-free construction. We then construct the sequences of actions used to attain the target equilibrium payoff.

Finally, we specify how play in each block unfolds over time.

3.3.1 Block Belief-Free Equilibrium Conditions

We view the repeated game as an infinite sequence of T^{**}-period blocks.[12] At the beginning of each block, each player i chooses a state $x_i \in \{G, B\}$. Given x_i, player i plays a behavior strategy $\sigma_i^*(x_i)$ within the block: in every period $t = 1, ..., T^{**}$ of a block, $\sigma_i^*(x_i)$ specifies a mixed action as a function of player i's *extended block history* $(\mathbb{L}_i, h_i^{t-1})$, where $\mathbb{L}_i$ encodes the result of private randomization conducted by player i at the beginning of the block (the details are specified in Section E.3), and $h_i^{t-1} = (a_{i,\tau}, \omega_{i,\tau})_{\tau=1}^{t-1} \in H_i^{t-1}$. Denote player i's strategy set in the T^{**}-period game by Σ_i.

Player i's payoff at the beginning of each block is determined solely by player $(i-1)$'s state, $x_{i-1} \in \{G, B\}$, and is denoted $v_i^*(x_{i-1}) \in \mathbb{R}$. Moreover, the distribution over player $(i-1)$'s state for the following block depends only on player $(i-1)$'s state and extended history in the current block. Therefore, player i's continuation payoff at the end of a block is a function only of player $(i-1)$'s state and extended history. Denote this continuation payoff by $w_i^*(x_{i-1}, h_{i-1}^{T^{**}})$.

We now present conditions under which a given payoff vector $\mathbf{v} \in \mathbb{R}^N$ is attainable in a block belief-free equilibrium. These are similar to the conditions in Hörner and Olszewski (2006), with one significant difference: Hörner and Olszewski assume monitoring has full support, so in their model Nash and sequential equilibrium coincide, and there is no need to keep track of players' beliefs. In contrast, our model does not have full support, so we must introduce beliefs, verify Kreps-Wilson consistency, and—most subtly—ensure that beliefs respect the block belief-free equilibrium structure, in that sequential rationality is satisfied conditional on each possible state vector $x_{-i} \in \{G, B\}^{N-1}$. To do this, we keep track of players' beliefs conditional on each vector $x_{-i} \in \{G, B\}^{N-1}$. This approach implicitly determines a complete, unconditional belief system, but since sequential rationality is always imposed conditional on x_{-i}, these unconditional beliefs do not enter into our analysis.

Formally, an *ex post belief system* $\beta = (\beta_i)_{i \in I}$ consists of, for each player $i \in I$, opposing state vector $x_{-i} \in \{G, B\}^{N-1}$, period $t \in \{1, \ldots, T^{**}\}$, and block history $h_i^{t-1} \in H_i^{t-1}$, a

[12] We reserve the notation T^* for the length of a particular subset of a block, defined in Section 3.3.3.

probability distribution $\beta_i\left(\cdot|x_{-i}, h_i^{t-1}\right) \in \Delta\left(H_{-i}^{t-1}\right)$. Together with a block strategy profile $(\sigma_i(x_i))_{i \in I, x_i \in \{G,B\}}$, an ex post belief system is *consistent* if there exists a sequence of completely mixed block strategy profiles $\left(\left(\sigma_i^k(x_i)\right)_{i \in I, x_i \in \{G,B\}}\right)_{k \in \mathbb{N}}$ converging pointwise to $(\sigma_i(x_i))_{i \in I, x_i \in \{G,B\}}$ such that, for each $i \in I$, $x_{-i} \in \{G, B\}^{N-1}$, $t \in \{1, ..., T^{**}\}$, and $h^{t-1} \in H^{t-1}$, we have

$$\beta(h_{-i}^{t-1}|x_{-i}, h_i^{t-1}) = \lim_{k \to \infty} \Pr^{\left(\sigma_j^k(x_j)\right)_{j \neq i}}\left(h_{-i}^{t-1}|x_{-i}, h_i^{t-1}\right).$$ [13]

We are now ready to present the equilibrium conditions. In what follows, $\mathbb{E}^{\sigma}[\cdot]$ denotes expectation with respect to strategy profile σ, and $\mathbb{E}^{(\sigma,\beta)}[\cdot|\cdot]$ denotes conditional expectation with respect to assessment (strategy profile and beliefs) (σ, β).

For all $\mathbf{v} \in \mathbb{R}^N$ and $\delta < 1$, if there exist $T^{**} \in \mathbb{N}$, strategies $(\sigma_i^*(x_i))_{i \in I, x_i \in \{G,B\}}$, consistent ex post belief system β^*, values $(v_i^*(x_{i-1}))_{i \in I, x_{i-1} \in \{G,B\}}$, and continuation payoffs $\left(w_i^*(x_{i-1}, h_{i-1}^{T^{**}})\right)_{i \in I, x_{i-1} \in \{G,B\}, h_{i-1}^{T^{**}} \in H_{i-1}^{T^{**}}}$ such that the following conditions hold for all $i \in I$, then $\mathbf{v} \in E(\delta)$:

1. [Sequential Rationality] For all $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\sigma_i^*(x_i) \in \underset{\sigma_i \in \Sigma_i}{\operatorname{argmax}}\, \mathbb{E}^{\left(\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right), \beta^*\right)}\left[(1-\delta)\sum_{\tau=1}^{T^{**}} \delta^{\tau-1} \hat{u}_i(\mathbf{a}_\tau) + \delta^{T^{**}} w_i^*(x_{i-1}, h_{i-1}^{T^{**}})|x_{-i}, h_i^{t-1}\right]. \quad (7)$$

(Here, the sum $\sum_{\tau=1}^{T^{**}}$ could alternatively be written as $\sum_{\tau=t}^{T^{**}}$, since payoffs already incurred in h_i^{t-1} are sunk. In addition, sequential rationality is imposed for every vector $x_{-i} \in \{G, B\}^{N-1}$. This is the defining feature of a block belief-free construction.)

2. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$v_i^*(x_{i-1}) = \mathbb{E}^{\sigma^*(x)}\left[(1-\delta)\sum_{t=1}^{T^{**}} \delta^{t-1} \hat{u}_i(\mathbf{a}_t) + \delta^{T^{**}} w_i^*(x_{i-1}, h_{i-1}^{T^{**}})\right]. \quad (8)$$

[13] With this definition, it is clear that, whenever an ex post belief system is consistent, the corresponding unconditional belief system is consistent in the usual Kreps-Wilson sense.

3. [Self-Generation] For all $x_{i-1} \in \{G, B\}$ and $h_{i-1}^{T^{**}}$,

$$w_i^*(x_{i-1}, h_{i-1}^{T^{**}}) \in [v_i^*(B), v_i^*(G)]. \tag{9}$$

4. [Full Dimensionality]

$$v_i^*(B) < v_i < v_i^*(G). \tag{10}$$

(I.e., player $i-1$ can randomize her initial state to deliver player i's target payoff v_i.)

Defining $\pi_i^*(x_{i-1}, h_{i-1}^{T^{**}}) := \frac{\delta^{T^{**}}}{1-\delta}\left(w_i(x_{i-1}, h_{i-1}^{T^{**}}) - v_i^*(x_{i-1})\right)$, we rewrite (7)–(10) as follows:

1. [Sequential Rationality] For all $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\sigma_i^*(x_i) \in \underset{\sigma_i \in \Sigma_i}{\operatorname{argmax}} \mathbb{E}^{\left(\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right), \beta^*\right)} \left[\sum_{\tau=1}^{T^{**}} \delta^{\tau-1} \hat{u}_i(\mathbf{a}_\tau) + \pi_i^*(x_{i-1}, h_{i-1}^{T^{**}}) | h_i^{t-1}\right]. \tag{11}$$

2. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$v_i^*(x_{i-1}) = \mathbb{E}^{\sigma^*(x)} \left[\frac{1-\delta}{1-\delta^{T^{**}}} \sum_{t=1}^{T^{**}} \delta^{t-1} \hat{u}_i(\mathbf{a}_t) + \pi_i^*(x_{i-1}, h_{i-1}^{T^{**}})\right]. \tag{12}$$

3. [Self-Generation] For all $x_{i-1} \in \{G, B\}$ and $h_{i-1}^{T^{**}}$,

$$\frac{1-\delta}{\delta^{T^{**}}}\pi_i^*(G, h_{i-1}^{T^{**}}) \leq 0, \frac{1-\delta}{\delta^{T^{**}}}\pi_i^*(B, h_{i-1}^{T^{**}}) \geq 0, \left|\frac{1-\delta}{\delta^{T^{**}}}\pi_i^*(x_{i-1}, h_{i-1}^{T^{**}})\right| \leq v_i^*(G) - v_i^*(B). \tag{13}$$

4. [Full Dimensionality] The same as (10).

Lemma 4 (Hörner and Olszewski (2006)) *For all* $\mathbf{v} \in \mathbb{R}^N$ *and* $\delta \in [0, 1)$, *if there exist* $T^{**} \in \mathbb{N}$, $(\sigma_i^*(x_i))_{i \in I, x_i \in \{G,B\}}$, β^*, $(v_i^*(x_{i-1}))_{i \in I, x_{i-1} \in \{G,B\}}$, *and* $\left(\pi_i^*(x_{i-1}, h_{i-1}^{T^{**}})\right)_{i \in I, x_{i-1} \in \{G,B\}, h_{i-1}^{T^{**}} \in H_{i-1}^{T^{**}}}$ *such that Conditions (10)–(13) are satisfied, then* $\mathbf{v} \in E(\delta)$.

3.3.2 Target Payoff and Actions

For all $\mathbf{v} \in \text{int}(F^*)$, there exist payoff vectors $(\bar{v}_i(x_{i-1}))_{i \in I, x_{i-1} \in \{G,B\}} \in \mathbb{R}^{2N}$ such that $(\bar{v}_i(x_{i-1}))_{i \in I} \in \text{int}(F^*)\ \forall (x_{i-1})_{i \in I} \in \{G,B\}^N$ and $\underline{u} < \bar{v}_i(B) < v_i < \bar{v}_i(G)\ \forall i \in I$. Define

$$\varepsilon^* := \frac{1}{10} \min_i \min \{\bar{v}_i(G) - v_i, v_i - \bar{v}_i(B), \bar{v}_i(B) - \underline{u}\}.$$

We approximate the payoff vectors $(\bar{v}_i(x_{i-1}))_{i \in I, x_{i-1} \in \{G,B\}}$ by sequences of action profiles: for all $\varepsilon^* > 0$, there exist $K_{\mathbf{v}} \in \mathbb{N}$ and a sequence of action profiles $\left(\mathbf{a}^k(x)\right)_{k=1}^{K_{\mathbf{v}}} \in A^{NK_{\mathbf{v}}}\ \forall x \in \{G,B\}^N$ such that, for all $i \in I$ and $x_{i-1} \in \{G,B\}$, we have $\left|\frac{1}{K_{\mathbf{v}}} \sum_{k=1}^{K_{\mathbf{v}}} \hat{u}_i\left(\mathbf{a}^k(x)\right) - \bar{v}_i(x_{i-1})\right| < \varepsilon^*$. Let $\hat{u}_i(x) = \frac{1}{K_{\mathbf{v}}} \sum_{k=1}^{K_{\mathbf{v}}} \hat{u}_i\left(\mathbf{a}^k(x)\right)$.

Next, fix $(v_i(x_{i-1}))_{i \in I, x_{i-1} \in \{G,B\}} \in \mathbb{R}^{2N}$ and sequences of action profile $\left((\mathbf{a}^k(x))_{k=1}^{K_{\mathbf{v}}}\right)_{x \in \{G,B\}^N} \in A^{2^N K_{\mathbf{v}}}$ such that, for all $i \in I$,

$$\begin{aligned} v_i(G) &= \min_{x: x_{i-1}=G} \hat{u}_i(x), \\ v_i(B) &= \max_{x: x_{i-1}=B} \hat{u}_i(x) > \underline{u} + 9\varepsilon^*, \text{ and} \\ v_i(B) + 9\varepsilon^* &< v_i < v_i(G) - 9\varepsilon^*. \end{aligned} \tag{14}$$

Players will repeat the target action sequence $(\mathbf{a}^k(x))_{k=1}^{K_{\mathbf{v}}}$ over L "sub-blocks," where

$$L := \left\lceil \frac{2\bar{u}(1 + K_{\mathbf{v}})}{\varepsilon^*} \right\rceil. \tag{15}$$

For $l > K_{\mathbf{v}}$, let $a_i^l(x) = a_i^{l (\text{mod}\, K_{\mathbf{v}})}(x)$.[14]

3.3.3 Calendar Time Structure

We now specify the calendar time structure of a block. The length of a block is parameterized by $T_0 \in \mathbb{N}$. A block unfolds in the following consecutive phases. For most phases, we give a precise description of play followed by a more intuitive description in parentheses.

[14] Hörner and Olszewski (2006) and several subsequent papers present their constructions under the assumption that $K_{\mathbf{v}} = 1$. With random matching, this assumption is usually with loss. For example, in the prisoner's dilemma, to punish player 1 while keeping her opponents' payoffs close to $u(C,C)$, we must cycle through action profiles where player 1 and most of her opponents cooperate, while different subsets of her opponents take turns defecting. We thus present our construction for arbitrary $K_{\mathbf{v}}$.

1. *Jamming Coordination Phase:* The jamming coordination protocol is played. This takes 2 periods. ("The players coordinate on who will serve as jamming players.")

2. *Initial Communication Phase:* Each player $i \in I$ sends $x_i \in \{G, B\}$ using the verified communication protocol with repetition T_0. As the verified protocol with repetition T and message set M_i takes $\mathcal{T}(|M_i|, T)$ periods, this phase takes a total of $N\mathcal{T}(2, T_0)$ periods. ("The players coordinate on x.")

3. *Initial Contagion Phase ("Contagion Phase 0"):* For each $i \in I$, using the verified communication protocol with repetition T_0, player i communicates whether or not she has detected a deviation from equilibrium play. This takes $N\mathcal{T}(2, T_0)$ periods. ("Any suspicion spreads.")

4. *Sub-Block $l \in \{1, ..., L\}$:* Each sub-block l consists of

   (a) *Main Phase l:* The main phase takes $(T_0)^6$ periods. Let $\mathbb{T}(\text{main}(l))$ denote the set of periods in main phase l. Play is described in Section F. ("If player i has not detected a deviation, she plays $a_i^l(x(i))$ in every period, where $x(i)$ is her inference of x in the initial communication phase. If player i has detected a deviation, she plays $\alpha^{\min}$ in every period.")

   (b) *Communication Phase l, Part 1:* For each $i \in I$, player $i-1$ selects $t_{i-1}(l) \in \mathbb{T}(\text{main}(l))$ uniformly at random and sends the number $t_{i-1}(l)$ using the verified communication protocol with repetition T_0. This takes $N\mathcal{T}((T_0)^6, T_0)$ periods. ("Players select random periods to monitor.")

   (c) *Communication Phase l, Part 2:* For each $i, n \in I$, player n sends $\left(a_{n,t_{i-1}(l)(n)}, \omega_{n,t_{i-1}(l)(n)}\right)$ using the verified communication protocol with repetition T_0, where $t_{i-1}(l)(n)$ is player n's inference of $t_{i-1}(l)$ in part 1 of Communication Phase l. This takes $N\mathcal{T}(|A|^2, T_0)$ periods. ("Players share information about the selected periods.")

   (d) *Contagion Phase l:* The same as Contagion Phase 0.

It will be useful to denote the last period of contagion phase L by T^*. That is,

$$\begin{aligned} T^*(T_0) &= \underbrace{2 + 2N\mathcal{T}(2, T_0)}_{\text{jamming coordination, initial communication, and contagion}} + \underbrace{LT_0^6}_{\text{main phases}} \\ &+ \underbrace{LN\left[\mathcal{T}((T_0)^6, T_0) + N\mathcal{T}(|A|^2, T_0) + \mathcal{T}(2, T_0)\right]}_{\text{communication phases part (1)'s, communication phases part (2)'s, and contagion}}. \end{aligned}$$

Note that the main phases comprise almost the entirety of the first $T^*(T_0)$ periods when T_0 is sufficiently large.

5. *Final Communication Phase to Share Information from Main Phases:* This additional communication phase is described in Sections E.3 and J.7. It uses a combination of the secure and verified communication protocols. In this phase, players $-i$ share their main-phase histories to construct the reward function for player i. As in the earlier communication phases, players communicate only about randomly chosen periods.

   Let $T_1(T_0)$ denote the last period of this phase, so the phase takes $T_1(T_0) - T^*(T_0)$ periods. As we will see, for all $\varepsilon > 0$, for sufficiently large T_0 we have

$$T_0^3 < T_1(T_0) - T^*(T_0) < T_0^{3+\varepsilon}. \tag{16}$$

6. *Final Communication Phase to Share Information from Non-Main Phases:* This additional communication phase is described in Sections E.2 and J.6. It uses the secure communication protocol. In this phase, players $-i$ share their non-main-phase histories. Since non-main phases are much shorter than main phases, players can take the time to communicate about all of them. This renders the verified protocol unnecessary.

   Let $T_2(T_0)$ denote the last period of this phase, so the phase takes $T_2(T_0) - T_1(T_0)$ periods. For all $\varepsilon > 0$, for sufficiently large T_0 we will have

$$T_0^{\frac{11}{2}} < T_2(T_0) - T_1(T_0) < T_0^{\frac{11}{2}+\varepsilon}. \tag{17}$$

7. *Final Communication Phase to Cancel Discounting:* This additional communication

phase is described in Sections E.1 and J.5. It uses the basic communication protocol. In this phase, players $-i$ share their observations regarding another randomly chosen period. If they learn that an action profile for which player i has a high payoff was played earlier rather than later, player i's continuation payoff is slightly reduced to cancel out the effect of discounting.

Let $T^{**}(T_0)$ denote the last period of this phase, so the phase takes $T^{**}(T_0) - T_2(T_0)$ periods. For all $\varepsilon > 0$, for sufficiently large T_0 we will have

$$T_0^3 < T^{**}(T_0) - T_2(T_0) < T_0^{3+\varepsilon}. \tag{18}$$

In total, the length of a block as a function of T_0 is $T^{**}(T_0)$. For all $\varepsilon > 0$, for sufficiently large T_0 we have

$$LT_0^6 < T^{**}(T_0) < (1+\varepsilon) LT_0^6.$$

Note that, as $T_0 \to \infty$, block payoffs are almost entirely determined by main phase payoffs.

3.4 Reward Functions

Finally, we briefly preview some key features of the reward function $\pi_i\left(x_{i-1}, h_{i-1}^{T^{**}}\right)$ to be constructed (or equivalently the continuation payoff function $w_i\left(x_{i-1}, h_{i-1}^{T^{**}}\right)$).

On-Path Continuation Payoffs These are defined to satisfy self-generation and promise-keeping: given target payoff $v_i(x_{i-1})$ and main-phase payoff $\hat{u}_i(x)$, define continuation payoffs $w_i\left(x_{i-1}, h_{i-1}^{T^{**}}\right) \in [v_i(B), v_i(G)]$ such that

$$\left(1-\delta^{T^{**}}\right)\hat{u}(x) + \delta^{T^{**}}\mathbb{E}\left[w_i\left(x_{i-1}, h_{i-1}^{T^{**}}\right)|x\right] = v_i(x_{i-1}). \tag{19}$$

Let $w_i(x) = \mathbb{E}\left[w_i\left(x_{i-1}, h_{i-1}^{T^{**}}\right)|x\right]$.

Final Communication Phases In the final communication phases, each player $i-1$ collects information from players $-i$ to construct player i's continuation payoff. As players $-i$ are indifferent to the result of such communication, we need only consider promise-keeping, self-generation, and incentive-compatibility for player i. For promise-keeping, we slightly adjust player i's continuation payoff to account for the possibility of erroneous match realizations or jamming in the final communication phases: this is achieved using two simple *reward adjustment lemmas* derived in Section D. Since the required adjustment is small when T_0 is large, this does not violate self-generation. Finally, since the probability that player i can successfully manipulate communication is very small given Lemma 3, a further small adjustment is sufficient to ensure incentive-compatibility.

Given these properties of the final communication phases, in the rest of the block the players can condition on the event that all messages transmit correctly in the final phases. In particular, players anticipate that all non-main-phase histories will be communicated, one random period from each main phase will be communicated (for each player), and any player who deviates in the block will "confess" her deviation.

Continuation Payoffs Following an Opponent's Deviation We require that, if in the final communication phases some player $j \neq i$ confesses to deviating in the block, player i is made indifferent over all play paths. That is, player i's continuation payoff is

$$w_i(x_{i-1}) - \frac{1-\delta}{\delta^{T^{**}}} \sum\nolimits_{t \in \text{non-main phase}} \delta^{t-1} \hat{u}_i(\mathbf{a}_t) - \frac{1-\delta}{\delta^{T^{**}}} \sum\nolimits_{t \in \text{main phase}} \frac{\mathbf{1}_{\{t \text{ is chosen}\}}}{\Pr(t \text{ is chosen})} \delta^{t-1} \hat{u}_i(\mathbf{a}_t), \tag{20}$$

where "t is chosen" means that, in the final communication phase to share information from main phases, players $-i$ aggregate information about $(a_{-i,t}, \omega_{-i,t})$ and identify $(a_{i,t}, \omega_{i,t}) = \varphi(a_{-i,t}, \omega_{-i,t})$. Here, we must ensure that $w_i(x_{i-1})$ is far enough from the boundary of $[v_i(B), v_i(G)]$ to satisfy self-generation. In expectation, player i's continuation payoff equals

$$w_i(x_{i-1}) - \frac{1-\delta}{\delta^{T^{**}}} \sum\nolimits_{t \in \text{non-main or main phase}} \delta^{t-1} \hat{u}_i(\mathbf{a}_t),$$

which leaves her indifferent over all play paths $(\mathbf{a}_t)_{t=1}^{T^{**}}$. Consequently, prior to the final communication phases, players condition on the event that their opponents do not deviate.

Initial Communication Phase and Phase $l \in \{1, \ldots, L\}$ The most important feature of the reward functions in these phases is that a player is made indifferent over all play paths if either an erroneous match realization or jamming occurs (as identified by the final communication about non-main phases). Therefore, players condition on the event that matching is non-erroneous and jamming does not occur.

Continuation Payoffs Following One's Own Deviation If the outcome of communication phase l is that player i is determined to have deviated from her prescribed action $a_i^l(x)$ in main phase l, then player i is minmaxed beginning in main phase $l+1$ and her continuation payoff is set to $v_i(B)$.

Let us verify that this punishment is sufficient to deter deviations. Suppose player i deviates in τ distinct periods in main phase l. This deviation yields a benefit of at most $(1-\delta)\,2\bar{u}\tau$. Meanwhile, it is detected if and only if one of these periods is chosen for monitoring, which occurs with probability $\tau/(T_0)^6$. The expected penalty associated with the deviation is therefore approximately

$$\frac{\tau}{(T_0)^6}\left[\left(1-\delta^{T^{**}-t_{l+1}}\right)(\hat{u}_i(x)-\underline{u})+\delta^{T^{**}-t_{l+1}}(w_i(x)-v_i(B))\right],$$

where t_{l+1} is the first period of main phase $l+1$ and we have ignored the negligible payoffs accrued during non-main phases. By (14) and (19), for sufficiently large δ we have $\hat{u}_i(x)-\underline{u}>9\varepsilon^*$ and $w_i(x)-v_i(B)>\left(1-\delta^{T^{**}}\right)/\delta^{T^{**}}\times 9\varepsilon^*$. The expected penalty is thus at least

$$\frac{\tau}{(T_0)^6}\left[1-\delta^{T^{**}-t_{l+1}}+\delta^{T^{**}-t_{l+1}}\left(1-\delta^{T^{**}}\right)/\delta^{T^{**}}\right]9\varepsilon^*.$$

Therefore, the ratio of deviation gain to expected penalty is at most

$$\frac{(1-\delta)(T_0)^6}{1-\delta^{T^{**}-t_{l+1}}+\delta^{T^{**}-t_{l+1}}\left(1-\delta^{T^{**}}\right)/\delta^{T^{**}}}\frac{2\bar{u}}{9\varepsilon^*}\underset{\delta\to 1}{\longrightarrow}\frac{(T_0)^6}{2T^{**}-t_{l+1}}\frac{2\bar{u}}{9\varepsilon^*}\leq\frac{1}{L}\frac{2\bar{u}}{9\varepsilon^*}.$$

Since $L>2\bar{u}/9\varepsilon^*$, deviations are deterred when δ close to 1.

Jamming Coordination Phase We have not described how to provide incentives to follow the jamming coordination protocol. Since the complete history of play in this phase is communicated during the final communication phases, this is fairly straightforward. The details are deferred to Section C.4.

3.5 Relation to the Private Monitoring Literature

Some readers may wish to understand in more detail how our construction relates to existing work on the folk theorem with private monitoring. Our goal is to construct a block belief-free equilibrium as in Hörner and Olszewski (2006). To allow accurate communication in the presence of random matching, we have players repeat actions and messages and apply a concentration inequality (Lemma 2). In this sense, our construction joins the line of research combining belief-free equilibria and review strategies, following Matsushima (2004). The closest papers in this literature are Yamamoto (2012) and Sugaya (2017).

Yamamoto shows how to combine belief-free equilibria and review strategies in general repeated games. There are two key differences with our approach. First, Yamamoto's construction relies on conditional independence: player i's signal and player j's signal are independent conditional on actions. Thus, player i cannot learn player j's inference from her own signals. In contrast, with random matching signals are not conditionally independent. For example, if player j's signals imply that she matched with the sender in every period in a communication phase, she can infer that her opponents did not match with the sender. We control this novel learning effect via the innovation of introducing jamming players.

Second, Yamamoto assumes pairwise identifiability (i.e., each player can unilaterally identify other players' deviations) and constructs a belief-free equilibrium (i.e., each player is indifferent among all actions that are ever played with positive probability, regardless of her opponents' histories). The former property ensures that communication is not necessary for monitoring, and the latter ensures that communication is also not necessary for providing incentives to punish. Indeed, in Yamamoto's construction, communication following main phases is used only to ensure that different punishers do not miscoordinate—somewhat like the contagion phase in our construction.

In contrast, our monitoring structure does not satisfy pairwise identifiability. Hence, to monitor deviations, players must aggregate information by communication. This necessitates the construction of secure communication protocols, so the deviator cannot manipulate communication. In addition, since our equilibrium is not belief-free, players may have strict incentives to infer each other's messages correctly. To control the resulting incentives to experiment, we must construct communication protocols that are accurate (if a player successfully infers a message, her inference is always correct) and error-proof (if she fails to infer a message, her continuation payoff is independent of her opponents' inferences).

Sugaya proves a general folk theorem by generalizing Yamamoto's construction to conditionally dependent monitoring. As in the current paper, mixed strategies are used to control incentives after erroneous histories that arise with small ex ante equilibrium probability. In particular, after observing such a history, a player believes this observation results from a rare realization of her opponents' mixed strategies. By specifying her continuation payoff to be constant after such erroneous realizations, the player is incentivized to adhere to the same continuation play as after non-erroneous histories.

However, the N-player version of Sugaya's construction still assumes pairwise identifiability. This makes communication straightforward, as when player i "sends a message" to player j, player j can construct a statistic whose distribution depends on player i's message but is independent of unilateral deviations by players $-i$. In the current paper, pairwise identifiability is robustly violated, so we must introduce novel communication protocols that let players share information securely.

4 Extensions

An advantage of our approach is that it is amenable to further extensions. We present three: imperfect monitoring within matches, non-pairwise matching, and non-i.i.d. matching. Our goal is not so much to establish the most general results possible but to illustrate the broader applicability of our proof technique—to this end, in this section we allow some slight simplifying assumptions, such as access to public randomization and modest restrictions on the stage game. Proofs are deferred to the Supplementary Appendix.

4.1 Almost-Perfect Within-Match Monitoring

We can allow almost-perfect monitoring within a match. This is not surprising since we build on Hörner and Olszewski (2006), who prove the folk theorem with almost-perfect monitoring.

The required modifications to our proof are relatively minor. First, we have jamming players mix over all actions, rather than just a^0 and a^1. This makes players attribute unexpected observations to randomization by jamming players rather than monitoring errors. Second, players' reward functions must be adjusted to account for the possibility of monitoring errors—this complicates matters slightly relative to the perfect within-match monitoring case, where Lemma 1 ensured that a player's opponents can perfectly identify her history once they aggregate their information. Third, it is useful to introduce a small probability that the block is extended to include a final "long communication phase" on which the required reward adjustments can be based.

Formally, a *within-match monitoring structure* (q, Ω) consists of a finite signal space Ω and a mapping $q : A \times A \to \Omega \times \Omega$, where $q(\omega_i, \omega_{\mu(i)}|a_i, a_{\mu(i)})$ is the probability that player i observes signal ω_i and her partner observes signal $\omega_{\mu(i)}$ when i plays a_i and her partner plays $a_{\mu(i)}$. Assume without loss of generality that q has full support. Let q_i denote the marginal distribution of q over i's signal. We say monitoring is *ϵ-perfect* if $\Omega = A$ and $q_i(a_{\mu(i)}|a_i, a_{\mu(i)}) \geq 1 - \epsilon \ \forall \left(a_i, a_{\mu(i)}\right) \in A^2$. Let $E(\delta, q)$ denote the sequential equilibrium payoff set with discount factor δ and monitoring structure q.

Theorem 2 *Suppose public randomization is available. For all $\mathbf{v} \in \text{int}(F^*)$, there exist $\bar{\delta} < 1$ and $\bar{\epsilon} > 0$ such that $\mathbf{v} \in E(\delta, q)$ for all $\delta > \bar{\delta}$ and all ϵ-perfect within-match monitoring structures q with $\epsilon \leq \bar{\epsilon}$.*

Note that Theorem 2 assumes public randomization, in contrast to both our main result and Hörner and Olszewski's folk theorem. In the proof, public randomization is used to decide when to extend the block by including a long communication phase.

4.2 Non-Pairwise Matching

The assumption that matching is pairwise is also restrictive. For example, this requires that all players "play the game" the same number of times, and thus rules out a distinction between frequent and infrequent participants. Our approach can however be extended to this setting, with some restrictions on the structure of the game and the target payoff set.

A matching μ is now an arbitrary partition of the population into *groups*, rather than pairs. (A group of size 1 means a player is "unmatched" in the current period.) We continue to assume that matches are drawn from a fixed i.i.d. distribution p. We also assume that there is an upper bound $M \leq N$ on the size of a group, and that any partition of the population into groups of size $\leq M$ occurs with probability at least $\bar{\varepsilon} > 0$.

Whenever $n^* \leq M$ players are matched together in a group, they play a finite game with action sets $(A_{i^*}[n^*])_{i^*=1}^{n^*}$ and payoff functions $(u_{i^*}^*[n^*])_{i^*=1}^{n^*}$, where $A_{i^*}[n^*] \geq 2 \ \forall i^*$. We allow two possible structures for the n^*-player games:

1. **Symmetric stage games**: For each $n^* \leq M$, the n^*-player game is symmetric: all players have the same action set $A[n^*]$ and payoff function $u[n^*] : A[n^*]^{n^*} \to \mathbb{R}$. At the end of each period, every player observes the number of her partners who take each action $a \in A$: letting $\mu_t(i)$ denote the set of player i's period-t partners, player i's period-t signal is $\omega_{i,t} = \left(n^*(i), (\omega_{i,t}(a))_{a \in A[n^*(i)]}\right)$, where $n^*(i) = 1 + |\mu_t(i)|$ and $\omega_{i,t}(a) = \left|\{\mu_t(i) : a_{\mu_t(i)} = a\}\right| \ \forall a \in A[n^*(i)]$.

   Each player i's strategy in the one-shot game is a mapping from $n^*(i)$ to an element of $A[n^*(i)]$. Let $\bar{A}$ denote the pure strategy set in the one-shot game (it is the same for every player). Let $\bar{A}^{\text{mix}}$ denote the mixed strategy set.

   Let $F = \text{co}\left(\left\{\left(u_i\left((\bar{a}_n)_{n \in I}\right)\right)_i\right\}_{\bar{a}_n \in \bar{A} \ \forall n}\right)$. Given a mixed strategy profile $(\bar{\alpha}_n)_{n \in I} \in \prod_{n \in I} \bar{A}^{\text{mix}}$, let $\underline{u}_i\left((\bar{\alpha}_n)_{n \in I}\right) := \max_{\bar{a}_i \in \bar{A}} \hat{u}_i\left(\bar{a}_i, (\bar{\alpha}_n)_{n \neq i}\right)$ denote the highest payoff player i can attain against $(\bar{\alpha}_n)_{n \neq i}$. Our target payoff set is

$$F^* = \left\{\mathbf{v} \in F : \exists \left(\bar{\alpha}_n^{\min}\right)_{n \in I} \in \prod_{n \in I} \bar{A}^{\text{mix}} \text{ such that } v_i \geq \underline{u}_i\left(\left(\bar{\alpha}_n^{\min}\right)_{n \in I}\right) \ \forall i \in I\right\}.$$

   In general, this set is smaller than the feasible and individually rational payoff set. However, it equals this set if the distribution over matches is symmetric across players. Moreover, for any match distribution, taking $\left(\bar{\alpha}_n^{\min}\right)_{n \in I}$ to be a symmetric Nash equilibrium yields a "Nash threat" folk theorem.

2. **Random player-roles**: For each $n^* \leq M$, the n^*-player game is arbitrary, but each player in I_{n^*} is randomly assigned one of the n^* player-roles. When player $i \in I_{n^*}$ is assigned player-role i^*, she has action set $A_{i^*}[n^*]$ and payoff function $u_{i^*}[n^*] : (A_{i^*}[n^*])_{i^*=1}^{n^*} \to \mathbb{R}$. Let $i^*(i)$ denote player i's assigned role. Player i's period-t

signal is $\omega_{i,t} = (n^*(i), i^*(i), (a_{i^*,t}(i))_{i^*=1}^{n^*(i)})$, where $a_{i^*,t}(i)$ is the period-t action of the player assigned to role i^* in i's match.

Each player i's strategy in the one-shot game is a mapping from $(n^*(i), i^*(i))$ to an element of $A_{i^*(i)}[n^*(i)]$. Given this definition, $\bar{A}$, $\bar{A}^{\text{mix}}$, F, $\underline{u}_i\left((\bar{\alpha}_n)_{n\in I}\right)$, and F^* are defined as in the symmetric stage game specification.

Theorem 3 *With non-pairwise matching and either symmetric stage games or random player-roles, for all* $\mathbf{v} \in \text{int}(F^*)$*, there exists* $\bar{\delta} < 1$ *such that* $\mathbf{v} \in E(\delta)$ *for all* $\delta > \bar{\delta}$*.*

Again, the required modifications to the proof are minor. A player must now report her group size and player-role (if applicable) in addition to her action and observation. Given this additional information, Lemma 1 generalizes to non-pairwise matching. In addition, the jamming coordination protocol must be modified to ensure that each player believes some of her opponents are jamming players with high enough probability, regardless of the sizes of the groups in which she herself matches.

4.3 Non-I.I.D. Matching

We can also extend our approach to situations where (pairwise) matching is determined by a Markov process that depends on both the current match and the current action profile. This encompasses models with endogenous match separation, such as finite population versions of Shapiro and Stiglitz (1984), Datta (1996), Kranton (1996), Carmichael and MacLeod (1997), Eeckhout (2006), Fujiwara-Greve and Okuno-Fujiwara (2009), and Peski and Szentes (2013).

Let the distribution over period-t matches $p(\cdot|\mathbf{a}_{t-1}, \mu_{t-1})$ depend on the previous action profile $\mathbf{a}_{t-1}$ and match μ_{t-1}. Assume $p(\cdot|\mathbf{a}_{t-1}, \mu_{t-1})$ has full support for each $\mathbf{a}_{t-1}, \mu_{t-1}$, and let $\bar{\varepsilon} > 0$ denote the minimum of $p(\mu_t|\mathbf{a}_{t-1}, \mu_{t-1})$ over all $\mathbf{a}_{t-1}$, μ_{t-1}, and μ_t.

We impose some identifiability conditions on $p(\cdot|\mathbf{a}_{t-1}, \mu_{t-1})$. Order the $N(N-1)/2$ pairs of distinct players $(i,j) \in I^2$, and denote the resulting sequence by C. Suppose in each period $t = 1, ..., N(N-1)/2$ players $i, j \in C_t$—the t^{th} element of C—play a^1 and other players play a^0. Call this strategy $\bar{\sigma}$. Let $y_t = 1$ denote the event that the pair of players in C_t match with each other in period t, and let $y_t = 0$ denote the complementary event. Let

$y_C = (y_t)_{t=1}^{N(N-1)/2}$. We assume y_C statistically identifies the period-1 match μ_1: letting P be the matrix with dimension

$$\underbrace{\prod_{k=0}^{N/2-1} (N-2k-1)}_{\#\text{ of possible matches}} \times \underbrace{2^{N(N-1)/2}}_{\#\text{ of possible values for } y_C}$$

whose (μ, y_C)-element corresponds to the probability of y_C when $\mu_1 = \mu$ and the players follow $\bar{\sigma}$, we assume P has full row rank.

We also assume that, for each $\mathbf{a} \in A^N$, the $\prod_{k=0}^{N/2-1}(N-2k-1) \times \prod_{k=0}^{N/2-1}(N-2k-1)$ matrix $Q(\mathbf{a})$ with (μ_{t-1}, μ_t)-element $p(\mu_t | \mathbf{a}, \mu_{t-1})$ has full rank. That is, μ_t statistically identifies μ_{t-1}.

The feasible payoff set is defined as follows: Let $F(\mu_1, \delta)$ be the set of payoffs $\mathbf{v} \in \mathbb{R}^N$ that are attained by some strategy profile in the repeated game with initial match μ_1 and discount factor δ, allowing public randomization. In the Supplementary Appendix, we show that $\lim_{\delta \to 1} F(\mu_1, \delta)$ exists and is independent of μ_1. The feasible payoff set is then defined as $F = \lim_{\delta \to 1} F(\mu_1, \delta)$ for arbitrary μ_1. We also show that $F = \lim_{\kappa \to \infty} \lim_{\delta \to 1} F^{\kappa}(\mu_1, \delta)$, where $F^{\kappa}(\mu_1, \delta)$ is the set of payoffs attainable by the infinite repetition of a strategy in the κ-period finitely repeated game with initial match μ_1, for any μ_1.[15]

The minmax payoff is the same as in the i.i.d. case: $\underline{u} = \min_{\alpha \in \Delta(A)} \max_{a \in A} u(a, \alpha)$. The set of feasible and individually rational payoffs is $F^* = \{\mathbf{v} \in F : v_i \geq \underline{u} \ \forall i \in I\}$.

Theorem 4 *With non-i.i.d. matching, for all* $\mathbf{v} \in \text{int}(F^*)$, *there exists* $\bar{\delta} < 1$ *such that* $\mathbf{v} \in E(\delta)$ *for all* $\delta > \bar{\delta}$.

The required modifications to the proof are now more substantial. The basic idea is to exploit the fact that, for large enough T, any two matches separated by T periods are almost independent. This lets us preserve the block belief-free structure of the main construction.

[15] Since players cannot observe the period-t match μ_t, we have a stochastic game with hidden state and private signals. Platzman (1980), Rosenberg, Solan, and Vieille (2002), and Yamamoto (2017) have shown the same result with public signals. In this case, the feasible payoff set is the solution to a single-agent partially observable Markov decision problem, and can be characterized by dynamic programming. This is no longer possible with private signals, and we use a novel argument based on the minmax theorem.

5 Discussion

Multiple player-roles and multiple communities: As seen in Section 4.2, our approach allows multiple player-roles. This accommodates settings with one-sided moral hazard within a match. We can also extend our result to allow the population to be divided into multiple communities, where each community is assigned a fixed role. For example, in a stage-game between a buyer and a seller, we can accommodate both the case where each player is always either a buyer or a seller, and the case where each player can play different roles.

Cheap talk and public randomization: The folk theorem would be easy to prove if we allowed public ("broadcast") cheap talk communication. This would make detecting deviations straightforward, and then cooperation could be sustained by punishing deviations through mutual minmaxing. Deb (2017) considers a setting with private (within-match) cheap talk and shows that it is possible to partially detect deviations, and then applies the perfect monitoring version of Hörner and Olszewski. On the other hand, allowing public randomization would not simplify our construction much.[16]

Incomplete information: A concern in random matching models is that equilibria may not be robust to incomplete information. For example, the contagion strategies of Kandori, Ellison, and Harrington perform poorly in the presence of a few "commitment types" who always defect. Our approach of viewing the random matching game as a single N-player game and controlling each player's continuation payoff separately should be more robust to these considerations. This idea is hard to formalize, however, as incomplete information can undermine the communication modules in our construction. Nonetheless, we conjecture that our approach could be combined with that in Fudenberg and Yamamoto (2010) to give a partial folk theorem for ex post equilibria in settings with incomplete information.

Low discount factors: We have emphasized a range of advantages of the block belief-free approach over the contagion approach in constructing equilibria in anonymous random matching games. A relative *dis*advantage is that our construction requires a very high

[16] In Phase 5 of our construction, each player i randomly chooses period numbers $(t_i(l))_{l=1}^{L}$ and communicates them to her opponents. With public randomization, we could eliminate this phase by letting nature select these random periods.

discount factor as a function of the population size, while contagion strategies are remarkably effective (in the prisoners' dilemma) even for fairly low δ.[17] Nonetheless, following Hörner and Takahashi (2016), it can be shown that the asymptotic rate of convergence of our constructed equilibrium set to F^* is at least $(1-\delta)^{-1/2}$ for generic stage games. More broadly, formalizing and investigating performance criteria for low δ in general anonymous random matching games is an interesting direction for future research.

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Appendix: Proof of Theorem 1

A Overview of the Proof and Notation

- First, in **Section B,** we establish statistical properties of the secure, verified, and jamming coordination protocols.
- Next, in **Section C**, we augment the protocols with reward functions to construct the *communication modules*. We then establish incentive properties of the four modules: basic, secure, verified, and jamming coordination.
- Third, in **Section D,** we establish two *reward adjustment lemmas*, used to correct for the possibility of unlikely communication errors (which are caused by both erroneous match realizations and jamming players).
- Fourth, in **Section E**, we prove several *reduction lemmas* that simplify the discounted, infinitely repeated game: (i) we reduce the game to an undiscounted, finitely repeated game with final-period reward functions, (ii) we show that, as a result of communication, the final-period reward functions can exhibit some dependence on other players' histories, and (iii) we show that it suffices to establish optimality of each player's strategy only at histories consistent with her opponents' equilibrium strategies. Proving these lemmas involves the basic, secure, and verified communication modules, as well as the reward adjustment lemmas.
- In **Section F**, we use the reduction lemmas and the verified and jamming coordination modules to construct the equilibrium strategies.
- In **Section G**, we construct the final reward function, which sums the rewards for the main phases, communication phases, and contagion phases.
- Finally, in **Sections H and I**, we verify the equilibrium conditions.
- **Section J** collects the proofs of several lemmas used earlier.

The proof uses a range of terminology to refer to sets of consecutive periods that carry meaning in the construction. The following glossary collects this terminology, ordered from the longest set of periods (a block) to the shortest (a single period).

Terminology	**Meaning**
Block	T^{**} periods, structured as in Section 3.3.3.
Sub-Block	Consecutive Main, Communication, and Contagion Phases. There are L sub-blocks in each block. See Section 3.3.3.
Phase	A major component of a block. See Section 3.3.3.
Sub-Phase	A complete play of a communication protocol within a phase. See Section F.
Round	A major component of the verified protocol. See Section 3.2.3.
Interval	$2T$ consecutive periods in the basic, secure, or verified protocol.
Half-Interval	T consecutive periods in the basic, secure, or verified protocol.
Period	A single play of the game.

Table 1: Glossary of Terminology Describing Timing

We also collect some notation that will be used repeatedly in the proof, indicating where the definitions may be found.

Notation	**Meaning**
v_i	The target payoff.
$v_i(G)$	The lowest payoff when players coordinate on x with $x_{i-1} = G$ (see (14)).
$v_i(B)$	The highest payoff when players coordinate on x with $x_{i-1} = B$ (see (14)).
$\underline{u}$	The minmax payoff (see Section 2).
$\bar{u}$	The greatest magnitude of any feasible payoff (see Section 2).
u^G	The smallest feasible payoff (see (63)).
u^B	The largest feasible payoff (see (63)).

Table 2: Glossary of Notation for Payoffs

In addition, recall that, by (14), $\underline{u} + 18\varepsilon^* < v_i(B) + 9\varepsilon^* < v_i < v_i(G) - 9\varepsilon^*$.

Notation	Meaning
$\pi_i^{\text{cancel}}(a_{-i}, \omega_{-i})$	Reward to make player indifferent over actions with payoff 0 (see (28)).
$\pi_i^a(a_{-i}, \omega_{-i})$	Reward to give payoff 0 if $a_i = a$ and -1 otherwise (see (29)).
$\pi_{i,t}(h_{-i}^{\mathbb{T}'})$	Reward to give payoff 0 if player follows verified protocol in checking rounds, and give payoff -1 otherwise (see (34)).
$\pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i})$	Reward to make player indifferent over actions with payoff $u^{x_{i-1}}$, while satisfying self-generation (see (64)).
$\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i})$	Reward to make player indifferent over actions with payoff $v_i(x_{i-1})$, while satisfying self-generation if all players play $a^k(x)$ (see (64)).
$\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i} \| \alpha^{\min})$	Reward to make player i indifferent over actions with payoff $v_i(x_{i-1})$ when opponents play $\alpha^{\min}$ (see (64)).

Table 3: Glossary of Notation for Reward Functions

Finally, we use standard asymptotic notation: "$f(T_0) = O(g(T_0))$" means "$\exists C > 0, \exists \bar{T} > 0 : \forall T_0 > \bar{T}, |f(T_0)| \leq Cg(T_0)$."

B Communication Protocols

In this section and Section C, we view each protocol as a distinct, finitely-repeated game. A "protocol history" is a history in such a game: if $\mathbb{T}$ is the set of periods comprising a protocol, a *protocol history* for player j is a vector $h_j = (a_{j,t}, \omega_{j,t})_{t \in \mathbb{T}} \in H_j$. Denote the set of protocol history profiles by $H = \prod_{j \in I} H_j$.

B.1 Secure Communication Protocol

Lemma 3 in the text provides the required properties. Here, we provide the proof.

Fix $j \neq i$ with $I_{\text{jam}} \setminus \{j\} \neq \emptyset$. We claim that (3) holds if $\omega_{j,t} = a^1$ for more than $(1 - \bar{\varepsilon}^3) T$ periods in some half-interval, while (4) and (5) hold if $\omega_{j,t} = a^1$ for at most $(1 - \bar{\varepsilon}^3) T$ periods in every half-interval.

First suppose there is an half-interval $\mathbb{S}$ in which $\omega_{j,t} = a^1$ for γ periods, with $\gamma > (1-\bar{\varepsilon}^3)T$. Fix a player $j' \in I_{\text{jam}} \backslash \{j\}$. Let j'JAMS denote the event that, in half-interval $\mathbb{S}$, player j' plays JAM and all other jamming players play REG. Let $(a_{j,t},\omega_{j,t})_{t=1}^{2T\lceil \log_2|M_i|\rceil}|_{\mathbb{S}}$ denote the restriction of $(a_{j,t},\omega_{j,t})_{t=1}^{2T\lceil \log_2|M_i|\rceil}$ to half-interval $\mathbb{S}$. Then

$$\begin{aligned}\frac{\Pr\left((a_{j,t},\omega_{j,t})_{t=1}^{2T\lceil \log_2|M_i|\rceil}|_{\mathbb{S}}|j'JAMS\right)}{\Pr\left((a_{j,t},\omega_{j,t})_{t=1}^{2T\lceil \log_2|M_i|\rceil}|_{\mathbb{S}}|ALLREG\right)} &= \left(\frac{p_{i,j}+p_{j',j}}{p_{i,j}}\right)^{\gamma}\left(\frac{1-p_{i,j}-p_{j',j}}{1-p_{i,j}}\right)^{T-\gamma} \\ &\geq \exp\left(\left(\left(1-\bar{\varepsilon}^3\right)\log\frac{p_{i,j}+p_{j',j}}{p_{i,j}}+\bar{\varepsilon}^3\log\frac{1-p_{i,j}-p_{j',j}}{1-p_{i,j}}\right)T\right).\end{aligned}$$

Since $\log\frac{p_{i,j}+p_{j',j}}{p_{i,j}} \geq \log(1+\bar{\varepsilon}) \geq \bar{\varepsilon}-\frac{1}{2}\bar{\varepsilon}^2 \geq \frac{1}{2}\bar{\varepsilon}$ and $\log\frac{1-p_{i,j}-p_{j',j}}{1-p_{i,j}} \geq -\frac{p_{j',j}}{1-p_{i,j}} \geq -\frac{1-\bar{\varepsilon}}{\bar{\varepsilon}}$ (and $\bar{\varepsilon} \leq \frac{1}{3}$), we have

$$\left(1-\bar{\varepsilon}^3\right)\log\frac{p_{i,j}+p_{j',j}}{p_{i,j}}+\bar{\varepsilon}^3\log\frac{1-p_{i,j}-p_{j',j}}{1-p_{i,j}} \geq \left(1-\bar{\varepsilon}^3\right)\frac{1}{2}\bar{\varepsilon}+\bar{\varepsilon}^3\left(-\frac{1-\bar{\varepsilon}}{\bar{\varepsilon}}\right) \geq \frac{1}{4}\bar{\varepsilon}.$$

Hence, by Bayes' rule,

$$\begin{aligned}&\Pr\left(ALLREG|\,(a_{j,t},\omega_{j,t})_{t=1}^{2T\lceil \log_2|M_i|\rceil}\right)\\ \leq\ &\left[1+\frac{\Pr(j'JAMS)\Pr\left((a_{j,t},\omega_{j,t})_{t=1}^{2T\lceil \log_2|M_i|\rceil}|_{\mathbb{S}}|j'JAMS\right)}{\Pr(ALLREG)\Pr\left((a_{j,t},\omega_{j,t})_{t=1}^{2T\lceil \log_2|M_i|\rceil}|_{\mathbb{S}}|ALLREG\right)}\right]^{-1}\\ \leq\ &\left[1+T^{-9}\frac{\Pr\left((a_{j,t},\omega_{j,t})_{t=1}^{2T\lceil \log_2|M_i|\rceil}|_{\mathbb{S}}|j'JAMS\right)}{\Pr\left((a_{j,t},\omega_{j,t})_{t=1}^{2T\lceil \log_2|M_i|\rceil}|_{\mathbb{S}}|ALLREG\right)}\right]^{-1}\\ \leq\ &\left[1+T^{-9}\exp\left(\frac{1}{4}\bar{\varepsilon}T\right)\right]^{-1} \leq T^9\exp\left(-\frac{1}{4}\bar{\varepsilon}T\right).\end{aligned}$$

This establishes (3).

Next suppose $\omega_{j,t} = a^1$ for at most $(1-\bar{\varepsilon}^3)T$ periods in every half-interval. Then, in each half-interval where player i plays a^1, player i matches with a player other than j in at least $\bar{\varepsilon}^3 T_0$ periods. Suppose player j plays a^0 throughout the protocol. For all $j' \notin \{i,j\}$, if player i matches with player j' at least once in each half-interval where player i plays a^1, and ALLREG occurs, then $m_i(j') = m_i$. Hence, by Lemma 2,

$$\Pr\left(m_i(j') = m_i \middle| \left(a^0, \omega_{j,t}\right)_{t=1}^{2T\lceil \log_2 |M_i| \rceil}, ALLREG\right) \geq 1 - \lceil \log_2 |M_i| \rceil \exp\left(-\bar{\varepsilon}\bar{\varepsilon}^3 T\right).$$

Applying this bound repeatedly for each $j' \neq i, j$, we obtain

$$\Pr\left(m_i(j') = m_i \ \forall j' \notin \{i, j\} \mid \left(a^0, \omega_{j,t}\right)_{t=1}^{2T\lceil \log_2 |M_i| \rceil}, ALLREG\right) \geq 1 - N \lceil \log_2 |M_i| \rceil \exp\left(-\bar{\varepsilon}^4 T\right).$$

This establishes (5). Similarly—regardless of player j's behavior—if player i matches with player $j' \neq i, j$ in some period in each half-interval where player i plays a^1, then $m_i(j') \in \{m_i, 0\}$. (In particular, $m_i(j') = 0$ if j ever matches with j' while playing $a_j \notin \{a^0, a^1\}$, or if i and j match with j' while playing a^1 in different halves of the same interval, and $m_i(j') = 1$ otherwise.) Hence, (4) also holds.

B.2 Verified Communication Protocol

B.2.1 Notation

Let σ_i^{*,m_i} denote player i's prescribed protocol strategy. For $j \neq i$, let σ_j^* denote player j's prescribed protocol strategy. Let $\sigma^{*,m_i} = \left(\sigma_i^{*,m_i}, \sigma_{-i}^*\right)$. For $j \in I$, let Σ_j denote the set of possible protocol strategies for j.

For each $j, j' \in I$, player j's equilibrium strategy in the j'-checking round is determined by $(a_{j,t}, \omega_{j,t})_{t \in \mathbb{T}(\text{msg})}$ and $\zeta_j \in \{\text{reg}, \text{jam}\}$ (independently of m_i). We say player j *follows* σ_j^* *in the* j'*-checking round* if, for each $\tau \in \mathbb{T}(j')$, her action $a_{i,\tau}$ is in the support of σ_j^* given $(a_{j,t}, \omega_{j,t})_{t \in \mathbb{T}(\text{msg})}$, $\zeta_j \in \{\text{reg}, \text{jam}\}$, and $(a_{j,t}, \omega_{j,t})_{t \in \mathbb{T}(j'), t \leq \tau - 1}$.

Let $H^{<j'}$ denote the set of protocol history profiles at the beginning of $\mathbb{T}(j')$ that arise with positive probability under some strategy profile σ. Given $h^{<j'} \in H^{<j'}$, let $H_j^{\mathbb{T}(j')}|_{h^{<j'}}$ denote the set of protocol history profiles during $\mathbb{T}(j')$ that are reached from $h^{<j'}$ with positive probability under some strategy profile $(\sigma_j, \sigma_{-j}^*)$ with $\sigma_j \in \Sigma_j$ (i.e., when players $-j$ follow the protocol).

B.2.2 Suspicious Histories

For each $j \in I$, say that player j is *suspicious* at protocol history h_j, denoted $\text{susp}(h_j) = 1$, if $m_i(j) = 0$. Otherwise, $\text{susp}(h_j) = 0$. Note that $\text{susp}(h_j) = 1$ only if some player deviates, some jamming player plays JAM, or the realized matching process is erroneous. (Recall that the matching process is erroneous if, for some half-interval, some pair of players do not match with each other even once.)

B.2.3 Regular and Erroneous Opponents' Histories

We classify each of player j's opponents' history profiles as *regular* or *erroneous*, $\theta_j(h_{-j}, \zeta) \in \{R, E\}$. (Note that this classification can also depend on the type profile $\zeta = (\zeta_n)_{n \in I}$.) When we construct a module based on this protocol, this variable will be used to construct continuation payoffs. In particular, $\theta_j(h_{-j}, \zeta) = E$ will imply that player j's continuation payoff does not depend on her opponents' inferences.

For $j, j' \in I$, we first define $\theta_j(h_{-j}, \zeta, j') = E$ ("j's opponents' histories in the j'-checking round are erroneous") if and only if one or more of the following four conditions holds:

1. $\zeta_j = \text{jam}$.

2. There exists $n \in \mathcal{I}_{\text{jam}} \setminus \{j, j'\}$ who plays JAM in some half-interval in $\mathbb{T}(j')$.

3. *[Condition FAIL]* $j \neq j'$ and there exist a half-interval $\mathbb{S}$ in $\mathbb{T}(j')$ and a player $n \neq j'$ such that player j' plays a^1 throughout $\mathbb{S}$ but $\omega_{n,t} = a^0$ for all $t \in \mathbb{S}$. (Note that whether this event occurs is determined by h_{-j}, as Lemma 1 implies that h_j is uniquely determined by h_{-j}.)

4. *[Condition FAILj']* $j = j'$, player j' follows $\sigma^*_{j'}$ in the j'-checking round, and there exist a half-interval $\mathbb{S}$ in $\mathbb{T}(j')$ and a player $n \neq j'$ such that player j' plays a^1 throughout $\mathbb{S}$ but $\omega_{n,t} = a^0$ for all $t \in \mathbb{S}$. (Again, this event is determined by h_{-j}, by Lemma 1.)

(Note that $\theta_j(h_{-j}, \zeta, j')$ depends on h_{-j} only through $h_{-j}^{\mathbb{T}(j')}$ and $h_{-j}^{\mathbb{T}(\text{msg})}$, the latter because whether player j' follows $\sigma^*_{j'}$ in the j'-checking round (in [Condition FAILj']) depends on $(a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$.)

We define $\theta_j(h_{-j}, \zeta) = E$ if and only if either $\theta_j(h_{-j}, \zeta, j') = E$ for some $j' \in I$ or some player $j' \neq j$ deviates from $\sigma^*_{j'}$ in any checking round. Otherwise, define $\theta_j(h_{-j}, \zeta) = R$. In addition, for each $j' \in I$, let $JAM_{j',-j}$ denote the event that there exists $n \in \mathcal{I}_{\text{jam}} \backslash \{j, j'\}$ who plays JAM in some half-interval in $\mathbb{T}(j')$. Let $REG_{j',-j}$ denote the complementary event.

Lemma 5 *For each player $j \in I$, each type profile $\zeta \in \{\text{reg}, \text{jam}\}^N$, and each history profile $h^{<j'} \in H^{<j'}$,*

1. *If all players follow σ^* in the j'-checking round, then $\Pr\left(\theta_j(h_{-j}, \zeta, j') = E | h^{<j'}, \zeta\right)$ is the same for every $h^{<j'} \in H^{<j'}$.*

2. $\sigma^*_{j'} \in \operatorname{argmax}_{\sigma_{j'} \in \Sigma^{\mathbb{T}}_{j'}} \Pr^{\left(\sigma_{j'}, \sigma^*_{-j'}\right)}\left(\theta_{j'}(h_{-j'}, \zeta, j') = E | \zeta, h^{<j'}\right)$.

3. *If all players follow σ^* in the j'-checking round and $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ for some $n \in I$, then $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = 0$ and $\theta_j(h_{-j}, \zeta, j') = E$.*

4. *If player j' follows $\sigma^*_{j'}$ in the j'-checking round, $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ for some $n \in I$, and $\theta_j(h_{-j}, \zeta, j') = R$, then $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = 0$.*

5. *If $j \neq j'$, players $-j$ follow σ^*_{-j} in the j'-checking round, and $(a_{j',t}(j), \omega_{j',t}(j))_{t \in \mathbb{T}(\text{msg})} \neq m_i(j')$, then $\theta_j(h_{-j}, \zeta, j') = E$.*

Proof.

1. For any message $(a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$, player j' plays a^1 the same number of times in each interval. Hence, the probability that FAIL (or FAILj') holds is independent of $(a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$.

2. If player j' deviates from $\sigma^*_{j'}$ then FAILj' does not hold. Moreover, Conditions 1 and 2 for $\theta_j(h_{-j}, \zeta, j') = E$ are independent of σ_j, and FAIL only applies when $j \neq j'$. Hence, the desired inequality holds.

3. If $j \in \mathcal{I}_{\text{jam}}$ or a player in $\mathcal{I}_{\text{jam}} \backslash \{j, j'\}$ plays JAM in some half-interval, then $\theta_j(h_{-j}, \zeta, j') = E$ by construction. If $j \notin \mathcal{I}_{\text{jam}}$ and all players $\mathcal{I}_{\text{jam}} \setminus \{j, j'\}$ play REG in every half-interval, then $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ only if player n does not observe a^1 in some half-interval where player j' plays a^1. Hence, $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = 0$ and FAIL or FAILj' holds.

4. If $\theta_j(h_{-j}, \zeta, j') = R$ then each player $n \neq j'$ observes a^1 in each half-interval where player j' plays a^1. Hence, $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ implies $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = 0$.

5. When players $-j$ follow σ^*_{-j}, $(a_{j',t}(j), \omega_{j',t}(j))_{t \in \mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ only if player j does not observe a^1 in some half-interval where player j' plays a^1. Hence, FAIL holds.

■

B.2.4 Statistical Properties

The next lemma establishes the key properties of the verified protocol.

Lemma 6 *Suppose that*

$$2N^2 \left\lceil \log_2 |A|^{4\lceil \log_2 |M_i| \rceil} \right\rceil \left(1 + T^9 \exp(-\bar{\varepsilon} T)\right) \leq T. \tag{21}$$

Then the following claims hold for every $m_i \in M_i$ and every type profile $\zeta \in \{\text{reg}, \text{jam}\}^N$:

1. *For any $j \neq i$ and any $\sigma_j \in \Sigma_j^{\mathbb{T}}$, given strategy profile $\left(\sigma_j, \sigma_{-j}^{*,m_i}\right)$, either (i) $m_i(n) = m_i$ for all $n \in I$, (ii) $\text{susp}(h_n) = 1$ for some $n \neq j$, or (iii) $\theta_j(h_{-j}, \zeta) = E$. Moreover, $\text{susp}(h_j) = 1$ implies $\theta_j(h_{-j}, \zeta) = E$.*

2. *For any $\sigma_i \in \Sigma_i^{\mathbb{T}}$, given $\left(\sigma_i, \sigma_{-i}^*\right)$, either (i) there exists $\hat{m}_i \in M_i$ with $m_i(n) = \hat{m}_i$ for all $n \in I$, (ii) $\text{susp}(h_n) = 1$ for some $n \neq i$, or (iii) $\theta_i(h_{-i}, \zeta) = E$. Moreover, $\text{susp}(h_i) = 1$ implies $\theta_i(h_{-i}, \zeta) = E$.*

3. *Given σ^{*,m_i}, for any $j \in I$, either (i) $m_i(n) = m_i$ and $\text{susp}(h_n) = 0$ for all $n \in I$, or (ii) $\theta_j(h_{-j}, \zeta) = E$.*

4. *Given σ^{*,m_i}, with probability at least $1 - T^{-8}$, all of the following events occur: (i) $m_i(n) = m_i$ for all $n \in I$, (ii) $\text{susp}(h_n) = 0$ for all $n \in I$, and (iii) $\theta_n(h_{-n}, \zeta) = R$ for all $n \notin \mathcal{I}_{\text{jam}}$.*

5. *For any $m_i, m_i' \in M_i$ and $j \in I$, $\Pr^{\sigma^{*,m_i}}(\theta_j(h_{-j}, \zeta) = R|\zeta) = \Pr^{\sigma^{*,m_i'}}(\theta_j(h_{-j}, \zeta) = R|\zeta)$.*

The proof is relegated to Section J (as are all other proofs that do not immediately follow the corresponding claims). Intuitively, $\theta_j(h_{-j}, \zeta) = E$ only if some player plays JAM or matching is erroneous, which is unlikely. Moreover, since the sender plays a^1 with the same frequency for all m_i, the probability of this event is independent of m_i.

The next lemma is analogous to Lemma 3 and is used later in the proof. Unlike Lemmas 5–6, this lemma involves conditions on players' beliefs about the type profile $(\zeta_n)_{n\in I} \in \{\text{reg}, \text{jam}\}^N$. To express these conditions, we assume each player n has a prior probability distribution over $(\zeta_n)_{n\in I}$ at the beginning of the protocol. Let $\Pr_n(\cdot|\cdot)$ denote conditional probability under player n's prior.

Lemma 7 *Fix any $j \in I$, $j' \neq j$, and $h^{<j'} \in H^{<j'}$. Suppose that, for all $h_j^{\mathbb{T}(j')} \in H_j^{\mathbb{T}(j')}|_{h^{<j'}}$,*

$$\Pr_j\left(\zeta_{j'} = \text{jam } \forall j' \neq j | m_i, h^{<j'}, h_j^{\mathbb{T}(j')}\right) \geq T^{-4(N-1)-1}.$$

Then, for all $h_j^{\mathbb{T}(j')} \in H_j^{\mathbb{T}(j')}|_{h^{<j'}}$, at least one of the following two conditions holds:

1. *We have*

$$\Pr_j\left(JAM_{j',-j} | m_i, h^{<j'}, h_j^{\mathbb{T}(j')}\right) \geq 1 - T^{4(N-1)+10} \exp\left(-\frac{1}{4}\bar{\varepsilon}T\right). \quad (22)$$

2. *The following two conditions hold:*

   (a) *For all $(a_{j,t})_{t\in\mathbb{T}(j')}$,*

$$\begin{aligned} &\Pr_j\left(\begin{array}{c} (a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} \in \left\{0, (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}\right\} \ \forall n \neq j \\ |m_i, h^{<j'}, h_j^{\mathbb{T}(j')}, REG_{j',-j} \end{array}\right) \\ \geq\ & 1 - N\left\lceil \log_2 |A|^{2\lceil \log_2 |M_i| \rceil} \right\rceil \exp\left(-\bar{\varepsilon}^4 T\right). \end{aligned} \quad (23)$$

   (b) *If $a_{j,t} = a^0$ for all $t \in \mathbb{T}(j')$, then*

$$\begin{aligned} &\Pr_j\left(\begin{array}{c} (a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})} \ \forall n \neq j \\ |m_i, h^{<j'}, h_j^{\mathbb{T}(j')}, REG_{j',-j} \end{array}\right) \\ \geq\ & 1 - N\left\lceil \log_2 |A|^{2\lceil \log_2 |M_i| \rceil} \right\rceil \exp\left(-\bar{\varepsilon}^4 T\right). \end{aligned} \quad (24)$$

Proof. The same as Lemma 3, except that $T^{4(N-1)+10}$ replaces T^9 in (3), as now $\mathcal{I}_{\text{jam}} \backslash \{j\}$ is non-empty with probability at least $T^{-4(N-1)-1}$ rather than 1. ■

B.3 Jamming Coordination Protocol

This protocol is used to coordinate on the identities of the jamming players $\mathcal{I}_{\text{jam}} \subseteq I$. The protocol is again parametrized by $T \in \mathbb{N}$. It takes 2 periods.

Jamming Coordination Protocol with Parameter T:

- In each of the two periods, each player i plays a^1 with probability T^{-2} and plays each $a \neq a^1$ with probability $(1 - T^{-2}) / (|A| - 1)$, independently across periods.

Given a protocol history h_i, we define $\zeta_i(h_i) = \text{jam}$ if $\omega_{i,t} = a^1$ for some $t \in \{1, 2\}$. That is, a player becomes a jamming player if she observes a^1 in either period.

Let $P_i(h_i) = \Pr\left(\zeta_j(h_j) = \text{jam } \forall j \neq i | h_i\right)$. For every protocol history h_i, the probability that all players in $I \backslash \{i, \mu_t(i)\}$ play a^1 in both periods t and $\mu_1(i) \neq \mu_2(i)$ is at least $\bar{\varepsilon} T^{-4(N-2)}$. Conditional on this event, the probability that $\zeta_j(h_j) = \text{jam } \forall j \neq i$ is 1. Hence,

$$P_i(h_i) \geq \bar{\varepsilon} T^{-4(N-2)}. \tag{25}$$

C Communication Modules

A communication module is a finitely repeated game derived by augmenting a communication protocol with a reward function that makes following the communication protocol a sequential equilibrium or belief-free equilibrium. As a module is just a protocol augmented with a payoff function, we use the term *module history* interchangeably with *protocol history*.

C.1 Basic Communication Module

Our first module augments the basic communication protocol. It thus defines a $2T \lceil \log_2 |M_i| \rceil$-period game, where i, M_i, and T are parameters. For each player $n \in I$, payoff functions in the module take the form

$$\sum_{t\in\mathbb{T}}\delta^{t-1}\hat{u}_n(\mathbf{a}_t)+\pi_n(h_{n-1})+w_n(h), \tag{26}$$

where $\hat{u}_n$ is the stage-game payoff function; π_n is a *reward function* that depends only on player $n-1$'s module history; and w_n is a *continuation payoff function* that depends on the entire module history. We wish to construct a reward function such that, when viewed as a strategy profile in this finitely repeated game, the basic communication protocol is a belief-free equilibrium.

Definition 1 *A strategy profile σ is a belief-free equilibrium (BFE) if, for each player i and history h_i, the continuation strategy $\sigma_i|_{h_i}$ is a best response against $\sigma_{-i}|_{h_{-i}}$ for every opposing history profile h_{-i}.*

We say that *the premise for basic communication with magnitude K is satisfied* if the following conditions hold:

1. Player i is indifferent about the result of communication: $w_i(h)=0$ for all h.

2. For each player $n\neq i$, the range of $w_n(h)$ is bounded by K:

$$\max_{h,\tilde{h}}\left|w_n(h)-w_n(\tilde{h})\right|<K.$$

Lemma 8 *For each $i\in I$, M_i, T, w, and $K\geq 2\bar{u}/\bar{\varepsilon}$ satisfying the premise for basic communication with magnitude K, there exists a family of functions $\left(\pi_n:H_{n-1}^{\mathbb{T}}\to\mathbb{R}\right)_{n\in I}$ such that the following hold:*

1. *With payoff functions (26), the basic communication protocol is a BFE for every $\delta\in[0,1)$.*

2. *For each $n\in I$ and $m_i\in M_i$, $\mathbb{E}\left[\sum_{t\in\mathbb{T}}\delta^{t-1}\hat{u}_n(\mathbf{a}_t)+\pi_n(h_{n-1})\right]=0$.*

3. *For each $n\in I$ and $t\in\mathbb{T}$,*

$$\max_{h_{n-1},\tilde{h}_{n-1}}\left|\pi_n(h_{n-1})-\pi_n\left(\tilde{h}_{n-1}\right)\right|<2\frac{\bar{u}+K}{\bar{\varepsilon}}|\mathbb{T}|. \tag{27}$$

Intuitively, for each receiver $n \neq i$, player $n-1$ rewards player n every time she observes a^0, which incentivizes player n to play a^0 throughout the module. Although whether player i (the sender) plays a^0 or a^1 also affects the probability that player $n-1$ observes a^0 in a given period (since i and $n-1$ may match), the expected number of rewards is independent of m_i because player i plays a^0 and a^1 with the same frequency for every m_i. In addition, whether player i plays a^0 in the first or second half-interval affects player n's instantaneous utility through discounting, and we must adjust the rewards to cancel this effect.

For player i, player $i-1$ makes her indifferent between playing a^0 and a^1 in every period. This is straightforward since player $i-1$'s observations statistically identify player i's actions.

Finally, note that Lemma 8 concerns the complete information game where the continuation payoff functions $(w_n)_{n\in I}$ are known. However, as the statement of the lemma holds for each realization of $(w_n)_{n\in I}$, the same argument applies for the incomplete information game where $(w_n)_{n\in I}$ is unknown but the premise for communication is satisfied for each $(w_n)_{n\in I}$. The same remark applies for Lemmas 9, 10, and 11.

C.2 Secure Communication Module

We now augment the secure communication protocol in the case where I_{jam} is a singleton. Fix the sender i and another player i^* with $i \neq i^*, i^*-1$. Let $I_{\text{jam}} = \{i^*-1\}$.

Recalling the specification of the protocol in Section 3.2, player i's strategy is determined by m_i, and others' strategies are independent of m_i. Let $(\sigma_i^{m_i}, \sigma_{-i})_{m_i \in M_i}$ denote the specified collection of strategy profiles (one for each $m_i \in M_i$).

We introduce some building blocks of the reward functions. (We will use these in later modules as well.) By Lemma 1, there exists a function $\pi_i^{\text{cancel}}(a_{-i}, \omega_{-i}) : A^{N-1} \times A^{N-1} \to \mathbb{R}$ such that, for each $\mathbf{a} \in A^N$, we have

$$\hat{u}_i(\mathbf{a}) + \pi_i^{\text{cancel}}(a_{-i}, \omega_{-i}) = 0. \tag{28}$$

Thus, the function $\pi_i^{\text{cancel}}(a_{-i,}, \omega_{-i})$ cancels player i's instantaneous utility.

Similarly, there exists a function $\pi_i^{a^0}(a_{-i}, \omega_{-i}) : A^{N-1} \times A^{N-1} \to \mathbb{R}$ such that, for each

$\mathbf{a} \in A^N$, we have

$$\pi_i^{a^0}(a_{-i}, \omega_{-i}) = \begin{cases} 0 & a_i = a^0 \\ -1 & a_i \neq a^0 \end{cases} . \tag{29}$$

Thus, the function $\pi_i^{a^0}(a_{-i,}, \omega_{-i})$ rewards player i for playing a^0.

For each $n \in I$, payoff functions in the secure communication module are given by

$$\begin{aligned} & \sum_{t \in \mathbb{T}} \hat{u}_n(\mathbf{a}_t) + \sum_{t \in \mathbb{T}} \left(\pi_n^{\text{cancel}}(a_{-n,t}, \omega_{-n,t}) + \mathbf{1}_{\{n=i^*\}} \pi_n^{a^0}(a_{-n,t}, \omega_{-n,t}) \right) + w_n(h) \\ = & \sum_{t \in \mathbb{T}} \mathbf{1}_{\{n=i^*\}} \pi_n^{a^0}(a_{-n,t}, \omega_{-n,t}) + w_n(h), \end{aligned} \tag{30}$$

for some function $w_n : H^{\mathbb{T}} \to \mathbb{R}$. (Note that we neglect discounting in this equation, in contrast to equation (26) for the basic communication module. The reason is that, as described in the text, the basic communication module will be used at the very end of each block to cancel the effects of discounting in the remainder of the block. So we must account for discounting directly only in the basic communication module.)

We will give conditions on $(w_n)_{n \in I}$ under which $(\sigma_i^{m_i}, \sigma_{-i})_{m_i \in M_i}$ is an "i^*-quasi-belief-free equilibrium" of the resulting finitely repeated game. Intuitively, this means that the strategy of each player $n \neq i^*$ is sequentially rational for every opposing history profile, and player i^*'s strategy is sequentially rational for some consistent belief system. In addition, sequential rationality for player i^* is imposed ex post with respect to m_i. This ensures that the module will remain incentive compatible when viewed as one part of the infinitely repeated game.

Definition 2 *A family of strategy profiles* $(\sigma_i^{m_i}, \sigma_{-i})_{m_i \in M_i}$ *is an* i^*-quasi-belief-free equilibrium *(*i^*-QBFE*) if (i) for each player* $n \neq i^*$ *and history* h_n*, the continuation strategy* $\sigma_n|_{h_n}$ *is a best response against* $\sigma_{-n}|_{h_{-n}}$ *for every opposing history profile* h_{-n}*, and (ii) for player* i^**, there exists a sequence of families of completely mixed strategy profiles* $\left((\sigma_i^{m_i,k}, \sigma_{-i}^k)_{m_i \in M_i} \right)_{k=1}^{\infty}$ *and a corresponding family of belief systems* $\beta(h_{-i^*}|m_i, h_{i^*})$ *(where* $\beta(h_{-i^*}|m_i, h_{i^*})$ *is the limit of conditional probabilities derived from* $\left(\left(\sigma_i^{m_i,k}, \sigma_{-i}^k \right) \right)_{k=1}^{\infty}$*) such that, for each* m_i *and* $h_{i^*}^{t-1}$*,*

$$\sigma_{i^*} \in \operatorname*{argmax}_{\tilde{\sigma}_{i^*} \in \Sigma_{i^*}} \mathbb{E}^{\left(\tilde{\sigma}_{i^*}, \sigma_{-i^*}^{m_i}\right)} \left[\sum_{t \in \mathbb{T}} \pi_{i^*}^{a^0}(a_{-i^*,t}, \omega_{-i^*,t}) + w_{i^*}(h) \,|m_i, h_{i^*}^{t-1} \right],$$

where the expectation is taken with respect to $\beta(h_{-i^*}^{t-1}|m_i, h_{i^*}^{t-1})$.

We say that *the premise for secure communication for player* i^* *with magnitude* T^K *is satisfied* if the following conditions hold:

1. All players but player i^* are indifferent about the result of communication: $w_n(h) = 0$ for all h and $n \neq i^*$.

2. If player $i^* - 1$ deviates from σ_{i^*-1} or $ALLREG$ does not occur,[18] then $w_{i^*}(h) = 0$ for all h.

3. If player $i^* - 1$ follows σ_{i^*-1} and $ALLREG$ occurs, then the following conditions hold:

   (a) If $m_i(i^* - 1) \in M_i \cup \{0\}$ is the same under protocol histories h and $\tilde{h}$, then $w_{i^*}(h) = w_{i^*}(\tilde{h})$. Under this condition, we abuse notation and write $w_{i^*}(h) = w_{i^*}(m_i(i^* - 1))$.

   (b) The range of $w_{i^*}(m_i(i^* - 1))$ is bounded by T^K:

$$\max_{m_i, \tilde{m}_i \in M_i \cup \{0\}} |w_{i^*}(m_i) - w_{i^*}(\tilde{m}_i)| < T^K. \tag{31}$$

   (c) $w_{i^*}(0) \leq w_{i^*}(m_i(i^* - 1))$ for all $m_i(i^* - 1) \in M_i$.

We now specify player i^*'s beliefs. In particular, we specify that, after any off-path observation, she assigns probability 1 to the event that player $i^* - 1$ deviated (and hence, if the above premise holds, $w_{i^*}(h) = 0$). This belief is clearly consistent: for concreteness, define $((\sigma_i^{m_i,k}, \sigma_{-i}^k)_{m_i \in M_i})_{k=1}^{\infty}$ by letting player $i^* - 1$ tremble uniformly over all actions with probability k^{-1} at each history, and letting every other player tremble uniformly over all actions with probability k^{-k} at each history.

Lemma 9 *For each* $i^* \in I$, $i \in I \backslash \{i^* - 1, i^*\}$, M_i, w, *and* K *satisfying the premise for secure communication for player* i^* *with magnitude* T^K, *if*

$$\lceil \log_2 |M_i| \rceil T^{9+K} \exp\left(-\bar{\varepsilon}^4 T\right) \leq 1, \tag{32}$$

[18] As in the verified protocol, player $i^* - 1$ *follows* σ_{i^*-1} if, for each τ, her action $a_{i^*-1,\tau}$ is in the support of σ_{i^*-1} given $(a_{i^*-1,t}, \omega_{i^*-1,t})_{t \leq \tau-1}$. Since $i^* - 1 \neq i$, the support is independent of m_i. Player $i^* - 1$ *deviates from* σ_{i^*-1} if she does not follow σ_{i^*-1}.

then with payoff functions (30) the secure communication protocol, together with the above belief system for player i, is an i^*-QBFE.

Proof. By construction, players other than i^* are indifferent over all actions throughout the module.

For player i^*, fix a period $t \in \mathbb{T}$ and history $(a_{i^*,\tau}, \omega_{i^*,\tau})_{\tau\in\mathbb{T},\tau\leq t-1}$. Suppose $\omega_{i^*,\tau} \in \{a^0, a^1\}$ for each $\tau \leq t-1$. By the same argument as for Lemma 3, for every possible continuation history $(a_{i^*,\tau}, \omega_{i^*,\tau})_{\tau\in\mathbb{T},\tau\geq t}$, with probability at least

$$1 - \lceil \log_2 |M_i| \rceil T^9 \exp\left(-\bar{\varepsilon}^4 T\right) \tag{33}$$

conditional on $(a_{i^*,\tau}, \omega_{i^*,\tau})_{\tau\in\mathbb{T}}$, either $ALLREG$ does not occur or $[m_i(i^*-1) \in \{m_i, 0\}$, and $m_i(i^*-1) = m_i$ if $a_{i^*,\tau} = a^0$ for all $\tau \in \mathbb{T}]$. Moreover, if $(\omega_{i^*,\tau})_{\tau\in\mathbb{T}}$ is such that $[m_i(i^*-1) \in \{m_i, 0\}$ and $m_i(i^*-1) = m_i$ if $a_{i^*,\tau} = a^0$ for all $\tau \in \mathbb{T}]$, then by definition of $m_i(i^*-1)$, we have $m_i(i^*-1) = m_i$ if and only if $a_{i^*,\tau} = a^0$ for each $\tau \in \mathbb{T}$ such that $\mu_\tau(i^*) = i^*-1$ and τ is in a half-interval where player i plays a^0. Hence, since $w_{i^*}(0) \leq w_{i^*}(m_i(i^*-1))$ for all $m_i(i^*-1) \in M_i$, taking $a_{i^*,\tau} = a^0$ for each $\tau \geq t$ maximizes $w_{i^*}(h)$ with probability at least (33). Given this, (31) and (32) imply that the reward term $\pi_{i^*}^{a^0}(a_{-i^*,\tau}, \omega_{-i^*,\tau})$ in (30) outweighs any possible benefit to player i^* from playing $a \neq a^0$ in an attempt to manipulate $m_i(i^*-1)$.

If instead $\omega_{i^*,\tau} \notin \{a^0, a^1\}$ for some $\tau \leq t-1$, then by construction of the belief system player i^* believes $w_{i^*}(h) = 0$ with probability 1. Hence, player i^* maximizes the reward term $\pi_{i^*}^{a^0}(a_{-i^*,\tau}, \omega_{-i^*,\tau})$ in (30), so playing a^0 as prescribed is optimal. ■

C.3 Verified Communication Module

We now augment the verified communication protocol. Throughout this subsection, fix $m_i^* \in M_i$ and let σ^* denote the prescribed protocol strategy profile given m_i^*. As in (29), for each $j \in I$ and $t \in \mathbb{T}(j)$, given $(a_{j,t}, \omega_{j,t})_{t\in\mathbb{T}(\text{msg})}$ identified from h_{-j} by Lemma 1, calculate the equilibrium action $\bar{a}_{j,t}$ and define

$$\pi_j^{\bar{a}_{j,t}}(h_{-j}) = \begin{cases} 0 & a_{j,t} = \bar{a}_{j,t} \\ -1 & a_{j,t} \neq \bar{a}_{j,t} \end{cases} . \tag{34}$$

Suppose each player j's payoff equals

$$\pi_j\left(h_{-j}, \zeta_j\right) + w_j(h, \zeta), \tag{35}$$

where the reward function $\pi_j\left(h_{-j}, \zeta_j\right)$ is given by

$$\pi_j\left(h_{-j}, \zeta_j\right) = \mathbf{1}_{\left\{\zeta_j = \text{reg}\right\}} \sum_{t \in \mathbb{T} \setminus \mathbb{T}(j)} \pi_j^{a^0}(a_{-j,t}, \omega_{-j,t}) + \sum_{t \in \mathbb{T}(j)} \pi_j^{\bar{a}_{j,t}}(h_{-j}). \tag{36}$$

As in (30), we ignore player j's instantaneous payoffs, as these can be cancelled by adding π_i^{cancel} to the reward function.

We say that *the premise for verified communication to send message $m_i^* \in M_i$ with magnitude T^K is satisfied* if there exist $\left(v_j^E\right)_{j \in I} \in \mathbb{R}^N$, and $\left(v_j^{m_i}\right)_{j \in I, m_i \in M_i \cup \{0\}} \in \mathbb{R}^N$ such that, for all $j \in I$ and $h \in H$, the following conditions hold:

1. If $\theta_j(h_{-j}, \zeta) = E$, then $w_j(h, \zeta) = T^K v_j^E$.

2. If $\theta_j(h_{-j}, \zeta) = R$ and $\text{susp}(h_n) = 1$ for some $n \neq j$, then $w_j(h, \zeta) = T^K v_j^0$.

3. If $\theta_j(h_{-j}, \zeta) = R$, $\text{susp}(h_n) = 0$ for all $n \neq j$, and $\exists \hat{m}_i \in M_i$ such that $m_i(n) = \hat{m}_i$ for all $n \in I$, then $w_j(h, \zeta) = T^K v_j^{\hat{m}_i}$.

4. $v_j^0 \leq \min\left\{\min_{m_i \in M_i} v_j^{m_i}, v_j^E\right\}$.

5. $v_i^{m_i^*} \geq v_i^{\hat{m}_i}$ for all $\hat{m}_i \in M_i \cup \{0\}$.

The interpretation of the above variables is that v_j^E is player j's normalized continuation payoff after erroneous opposing histories; v_j^0 is player j's normalized punishment payoff (which results if $\theta_j(h_{-j}, \zeta) = R$ and $\text{susp}(h_n) = 1$ for some $n \neq j$); and $v_j^{m_i}$ is player j's normalized continuation payoff after players coordinate on message m_i. Denote the range of

$w_j(h,\zeta)/T^K$ by

$$\bar{v} := \max_{j\in I}\left\{\max\left\{v_j^E, \left(v_j^{m_i}\right)_{m_i\in M_i}\right\} - v_j^0\right\}.$$

We modify player i's strategy in the message round after she herself deviates as follows: Recall that we define $m_i(n) = 1$ if player n infers some $(a_{i,t})_{t\in\mathbb{T}(\text{msg})}$ not corresponding to the binary expansion of any message. We can thus view the play of such $(a_{i,t})_{t\in\mathbb{T}(\text{msg})}$ as sending message $m_i = 1$. With this interpretation, for each h_i^{t-1}, let $M_i(h_i^{t-1}) \subset M_i$ be the (non-empty) set of messages $\tilde{m}_i$ such that $(a_{i,\tau})_{\tau=1}^{t-1}$ is consistent with the binary expansion of $\tilde{m}_i$; and let $M_i^*(h_i^{t-1}) = \arg\max_{m_i\in M_i(h_i^{t-1})} v_i^{m_i}$ be the elements that maximize $v_i^{m_i}$. Given h_i^{t-1}, if $m_i^* \in M_i^*(h_i^{t-1})$, player i plays $a_{i,t}$ corresponding to the binary expansion of m_i^*; otherwise, she plays $a_{i,t}$ corresponding to the binary expansion of some $m_i \in M_i^*(h_i^{t-1})$.

Call a history σ-*consistent* if it is reached with positive probability under strategy profile σ. Recall that $\Pr_j(\cdot|\cdot)$ denotes conditional probability under player j's prior on $(\zeta_n)_{n\in I}$. Recall that $H^{<j'}$ is the set of module history profiles at the beginning of $\mathbb{T}(j')$ that are σ-consistent for some $\sigma\in\Sigma$, and let $H_j^{\mathbb{T}(j')}|_{h^{<j'}}$ be the set of module histories during $\mathbb{T}(j')$ that are $(\sigma_j, \sigma_{-j}^*)$-consistent for some $\sigma_j \in \Sigma_j$ given $h^{<j'}$. We assume that, for every player $j, j' \in I$, module strategy σ_j, $h^{<j'} \in H^{<j'}$, and $h_j \in H_j^{\mathbb{T}(j')}|_{h^{<j'}}$,

$$\Pr_j\left(n\in\mathcal{I}_{\text{jam}}\ \forall n\neq j|h^{<j'}, h_j\right) \geq T^{-4(N-1)}. \tag{37}$$

Lemma 10 *Suppose that*

$$\begin{gathered}4\bar{v}N\left\lceil\log_2|A|^{2\lceil\log_2|M_i|\rceil}\right\rceil T^{4(N-1)+10+K}\exp(-\bar{\varepsilon}^4 T) \leq 1 \text{ and} \\ \bar{v}T^{K-8} \leq \tfrac{1}{2}.\end{gathered} \tag{38}$$

If the premise for verified communication with magnitude T^K and (37) are satisfied, then with payoff functions (35) the verified communication protocol is a sequential equilibrium. In addition, if there exists $i^* \in I\setminus\{i\}$ such that $\mathcal{I}_{\text{jam}} = I\setminus\{i^*\}$ and $v_j^E = v_j^{m_i}$ for all $j\neq i^*$ and $m_i \in M_i\cup 0$, while for player i^* the premise for verified communication and (37) are satisfied, then with payoff functions (35) the verified communication protocol is an i^*-QBFE.

Intuitively, so long as the prior probability that players jam is not too low, whenever player j observes an erroneous history she believes that JAM is played and $\theta_j(h_{-j}, \zeta) = E$. Otherwise, she believes that all other players match with the sender at least once in each half-interval. Hence, if she deviates and changes some player's inference, this induces $\text{susp}(h_n) = 1$ and yields the punishment payoff v_j^0.

C.4 Jamming Coordination Module

We now augment the jamming coordination protocol. For each $i \in I$, payoff functions take the form

$$\sum_{t=1}^{2} \pi_{i,t}^{\text{indiff}}(h_{-i}|T) + w_i(h|T), \tag{39}$$

where we have made explicit the dependence of the reward function and continuation payoff function on T. Again, as in (30), we ignore player i's instantaneous payoffs.

We say that *the premise for jamming coordination is satisfied* if there exist $K \geq 1$ and $\bar{T} \geq 1$ such that, for all $T > \bar{T}$, there exist $\bar{w}_i(T) \in \mathbb{R}$ and $(v_i(\mathcal{I}_{\text{jam}}\backslash\{i\}|T))_{\mathcal{I}_{\text{jam}}\backslash\{i\}\subset I\backslash\{i\}} \in \mathbb{R}^{2^{N-1}}$ satisfying the following conditions:

1. $w_i(h|T) = \bar{w}_i(T)$ for every protocol history h such that $\omega_{i,t} = a^1$ for some $t \in \{1, 2\}$.
2. $w_i(h|T) = v_i(\mathcal{I}_{\text{jam}}\backslash\{i\}|T)$ for every protocol history h such that $\omega_{i,t} \neq a^1$ for each t.
3. The range of $v_i(\mathcal{I}_{\text{jam}}\backslash\{i\}|T)$ is at most 1:
$$\max_{i\in I, \mathcal{I}_{\text{jam}}\backslash\{i\}, \widetilde{\mathcal{I}_{\text{jam}}\backslash\{i\}}\subset I\backslash\{i\}} \left| v_i(\mathcal{I}_{\text{jam}}\backslash\{i\}|T) - v_i(\widetilde{\mathcal{I}_{\text{jam}}\backslash\{i\}}|T) \right| \leq 1. \tag{40}$$
4. The difference between $v_i(\mathcal{I}_{\text{jam}}\backslash\{i\}|T)$ and $\bar{w}_i(T)$ is bounded by $T^6 K$:
$$\max_{i\in I, \mathcal{I}_{\text{jam}}\backslash\{i\}\subset I\backslash\{i\}} |v_i(\mathcal{I}_{\text{jam}}\backslash\{i\}|T) - \bar{w}_i(T)| \leq T^6 K. \tag{41}$$

Lemma 11 *There exists a family of functions $\left(\pi_{i,t}^{\text{indiff}}(h_{-i}|T)\right)_{t\in\{1,2\}, T\in\mathbb{N}}$ indexed by T such that*

1. *We have*
$$\lim_{T\to\infty} \max_{h_{-i}} \frac{\left|\sum_{t=1}^{2} \pi_{i,t}^{\text{indiff}}(h_{-i}|T)\right|}{T} = 0. \tag{42}$$

2. *If the premise for jamming coordination is satisfied, then there exists $\bar{T} > 0$ such that, with payoffs (39), the jamming coordination protocol is a sequential equilibrium for all $T > \bar{T}$.*

Intuitively, whether player i observes $\omega_{i,t} = a^1$ or $\omega_{i,t} \neq a^1$ for $t = 1, 2$ is independent of her own strategy. Hence, incentives come solely from the fact that playing a^1 changes $\mathcal{I}_{\text{jam}} \backslash \{i\}$. Since the effect of changing $\mathcal{I}_{\text{jam}} \backslash \{i\}$ on continuation payoffs is bounded independent of T, this effect can be cancelled by a reward function of magnitude less than T.

D Reward Adjustment Lemmas

We now introduce two lemmas that will let us adjust the reward functions to correct for unlikely errors in communication. Given a parameter $T \in \mathbb{N}$, let $M(T) \subset \mathbb{N}$ be a finite set, let $F : \mathbb{N} \to \mathbb{R}_+$ be a function of T satisfying $\liminf_{T\to\infty} F(T) > 0$, let $f_T : M(T) \to [-F(T), F(T)]$ be a function of $m_i \in M(T)$, and let $\tilde{m}_i \in M(T) \cup \{0\}$ be a random variable such that, for each $m_i \in M(T)$, $\Pr(\tilde{m}_i = m_i | m_i) = p_T(m_i)$ and $\Pr(\tilde{m}_i = 0 | m_i) = 1 - p_T(m_i)$. Applied to the remainder of the proof, T will index the length of an interval, $M(T)$ will be a message set, f_T will be a reward function bounded by $F(T)$, and $p_T(m_i)$ will be the probability that message m_i is received when message m_i is sent.

Lemma 12 *Suppose that* $\lim_{T\to\infty} \min_{m_i \in M(T)} p_T(m_i) = 1$. *For all $\varepsilon > 0$, there exists $\bar{T} > 0$ such that, for all $T > \bar{T}$, there exists a function $g_T : M(T) \cup \{0\} \to [-(1+\varepsilon) F(T), (1+\varepsilon) F(T)]$ such that* $\max_{m_i \in M(T)} |f_T(m_i) - g_T(m_i)| \leq \varepsilon F(T)$ *and* $\mathbb{E}[g_T(\tilde{m}_i)|m_i] = f_T(m_i)$ *for all $m_i \in M(T)$.*

Proof. Define $g_T(0) = 0$ and $g_T(m_i) = \frac{1}{p_T(m_i)} f_T(m_i)\ \forall m_i \in M(T)$. Then $\mathbb{E}[g_T(\tilde{m}_i)|m_i] = f_T(m_i)\ \forall T \in \mathbb{N}, m_i \in M(T)$, and $\liminf_{T\to\infty} F(T) > 0$ and $\lim_{T\to\infty} \min_{m_i \in M(T)} p_T(m_i) = 1$ imply that, for each $\varepsilon > 0$, for sufficiently large T, $g_T(m_i) \in [-(1+\varepsilon) F(T), (1+\varepsilon) F(T)]$ and $|f_T(m_i) - g_T(m_i)| \leq \varepsilon F(T)\ \forall m_i \in M(T)$. ■

A similar result holds if we account for self-generation. For $x_{i-1} \in \{G, B\}$, define

$$\text{sign}(x_{i-1}) := \begin{cases} -1 & \text{if } x_{i-1} = G \\ 1 & \text{if } x_{i-1} = B \end{cases}$$

For each $T \in \mathbb{N}$ and $x_{i-1} \in \{G, B\}$, let $f_T^{x_{i-1}} : M(T) \to [-F(T), F(T)]$ be a function of $m_i \in M(T)$ such that there exists $c \geq 0$ satisfying

$$\max_{m_i \in M(T), x_{i-1} \in \{G,B\}} \operatorname{sign}(x_{i-1}) f_T^{x_{i-1}}(m_i) \geq -cT \quad \forall T \in \mathbb{N}. \tag{43}$$

Lemma 13 *Suppose that*

$$\lim_{T\to\infty} \min_{m_i \in M(T)} p_T(m_i) = 1 \textit{ and } \lim_{T\to\infty} \max_{m_i \in M(T)} (1 - p_T(m_i)) \max\{F(T), cT\} = 0. \tag{44}$$

For all $\varepsilon > 0$, there exists $\bar{T} > 0$ such that, for all $T > \bar{T}$ and $x_{i-1} \in \{G, B\}$, there exists a function $g_T^{x_{i-1}} : M(T) \cup \{0\} \to [-(1+\varepsilon)F(T), (1+\varepsilon)F(T)]$ such that

(i) $\max_{x_{i-1} \in \{G,B\}, m_i \in M(T)} |f_T^{x_{i-1}}(m_i) - g_T^{x_{i-1}}(m_i)| < \varepsilon F(T)$,

(ii) $\mathbb{E}[g_T^{x_{i-1}}(\tilde{m}_i)|m_i] = f_T^{x_{i-1}}(m_i)$ *for all* $m_i \in M(T)$,

(iii) $\min_{m_i \in M(T)} \operatorname{sign}(x_{i-1}) g_T^{x_{i-1}}(m_i) \geq -(1+\varepsilon)cT$, *and*

(iv) $\min_{m_i \in M(T)} g_T^{x_{i-1}}(m_i) \geq g_T^{x_{i-1}}(0)$.

Applied to the remainder of the proof, condition (iii) helps satisfy self-generation, and condition (iv) helps satisfy the premises for the secure and verified modules.

Proof. Without loss, assume $F(T) \geq cT \ \forall T$ (otherwise, for each T with $F(T) < cT$, redefine $F(T) := cT$). Define

$$c_T^{x_{i-1}} = \begin{cases} (1+\varepsilon)c \text{ if } x_{i-1} = B \\ (1+\varepsilon)\frac{F(T)}{T} \text{ if } x_{i-1} = G \end{cases}.$$

By (43), $f_T^{x_{i-1}}(m_i) + c_T^{x_{i-1}}T \geq 0 \ \forall T \in \mathbb{N}, m_i \in M(T), x_{i-1} \in \{G, B\}$.

Now define $g_T^{x_{i-1}}(0) = -c_T^{x_{i-1}}T$ and

$$g_T^{x_{i-1}}(m_i) = \frac{1}{p_T(m_i)} f_T^{x_{i-1}}(m_i) + \frac{1 - p_T(m_i)}{p_T(m_i)} c_T^{x_{i-1}}T \quad \forall m_i \in M_i(T).$$

Then $\mathbb{E}[g_T^{x_{i-1}}(\tilde{m}_i)|m_i] = f_T^{x_{i-1}}(m_i) \ \forall T, m_i, x_{i-1}$, and $\liminf_{T\to\infty} F(T) > 0$ and (44) imply that, for sufficiently large T, $g_T^{x_{i-1}}(m_i) \in [-(1+\varepsilon)F(T), (1+\varepsilon)F(T)]$, $|f_T^{x_{i-1}}(m_i) - g_T^{x_{i-1}}(m_i)| \leq$

$\varepsilon F(T)\ \forall m_i$, and $\operatorname{sign}(x_{i-1})\, g_T^{x_{i-1}}(\tilde{m}_i) \geq (1+\varepsilon)cT\ \forall m_i$. Finally condition (iv) holds, as

$$g_T^{x_{i-1}}(m_i) - g_T^{x_{i-1}}(0) = \frac{1}{p_T(m_i)}\left(f_T^{x_{i-1}}(m_i) + c_T^{x_{i-1}}T\right) \geq 0 \quad \forall T, m_i, x_{i-1}.$$

■

E Equilibrium Conditions: Reduction Lemmas

This section uses the communication modules, the calendar time structure of a block, and Lemmas 12 and 13 to simplify Conditions (10)–(13). It also describes play in Phases 5–7 of the equilibrium strategies described in Section 3.3.3. Given the results of this section, for the remainder of the proof it suffices to consider the simplified versions of Conditions (10)–(13) derived below and ignore Phases 5–7 of the equilibrium strategies.

E.1 Reduction to an Undiscounted, Finitely Repeated Game

Naïvely taking the limit $\delta \to 1$ and ignoring Phase 7 of the equilibrium strategies suggests that Conditions (10)–(13) are satisfied for sufficiently high δ if there exists a block length T_2 and reward functions $\left(\pi_i^*(x_{i-1}, h_{i-1}^{T_2})\right)_{i\in I, x_{i-1}\in\{G,B\}, h_{i-1}^{T_2}\in H_{i-1}^{T_2}}$ such that

1. [Sequential Rationality] For all $x \in \{G,B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\sigma_i^*(x_i) \in \arg\max_{\sigma_i\in\Sigma_i} \mathbb{E}^{\left((\sigma_i, \sigma_{-i}^*(x_{-i})), \beta^*\right)}\left[\sum_{\tau=1}^{T_2} \hat{u}_i(\mathbf{a}_\tau) + \pi_i^*(x_{i-1}, h_{i-1}^{T_2}) | x_{-i}, h_i^{t-1}\right]. \tag{45}$$

2. [Promise Keeping] For all $x \in \{G,B\}^N$,

$$v_i(x_{i-1}) + \operatorname{sign}(x_{i-1})4\varepsilon^* = \frac{1}{T_2}\mathbb{E}^{\sigma^*(x)}\left[\sum_{t=1}^{T_2} \hat{u}_i(\mathbf{a}_t) + \pi_i^*(x_{i-1}, h_{i-1}^{T_2})\right]. \tag{46}$$

3. [Self-Generation] For all $x_{i-1} \in \{G,B\}$ and $h_{i-1}^{T_2} \in H_{i-1}^{T_2}$,

$$\operatorname{sign}(x_{i-1})\pi_i^*(x_{i-1}, h_{i-1}^{T_2}) \geq -4\varepsilon^* T_2. \tag{47}$$

Recall that $v_i(x_{i-1})$ is fixed by (14), so we omit (10). In addition, we have allowed $4\varepsilon^* T_2$ slack in (46) and (47), using the slack in (14).

We show that replacing (10)–(13) with (45)–(47) is indeed valid.

Lemma 14 *Suppose that, for all $\bar{T} > 0$, there exist $T_0 > \bar{T}$, $T_2 \geq T^*(T_0)$, strategies $(\sigma_i^*(x_i))_{i,x_i}$ and consistent ex post belief system β^* in the T_2-period finitely repeated game, and reward functions $\left(\pi_i^*(x_{i-1}, h_{i-1}^{T_2})\right)_{i,x_{i-1},h_{i-1}^{T_2}}$ such that Conditions (45)–(47) are satisfied. Then there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.*

To prove Lemma 14, we show that players can "cancel the effects of discounting" by using the communication phase at the very end of each block (Phase 7 of the equilibrium strategy profile). During this communication phase, players use the basic communication protocol to aggregate information regarding a random period in the block. A player then receives a small reward if she took an action yielding a higher payoff later in the block, so as to leave her indifferent to the timing of her actions within the block. In the construction, we use Lemma 8 to show that truthful communication is sequentially rational, and we use Lemma 12 to adjust the reward functions to correct for errors in communication.

E.2 Allowing Dependence on Other Players' Non-Main Phase Histories

We now show that player i's reward function can be made to depend on players $-i$'s histories in the non-main phases, so long as the magnitude of this dependence is bounded by $(T_1)^3/2$. In particular, the reward can depend on (i) players $-i$'s state profile $x_{-i} \in \{G, B\}^{N-1}$, and (ii) players $-i$'s history during non-main phases $h_{-i}^{\mathbb{T}''}$, where

$$\mathbb{T}'' := \{1, ..., T_1\} \setminus \bigcup_{l=1}^{L} \mathbb{T}(\text{main}(l)).$$

Recalling the calendar time structure in Section 3.3.3, note that, for sufficiently large T_0,

$$|\mathbb{T}''| < (T_0)^{\frac{3}{2}} < (T^*)^{\frac{1}{4}}. \tag{48}$$

The reward function thus takes the form $\pi_i^*\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right)$. We require that the range of the reward function is bounded by $(T_1)^3/2$:

$$\sup_{x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}} \left|\pi_i^*\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right)\right| < \frac{(T_1)^3}{2}. \tag{49}$$

We wish to replace $\pi_i^*(x_{i-1}, h_{i-1}^{T_2})$ with $\pi_i^*\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right)$ in (45)–(47), yielding the following conditions:

1. [Sequential Rationality] For all $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\sigma_i^*(x_i) \in \arg\max_{\sigma_i \in \Sigma_i} \mathbb{E}^{\left(\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right), \beta^*\right)}\left[\sum_{\tau=1}^{T_1} \hat{u}_i(\mathbf{a}_\tau) + \pi_i^*\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right) | h_i^{t-1}\right]. \tag{50}$$

2. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$v_i(x_{i-1}) + \text{sign}(x_{i-1}) 3\varepsilon^* = \frac{1}{T_1} \mathbb{E}^{\sigma^*(x)}\left[\sum_{t=1}^{T_1} \hat{u}_i(\mathbf{a}_t) + \pi_i^*\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right)\right]. \tag{51}$$

3. [Self-Generation] For all x_{-i}, $h_{i-1}^{T^*}$, and $h_{-i}^{\mathbb{T}''}$,

$$\text{sign}(x_{i-1}) \pi_i^*\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right) \geq -3\varepsilon^* T_1. \tag{52}$$

Note that we have reduced the allowable slack in the promise keeping and self-generation constraints to $3\varepsilon^* T_1$. This is because we "use up" $\varepsilon^* T_1$ slack when replacing $\pi_i^*(x_{i-1}, h_{i-1}^{T_2})$ with $\pi_i^*\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right)$.

Lemma 15 *Suppose that, for all $\bar{T} > 0$, there exist $T_0 > \bar{T}$, $T_1 \geq T^*(T_0)$, strategies $(\sigma_i^*(x_i))_{i, x_i}$ and consistent ex post belief system β^* in the T_1-period repeated game, and reward functions $\left(\pi_i^*\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right)\right)_{i, x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}}$ such that (49)–(52) are satisfied. Then, for all $\bar{T} \geq 0$, there exist $T_0 \geq \bar{T}$, $T_2(T_0) \geq T_1$ (satisfying (17)), strategies $(\sigma_i^{**}(x_i))_{i \in I}$ and consistent ex post belief system β^{**} in the $T_2(T_0)$-period repeated game, and reward functions $\left(\pi_i^{**}\left(x_{i-1}, h_{i-1}^{T_2}\right)\right)_{i, x_{i-1}, h_{i-1}^{T_2}}$ such that (45)–(47) are satisfied with $T_2 = T_2(T_0)$.*

To prove Lemma 15, we use the secure communication module with repetition $T = (T_1)^{\frac{1}{2}}$ and magnitude $(T_1)^3$ for the sender. Intuitively, for each $i \in I$, players $-(i-1, i)$ sequentially send their histories to player $i-1$, who uses this information to construct player i's reward. This corresponds to Phase 6 of the equilibrium strategy profile. We then use Lemma 13 to adjust for errors in communication. Note that the cardinality of $\left(x_{-i}, h_{-i}^{\mathbb{T}''}\right) \in \{G, B\}^N \times H_{-i}^{\mathbb{T}''}$ is of order at most $2\,|A|^{2(T_1)^{\frac{1}{4}}}$, so communicating $\left(x_{-i}, h_{-i}^{\mathbb{T}''}\right)$ with repetition $(T_1)^{\frac{1}{2}}$ takes a number of periods of order at most $(T_1)^{\frac{3}{4}} \ll T_1$. Hence, payoffs are still determined only by the main phase of each sub-block.

E.3 Allowing Dependence on Other Players' Main Phase Histories

We now define player i's private randomization $\mathbb{L}_i$ at the beginning of the block. In particular, player i randomly selects a period $t_i(l) \in \mathbb{T}(\text{main}(l))$ in each main phase $l = 1, ..., L$, according to

$$\Pr\left((t_i(l))_{l=1}^{L} = (t_l)_{l=1}^{L}\right) = (T_0)^{-6L}$$

for all $(t_l)_{l=1}^{L}$ such that $t_l \in \mathbb{T}(\text{main}(l))$ for each $l = 1, ..., L$. Let $\mathbb{L}_i = (t_i(l))_{l=1}^{L}$.

We show that player i's reward function in the T^*-period repeated game can be made to depend on players $-i$'s histories in periods in $\mathbb{L}_{i-1}$: that is, on

$$h_{-i}^{\mathbb{L}_{i-1}} := \left(a_{-i,t_{i-1}(l)}, \omega_{-i,t_{i-1}(l)}\right)_{l=1,...,L}. \tag{53}$$

Define

$$\mathbb{T}' := \{1, ..., T^*\} \setminus \bigcup_{l=1}^{L} \mathbb{T}(\text{main}(l)).$$

The difference between $\mathbb{T}'$ and $\mathbb{T}''$ is that $\mathbb{T}'$ does not include Phase 5 of the equilibrium strategies.

The reward function takes the form $\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$.[19] We require that the range of

[19] Relative to the previous subsection, the argument $h_{-i}^{\mathbb{L}}$ has been added to the reward function and the argument $h_{i-1}^{T^*}$ has been removed, as $h_{-i}^{\mathbb{L}}$ contains enough information about player $i-1$'s main phase history to provide incentives for player i.

the reward function is bounded by $(T^*)^3/2$:

$$\sup_{x_{-i},h^{\mathbb{T}'}_{-i},h^{\mathbb{L}_{i-1}}_{-i}} \left|\pi_i^*\left(x_{-i}, h^{\mathbb{T}'}_{-i}, h^{\mathbb{L}_{i-1}}_{-i}\right)\right| < \frac{(T^*)^3}{2}. \tag{54}$$

We wish replace $\pi_i^*(x_{i-1}, h^{T^*}_{i-1}, h^{\mathbb{T}''}_{-i})$ with $\pi_i^*\left(x_{-i}, h^{\mathbb{T}'}_{-i}, h^{\mathbb{L}_{i-1}}_{-i}\right)$ in (50)–(52). In the following conditions, we also cancel the instantaneous utilities outside of the main phases (which can be accomplished by using the reward function (28)).

1. [Sequential Rationality] For all $x \in \{G,B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\sigma_i^*(x_i) \in \arg\max_{\sigma_i \in \Sigma_i} \mathbb{E}^{\left(\left(\sigma_i, \sigma^*_{-i}(x_{-i})\right), \beta^*\right)}\left[\sum_{t \in \bigcup_{l=1}^{L} \mathbb{T}(\text{main}(l))} \hat{u}_i(\mathbf{a}_t) + \pi_i^*\left(x_{-i}, h^{\mathbb{T}'}_{-i}, h^{\mathbb{L}_{i-1}}_{-i}\right) | h_i^{t-1}\right]. \tag{55}$$

2. [Promise Keeping] For all $x \in \{G,B\}^N$,

$$v_i(x_{i-1}) + \text{sign}(x_{i-1})2\varepsilon^* = \frac{1}{T^*}\mathbb{E}^{\sigma^*(x)}\left[\sum_{t \in \bigcup_{l=1}^{L} \mathbb{T}(\text{main}(l))} \hat{u}_i(\mathbf{a}_t) + \pi_i^*\left(x_{-i}, h^{\mathbb{T}'}_{-i}, h^{\mathbb{L}_{i-1}}_{-i}\right)\right]. \tag{56}$$

3. [Self-Generation] For all x_{-i}, $h^{\mathbb{T}'}_{-i}$, and $h^{\mathbb{L}_{i-1}}_{-i}$,

$$\text{sign}(x_{i-1})\pi_i^*\left(x_{-i}, h^{\mathbb{T}'}_{-i}, h^{\mathbb{L}_{i-1}}_{-i}\right) \geq -2\varepsilon^* T^*. \tag{57}$$

Lemma 16 *Suppose that, for all $\bar{T} > 0$, there exist $T_0 > \bar{T}$, strategies $(\sigma_i^*(x_i))_{i,x_i}$ and consistent ex post belief system β^* in the $T^*(T_0)$-period repeated game, and reward functions $\left(\pi_i^*\left(x_{-i}, h^{\mathbb{T}'}_{-i}, h^{\mathbb{L}_{i-1}}_{-i}\right)\right)_{i,x_{-i},h^{\mathbb{T}'}_{-i},h^{\mathbb{L}_{i-1}}_{-i}}$ such that (54)–(57) are satisfied. Then there exist $T_1(T_0)$ (satisfying (16)), strategies $(\sigma_i^{**}(x_i))_{i,x_i}$ and consistent ex post belief system β^{**} in the $T_1(T_0)$-period repeated game, and reward functions $\left(\pi_i^{**}\left(x_{-i}, h^{\mathbb{T}'}_{-i}\right)\right)_{i,x_{-i},h^{\mathbb{T}'}_{-i}}$ such that (49)–(52) are satisfied with $T_1 = T_1(T_0)$.*

The proof of Lemma 16 is similar to that of Lemma 15, but using the verified communication module with repetition $T = (T^*)^{\frac{1}{2}}$ and magnitude $(T^*)^3$. This corresponds to Phase 5 of the equilibrium strategy profile. Note that the cardinality of $(\mathbb{L}_{i-1}, h_{-i}^{\mathbb{L}_{i-1}}) \in ((T_0)^6, A^2)^L$ is of order at most $(T_0)^{6L}$, so communicating $(\mathbb{L}_{i-1}, h_{-i}^{\mathbb{L}_{i-1}})$ with repetition $(T^*)^{\frac{1}{2}}$ takes a number of periods of order at most $(T^*)^{\frac{1}{2}+\varepsilon} \ll T^*$ (for any $\varepsilon > 0$). We can also adjust the reward based on x_{-i} by letting players $-i$ communicate x.

E.4 "Ignoring" Other Players' Deviations

We further simplify Lemma 16. Consider the following conditions:

1. [$t_i(l)$ Not Revealed Until End of Main Phase l] For all $x_i \in \{G, B\}$, $l \in \{1, ..., L\}$, $t \in \{1, ..., T^*\}$, $(\mathbb{L}_i, h_i^{t-1})$, and $(\tilde{\mathbb{L}}_i, \tilde{h}_i^{t-1})$, if $t \leq \tau$ for some $\tau \in \mathbb{T}(\text{main}(l))$, $t_i(\hat{l}) = \tilde{t}_i(\hat{l})$ for each $\hat{l} = 1, ..., l-1$, and $h_i^{t-1} = \tilde{h}_i^{t-1}$, then

$$\sigma_i(x_i)|_{(\mathbb{L}_i, h_i^{t-1})} = \sigma_i(x_i)|_{(\tilde{\mathbb{L}}_i, \tilde{h}_i^{t-1})}. \tag{58}$$

2. [Reward Bound]

$$\sup_{x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}} \left| \pi_i^* \left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}} \right) \right| < \frac{(T^*)^2}{2}. \tag{59}$$

3. [Incentive Compatibility] Let $H_i(x_{-i})$ denote the set of histories that arise with positive probability under some strategy profile $(\sigma_i, \sigma_{-i}(x_{-i}))$ with $\sigma_i \in \Sigma_i^{T^*}$. For all $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i(x_{-i})$,

$$\sigma_i^*(x_i) \in \arg\max_{\sigma_i \in \Sigma_i} \mathbb{E}^{\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right)} \left[\sum_{t \in \bigcup_{l=1}^L \mathbb{T}(\text{main}(l))} \hat{u}_i(\mathbf{a}_t) + \pi_i^* \left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}} \right) | h_i^{t-1} \right]. \tag{60}$$

   Note that we do not need to define the trembling sequence to define $\mathbb{E}[\cdot|\cdot]$ for (60).

4. [Promise Keeping] For all $x \in \{G, B\}^N$,

$$\left.\begin{array}{l} v_i(G) - 2\varepsilon^* \leq \\ v_i(B) + 2\varepsilon^* \geq \end{array}\right\} \frac{1}{T^*} \mathbb{E}^{\sigma^*(x)} \left[\sum_{t \in \bigcup_{l=1}^L \mathbb{T}(\text{main}(l))} \hat{u}_i(\mathbf{a}_t) + \pi_i^* \left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}} \right) \right]. \tag{61}$$

5. [Self-Generation] The same as (57).

Lemma 17 *Suppose that, for all $\bar{T} > 0$, there exist $T_0 > \bar{T}$, strategies $(\sigma_i^*(x_i))_{i,x_i}$ in the $T^*(T_0)$ period repeated game, and reward functions $\left(\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)\right)_{i,x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}}$ such that (57)–(61) are satisfied. Then there exist $T_1(T_0) \geq T^*$ (satisfying (16)), strategies $(\sigma_i^{**}(x_i))_{i,x}$ and consistent ex post belief system β^{**} in the $T_1(T_0)$-period repeated game, and reward functions $\left(\pi_i^{**}\left(x_{-i}, h_{-i}^{\mathbb{T}'}\right)\right)_{i,x_{-i},h_{-i}^{\mathbb{T}'}}$ such that (49)–(52) are satisfied with $T_1 = T_1(T_0)$.*

As in Lemma 16, players $-i$ communicate their history profile in $\mathbb{L}_{i-1}$. Since $\mathbb{L}_{i-1}$ is random and is not revealed until main phase l is over, by giving a reward based on the history profile in $\mathbb{L}_{i-1}$, player i can be made indifferent over actions after another player "confesses" that she deviated in or before main phase l.

F Equilibrium Strategies

We now define the equilibrium strategies $(\sigma_i(x_i))_{i\in I}$. By Lemma 17, we focus on the first four phases of the calendar time description in Section 3.3.3—a T^*-period finitely repeated game—since Phases 5–7 were addressed in Section E.

From now on, we abbreviate "the verified communication protocol with repetition T_0" to simply "the communication protocol."

It will be useful to introduce the notion of a *sub-phase*—corresponding to one complete play of the communication protocol within a phase—and to introduce notation that can stand for a generic main phase or sub-phase. As detailed below, for $i, n \in I$, we denote a sub-phase within the initial communication phase by $(0, i)$; denote a sub-phase within Communication Phase l, Part 1 by (l, i); denote a sub-phase within Communication Phase l, Part 2 by (l, i, n); and denote a sub-phase within contagion phase l by (l, i, con). We thus introduce notation

$$\lambda \in \{0 \times (\{\text{jam}\} \cup I \cup (I \times \{\text{con}\}))\} \cup \left\{\{1, \ldots L\} \times \{\text{main}\} \cup I \cup I^2 \cup (I \times \{\text{con}\})\right\}.$$

In this notation, the first coordinate of λ is 0 for the jamming coordination phase, the initial communication phase, and contagion phase 0, and it is l throughout sub-block $l \in \{1, \ldots, L\}$. The second coordinate of λ is (i) jam for the jamming coordination phase (for $l = 0$), (ii)

$i \in I$ for sub-phase (l, i) (for $l \geq 0$), (iii) (i,con) for sub-phase (l, i,con) (for $l \geq 0$), (iv) main for main phase l (for $l \geq 1$), or (v) (i, n) for sub-phase (l, i, n) (for $l \geq 1$).

For $l \in \{0, \dots, L\}$ we write $\lambda \leq l$ (resp., $\lambda < l$) if the first coordinate of λ is $\leq l$ (resp., $< l$), and similarly for $\lambda \geq l$ and $\lambda > l$. Similarly, for two sub-phases λ and λ', we say $\lambda \leq \lambda'$ if and only if sub-phase λ precedes or equals sub-phase λ'.

Given λ, let $h_i^{<\lambda}$ and $h_i^{\leq\lambda}$ be player i's history at the beginning and the end of sub-phase λ, respectively. Define $h^{<\lambda}$, $h^{\leq\lambda}$, $h_{-i}^{<\lambda}$, and $h_{-i}^{\leq\lambda}$ similarly.

F.1 Jamming Coordination Phase

At the beginning of the block, player i randomly selects a period $t_i(l) \in \mathbb{T}(\text{main}(l))$ for each $l = 1, ..., L$. This is encoded in $\mathbb{L}_i$ as defined in Section E.3.

Then the jamming coordination protocol is played. We refer to the set of the two periods consisting of the jamming coordination phase as *sub-phase* $(0, \text{jam})$. Denote player i's protocol history by $h_i^{(0,\text{jam})} = (a_{i,t}, \omega_{i,t})_{t=1}^2$. Recall from Section B.3 that $\zeta_i(h_i^{(0,\text{jam})}) = \text{jam}$ if $\omega_{i,t} = a^1$ for some $t \in \{1, 2\}$; otherwise, $\zeta_i(h_i^{(0,\text{jam})}) = \text{reg}$. In subsequent communication protocols, let $i \in \mathcal{I}_{\text{jam}}$ if and only if $\zeta_i(h_i^{(0,\text{jam})}) = \text{jam}$. Since Lemma 1 implies that $h_{-i}^{(0,\text{jam})}$ uniquely identifies $h_i^{(0,\text{jam})}$, we can equally view $(\zeta_n)_{n\in I}$ as a function of $h_{-i}^{(0,\text{jam})}$, denoted by $\zeta(h_{-j}^{(0,\text{jam})})$. Let $\theta_i(h_{-j}^{(0,\text{jam})}) = R$ if $\zeta_i(h_{-i}^{(0,\text{jam})}) = \text{reg}$ and $\theta_i(h_{-j}^{(0,\text{jam})}) = E$ if $\zeta_i(h_{-i}^{(0,\text{jam})}) = \text{jam}$. By Lemma 16, player i's reward function can be conditioned on $\zeta(h_{-j}^{(0,\text{jam})})$ and $\theta_i(h_{-j}^{(0,\text{jam})})$.

F.2 Initial Communication Phase

For each $i \in I$, player i sends x_i by the communication protocol. We refer to the set of $N + 1$ rounds consisting of the message round where player i sends x_i and the subsequent j-checking rounds for each $j \in I$ as *sub-phase* $(0, i)$.

For each $j \in I$, player j's history $h_j^{(0,i)}$ in sub-phase $(0, i)$ determines an inference $x_i(j)$ and a realization $\text{susp}(h_j^{(0,i)}) \in \{0, 1\}$. Collectively, the history profile $h_{-j}^{(0,i)}$ determines the variables $\theta_j(h_{-j}^{(0,i)}, \zeta(h_{-j}^{(0,\text{jam})}))$.

After sub-phase $(0, i)$ has been concluded for all $i \in I$, the history of each player $j \in I$ determines an inferred state profile $x(j) = (x_i(j))_{i\in I} \in \{G, B, 0\}^N$. In addition, for $i \in I$, given $h^{\leq(0,i)}$, let

$$I^D(h^{\leq(0,i)}) := \left\{ j \in I : \text{susp}(h_j^\lambda) = 1 \text{ for some sub-phase } \lambda \leq (0,i) \right\}$$

be the set of players who reach suspicious histories by the end of the sub-phase $(0,i)$.[20] Moreover, let $\theta_j(h_{-j}^{\leq(0,i)}) = E$ if there exists a sub-phase $\lambda \leq (0,i)$ such that $\theta_j(h_{-j}^\lambda, \zeta(h_{-j}^{(0,\text{jam})})) = E$.[21] Otherwise, let $\theta_j(h_{-j}^{\leq(0,i)}) = R$.

In general, for each sub-phase λ, we will define $\theta_j(h_{-j}^\lambda, \zeta(h_{-j}^{(0,\text{jam})})) \in \{E, R\}$ as a function of h^λ. Given the history $h^{\leq\lambda}$ at the end of sub-phase λ, we define $\theta_j(h_{-j}^{\leq\lambda}) = E$ if there exists a sub-phase $\lambda' \leq \lambda$ such that $\theta_j(h_{-j}^{\lambda'}, \zeta(h_{-j}^{(0,\text{jam})})) = E$. Otherwise, define $\theta_j(h_{-j}^{\leq\lambda}) = R$.

F.3 Contagion Phase 0

For each $i \in I$, player i communicates whether her history is suspicious. We refer to the $N+1$ rounds where player i sends this message and checking occurs as *sub-phase* $(0, i, \text{con})$.

In particular, given $I^D(h^{<(0,1,\text{con})})$ (which equals $I^D(h^{\leq(0,N)})$), in sub-phase $(0,i,\text{con})$ player i sends $m_i^{(0,i,\text{con})} = 1$ if $i \in I^D(h^{<(0,i,\text{con})})$ and $m_i^{(0,i,\text{con})} = 0$ otherwise. For each $j \in I$, player j's history $h_j^{(0,i,\text{con})}$ determines an inference $m_i^{(0,i,\text{con})}(j) \in \{0,1\}$ and a realization $\text{susp}(h_j^{(0,i,\text{con})}) \in \{0,1\}$. Collectively, the history profile $h_{-j}^{(0,i,\text{con})}$ determines the variables $\theta_j(h_{-j}^{(0,i,\text{con})}, \zeta(h_{-j}^{(0,\text{jam})}))$.

For the history $h^{\leq(0,i,\text{con})}$ at the end of sub-phase $(0,i,\text{con})$, let

$$I^D(h^{\leq(0,i,\text{con})}) := I^D(h^{<(0,i,\text{con})}) \cup \left\{ j \in I : m_i^{(0,i,\text{con})}(j) = 1 \text{ or susp} \left(h_j^{(0,i,\text{con})} \right) = 1 \right\}. \quad (62)$$

F.4 Sub-Block l

For $l = 1, \ldots, L$, strategies in sub-block l depend on the variables $I^D\left(h^{<(l,\text{main})}\right) \subset I$ and $(\theta_i(h_{-i}^{<(l,\text{main})}))_{i \in I} \in \{R, E\}^N$. We have already defined $I^D\left(h^{<(l,\text{main})}\right)$ and $\left(\theta_i\left(h^{<(l,\text{main})}\right)\right)_{i \in I}$. As we will see, the outcome of sub-block l determines $I^D\left(h^{<(l+1,\text{main})}\right)$ and $(\theta_i(h_{-i}^{<(l+1,\text{main})}))_{i \in I}$.

[20] If $\lambda = (0, \text{jam})$, define susp $\left(h_j^\lambda\right) = 0$

[21] If $\lambda = (0, \text{jam})$, define $\theta_j\left(h_{-j}^\lambda, \zeta(h_{-j}^{\text{jam}})\right) = \theta_i(h_{-j}^{(0,\text{jam})})$.

F.4.1 Main Phase l

If $i \in I^D\left(h^{<(l,\text{main})}\right)$, player i plays $\alpha^{\min}$ in every period. If $i \notin I^D\left(h^{<(l,\text{main})}\right)$, then $x_j(i) \in \{G, B\}$ for all $j \in I$, and hence the action profile $\mathbf{a}^l(x(i))$ is well-defined. In this case, in every period player i plays $a_i^l(x(i))$, the i-th component of action profile $\mathbf{a}^l(x(i))$.

Given a history profile $h^{\leq(l,\text{main})}$ at the end of main phase l, let $\theta_j(h_{-j}^{\leq(l,\text{main})}) = \theta_j(h_{-j}^{<(l,\text{main})})$ and $I^D\left(h^{\leq(l,\text{main})}\right) = I^D\left(h^{<(l,\text{main})}\right)$. That is, θ_j and I^D remain constant in main phase l.

F.4.2 Communication Phase l, Part 1

For each $i \in I$, player $i-1$ sends the number $t_{i-1}(l)$ by the communication protocol. We refer to the rounds where player $i-1$ sends $t_{i-1}(l)$ and checking occurs as *sub-phase* (l, i).

For each $j \in I$, player j's history $h_j^{(l,i)}$ in sub-phase (l, i) determines $t_{i-1}(l)(j) \in \mathbb{T}(\text{main}(l)) \cup \{0\}$ and $\text{susp}(h_j^{(l,i)}) \in \{0, 1\}$. Collectively, the history profile $h_{-j}^{(l,i)}$ determines $\theta_j(h_{-j}^{(l,i)}, \zeta(h_{-j}^{(0,\text{jam})}))$.

F.4.3 Communication Phase l, Part 2

For each $i \in I$ and $n \in I$, player i sends the message $(a_{i,t_{n-1}(l)(i)}, \omega_{i,t_{n-1}(l)(i)})$ by the communication protocol. (If $t_{n-1}(l)(i) = 0$, she sends an arbitrary pair $(a, \omega) \in A^2$.) We refer to the rounds where player i sends $(a_{i,t_{n-1}(l)(i)}, \omega_{i,t_{n-1}(l)(i)})$ and checking occurs as *sub-phase* (l, i, n).

For each $j \in I$, player j's history $h_j^{(l,i,n)}$ in sub-phase (l, i, n) determines an inference $\left(a_{i,t_{n-1}(l)}(j), \omega_{i,t_{n-1}(l)}(j)\right) \in A^2 \cup \{0\}$ and a realization $\text{susp}(h_j^{(l,i,n)}) \in \{0, 1\}$. Collectively, the history profile $h_{-j}^{(l,i,n)}$ determines $\theta_j(h_{-j}^{(l,i,n)}, \zeta(h_{-j}^{(0,\text{jam})}))$.

After sub-phase (l, i, n) has concluded for each $i \in I$ and $n \in I$, the history of each player $j \in I$ determines an inferred vector of outcomes $(a_{i,t_{n-1}(l)}(j), \omega_{i,t_{n-1}(l)}(j))_{i \in I} \in \prod_{n \in I}(A^2 \cup \{0\})$.

Players identify deviations as follows: Given $n \in I$, $x \in \{G, B\}^N$, and $(\mathbf{a}, \boldsymbol{\omega}) \in A^{2N}$, let $\text{dev}_n^l(x, \mathbf{a}, \boldsymbol{\omega}) = 1$ denote the event that either $(a_n, \omega_n) \neq \varphi(a_{-n}, \omega_{-n})$ (Lemma 1 implies (a_n, ω_n) is infeasible given players $-n$'s history) or $a_n \neq a_n^l(x)$. In addition, let $\text{dev}_n^l\left(x(i), \mathbf{a}_{t_{n-1}(l)}(i), \boldsymbol{\omega}_{t_{n-1}(l)}(i)\right) = 1$ if $x(i) \notin \{G, B\}^N$ or $\left(\mathbf{a}_{t_{n-1}(l)}(i), \boldsymbol{\omega}_{t_{n-1}(l)}(i)\right) \notin A^{2N}$.

Thus, $\text{dev}_n^l\left(x(i), \mathbf{a}_{t_{n-1}(l)}(i), \boldsymbol{\omega}_{t_{n-1}(l)}(i)\right) = 1$ means that the outcome of the communication in sub-phases $(l, j, n)_{j \in I}$ implies that player n deviated in the main phase, some player deviated in the communication phase, or the players failed to coordinate on some message.

Let h be a history at the end of sub-phase (l, i) or (l, i, n). Let $I^D(h)$ be the set of players who infer susp $= 1$ or dev $= 1$ by the end of the sub-phase: that is, for sub-phase (l, i),

$$I^D(h) := I^D(h^{\leq (l,\text{main})}) \cup \left\{ j \in I : \max_{\lambda \leq (l,i)} \text{susp}\left(h_j^\lambda\right) = 1 \right\},$$

and for sub-phase (l, i, n), $I^D(h)$ is defined as

$$I^D(h^{\leq (l,\text{main})}) \cup \left\{ j \in I : \max \left\{ \begin{array}{c} \max_{\lambda \leq (l,i,n)} \text{susp}(h_j^\lambda), \\ \max_{(l,N,n') \leq (l,i,n)} \text{dev}_{n'}^l\left(x(j), \mathbf{a}_{t_{n'-1}(l)}(j), \boldsymbol{\omega}_{t_{n'-1}(l)}(j)\right) \end{array} \right\} = 1 \right\}.$$

F.4.4 Contagion Phase l

For each $i \in I$, in sub-phase (l, i, con), player i sends whether $i \in I^D(h^{<(l,i,\text{con})})$, as in sub-phase $(0, i, \text{con})$. We define $\theta_j(h_{-j}^{(l,i,\text{con})}, \zeta(h_{-j}^{(0,\text{jam})}))$ and $I^D\left(h^{\leq (l,i,\text{con})}\right)$ as in sub-phase $(0, i, \text{con})$.

Finally, let $I_{-i}^D(h_{-i}) = I^D(h) \setminus \{i\}$. Note that I_{-i}^D is a function of players $-i$'s histories only, since whether $j \in I^D(h)$ is determined by h_j.

F.5 Observations

We close this section with some immediate implications of the construction. For each player $i \in I$, regardless of her strategy, either all her opponents successfully infer the state x, or they all become suspicious, or $\theta_i(h_{-i}) = E$. In addition, if some player became suspicious in one sub-block, then either everyone becomes suspicious or $\theta_i(h_{-i}) = E$ in the next sub-block. Finally, a deviation by player i from $a_i(x(i))$ in period $t_{i-1}(l)$ is detected for sure.

Lemma 18 *For any $i \in I$, $x \in \{G, B\}$, $\sigma_i \in \Sigma_i$, $l \in \{1, \ldots, L\}$, $l \leq \lambda < l + 1$, and $(\sigma_i, \sigma_{-i}(x_{-i}))$-consistent history $h^{<\lambda}$ at the beginning of (sub-) phase λ, the following claims hold:*

1. *Either (i)* $x(n) = x(i-1)$ $\forall n \in I$ *with* $x_j(n) = x_j$ *for each* $j \neq i$*, (ii)* $I_{-i}^D(h_{-i}^{<\lambda}) = I \backslash \{i\}$*, or (iii)* $\theta_i\left(h_{-i}^{<\lambda}\right) = E$.

2. *If* $I_{-i}^D(h_{-i}^{<(\tilde{l},\text{main})}) \neq \emptyset$ *for some* $\tilde{l} \leq l-1$*, then either* $I_{-i}^D(h_{-i}^{<\lambda}) = I \backslash \{i\}$ *or* $\theta_i\left(h_{-i}^{<\lambda}\right) = E$.

3. *If* $a_{i,t_{i-1}(l)} \neq a_i(x(i))$*, then either* $I_{-i}^D(h_{-i}^{<(l+1,\text{main})}) = I \backslash \{i\}$ *or* $\theta_i(h_{-i}^{<(l+1,\text{main})}) = E$.

Proof. Claims 1 and 2: By Claims 1 and 2 of Lemma 6, either (i) $x(n) = \hat{x} \in \{G, B\}^N$ $\forall n \in I$ with $\hat{x}_j = x_j$ for each $j \neq i$, (ii) $\text{susp}_n(h_n^{(0,j)}) = 1$ for some $n \neq i$ and $j \in I$, or (iii) $\theta_i(h_{-i}^{(0,j)}, \zeta(h_{-i}^{(0,\text{jam})}))$ for some $j \in I$. By the same claim applied to the contagion phase, if $I_{-i}^D(h_{-i}^{<(\tilde{l},\text{main})}) \neq \emptyset$ for some $\tilde{l} \leq l-1$, then $I_{-i}^D(h_{-i}) = I \backslash \{i\}$ or $\theta_i(h_{-i}) = E$ at the end of contagion phase $\tilde{l}$.

Claim 3: Suppose $a_{i,t_{i-1}(l)} \neq a_i(x(i))$. By Claim 1, $a_{i,t_{i-1}(l)} \neq a_i(x(i-1))$, $I_{-i}^D(h_{-i}^{<(l,\text{main})}) = I \backslash \{i\}$, or $\theta_i(h_{-i}^{<(l,\text{main})}) = E$. If $a_{i,t_{i-1}(l)} \neq a_i(x(i-1))$, by Claim 1 of Lemma 6, either (i) $\text{dev}_i^l\left(x(i-1), \mathbf{a}_{t_{i-1}(l)}(i-1), \boldsymbol{\omega}_{t_{i-1}(l)}(i-1)\right) = 1$, (ii) $\text{susp}_n(h_n^{\tilde{\lambda}}) = 1$ for some $n \neq i$ and $\tilde{\lambda} \in (l, i) \cup \{(l, n', i)\}_{n' \in I}$, or (iii) $\theta_i(h_{-i}) = E$ at the beginning of contagion phase l. Since the former two conditions imply $I_{-i}^D(h_{-i}) \neq \{\emptyset\}$ at the beginning of contagion phase l, we have $I_{-i}^D(h_{-i}^{<(l+1,\text{main})}) = I \backslash \{i\}$ or $\theta_i(h_{-i}^{<(l+1,\text{main})}) = E$ as a result of contagion phase l by Claim 1 of Lemma 6. ■

G Reward Function

This section constructs the reward function (ignoring for the moment the jamming coordination phase, which is addressed in Lemma 20). The reward function is the sum of rewards for the main phases, π_i^{main}, and rewards for the communication and contagion phases, $\pi_i^{\text{non-main}}$.

Let $u^G = \min_{(a,a') \in A^2} u(a, a')$ and $u^B = \max_{(a,a') \in A^2} u(a, a')$. By (14), for all $i \in I$, we have

$$\max\left\{v_i(G), u^B\right\} - \min\left\{u^G, v_i(B)\right\} \leq 2\bar{u}. \tag{63}$$

Recall that, by Lemma 1, (a_{-i}, ω_{-i}) perfectly identifies $\mathbf{a}$. Hence, defining

$$\begin{aligned}
\pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) &= u^{x_{i-1}} - \hat{u}_i(\mathbf{a}), \\
\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i}) &= v_i(x_{i-1}) - \hat{u}_i(\mathbf{a}), \\
\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i}|\alpha^{\min}) &= v_i(x_{i-1}) - u(a_i, \alpha^{\min})
\end{aligned}$$

for each $\mathbf{a} \in A^N$, we have

$$\begin{aligned}
\mathbb{E}\left[\hat{u}_i(\mathbf{a}) + \pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i})|\mathbf{a}\right] = u^{x_{i-1}}, \\
\mathbb{E}\left[\hat{u}_i(\mathbf{a}) + \pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i})|\mathbf{a}\right] = v_i(x_{i-1}), \\
\mathbb{E}\left[\hat{u}_i(\mathbf{a}) + \pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i}|\alpha^{\min})|a_i, \alpha_{-i}^{\min}\right] = v_i(x_{i-1}),
\end{aligned} \tag{64}$$

and

$$\begin{aligned}
\text{sign}(x_{i-1})\pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) \geq 0, \\
\max_{x_{i-1}, a_{-i}, \omega_{-i}} \max\left\{\left|\pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i})\right|, \left|\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i})\right|, \left|\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i}|\alpha^{\min})\right|\right\} \leq 2\bar{u}.
\end{aligned} \tag{65}$$

Moreover, letting $\varphi_A(a_{-i}, \omega_{-i})$ be the unique action $a_i \in A$ such that $\varphi(a_{-i}, \omega_{-i}) = (a_i, \omega_i)$ for some $\omega_i \in A$, we have, by (14),

$$\begin{aligned}
&\text{sign}(x_{i-1})\tfrac{1}{K_v}\textstyle\sum_{k=1}^{K_{\mathbf{v}}} \pi_i^{v_i}(a_{-i}^k(x), \omega_{-i,k}) \geq 9\varepsilon^* && \text{if } \varphi_A(a_{-i}^k(x), \omega_{-i,k}) = a_i^k(x)\ \forall k \in \{1, \ldots, K_{\mathbf{v}}\}, \\
&2\bar{u} \geq \pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i}|\alpha^{\min}) \geq 9\varepsilon^* && \text{for all } (x_{i-1}, a_{-i}, \omega_{-i}).
\end{aligned} \tag{66}$$

We now define $\pi_i^{\text{non-main}}$ and π_i^{main}. Define

$$\pi_i^{\text{non-main}}(h_{-i}^{\mathbb{T}'}) = \mathbf{1}_{\left\{\zeta_i(h_{-i}^{(0,\text{jam})}) = \text{reg}\right\}} \sum_{t \in \mathbb{T}'} \pi_{i,t}(h_{-i}^{\mathbb{T}'}), \tag{67}$$

where $\pi_{i,t}(h_{-i}^{\mathbb{T}'})$ corresponds to the reward for the verified communication module in (36).

Next, define

$$\pi_i^{\text{main}}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) = \sum_{l=1}^{L} \pi_i^{\text{main}}(l, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}),$$

where, for each l, we define

$$\pi_i^{\text{main}}(l, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) \tag{68}$$

$$= \sum_{t \in \mathbb{T}(\text{main}(l))} \mathbf{1}_{\{t_{i-1}(l)=t\}} (T_0)^6 \left(\begin{array}{c} \mathbf{1}_{\left\{\theta_i(h_{-i}^{\leq(l,\text{main})})=E\right\}} \pi_i^{\text{cancel}}(x_{i-1}, a_{-i,t}, \omega_{-i,t}) \\ +\mathbf{1}_{\left\{\theta_i(h_{-i}^{\leq(l,\text{main})})=R\right\}} \mathbf{1}_{\left\{I_{-i}^D(h_{-i}^{\leq(l,\text{main})})\neq I\setminus\{i\}\right\}} \pi_i^{v_i}(x_{i-1}, a_{-i,t}, \omega_{-i,t}) \\ +\mathbf{1}_{\left\{\theta_i(h_{-i}^{\leq(l,\text{main})})=R\right\}} \mathbf{1}_{\left\{I_{-i}^D(h_{-i}^{\leq(l,\text{main})})=I\setminus\{i\}\right\}} \pi_i^{v_i}(x_{i-1}, a_{-i,t}, \omega_{-i,t} | \alpha^{\min}) \\ -\mathbf{1}_{\left\{\theta_i(h_{-i}^{\leq(l,\text{main})})=R\right\}} \mathbf{1}_{\left\{I_{-i}^D(h_{-i}^{\leq(l,\text{main})})\neq\emptyset\right\}} \mathbf{1}_{\{x_{i-1}=G\}} 2\bar{u} \end{array} \right).$$

In total, the reward function following the jamming coordination phase is defined as

$$\pi_i^{\geq 3}\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right) = \pi_i^{\text{main}}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) + \pi_i^{\text{non-main}}(h_{-i}^{\mathbb{T}'}).$$

H Equilibrium Conditions: Final Statement

The main remaining step in the proof is verifying the equilibrium conditions given each history in the jamming coordination phase. It suffices to establish incentive compatibility and promise keeping, as self-generation is addressed in the proof of Lemma 20.

Lemma 19 *There exists $\bar{T} > 0$ such that, for all $T_0 > \bar{T}$, all $i \in I$, all $x \in \{G, B\}^N$, and all jamming coordination phase histories $h_i^{(0,\text{jam})}$, we have*

1. *[Incentive Compatibility] For each $t \geq 3$ and $h_i^{t-1} \in H_i(x_{-i})$,*

$$\sigma_i(x_i) \in \arg\max_{\sigma_i \in \Sigma_i} \mathbb{E}^{(\sigma_i, \sigma_{-i}(x_{-i}))} \left[\sum_{t \in \bigcup_{l=1}^{L} \mathbb{T}(\text{main}(l))} \hat{u}_i(\mathbf{a}_t) + \pi_i^{\geq 3}\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right) | h_i^{(0,\text{jam})}, h_i^{t-1} \right]. \tag{69}$$

2. *[Promise Keeping after $\zeta_i(h_{-i}^{(0,\text{jam})}) = \text{reg}$] If $\zeta_i(h_{-i}^{(0,\text{jam})}) = \text{reg}$ and*

$$v_i(x_{-i}, \mathcal{I}_{\text{jam}} \setminus \{i\}) := \frac{1}{T^*} \mathbb{E}^{\sigma(x)} \left[\sum_{t \in \bigcup_{l=1}^{L} \mathbb{T}(\text{main}(l))} \hat{u}_i(\mathbf{a}_t) + \pi_i^{\geq 3}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) | \mathcal{I}_{\text{jam}} \right], \tag{70}$$

then, for all $\mathcal{I}_{\text{jam}}\backslash\{i\}, \widetilde{\mathcal{I}_{\text{jam}}\backslash\{i\}} \subset I\backslash\{i\}$*, we have*

$$v_i\left(x_{-i}, \mathcal{I}_{\text{jam}}\backslash\{i\}\right) \begin{cases} \geq v_i(x_{i-1}) - \varepsilon^* & \text{if } x_{i-1} = G \\ \leq v_i(x_{i-1}) + \varepsilon^* & \text{if } x_{i-1} = B \end{cases}, \text{ and} \tag{71}$$

$$\left| v_i\left(x_{-i}, \mathcal{I}_{\text{jam}}\backslash\{i\}\right) - v_i(x_{-i}, \widetilde{\mathcal{I}_{\text{jam}}\backslash\{i\}})\right| \leq \frac{1}{T^*}. \tag{72}$$

The theorem now follows easily from Lemmas 11, 17, and 19.

Lemma 20 *Suppose Lemma 19 holds. Then there exists* $\bar{\delta} < 1$ *such that* $\mathbf{v} \in E(\delta)$ *for all* $\delta > \bar{\delta}$.

Proof. By definition of the strategies $\sigma(x)$ in Section F, (58) holds. Hence, putting together Lemmas 14–17, it suffices to construct reward functions π_i^* that, together with $\sigma(x)$, satisfy equations (57) and (59)–(61).

We first construct the reward for the jamming coordination phase, denoted $\pi_i^{\text{indiff}}(x_{-i}, h_{-i}^{(0,\text{jam})})$, using Lemma 11. To this end, we verify the premise for jamming coordination.

Given (68), if $\zeta_i(h_{-i}^{(0,\text{jam})}) = \text{jam}$ then player i is indifferent among all actions from period 3 on, since $\zeta_i(h_{-i}^{(0,\text{jam})}) = \text{jam}$ implies $\theta_i(h_{-i}^{<(l,\text{main})}) = E$ for each $l = 0, ..., L$. Hence, if $\zeta_i(h_{-i}^{(0,\text{jam})}) = \text{jam}$ then

$$\frac{1}{T^*}\mathbb{E}^{\sigma(x)}\left[\sum_{t \in \bigcup_{l=1}^{L} \mathbb{T}(\text{main}(l))} \hat{u}_i\left(\mathbf{a}_t\right) + \pi_i^{\geq 3}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) | \mathcal{I}_{\text{jam}}\right] = u^{x_{i-1}}. \tag{73}$$

In addition, (71) and (72) hold. Therefore, the premise for jamming coordination is satisfied. By Lemma 11, there exists $\pi_i^{\text{indiff}}(x_{-i}, h_{-i}^{(0,\text{jam})})$ such that the jamming coordination protocol is incentive compatible and

$$\lim_{T_0 \to \infty} \max_{x_{-i}, h_{-i}^{(0,\text{jam})}} \frac{\left|\pi_i^{\text{indiff}}(x_{-i}, h_{-i}^{(0,\text{jam})})\right|}{(T_0)^6} = 0. \tag{74}$$

We now define the total reward function as

$$\pi_i(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) = \pi_i^{\text{indiff}}(x_{-i}, h_{-i}^{(0,\text{jam})}) + \pi_i^{\geq 3}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}).$$

It remains to verify (57)–(61).

Note that, by the construction of $\pi_{i,t}(h_{-i}^{\mathbb{T}'})$ in (34), for all $x \in \{G, B\}^N$ and $h_{-i}^{\mathbb{T}'}$, we have

$$\operatorname{sign}(x_{i-1})\, \pi_i^{\text{non-main}}(h_{-i}^{\mathbb{T}'}) \geq -4\, |\mathbb{T}'|\, \bar{u}. \tag{75}$$

To derive a similar equation for π_i^{main}, if $\theta_i(h_{-i}^{<(l,\text{main})}) = E$, then (65) implies that π_i^{main} is non-positive if $x_{i-1} = G$ and non-negative if $x_{i-1} = B$. If $\theta_i(h_{-i}^{<(l,\text{main})}) = R$ and $I_{-i}^D(h_{-i}^{<(l,\text{main})}) = I \backslash \{i\}$, then the same conclusion holds by (66).

We now show that, in all other cases, $\operatorname{sign}(x_{i-1})\, \pi_i^{\text{main}}(l, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) < 0$ in at most $(1 + K_{\mathbf{v}})$ sub-blocks. To see this, note that if $I_{-i}^D(h_{-i}^{<(l,\text{main})}) \neq \emptyset$ then Lemma 18 implies that, as a result of contagion phase $l+1$, either $I_{-i}^D(h_{-i}^{<(l+1,\text{main})}) = I \backslash \{i\}$ or $\theta_i(h_{-i}^{<(l+1,\text{main})}) = E$ (regardless of player i's behavior). If $\theta_i(h_{-i}^{<(l,\text{main})}) = R$ and $I_{-i}^D(h_{-i}^{<(l,\text{main})}) = \emptyset$, then Lemma 18 implies that, for each $n \in I$, $x(n) = \hat{x}$ for some $\hat{x} \in \{G, B\}^N$ with $\hat{x}_{i-1} = x_{i-1}$. Hence, by (66),

$$\operatorname{sign}(x_{i-1}) \frac{1}{K_v} \sum_{k=1}^{K_v} \pi_i^{v_i}(x_{i-1}, a_{-i,t_{i-1}(l)}, \omega_{-i,t_{i-1}(l)}) \geq 0$$

as long as $a_i^l(x(i-1)) = \varphi_A(a_{-i,t_{i-1}(l)}, \omega_{-i,t_{i-1}(l)}) = a_{i,t_{i-1}(l)}$. Moreover, if $a_{i,t_{i-1}(l)} \neq a_i^l(x(i-1))$, then Lemma 18 implies that either $I_{-i}^D(h_{-i}^{<(l+1,\text{main})}) = I \backslash \{i\}$ or $\theta_i(h_{-i}^{<(l+1,\text{main})}) = E$.

Combining these observations, there exists a subset $\mathcal{L} \subset \{1, ..., L\}$ with $|\mathcal{L}| \geq L - (K_{\mathbf{v}} + 1)$ such that

$$\sum_{l \in \mathcal{L}} \operatorname{sign}(x_{i-1})\, \pi_i^{\text{main}}(l, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) \geq 0.$$

Since π_i^{u} and $\pi_i^{v_i}$ are bounded by (65), we have

$$\operatorname{sign}(x_{i-1})\, \pi_i^{\text{main}}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) \geq -2\bar{u}\,(1 + K_{\mathbf{v}})\,(T_0)^6 \geq_{\text{by (15)}} -\varepsilon^* L\,(T_0)^6 \quad \forall x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}. \tag{76}$$

Now, by (75) and (76), for all $x_{i-1}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}$, we have

$$\lim_{T_0 \to \infty} \frac{\operatorname{sign}(x_{i-1}) \pi_i^{\geq 3}(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}})}{L\,(T_0)^6} \geq -\varepsilon^*.$$

Together with (74), this implies (57) for sufficiently large T_0. Since the range of $\pi_i^{\geq 3}$ is $O\left((T_0)^{6+\varepsilon}\right)$ $\forall\varepsilon > 0$, (59) also holds for sufficiently large T_0.

Next, Lemma 19 implies that there is no profitable deviation from $\sigma_i(x_i)$ after the jamming coordination phase. Given this, Lemma 11 implies that there is also no profitable deviation from $\sigma_i(x_i)$ during the jamming coordination phase. Hence, (60) holds.

Finally, since $\mathcal{I}_{\text{jam}} \neq \emptyset$ with probability no more than $2N(T_0)^{-2}$, by (71), (73), and (74), the total payoff satisfies

$$\lim_{T_0\to\infty} \frac{1}{T^*}\mathbb{E}^{\sigma(x)}\left[\sum_{t\in\bigcup_{l=1}^{L}\mathbb{T}(\text{main}(l))} \hat{u}_i(\mathbf{a}_t) + \pi_i(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}})\right] = v_i(x_{-i}, \emptyset).$$

Hence, (70) implies (61) for sufficiently large T_0. ■

I Proof of Lemma 19

I.1 Notation

In this section, when considering any strategy σ_i and history h, h is assumed to be $(\sigma_i, \sigma_{-i}(x_{-i}))$-consistent.

For $l \in \{0, \ldots, L\}$ and $l \leq \lambda < l+1$, let $\mathbb{L}^{\leq\lambda} := (t_n(\tilde{l}))_{n\in I, \tilde{l}\leq l}$. Similarly, let $\mathbb{L}^{<\lambda} := (t_n(\tilde{l}))_{n\in I, \tilde{l}\leq l}$ if $l < \lambda$ and $\mathbb{L}^{<\lambda} := (t_n(\tilde{l}))_{n\in I, \tilde{l}<l}$ if $\lambda = (l, \text{main})$.

For each λ, at the end of (sub-) phase λ, if player i knew $\mathbb{L}^{\leq\lambda}$ and $h^{\leq\lambda}$, she could attain a continuation payoff of

$$\begin{aligned} & w_i(x_{-i}, \mathbb{L}^{\leq\lambda}, h^{\leq\lambda}) \qquad\qquad (77) \\ := & \max_{\sigma_i\in\Sigma_i} \mathbb{E}^{(\sigma_i, \sigma_{-i}(x_{-i}))}\left[\begin{array}{c} \sum_{\tilde{l}=l+1}^{L}\sum_{t\in\mathbb{T}(\text{main}(\tilde{l}))}\hat{u}_i(\mathbf{a}_t) + \sum_{\tilde{l}=l+1}^{L}\pi_i^{\text{main}}(\tilde{l}, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) \\ +\mathbf{1}_{\left\{\zeta_i(h_{-i}^{(0,\text{jam})})=\text{reg}\right\}}\sum_{t\in\mathbb{T}': t\succsim\lambda}\pi_{i,t}(h_{-i}^{\mathbb{T}'}) \end{array} |\mathbb{L}^{\leq\lambda}, h^{\leq\lambda}\right], \end{aligned}$$

where $t \succsim \lambda$ means period t follows or is within (sub-) phase λ. On the other hand, let $v_i(x, \mathbb{L}^{\leq\lambda}, h^{\leq\lambda})$ denote player i's continuation payoff from strategy $\sigma_i(x_i)$. We will show that, for any (sub-) phase λ and history $(\mathbb{L}^{\leq\lambda}, h^{\leq\lambda})$, $w_i(x_{-i}, \mathbb{L}^{\leq\lambda}, h^{\leq\lambda}) = v_i(x, \mathbb{L}^{\leq\lambda}, h^{\leq\lambda})$.

I.2 Equilibrium Properties

First, we show that there is no instantaneous deviation gain from $\sigma_i(x_i)$:

Lemma 21 *There exists $\bar{T} > 0$ such that, for any $T_0 > \bar{T}$, $i \in I$, $x \in \{G, B\}^N$, $\sigma_i \in \Sigma_i$, $l \in \{1, ..., L\}$, $\mathbb{L}^{<(l,\text{main})}$, and history $h^{<(l,\text{main})}$ at the beginning of phase (l, main),*

$$\begin{aligned}
& \max_{\sigma_i \in \Sigma_i} \mathbb{E}^{(\sigma_i, \sigma_{-i}(x_{-i}))} \left[\begin{array}{c} \sum_{t \in \mathbb{T}(\text{main}(l))} \hat{u}_i(\mathbf{a}_t) + \pi_i^{\text{main}}(l, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) \\ +\mathbf{1}_{\left\{\zeta_i(h_{-i}^{(0,\text{jam})}) = \text{reg}\right\}} \sum_{t \in \mathbb{T}_i^{a^0}: t \text{ in sub-block } l} \pi_i^{a^0}(a_{-i,t}, \omega_{-i,t}) \end{array} |\mathbb{L}^{<(l,\text{main})}, h^{<(l,\text{main})} \right] \\
= \;& \mathbb{E}^{\sigma(x)} \left[\sum_{t \in \mathbb{T}(\text{main}(l))} \hat{u}_i(\mathbf{a}_t) + \pi_i^{\text{main}}(l, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) |\mathbb{L}^{<(l,\text{main})}, h^{<(l,\text{main})} \right] \\
= \;& \begin{cases} (T_0)^6 \left(v_i(x_{i-1}) - \mathbf{1}_{\{x_{i-1} = G\}} \mathbf{1}_{\left\{I_{-i}^D(h_{-i}^{<(l,\text{main})}) \neq \emptyset\right\}} 2\bar{u} \right) & \text{if } \theta_i(h_{-i}^{<(l,\text{main})}) = R, \\ (T_0)^6 \, u^{x_{i-1}} & \text{if } \theta_i(h_{-i}^{<(l,\text{main})}) = E. \end{cases}
\end{aligned}$$

Proof. Playing $\sigma_i(x_i)$ yields the highest value of $\pi_{i,t}(h_{-i}^{\mathbb{T}'})$: 0. Hence, we focus on $\sum_{t \in \mathbb{T}(\text{main}(l))} \hat{u}_i(\mathbf{a}_t)$ and π_i^{main}. If $\theta_i(h_{-i}^{<(l,\text{main})}) = R$ then, by (68), the reward function satisfies

$$\begin{aligned}
& \pi_i^{\text{main}}(l, x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}) \\
= \;& (T_0)^6 \times \begin{cases} v_i^{v_i}(x_{i-1}, a_{-i,t}, \omega_{-i,t}) - \mathbf{1}_{\{x_{i-1} = G\}} \mathbf{1}_{\left\{I_{-i}^D(h_{-i}^{<(l,\text{main})}) \neq \emptyset\right\}} 2\bar{u} & \text{if } I_{-i}^D(h_{-i}^{<(l,\text{main})}) \neq I \setminus \{i\}, \\ \pi_i^{v_i}(x_{i-1,t}, a_{-i,t}, \omega_{-i,t} | \alpha^{\min}) - \mathbf{1}_{\{x_{i-1} = G\}} 2\bar{u} & \text{if } I_{-i}^D(h_{-i}^{<(l,\text{main})}) = I \setminus \{i\} \end{cases}
\end{aligned}$$

for $t = t_{i-1}(l)$ (and 0 for other t's). For each $t \in \mathbb{T}(\text{main}(l))$ and $a_{i,t}$, since $t_{i-1}(l) = t$ with probability $(T_0)^{-6}$ (recall that $\mathbb{L}^{<(l,\text{main})}$ does not include $t_{i-1}(l)$ and (58) holds) and since players $-i$ play $a_{-i}(x(i-1))$ when $I_{-i}^D(h_{-i}^{<(l,\text{main})}) = \emptyset$ (by Lemma 18) and play $\alpha^{\min}$ when $I_{-i}^D(h_{-i}^{<(l,\text{main})}) = I \setminus \{i\}$, the per-period expected payoff is $v_i(x_{i-1}) - \mathbf{1}_{\{x_{i-1} = G\}} \mathbf{1}_{\left\{I_{-i}^D(h_{-i}^{<(l,\text{main})}) \neq \emptyset\right\}} 2\bar{u}$, by (64). If instead $\theta_i(h_{-i}^{<(l,\text{main})}) = E$ then the result follows from (68) and (64). ■

Second, for each sub-phase λ, if $i \in I^D(h^\lambda)$ then $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$ or $\theta_i(h_{-i}^{<\lambda}) = E$.

Lemma 22 *There exists $\bar{T} > 0$ such that, for any $T_0 > \bar{T}$, $i \in I$, λ, and history $h^{<\lambda}$ at the beginning of (sub-) phase λ, if $i \in I^D(h^{<\lambda})$ then $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$ or $\theta_i(h_{-i}^{<\lambda}) = E$.*

Proof. By definition, $i \in I^D(h^{<\lambda})$ only if $\text{susp}_i(h_i) = 1$ or $\text{dev}_n^l\left(x(i), \mathbf{a}_{t_{n-1}(l)}(i), \boldsymbol{\omega}_{t_{n-1}(l)}(i)\right) = 1$ for some $n \in I$ as the result of communication sub-phases preceding λ. We will show that both of these two cases imply $I^D_{-i}(h^{<\lambda}_{-i}) \neq \emptyset$ or $\theta_i(h^{<\lambda}_{-i}) = E$.

In each communication sub-phase, by Claims 1 and 2 of Lemma 6, $\text{susp}_i(h_i) = 1$ implies $\theta_i(h^{<\lambda}_{-i}) = E$ for each subsequent sub-phase.

In addition, by Claims 1 and 2 of Lemma 6, all players infer the same message, $\text{susp}_n(h_n) = 1$ for some $n \neq i$, or $\theta_i(h_{-i}) = E$. If $\text{dev}^l_n \left(x\left(i\right), \mathbf{a}_{t_{n-1}(l)}\left(i\right), \boldsymbol{\omega}_{t_{n-1}(l)}\left(i\right)\right) = 1$ for some $n \in I$, then each of these three cases implies either $I^D_{-i}(h^{<\lambda}_{-i}) \neq \emptyset$ or $\theta_i(h^{<\lambda}_{-i}) = E$. ■

Third, the distribution of $\theta_i(h_{-i})$ is independent of the history in previous sub-phases, and $\theta_i(h_{-i}) = E$ is rare.

Lemma 23 *There exists $\bar{T} > 0$ such that, for any $T_0 > \bar{T}$, $i \in I$, λ, and $l \geq \lambda$, there exists $p(\mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda, \theta_i(h^{\leq\lambda}_{-i}), l)$ such that, for any $x \in \{G, B\}^N$, $\mathbb{L}^{\leq\lambda}$, and history $h^{\leq\lambda}$ at the end of (sub-) phase λ, we have*

$$\Pr{}^{\sigma(x)}\left(\theta_i(h^{<(l,\text{main})}_{-i}) = E | \mathbb{L}^{\leq\lambda}, h^{\leq\lambda}\right) = p_i(\mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda, \theta_i(h^{\leq\lambda}_{-i}), l).$$

Moreover, for $\theta_i(h^{\leq\lambda}_{-i}) = R$, we have $p_i(\mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda, \theta_i(h^{\leq\lambda}_{-i}), l) \leq (4\bar{u}L)^{-1} (T_0)^{-7}$.

Proof. By Claim 5 of Lemma 6, the distribution of θ_i in each communication sub-phase is determined by $\mathcal{I}_{\text{jam}} \backslash \{i\}$, independent of the message sent. In addition, since $\theta_i(h^{\leq\lambda}_{-i}) = R$ implies $\zeta_i(h^{(0,\text{jam})}_{-i}) = \text{reg}$, for sufficiently large T_0, in each communication sub-phase the probability of $\theta_i(h_{-i}, \zeta_i(h^{(0,\text{jam})}_{-i})) = E$ is at most $(T_0)^{-8}$ (by Claim 4 of Lemma 6). Hence the lemma holds. ■

I.3 Verification of Promise Keeping and Incentive Compatibility

In equilibrium, by Lemma 21, for each λ with $l \leq \lambda < l+1$, $\mathbb{L}^{\leq\lambda}$, and $h^{\leq\lambda}$, we have

$$v_i(x, \mathbb{L}^{\leq\lambda}, h^{\leq\lambda}) = \sum_{\tilde{l} \geq l+1} (T_0)^6 \left\{ \begin{array}{c} p_i(\mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda, \theta_i(h^{\leq\lambda}_{-i}), \tilde{l}) u^{x_{i-1}} \\ +(1 - p_i(\mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda, \theta_i(h^{\leq\lambda}_{-i}), \tilde{l})) \left(v_i(x_{i-1}) - \mathbf{1}_{\{x_{i-1}=G\}} \mathbf{1}_{\left\{I^D_{-i}(h^{<(\tilde{l},\text{main})}_{-i}) \neq \emptyset\right\}} 2\bar{u}\right) \end{array} \right. \tag{78}$$

By Claim 3 of Lemma 6, $I^D_{-i}(h^{<\lambda}_{-i}) \neq \emptyset$ implies $\theta_i(h^{\lambda}_{-i}) = E$ on the equilibrium path. Hence, with $\lambda = (0, \text{jam})$, by Lemma 23 and the fact that $T^* = O\left((T_0)^{6+\varepsilon}\right)$ $\forall \varepsilon > 0$, we have (70)–(72). It thus remains to verify (69).

The proof of (69) involves verifying the premise for verified communication, which requires the following simple lower bound on the probability of JAM:

Lemma 24 *There exists $\bar{T} > 0$ such that, for any $T_0 > \bar{T}$, $i \in I$, $x_{-i} \in \{G, B\}^{N-1}$, $\mathbb{L}$, $\sigma_i \in \Sigma_i$, h_i^t, and history $h^{3:t}$ from period 3 to period t, we have*

$$\Pr\left(\zeta_j(h_j^{(0,\text{jam})}) = \text{jam } \forall j \neq i | \mathbb{L}, h^{3:t}, h_i^t\right) \geq (T_0)^{-4(N-1)}. \tag{79}$$

Proof. By iterated expectations, it suffices to prove the lemma for $t = T^*$. For any jamming coordination phase history $h_i^{(0,\text{jam})}$, let $p_i(h_i^{(0,\text{jam})})$ denote the conditional probability that each player $j \neq i$ observes a^1 during the jamming coordination phase. By (25), we have $p_i(h_i^{(0,\text{jam})}) \geq \bar{\varepsilon}\,(T_0)^{-4(N-2)}$. It remains to account for updating from $h^{3:t}$ between periods 3 and T^* (recall that the jamming coordination phase ends in period 2).

Suppose player i could perfectly observe whether her opponents play REG or JAM in every half-interval. (Note that the other information in $(\mathbb{L}, h^{3:t})$ does not update the probability of $\zeta_j(h_j^{(0,\text{jam})})$). Then $\Pr\left(\zeta_j(h_j^{(0,\text{jam})}) = \text{jam } \forall j \neq i | h_i^{T^*}\right)$ would be minimized when REG is always played. As the probability that REG is always played is at least $1 - NT_0^2/T_0^9$ (conditional on any realization of $\left(\zeta_j(h_j^{(0,\text{jam})})\right)_{j \in I}$), we have

$$\Pr\left(\zeta_j(h_j^{(0,\text{jam})}) = \text{jam } \forall j \neq i | h_i^{T^*}\right) \geq \frac{\bar{\varepsilon}\,(T_0)^{-4(N-2)}\left(1 - N\frac{T_0^2}{T_0^9}\right)}{\bar{\varepsilon}\,(T_0)^{-4(N-2)}\left(1 - N\frac{T_0^2}{T_0^9}\right) + 1} = O\left((T_0)^{-4(N-2)}\right).$$

Hence, for sufficiently large T_0, (79) holds. ■

It will also be useful to simplify (78). By Lemma 23, there exists a payoff $v_i(x, \mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), D)$ (where D stands for "Deviation is Detected") such that, for each $h_{-i}^{<\lambda}$ with $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$, we have

$$v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) = v_i(x, \mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), D);$$

and for each $h_{-i}^{<\lambda}$ with $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$, we have (since $v_i(G) - 2\bar{u} \leq u^G$ and $v_i(B) \leq u^B$ by (63))

$$v_i(x, \mathbb{L}^{\leq \lambda}, h^{\leq \lambda}) \geq v_i(x, \mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda, \theta_i(h_{-i}^{\leq \lambda}), D). \tag{80}$$

In addition, on the equilibrium path, either $I_{-i}^{D}(h_{-i}^{<(l,\text{main})}) = \emptyset$ or $\theta_i(h_{-i}^{<(l,\text{main})}) = E$. Hence, for each λ with $l \leq \lambda < l+1$, $\mathbb{L}^{\leq\lambda}$, and $h^{\leq\lambda}$, on-path payoffs are given by

$$\begin{aligned} v_i(x, \mathbb{L}^{\leq\lambda}, h^{\leq\lambda}) &= v_i(x, \mathcal{I}_{\text{jam}}\backslash\{i\}, \lambda, \theta_i(h_{-i}^{\leq\lambda}), N) \\ : &= \sum_{\tilde{l}\geq l+1} (T_0)^6 \left\{ \begin{array}{c} p_i(\mathcal{I}_{\text{jam}}\backslash\{i\}, \lambda, \theta_i(h_{-i}^{\leq\lambda}), \tilde{l}) u^{x_{i-1}} \\ +(1 - p_i(\mathcal{I}_{\text{jam}}\backslash\{i\}, \lambda, \theta_i(h_{-i}^{\leq\lambda}), \tilde{l})) v_i(x_{i-1}) \end{array} \right\}. \end{aligned}$$

I.3.1 Proof of (69)

The proof is by induction. For $\lambda \geq L$, $v_i(x, \mathbb{L}^{\leq\lambda}, h^{\leq\lambda}) = w_i(x_{-i}, \mathbb{L}^{\leq\lambda}, h^{\leq\lambda}) = 0$, since there is no main phase following λ and playing $\sigma_i(x_i)$ yields $\pi_{i,t}(h_{-i}^{\mathbb{T}'}) = 0$. Given this observation, it suffices to establish the following claim:

Inductive hypothesis For each x, λ, $\mathbb{L}^{<\lambda}$, and $h^{<\lambda}$, if the equilibrium continuation payoff given $(\mathbb{L}^{\leq\lambda}, h^{\leq\lambda})$ equals $v_i(x, \mathbb{L}^{\leq\lambda}, h^{\leq\lambda})$, then $\sigma_i(x_i)$ is sequentially rational given $(\mathbb{L}^{\leq\lambda}, h^{\leq\lambda})$.

If $\theta_i(h_{-i}^{<\lambda}) = E$, then the claim follows from Lemma 21 and the fact that $\theta_i(h_{-i}^{<\lambda}) = E$ implies $\theta_i(h_{-i}^{<(l,\text{main})}) = E$ for all $l \geq \lambda$. So assume $\theta_i(h_{-i}^{<\lambda}) = R$.

For communication sub-phase λ, we use the notation v_i^E, $(v_i^{m_i})_{m_i \in M_i}$, and v_i^0 as in Section C.3. Note that (80) implies, for each $x, \mathbb{L}^{\leq\lambda}, h^{\leq\lambda}$,

$$v_i(x, \mathbb{L}^{\leq\lambda}, h^{\leq\lambda}) \geq v_i^0 = v_i(x, \mathcal{I}_{\text{jam}}\backslash\{i\}, \lambda, R, D).$$

Contagion Phase (l, i, con)**:** For the equilibrium message m_i (equal to 0 if $i \notin I^D(h^{<\lambda})$ and 1 if $i \in I^D(h^{<\lambda})$) and the alternative message $\hat{m}_i \in \{0,1\} \setminus \{m_i\}$, we have

$$\begin{aligned} v_i^{m_i} &\geq v_i^{\hat{m}_i} = v_i^0 \text{ if } I_{-i}^{D}(h_{-i}^{<\lambda}) = \emptyset \text{ and } i \notin I^D(h^{<\lambda}) \text{ (by (80))}, \\ v_i^{m_i} &= v_i^{\hat{m}_i} = v_i^0 \text{ if } I_{-i}^{D}(h_{-i}^{<\lambda}) \neq \emptyset \text{ or } i \in I^D(h^{<\lambda}), \end{aligned}$$

since $\theta_i(h_{-i}^{<\lambda}) = R$ and $i \in I^D(h^{<\lambda})$ imply $I_{-i}^{D}(h_{-i}^{<\lambda}) \neq \emptyset$ by Lemma 22. Given $v_i^E = u^{x_{i-1}}$, the premise holds. Hence, $\sigma_i(x_i)$ is sequentially rational.

Contagion Phase (l, j, con) **with** $j \neq i$**:** Since $v_i^{m_j} \geq v_i^0$ for all $m_j \in M_j$ by (80), the premise holds.

Communication Sub-Phase (l, i, n) **with** $n \neq i$**:** In sub-phases (l, n) and (l, j, n) with $j < i$, Claim 1 of Lemma 6 implies that either players coordinate on $t_n(l-1)$ and $(a_{j,t_n(l-1)}, \omega_{j,t_n(l-1)})_j$ or $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$ (given $\theta_i(h_{-i}^{<\lambda}) = R$). By the inductive hypothesis, players follow $\sigma(x)$ in later sub-phases, so by Claim 4 of Lemma 6 either players coordinate on $(a_{j,t_n(l-1)}, \omega_{j,t_n(l-1)})_{j>i}$ or $\theta_i(h_{-i}^{<(l+1,\text{main})}) = E$. Hence, given $\theta_i(h_{-i}^{<(l+1,\text{main})}) = R$, for each message $\hat{m}_i \neq (a_{i,t_n(l-1)(i)}, \omega_{i,t_n(l-1)(i)})$, coordinating on $\hat{m}_i$ induces $\text{dev}_n = 1$. Hence, $v_i^{m_i} \geq v_i^{\hat{m}_i} = v_i^0$. Given $v_i^E = u^{x_{i-1}}$, the premise holds.

Communication Sub-Phase (l, j, n) **with** $j \neq i$**:** The same as sub-phase (l, j, con).

Communication Sub-Phase (l, i)**:** If $I_{-i}^D(h_{-i}^{<\lambda}) \neq \emptyset$, then $v_i^{m_i} = v_i^0$ for each $m_i \in M_i$, so the premise holds. So assume $I_{-i}^D(h_{-i}^{<\lambda}) = \emptyset$..

Suppose first that $a_{i,t_{i-1}(l)} = a_i^l(x(i))$. Given $I_{-i}^D(h_{-i}^{<\lambda}) = \emptyset$ and $\theta_i(h_{-i}^{<\lambda}) = R$, by Claim 1 of Lemma 6, players coordinated on $t_j(l-1)$ with $j - 1 < i$. Since players follow $\sigma(x)$ in later sub-phases, Claim 4 of Lemma 6 implies that either players coordinate on the true message or $\theta_i(h_{-i}^{<(l+1,\text{main})}) = E$ in later sub-phases. Hence, for any $t \in \mathbb{T}(\text{main}(l))$, as long as $t_i(l-1)(n) = t$ for each $n \in I$, we have $I_{-i}^D(h_{-i}^{<(l+1,\text{main})}) = \emptyset$ or $\theta_i(h_{-i}^{<(l+1,\text{main})}) = E$. Therefore, for each message m_i, the continuation payoff is

$$v_i^{m_i} = v_i(x, \mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda + 1, R, N) \geq v_i^0 = v_i(x, \mathcal{I}_{\text{jam}} \backslash \{i\}, \lambda + 1, R, D),$$

so the premise holds.

Suppose instead $a_{i,t_{i-1}(l)} \neq a_i^l(x(i))$. Then Lemma 18 implies that $I_{-i}^D(h_{-i}^{<(l+1,\text{main})}) \neq \emptyset$ or $\theta_i(h_{-i}^{<(l+1,\text{main})}) = E$, regardless of player i's behavior. Hence, for each message m_i, the continuation payoff is $v_i^{m_i} = v_i^0$. Again, the premise holds.

Communication Sub-Phase (l, j) **with** $j \neq i$**:** The same as sub-phase (l, j, con).

Main Phase: If $I_{-i}^D(h_{-i}^{<(l,\text{main})}) \neq \emptyset$, then the continuation payoff is independent of player i's behavior in the main phase, so Lemma 21 implies the result. If $I_{-i}^D(h_{-i}^{<\lambda}) = \emptyset$, then given a history profile $(\mathbb{L}^{\leq\lambda}, h^{\leq\lambda})$ at the end of main phase l, by Lemma 18, the probability that

$I_{-i}^{D}(h_{-i}^{<(l+1,\text{main})}) \neq \emptyset$ is determined by and increasing in

$$\frac{\left|\{t \in \mathbb{T}(\text{main}(l)) : a_{i,t} \neq a_i^l(x(i))\}\right|}{(T_0)^6}.$$

Since the distribution of $\theta_i(h_{-i}^{<(l+1,\text{main})})$ is independent of player i's behavior in main phase l by Lemma 23, it is optimal to play $a_i^l(x(i))$ at each history.

J Omitted Proofs of Lemmas for Theorem 1

J.1 Proof of Lemma 6

Claim 1: If $\text{susp}\,(h_n) = 1$ for some $n \neq j$, then (ii) holds. If $\theta_j\,(h_{-j}, \zeta, j') = E$ for some $j' \in I$, then (iii) holds. So assume otherwise.

In light of the definition of FAIL, this implies that, for each $j' \neq j$ and $n \neq j'$, player n observes a^1 in each half-interval in $\mathbb{T}\,(j')$ where player j' plays a^1. Hence, $(a_{j',t}(j), \omega_{j',t}(j))_{t\in\mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$. Moreover, for each player $n \neq j, j'$, since $\text{susp}\,(h_n) = 0$, she does not observe a^0 in any other half-interval in $\mathbb{T}\,(j')$. Hence, $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$. Combining these observations, we have $(a_{j',t}(n), \omega_{j',t}(n))_{t\in\mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t\in\mathbb{T}(\text{msg})}$ for each $j', n \in I$. Therefore, $m_i(n) = m_i\,(n')$ for all $n \in I$. Finally, as player i follows the protocol, this message must equal m_i.

For the last part of the claim, consider each event that induces $\text{susp}\,(h_j) = 1$: If $(a_{n,t}(j), \omega_{n,t}(j))_{t\in\mathbb{T}(\text{msg})} = 0$ for some $n \neq j$, then $(a_{n,t}(j), \omega_{n,t}(j))_{t\in\mathbb{T}(\text{msg})} \neq (a_{n,t}, \omega_{n,t})_{t\in\mathbb{T}(\text{msg})}$. Hence, either some player $j' \neq n, j$ played JAM or player j did not match with player n in a half-interval in $\mathbb{T}\,(n)$ where player n played a^1. In either case, $\theta_j\,(h_{-j}, \zeta, n) = E$. If $(a_{j,t}(n), \omega_{j,t}(n))_{t\in\mathbb{T}(\text{msg}),j\in I}$ is not feasible, then again there exists $n \neq j$ with $(a_{n,t}(j), \omega_{n,t}(j))_{t\in\mathbb{T}(\text{msg})} \neq (a_{n,t}, \omega_{n,t})_{t\in\mathbb{T}(\text{msg})}$.

Claim 2: The same as Claim 1, except that the commonly inferred message $\hat{m}_i$ may differ from m_i.

Claim 3: Follows from Claim 3 of Lemma 5.

Claim 4: Given Claim 3, it suffices to show $\Pr^{\sigma^*, m_i}\,(\theta_j\,(h_{-j}, \zeta) = E) \leq T^{-8}$. For each

$j' \in I$, if no one plays JAM in $\mathbb{T}(j')$, then $\theta_j(h_{-j}, \zeta, j') = E$ only if some player $n \neq j'$ fails to observe a^1 in a half-interval in $\mathbb{T}(j')$ where player j' plays a^1. By Lemma 2, this event occurs with probability at most $(N-1)\left\lceil \log_2 |A|^{4\lceil \log_2 |M_i| \rceil} \right\rceil \exp(-\bar{\varepsilon}T)$. In total, $\theta_j(h_{-j}, \zeta) = E$ occurs with probability at most

$$\underbrace{2N(N-1)\left\lceil \log_2 |A|^{4\lceil \log_2 |M_i| \rceil} \right\rceil T^{-9}}_{\exists j' \in I, n \neq j: n \text{ plays JAM in } \mathbb{T}(j')} + \underbrace{N(N-1)\left\lceil \log_2 |A|^{4\lceil \log_2 |M_i| \rceil} \right\rceil \exp(-\bar{\varepsilon}T)}_{\exists j' \in I, n \neq i: n \text{ fails to observe } a^1 \text{ in } \mathbb{T}(j')}. \tag{81}$$

By (21), this sum is at most T^{-8}.

Claim 5: Follows from Claim 1 of Lemma 5.

J.2 Proof of Lemma 8

Let $\mathbf{a}^1 \in A^N$ be the action profile where player i plays a^1 and all other players play a^0. Let $\mathbf{a}^0 \in A^N$ be the action profile where all players play a^0. Let

$$\mathbb{T}^{1\text{st}} := \bigcup_{k=1}^{\lceil \log_2 |M_i| \rceil} \{2(k-1)T + 1, ..., 2(k-1)T + T\}$$

denote the set of periods in the first half of each interval. For $n \neq i$, define

$$\hat{\pi}_n(h_{n-1}) = \sum_{t \in \mathbb{T}} \frac{2K \mathbf{1}_{\{\omega_{n-1,t} = a^0\}}}{p_{n-1,n}} + \sum_{t \in \mathbb{T}^{1\text{st}}} \frac{\mathbf{1}_{\{\omega_{n-1,t} = a^1\}} \left(1 - \delta^T\right) \left(\hat{u}_n(\mathbf{a}^0) - \hat{u}_n(\mathbf{a}^1)\right)}{p_{n-1,i}}$$

and $\pi_n(h_{n-1}) = \hat{\pi}_n(h_{n-1}) - c_n$, where c_n is a constant to be determined. We will show that, for $n \neq i$, Claims 1 and 3 of the lemma hold for any c_n, and that $\mathbb{E}\left[\sum_{t \in \mathbb{T}} \delta^{t-1} \hat{u}_n(\mathbf{a}_t) + \hat{\pi}_n(h_{n-1})\right]$ is a constant independent of m_i. Setting $c_n = \mathbb{E}\left[\sum_{t \in \mathbb{T}} \delta^{t-1} \hat{u}_n(\mathbf{a}_t) + \hat{\pi}_n(h_{n-1})\right]$ then implies that Claim 2 also holds.

For Claim 1, we require that playing a^0 throughout the module is optimal with payoff function (26). This follows immediately from the facts that $K \geq \frac{2\bar{u}}{\bar{\varepsilon}}$ and $\max_{h, \tilde{h}} \left| w_n(h) - w_n(\tilde{h}) \right| < K$, which imply that the first term of $\hat{\pi}_n(h_{n-1})$ dominates any difference in $\sum_{t \in \mathbb{T}} \delta^{t-1} \hat{u}_n(\mathbf{a}_t)$ and $w_n(h)$. Claim 3 is also immediate.

To see that $\mathbb{E}\left[\sum_{t \in \mathbb{T}} \delta^{t-1} \hat{u}_n(\mathbf{a}_t) + \hat{\pi}_n(h_{n-1})\right]$ is independent of m_i, note that player i

plays a^1 the same number of times regardless of m_i. Therefore, $\mathbb{E}\left[\sum_{t\in\mathbb{T}}\frac{2K\mathbf{1}_{\{\omega_{n-1,t}=a^0\}}}{p_{n-1,n}}\right]$ is independent of m_i. It remains to show that

$$\sum_{t\in\mathbb{T}}\delta^{t-1}\hat{u}_n\left(\mathbf{a}_t\right)+\sum_{t\in\mathbb{T}^{1\text{st}}}\frac{\mathbb{E}\left[\mathbf{1}_{\{\omega_{n-1,t}=a^1\}}\right]\left(1-\delta^T\right)\left(\hat{u}_n\left(\mathbf{a}^0\right)-\hat{u}_n\left(\mathbf{a}^1\right)\right)}{p_{n-1,i}} \tag{82}$$

is independent of m_i.

We show that (82) is independent of m_i for each interval: that is, for each $k\in\{1,\ldots,\lceil\log_2|M_i|\rceil\}$, when the sums in (82) are restricted to $\tau\in\{2(k-1)T+1,\ldots,2kT\}$, they are the same when player i plays a^1 in the first half of the k^{th} interval as when she plays a^1 in the second half. For, when player i plays a^1 in the second half of the k^{th} interval, (82) equals

$$\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^0\right)+\sum_{\tau=2(k-1)T+T+1}^{2kT}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^1\right),$$

while when player i plays a^1 in the first half of the k^{th} interval, (82) equals

$$\begin{aligned}
&\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^1\right)+\sum_{\tau=2(k-1)T+T+1}^{2kT}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^0\right)+\left(1-\delta^T\right)\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\left(\hat{u}_n\left(\mathbf{a}^0\right)-\hat{u}_n\left(\mathbf{a}^1\right)\right)\\
=&\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^1\right)+\delta^T\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^0\right)+\left(1-\delta^T\right)\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\left(\hat{u}_n\left(\mathbf{a}^0\right)-\hat{u}_n\left(\mathbf{a}^1\right)\right)\\
=&\ \delta^T\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^1\right)+\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^0\right)\\
=&\sum_{\tau=2(k-1)T+1}^{2(k-1)T+T}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^0\right)+\sum_{\tau=2(k-1)T+T+1}^{2kT}\delta^{\tau-1}\hat{u}_n\left(\mathbf{a}^1\right).
\end{aligned}$$

Hence, the sum is the same either way.

Finally, for player i, define

$$\hat{\pi}_i\left(h_{i-1}\right)=\sum_{t\in\mathbb{T}}\frac{1}{p_{i-1,i}}\left(\delta^{t-1}\mathbf{1}_{\{\omega_{i-1,t}=a^1\}}\left(\hat{u}_i\left(\mathbf{a}^1\right)-\hat{u}_i\left(\mathbf{a}^0\right)\right)+\mathbf{1}_{\{\omega_{i-1,t}\in\{a^0,a^1\}\}}2\bar{u}\right).$$

The first term in the sum makes player i indifferent between playing a^0 and a^1, and the second term gives her an incentive not to play $a \notin \{a^0, a^1\}$. Since player i is indifferent between a^0 and a^1, it follows that $c_i = \mathbb{E}\left[\sum_{t\in\mathbb{T}} \delta^{t-1}\hat{u}_i(\mathbf{a}_t) + \hat{\pi}_i(h_{i-1})\right]$ is independent of m_i. Hence, letting $\pi_{i,t}(h_{i-1}) = \hat{\pi}_{i,t}(h_{i-1}) - c_i$, Claims 1-3 of the lemma hold for $n = i$.

J.3 Proof of Lemma 10

We prove the first part of the lemma by backward induction. We assume throughout that $\zeta_j = \text{reg}$; if instead $\zeta_j = \text{jam}$, then (35) equals $w_j(h,\zeta)$ and $\theta_j(h_{-j},\zeta) = E$, so player j is indifferent over all protocol strategies by Condition 1 of the premise for communication.

Final Checking Round Let j' be the index of the final checking round. Fix $h \in H^{<j'}$. The following lemma verifies the receivers' incentives, since $\hat{u}_j(\mathbf{a}_\tau) + \pi_j^{a^0}(a_{-j,\tau}, \omega_{-j,\tau})$ for $\tau \notin \mathbb{T}(j')$ is sunk.

Lemma 25 *Assume $j \neq j'$ and $\zeta_j = \text{reg}$. For every history $h^{<j'} \in H^{<j'}$ and h_j^{t-1} with $t \in \mathbb{T}(j')$, and every action $a_{j,t} \in A$, when player j follows her optimal continuation strategy after taking action $a_{j,t}$, we have*

$$\begin{aligned} & \mathbb{E}\left[\sum_{\tau\in\mathbb{T}(j')} \pi_j^{a^0}(a_{-j,\tau},\omega_{-j,\tau}) + w_j(h,\zeta)\,|h^{<j'}, h_j^{t-1}, a_{j,t} = a^0\right] \\ \geq\; & \mathbb{E}\left[\sum_{\tau\in\mathbb{T}(j')} \pi_j^{a^0}(a_{-j,\tau},\omega_{-j,\tau}) + w_j(h,\zeta)\,|h^{<j'}, h_j^{t-1}, a_{j,t} \neq a^0\right] + \frac{1}{2}. \end{aligned} \tag{83}$$

Proof. If $\theta_j(h_{-j},\zeta,j'') = E$ for some $j'' \neq j'$, the result follows immediately from (29), (36), and $\zeta_j = \text{reg}$. So suppose $\theta_j(h_{-j},\zeta,j'') = R$ for all $j'' \neq j'$. Since a deviation by any player $j'' \neq j$ induces $\theta_j(h_{-j},\zeta) = E$, we also assume players $-j$ follow σ_{-j}^* in every checking round. Hence, $\theta_j(h_{-j},\zeta,j') = E$ if and only if (i) some player $n \neq j'$ does not observe a^1 in a half-interval where player j' plays a^1 or (ii) some player $n \neq j, j'$ plays JAM in $\mathbb{T}(j')$. In particular, letting $R_{j',-j}$ denote the event that each player $n \neq j, j'$ is matched with player j' in every half-interval where player j' takes a^1, $\Pr(\theta_j(h_{-j},\zeta,j') = E|R_{j',-j}, h^{<j'}, h_j^{t-1})$ is independent of σ_j.

With i replaced by j', i^* replaced with j, $\mathbb{T}$ replaced with $\mathbb{T}(j')$, and Lemma 3 replaced with Lemma 7, by the same argument as for Lemma 9, with probability at least

$$1 - \max\left\{T^{4(N-1)+10}, N\left\lceil \log_2 |A|^{2\lceil \log_2 |M_i| \rceil}\right\rceil\right\} \exp\left(-\bar{\varepsilon}^4 T\right), \tag{84}$$

conditional on $(a_{j,\tau}, \omega_{j,\tau})_{\tau \in \mathbb{T}(j')}$, either $\theta_j(h_{-j}, \zeta, j') = E$ or [for each $n \neq j$, $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} \in \{(a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}, 0\}$, and $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ if and only if $a_{j,\tau} = a^0$ for each $\tau \in \mathbb{T}$ such that $\mu_\tau(j) = n$ and τ is in a half-interval where player j' plays a^0]. The latter event implies $R_{j',-j}$.

Since $\Pr(\theta_j(h_{-j}, \zeta, j') = E | R_{j',-j}, h^{<j'}, h_j^{t-1})$ independent of σ_j and $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = 0$ induces $\text{susp}_n(h_n) = 1$, playing $a_{j,\tau} = a^0$ for each $\tau \geq t$ maximizes $w_j(h, \zeta)$ with probability at least (84). Given this, the reward term $\pi_j^{a^0}(a_{-j,\tau}, \omega_{-j,\tau})$ outweighs any possible benefit to player j from playing $a \neq a^0$ in an attempt to manipulate $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg}), n \neq j}$. ■

We next verify the sender's incentive:

Lemma 26 *Assume $\zeta_{j'} = \text{reg}$. For every history $h^{<j'} \in H^{<j'}$ and $h_{j'}^{t-1}$ with $t \in \mathbb{T}(j')$, and every action $a_{j',t} \in A$, when player j' follows her optimal continuation strategy after taking action $a_{j',t}$, we have*

$$\begin{aligned} &\mathbb{E}\left[\sum_{\tau \in \mathbb{T}(j')} \pi_{j'}^{a_\tau}(a_{-j',\tau}, \omega_{-j',\tau}) + w_{j'}(h, \zeta) | h^{<j'}, h_{j'}^{t-1}, a_{j',t} = a_t\right] \\ \geq\ &\mathbb{E}\left[\sum_{\tau \in \mathbb{T}(j')} \pi_{j'}^{a_\tau}(a_{-j',\tau}, \omega_{-j',\tau}) + w_{j'}(h, \zeta) | h^{<j'}, h_{j'}^{t-1}, a_{j',t} \neq a_t\right] + \frac{1}{2}, \end{aligned} \tag{85}$$

where a_t is the action determined by $\sigma_{j'}^*$ given $(a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$.

Proof. Again, we assume $\theta_{j'}(h_{-j'}, \zeta, j'') = R$ for all $j'' \neq j'$ and players $-j'$ follow $\sigma_{-j'}^*$ in all checking rounds. In addition, assume $REG_{j',-j'}$, as otherwise $\theta_{j'}(h_{-j'}, \zeta, j') = E$. Given the reward $\pi_{j'}^{\bar{a}_{j',t}}(h_{-j'})$, it suffices to show that following $\sigma_{j'}^*$ maximizes $w_{j'}(h, \zeta)$.

By Claims 4 and 5 of Lemma 5, for each $j'' \neq j'$, we have $(a_{j'',t}(n), \omega_{j'',t}(n))_{t \in \mathbb{T}(\text{msg})} \in \{(a_{j'',t}, \omega_{j'',t})_{t \in \mathbb{T}(\text{msg})}, 0\}$ for all $n \in I$.

Fix $t \in \mathbb{T}(j')$, $h^{<j'}$, and $h_{j'}^{t-1}$. If $(a_{j'',t}(n), \omega_{j'',t}(n))_{t \in \mathbb{T}(\text{msg})} = 0$ for some $j'' \neq j'$ and $n \in I$, then Claim 1 of Lemma 6 implies that $\text{susp}_{n'}(h_{n'}) = 1$ for some $n' \neq j$. Hence, maximizing $w_{j'}(h, \zeta)$ is equivalent to maximizing the probability that $\theta_j(h_{-j}, \zeta, j') = E$. If player j' followed $\sigma_{j'}^*$ until period $t-1$ within $\mathbb{T}(j')$, then following $\sigma_{j'}^*$ maximizes $\theta_{j'}(h_{-j'}, \zeta, j') = E$. Otherwise, $\theta_{j'}(h_{-j'}, \zeta, j') = R$ given $REG_{j',-j'}$ and any strategy maximizes $w_{j'}(h, \zeta)$. In total, it is optimal to follow $\sigma_{j'}^*$.

Now suppose $(a_{j'',t}(n), \omega_{j'',t}(n))_{t \in \mathbb{T}(\text{msg})} = (a_{j'',t}, \omega_{j'',t})_{t \in \mathbb{T}(\text{msg})}$ for each $j'' \neq j'$ and $n \in I$. Suppose player j' followed $\sigma_{j'}^*$ until period $t-1$ within $\mathbb{T}(j')$. On the one hand, if player j' deviates from $\sigma_{j'}^*$ in period t, then $\theta_{j'}(h_{-j'}, \zeta, j') = R$ given $REG_{j',-j'}$. Since $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ for some $n \neq j'$ induces $\text{susp}(h_n) = 1$, player j''s payoff is

$$P(\sigma_{j'}|h^{<j'}, h_{j'}^{t-1}) T^K v_{j'}^{m_i} + (1 - P(\sigma_{j'}|h^{<j'}, h_{j'}^{t-1})) T^K v_{j'}^0,$$

where m_i corresponds to $(a_{i,t})_{t \in \mathbb{T}(\text{msg})}$ and $P(\sigma_{j'}|h^{<j'}, h_{j'}^{t-1})$ is the probability that $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ for all $n \neq j'$. On the other hand, if player j' follows $\sigma_{j'}^*$ in period t, then her equilibrium payoff is

$$P(\sigma_{j'}^*|h^{<j'}, h_{j'}^{t-1}) T^K v_{j'}^{m_i} + (1 - P(\sigma_{j'}^*|h^{<j'}, h_{j'}^{t-1})) T^K v_{j'}^E,$$

since $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} \neq (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ implies $\theta_{j'}(h_{-j'}, \zeta, j') = E$. As $\min\{v_{j'}^{m_i}, v_{j'}^E\} \geq v_{j'}^0$ by premise and $P(\sigma_{j'}^*|h^{<j'}, h_{j'}^{t-1}) \geq P(\sigma_{j'}|h^{<j'}, h_{j'}^{t-1})$ by definition, it is optimal to play $\sigma_{j'}^*$.

Suppose instead player j' deviated from σ_j^* within $\mathbb{T}(j')$ before period $t-1$. Then $\theta_{j'}(h_{-j'}, \zeta, j') = R$ given $REG_{j',-j'}$, so her payoff is

$$P(\sigma_{j'}|h^{<j'}, h_{j'}^{t-1}) T^K v_{j'}^{m_i} + (1 - P(\sigma_{j'}|h^{<j'}, h_{j'}^{t-1})) T^K v_{j'}^0.$$

Again, following $\sigma_{j'}^*$ for the rest of the round maximizes $P(\sigma_{j'}|h^{<j'}, h_{j'}^{t-1})$. ■

Backward Induction Given that players follow σ^* in subsequent rounds and Claim 1 of Lemma 5, we can assume $\theta_j(h_{-j}, \zeta, j'') = R$ for each j'' for which the j''-checking round follows the current round. Hence, the same proof as for Lemmas 25 and 26 establish each player's incentive to follow σ^* after any history.

Message Round Again, given that players follow σ^* in the checking rounds and Claim 1 of Lemma 5, we can assume $\theta_j(h_{-j}, \zeta, j') = R$ for each $j' \in I$, and so assume $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ and $\text{susp}_n(h_n) = 0$ for all $n, j' \in I$. Given this, the strategy of each player $j \neq i$ does not affect $w_j(h, \zeta)$, so incentives are satisfied. For player i, given $(a_{j',t}(n), \omega_{j',t}(n))_{t \in \mathbb{T}(\text{msg})} = (a_{j',t}, \omega_{j',t})_{t \in \mathbb{T}(\text{msg})}$ for all $n, j' \in I$, $m_i(n)$ will be equal to $\hat{m}_i$ if player i plays $(a_{i,t})_{t \in \mathbb{T}(\text{msg})}$ corresponding to the binary expansion of $\hat{m}_i$ (with the interpretation that, if $(a_{i,t})_{t \in \mathbb{T}(\text{msg})}$ does not correspond to the binary expansion of any $\hat{m}_i \in M_i$, then $m_i(n) = 1$). Hence, following σ_i^{*,m_i^*} is optimal after any history.

i^***-QBFE** The last part of the lemma is immediate: Since $v_j^E = v_j^{m_i} = v_j^{\text{punish}}$ for each $m_i \in M_i$ and $j \neq i^*$, players $-i^*$'s incentives are satisfied. For player i^*, the proof of the first part of the lemma applies.

J.4 Proof of Lemma 11

Definition of the Reward Function

We must define $\pi_{i,t}^{\text{indiff}}(h_{-i}|T)$. Given h_{-i}, fix h_i uniquely identified from h_{-i} by Lemma 1. Let H_i^0 be the set of histories for player i with $\omega_{i,1} \neq a^1$ and $\omega_{i,2} \neq a^1$. Given the resulting profile $h = (h_i, h_{-i})$, for $t = 2$, we define $\Delta v_{i,t}(h_{-i}|T)$ as follows:

1. If $\omega_{i,t-1} = a^1$, then $\Delta v_{i,t}(h_{-i}|T) := 0$.

2. Otherwise, define $\Pr(\mathcal{I}_{\text{jam}} \setminus \{i\}|h^{t-1}, H_i^0, a_i)$ as the conditional probability that the realized set of jamming players other than i at the end of the protocol equals $\mathcal{I}_{\text{jam}} \setminus \{i\}$, given that players $-i$ follow the protocol, $h_i \in H_i^0$, and player i plays a_i in period t. Let

$$\Delta v_{i,t}(h_{-i}|T) = \sum_{\mathcal{I}_{\text{jam}} \setminus \{i\}} \begin{pmatrix} \Pr(\mathcal{I}_{\text{jam}} \setminus \{i\}|h^{t-1}, H_i^0, a^1) \\ -\Pr(\mathcal{I}_{\text{jam}} \setminus \{i\}|h^{t-1}, H_i^0, a^0) \end{pmatrix} v_i(\mathcal{I}_{\text{jam}} \setminus \{i\}|T).$$

Note that $|\Delta v_{i,t}(h_{-i}|T)| \leq T^5$, by (40).

Finally, for $t = 2$, we define

$$\pi_{i,t}^{\text{indiff}}\left(h_{-i}|T\right) = -\mathbf{1}_{\{a_{i,t}=a^1\}}\Delta v_{i,t}\left(h_{-i}|T\right). \tag{86}$$

For $t = 1$, define $\Pr\left(\mathcal{I}_{\text{jam}} \setminus \{i\}|h^{t-1}, H_i^0, a_i\right)$ as the conditional probability that the realized set of jamming players other than i at the end of the protocol equals $\mathcal{I}_{\text{jam}} \setminus \{i\}$, given that players $-i$ follow the protocol, $h_i \in H_i^0$, and player i plays a_i in period t and a^0 in period $t+1$. The resulting definitions of $\Delta v_{i,t}\left(h_{-i}|T\right)$ and $\pi_{i,t}^{\text{indiff}}\left(h_{-i}|T\right)$ are the same as for $t = 2$.

Note that $\left|\pi_{i,t}^{\text{indiff}}\left(h_{-i}|T\right)\right| \leq T^5$ for $t = 1, 2$. Hence, (42) holds.

Incentive Compatibility

We show that, for every player i and period $t = 1, 2$, it is optimal for player i to follow the protocol in period t given that she follows the protocol in every later period.

Recall that $\Pr\left(h_i \in H_i^0\right)$ is independent of player i's strategy, and

$$w_i(h|T) = \begin{cases} \bar{w}_i(T) & \text{if } h_i \notin H_i^0 \\ v_i\left(\mathcal{I}_{\text{jam}} \setminus \{i\}|T\right) & \text{if } h_i \in H_i^0 \end{cases}.$$

Hence, player i maximizes her payoff by maximizing

$$\sum_{t=1}^{2} \pi_{i,t}^{\text{indiff}}\left(h_{-i}|T\right) + v_i\left(\mathcal{I}_{\text{jam}} \setminus \{i\}|T\right)$$

conditional on $h_i \in H_i^0$.

For $t = 2$, ignoring sunk payoffs, player i maximizes

$$\pi_{i,t}^{\text{indiff}}\left(h_{-i}|T\right) + v_i\left(\mathcal{I}_{\text{jam}} \setminus \{i\}|T\right) \tag{87}$$

conditional on $h_i \in H_i^0$. By (86), player i is indifferent between a^0 and a^1. Moreover, she is also indifferent between a^0 and any $a \notin \{a^0, a^1\}$, since (i) the distribution of $\mathcal{I}_{\text{jam}} \setminus \{i\}$ is the same whether she takes a^0 or $a \notin \{a^0, a^1\}$, and (ii) by (86), $\pi_i^{\text{indiff}}\left(h_{-i}|T\right)$ is the same as well. Hence, player i is indifferent over all actions.

For $t = 1$, noting that her period 1 action does not affect the distribution of anyone's action in period 2, player i again maximizes (87) conditional on $h_i \in H_i^0$. Again, (86) implies she is indifferent among all actions.

J.5 Proof of Lemma 14

Given the premise of the lemma, we construct a sequential equilibrium with value $\mathbf{v}$. By Lemma 4, it suffices to show that, for sufficiently large T_0 and $\delta < 1$, there exist T^{**}, $(\sigma_i^{**}(x_i))_{i,x_i}$, β^{**}, $(v_i^{**}(x_{i-1}))_{i,x_{i-1}}$ and $\left(\pi_i^{**}(x_{i-1}, h_{i-1}^{T^{**}})\right)_{i,x_{i-1},h_{i-1}^{T^{**}}}$ such that (10)–(13) are satisfied in the T^{**}-period discounted repeated game. Let

$$T^{**} = T_2 + N(T_2)^{\frac{1}{2}} \left(\lceil \log_2 T_2 \rceil + (N-2)\left(\lceil \log_2 (T_2+1) \rceil + \lceil \log_2 |A|^2 \rceil\right)\right).$$

Given $T_2 \geq T^*(T_0)$, (16), and (17), we have (18).

Construction of $\sigma_i^{**}(x_i)$

Play within each T_2-period block is given by $(\sigma_i^*(x_i))_{i \in I}$. After each T_2-period block, players communicate as follows for $T^{**} - T_2$ periods:

- For $i = 1$, player $i-1 \pmod N$ randomly chooses a period $t_{i-1} \in \{1, ..., T_2\}$ and sends t_{i-1} using the basic communication protocol with repetition $T = (T_2)^{\frac{1}{2}}$.
- Sequentially, each player $n \neq i, i-1$ sends her inferred message $t_{i-1}(n) \in \{1, ..., T_2\} \cup \{0\}$ and (if $t_{i-1}(n) \neq 0$) $h_{n,t_{i-1}(n)} = \left(a_{n,t_{i-1}(n)}, \omega_{n,t_{i-1}(n)}\right)$ using the basic communication protocol with repetition $T = (T_2)^{\frac{1}{2}}$. (If $t_{i-1}(n) = 0$, player n sends $t_{i-1}(n)$ together with an arbitrary pair $(a, \omega) \in A^2$.)
- For each $n \neq i, i-1$, player $i-1$ infers messages $t_{i-1}(n)(i-1) \in \{1, ..., T_2\} \cup \{0\}$ and $h_{n,t_{i-1}(n)}(i-1) \in A^2 \cup \{0\}$. We say that *communication succeeds* if $t_{i-1}(n)(i-1) = t_{i-1}$ and $h_{n,t_{i-1}(n)}(i-1) \neq 0$ for all $n \neq i, i-1$. Denote the event that communication succeeds (resp., fails) by $s_{i-1} = 1$ (resp., $s_{i-1} = 0$).
- Repeat this procedure for $i = 2, \ldots, N$.

Let $\mathbb{T}^{**}$ denote the set of these final $T^{**} - T_2$ periods, and let $\sigma_i^{T^{**}}|_{h_i^{T_2}}$ be the above strategy for player i, given $h_i^{T_2}$. Finally, note that, if $s_{i-1} = 1$ and players follow $\sigma^{T^{**}}$, then

$h_{-i,t_{i-1}}(i-1) = h_{-i,t_{i-1}}$.

Construction of β^{**}

As will be seen, for periods $T_2+1,...,T^{**}$, the equilibrium is belief-free. Hence, any consistent beliefs suffice. For periods $1,...,T_2$, let $\beta^{**}=\beta^{*}$.

Construction of $\pi_i^{**}(x_{i-1}, h_{i-1}^{T^{**}})$

Since $h_{-i,t_{i-1}}$ uniquely identifies $a_{i,t_{i-1}}$ by Lemma 1, there exists $\tilde{\pi}_{i,t}^{\delta}\left(t_{i-1}, h_{-i,t_{i-1}}\right)$ such that, for all $\mathbf{a}_t \in A^N$ and $t \in \{1,...,T_2\}$,

$$\tilde{\pi}_{i,t}^{\delta}\left(t_{i-1}, h_{-i,t_{i-1}}\right) = \mathbf{1}_{\{t_{i-1}=t\}} T_2 \left(1-\delta^{t-1}\right) \hat{u}_i\left(\mathbf{a}_t\right). \tag{88}$$

Note that

$$\lim_{\delta\to 1} \max_{t,t_{i-1},h_{-i,t_{i-1}}} \tilde{\pi}_{i,t}^{\delta}\left(t_{i-1}, h_{-i,t_{i-1}}\right) = 0. \tag{89}$$

We use Lemma 12 to adjust $\tilde{\pi}_{i,t}^{\delta}\left(t_{i-1}, h_{-i,t_{i-1}}\right)$ to account for errors in communication.

Claim 1 *There exist* $\left(\pi_{i,t}^{\delta}\left(t_{i-1}, s_{i-1}, h_{-i,t_{i-1}}(i-1)\right)\right)_{i,t,t_{i-1},s_{i-1},h_{-i,t_{i-1}}(i-1)}$ *such that*

1. *For all* $i \in I$, $t_{i-1} \in \{1,\ldots,T_2\}$, *and* $h^{T_2} \in H^{T_2}$,

$$\mathbb{E}\left[\pi_{i,t}^{\delta}\left(t_{i-1}, s_{i-1}, h_{-i,t_{i-1}}(i-1)\right) | h^{T_2}, t_{i-1}\right] = \tilde{\pi}_{i,t}^{\delta}\left(t_{i-1}, h_{-i,t_{i-1}}\right).$$

2. $\lim_{\delta\to 1} \max_{i,t,t_{i-1},s_{i-1},h_{-i,t_{i-1}}(i-1)} \pi_{i,t}^{\delta}\left(t_{i-1}, s_{i-1}, h_{-i,t_{i-1}}(i-1)\right) = 0$.

Proof. Let $\tilde{h}_{-i,t_{i-1}} = h_{-i,t_{i-1}}(i-1)$ if $s_{i-1}=1$ and $\tilde{h}_{-i,t_{i-1}}=0$ otherwise. Since $s_{i-1}=1$ implies $h_{-i,t_{i-1}}(i-1) = h_{-i,t_{i-1}}$, we have

$$\Pr\left(\tilde{h}_{-i,t_{i-1}} = h_{-i,t_{i-1}} | t_{i-1}\right) + \Pr\left(\tilde{h}_{-i,t_{i-1}} = 0 | t_{i-1}\right) = 1.$$

Moreover, by Lemma 2,

$$\Pr\left(\tilde{h}_{-i,t_{i-1}} = h_{-i,t_{i-1}} | t_{i-1}\right) \geq 1 - N\left(\lceil \log_2(T_2+1) \rceil + \lceil \log_2 |A|^2 \rceil\right) \exp(-\bar{\varepsilon}(T_2)^{\frac{1}{2}}).$$

Hence, the claim follows from (88), (89), and Lemma 12. ■

Let

$$\pi_i^\delta(x_{i-1}, h_{i-1}^{T^{**}}) := \sum_{t=1}^{T_2} \pi_{i,t}^\delta \left(t_{i-1}, s_{i-1}, h_{-i,t_{i-1}}(i-1)\right).$$

Note that, for all $j \neq i$, $\pi_j^\delta(x_{j-1}, h_{j-1}^{T^{**}})$ does not depend on the outcome of those periods in $\mathbb{T}^{**}$ used to construct $h_{-i,t_{i-1}}(i-1)$. Hence, by Lemma 8, there exist $\left(\pi_t\left(h_{i-1}^{\mathbb{T}^{**}}\right)\right)_{i\in I}$ such that $\sigma^{T^{**}}$ is a BFE in $\mathbb{T}^{**}$ when payoffs are given by

$$\mathbb{E}\left[\sum_{t\in\mathbb{T}^{**}} \delta^{t-1}\hat{u}_i(\mathbf{a}_t) + \pi_i^\delta(x_{i-1}, h_{i-1}^{\mathbb{T}^{**}}) + \pi_t\left(h_{i-1}^{\mathbb{T}^{**}}\right) \mid h_i^{T_2}\right]. \tag{90}$$

Finally, we define

$$\pi_i^{**}(x_{i-1}, h_{i-1}^{\mathbb{T}^{**}}) := \pi_i^*(x_{i-1}, h_{i-1}^{T_2}) + \pi_i^\delta(x_{i-1}, h_{i-1}^{\mathbb{T}^{**}}) + \pi_t\left(h_{i-1}^{\mathbb{T}^{**}}\right) + \text{sign}(x_{i-1})5\varepsilon^* T_2. \tag{91}$$

We now verify conditions (10)–(13).

[Sequential Rationality]

Ignoring sunk payoffs and the constant term $\text{sign}(x_{i-1})5\varepsilon^* T_2$, player i maximizes (90) in $\mathbb{T}^{**}$. By construction of $\left(\pi_t\left(h_{i-1}^{\mathbb{T}^{**}}\right)\right)_{i\in I}$, (11) holds for all $t \in \mathbb{T}^{**}$ for any consistent belief system, since by Lemma 8 the basic protocol is a BFE.

Next, by Lemma 8, $\mathbb{E}\left[\sum_{t\in\mathbb{T}^{**}} \delta^{t-1}\hat{u}_i(\mathbf{a}_t) + \pi_t\left(h_{i-1}^{\mathbb{T}^{**}}\right) | h^{T_2}\right]$ does not depend on h^{T_2}. Therefore, in period $t \leq T_2$, player i maximizes

$$\begin{aligned} & \mathbb{E}\left[\sum_{\tau=t}^{T_2} \delta^{t-1}\hat{u}_i(\mathbf{a}_\tau) + \pi_i^*(x_{i-1}, h_{i-1}^{T_2}) + \pi_i^\delta(x_{i-1}, h_{i-1}^{\mathbb{T}^{**}}) | h_i^{t-1}\right] \\ = \ & \mathbb{E}\left[\sum_{\tau=t}^{T_2} \delta^{t-1}\hat{u}_i(\mathbf{a}_\tau) + \pi_i^*(x_{i-1}, h_{i-1}^{T_2}) + \mathbb{E}\left[\pi_i^\delta(x_{i-1}, h_{i-1}^{\mathbb{T}^{**}}) | h^{T_2}\right] | h_i^{t-1}\right] \\ = \ & \mathbb{E}\left[\sum_{\tau=t}^{T_2} \hat{u}_i(\mathbf{a}_\tau) + \pi_i^*(x_{i-1}, h_{i-1}^{T_2}) + \mathbb{E}\left[\sum_{\tau=1}^{t-1} \pi_{i,\tau}^\delta\left(t_{i-1}, s_{i-1}, h_{-i,t_{i-1}}(i-1)\right) | h^{T_2}\right] | h_i^{t-1}\right], \end{aligned}$$

where the first equality follows by iterated expectation, and the second follows because $t_{i-1} = t$ with probability $(T_2)^{-1}$ for each $t \in \{1, ..., T_2\}$ and (88) holds.

Since $\mathbb{E}\left[\sum_{\tau=1}^{t-1} \pi_{i,\tau}^\delta\left(t_{i-1}, s_{i-1}, h_{-i,t_{i-1}}(i-1)\right) | h^{T_2}\right]$ does not depend on player i's continuation strategy given h_i^{t-1}, player i maximizes

$$\mathbb{E}\left[\sum_{\tau=t}^{T}\hat{u}_i\left(\mathbf{a}_\tau\right)+\pi_i^*(x_{i-1},h_{i-1}^{T_2})\mid h_i^{t-1}\right], \tag{92}$$

which equals the objective in (45) (ignoring sunk payoffs already incurred in h_i^{t-1}). Hence, (45) implies (11).

[Promise Keeping]

Promise keeping (equation (12)) is satisfied with $v_i^{**}(x_{i-1})$ defined by

$$\begin{aligned} v_i^{**}(x_{i-1}) \quad : \quad &= \frac{1-\delta}{1-\delta^{T^{**}}}\mathbb{E}\left[\begin{array}{c}\sum_{t=1}^{T^{**}}\delta^{t-1}\hat{u}_i\left(\mathbf{a}_t\right)+\pi_i^*(x_{i-1},h_{i-1}^{T_2})+\pi_i^\delta(x_{i-1},h_{i-1}^{\mathbb{T}^{**}}) \\ +\pi_t\left(h_{i-1}^{\mathbb{T}^{**}}\right)+\text{sign}(x_{i-1})5\varepsilon^* T_2\end{array}\right] \\ &= \frac{1-\delta}{1-\delta^{T^{**}}}\mathbb{E}\left[\sum_{t=1}^{T_2}\hat{u}_i\left(\mathbf{a}_t\right)+\pi_i^*(x_{i-1},h_{i-1}^{T_2})+\text{sign}(x_{i-1})5\varepsilon^* T_2\right] \end{aligned} \tag{93}$$

for $x_{i-1}\in\{G,B\}$.

[Self-Generation]

Since $\frac{|\mathbb{T}^{**}|}{T_2}\to 0$, for sufficiently large T_2 and δ, we have

$$\begin{aligned} &\text{sign}(x_{i-1})\left(\pi_i^*(x_{i-1},h_{i-1}^{T_2})+\pi_i^\delta(x_{i-1},h_{i-1}^{\mathbb{T}^{**}})+\pi_t\left(h_{i-1}^{\mathbb{T}^{**}}\right)+\text{sign}(x_{i-1})5\varepsilon^* T_2\right) \\ \geq\ &\text{sign}(x_{i-1})\pi_i^*(x_{i-1},h_{i-1}^{T_2})+4\varepsilon^* T_2\geq 0, \end{aligned}$$

where the first inequality follows by Claim 1 and (27) and the second follows by (47). Hence, (13) holds.

[Full Dimensionality]

By (14), we have $v_i\left(B\right)+9\varepsilon^*<v_i<v_i\left(G\right)-9\varepsilon^*$. Since $\frac{1-\delta}{1-\delta^{T^{**}}}\to\frac{1}{T^{**}}$ as $\delta\to 1$ and $\frac{|\mathbb{T}^{**}|}{T_2}\to 0$ as $T_0\to\infty$, for sufficiently large T_0 and δ, (46) and (93) imply that $v_i^{**}(x_{i-1})$ is sufficiently close to $v_i\left(x_{i-1}\right)+\text{sign}(x_{i-1})9\varepsilon^*$ compared to the slack between v_i and $v_i\left(x_{i-1}\right)+\text{sign}(x_{i-1})9\varepsilon^*$:

$$\left|v_i^{**}(x_{i-1})-\left(v_i\left(x_{i-1}\right)+\text{sign}(x_{i-1})9\varepsilon^*\right)\right|<\frac{1}{3}\min_{x_{i-1}\in\{G,B\}}\left|v_i\left(x_{i-1}\right)+\text{sign}(x_{i-1})9\varepsilon^*-v_i\right|.$$

Hence, $v_i^{**}(B)<v_i<v_i^{**}(G)$.

J.6 Proof of Lemma 15

Let $T_2 = T_1 + \tilde{T}_1 + \tilde{T}_2$, where

$$\tilde{T}_1 = 2N\left(N-2\right)\left\lceil \log_2(2(2\left|A\right|^2)^{|\mathbb{T}''|}+1)\right\rceil (T_1)^{\frac{1}{2}} \tag{94}$$

and

$$\tilde{T}_2 = 2N\left(N-2\right)\left\lceil \log_2 \left|A\right|^2\right\rceil \tilde{T}_1 (T_1)^{\frac{1}{12}}.$$

As $|\mathbb{T}''| < (T_1)^{\frac{1}{4}}$ (by (48) and $T_1 \geq T^*(T_0)$) and (16) holds, we have (17) and

$$\lim_{T_0\to\infty} \frac{\tilde{T}_1}{(T_1)^{\frac{5}{6}}} = \lim_{T_0\to\infty} \frac{\tilde{T}_1 + \tilde{T}_2}{(T_1)^{\frac{11}{12}}} = 0.$$

We construct strategies $\sigma_i^{**}(x_i)$, beliefs β^{**}, and reward functions $\pi_i^{**}\left(x_{i-1}, h_{i-1}^{T_2}\right)$ in the T_2-period game that satisfy the premise of Lemma 14.

Construction of $\sigma_i^{**}(x_i)$

Play within each T_1-period block is given by $(\sigma_i^*(x_i))_{i\in I}$. After each T_1-period block, players communicate as follows for $\tilde{T}_1 + \tilde{T}_2$ periods:

Communication for periods $T_1+1, \ldots, T_1+\tilde{T}_1$:

- Let $i = 1$. Sequentially, each player $n \neq i, i-1$ sends $(x_n, h_n^{\mathbb{T}''}) = \left(x_n, (a_{n,t}, \omega_{n,t})_{t\in\mathbb{T}''}\right)$ using the secure communication protocol with repetition $T = (T_1)^{\frac{1}{2}}$ and $\mathcal{I}_{\text{jam}} = \{i-1\}$. This takes $2\left(N-2\right)\left\lceil \log_2(2\left|A\right|^2)^{|\mathbb{T}''|}+1)\right\rceil (T_1)^{\frac{1}{2}}$ periods.

- For each $n \neq i, i-1$, player $i-1$ infers a message $m_{i-1}(n)(i-1)$. If $m_{i-1}(n)(i-1) = 0$ for some $n \neq i, i-1$, or if player $i-1$ plays JAM during a round where she receives a message via the secure protocol, let $s_{i-1} = 0$ ("communication fails"). Otherwise, $s_{i-1} = 1$ ("communication succeeds").

- Repeat this procedure for $i = 2, \ldots, N$.

Communication for periods $T_1+\tilde{T}_1+1, \ldots, T_1+\tilde{T}_1+\tilde{T}_2$:

- Let $i = 1$. Sequentially, each player $n \neq i, i-1$ sends $(a_{n,t}, \omega_{n,t})_{t\in\left\{T_1+1,\ldots,T_1+\tilde{T}_1\right\}}$

using the basic communication protocol with repetition $T = (T_1)^{\frac{1}{12}}$. This takes $2(N-2)\lceil \log_2 |A|^2 \rceil \tilde{T}_1 (T_1)^{\frac{1}{12}}$ periods.

- Repeat this procedure for $i = 2, \ldots, N$.

Construction of β^{**}

For periods $T_1 + \tilde{T}_1 + 1, \ldots, T_1 + \tilde{T}_1 + \tilde{T}_2$, the equilibrium is belief-free, so any consistent belief system suffices. For periods $T_1 + 1, \ldots, T_1 + \tilde{T}_1$, specify beliefs as in Lemma 9 given player n's equilibrium message. For periods $1, ..., T_1$, let $\beta^{**} = \beta^*$.

Construction of $\pi_i^{**}\left(x_{i-1}, h_{i-1}^{T_2}\right)$

If $s_{i-1} = 1$, we denote player $i-1$'s inference of player n's message during periods $T_1 + 1, \ldots, T_1 + \tilde{T}_1$ by $\left(x_n(i-1), h_n^{\mathbb{T}''}(i-1)\right)$. We first construct a function $\tilde{\pi}_i^*\left(x_{-i}(i-1), h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}(i-1)\right)$ as follows: Define $\left(\tilde{x}_{-i}, \tilde{h}_{-i}^{\mathbb{T}''}\right) = \left(x_{-i}(i-1), h_{-i}^{\mathbb{T}''}(i-1)\right)$ if $s_{i-1} = 1$ and $\left(\tilde{x}_{-i}, \tilde{h}_{-i}^{\mathbb{T}''}\right) = 0$ otherwise. Note that (i) for sufficiently large T_1, inequality (2) implies

$$\min_{x_{-i}, h_{-i}^{\mathbb{T}''}} \Pr\left(s_{i-1} = 1 | x_{-i}, h_{-i}^{\mathbb{T}''}\right) \geq 1 - (T_1)^{-\frac{8}{2}}, \tag{95}$$

(ii) $s_{i-1} = 1$ implies $\left(x_{-i}(i-1), h_{-i}^{\mathbb{T}''}(i-1)\right) = \left(x_{-i}, h_{-i}^{\mathbb{T}''}\right)$, and (iii) π_i^* satisfies (52). Hence, in the notation of Lemma 13,

$$\begin{aligned} T &= (T_1)^{\frac{1}{2}}, \\ p_T(m_i) &= 1 - (T_1)^{-4} = 1 - T^{-8} \quad \forall m_i \in M(T), \\ F(T) &= \frac{1}{2}(T_1)^3 = \frac{1}{2}T^6, \\ c &= 3\varepsilon^*. \end{aligned}$$

Since $\lim_{T\to\infty}(1 - p_T(m_i)) \max\{F(T), cT\} = 0$, Lemma 13 implies that, for sufficiently large T_0, there exists $\tilde{\pi}_i^*\left(\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}\right)$ such that

$$\max_{\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}} \left|\tilde{\pi}_i^*\left(\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}\right)\right| \leq T^6, \tag{96}$$

$$\mathbb{E}\left[\tilde{\pi}_i^*\left(\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}\right) | x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right] = \pi_i^*\left(x_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right), \tag{97}$$

$$\operatorname{sign}(x_{i-1})\,\tilde{\pi}_i^*\left(\tilde{x}_{-i}, h_{i-1}^{T^*}, h_{-i}^{\mathbb{T}''}\right) \geq -\frac{7}{2}\varepsilon^* T_1 \forall \tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}, \text{ and} \tag{98}$$

$$\tilde{\pi}_i^*\left(\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}\right) \text{ is minimized when } s_{i-1} = 0. \tag{99}$$

We next construct a function

$$\sum_{t=T_1+1}^{T_1+\tilde{T}_1} \tilde{\pi}_i^{\text{cancel}}\left(a_{-i,t}(i-1), \omega_{-i,t}(i-1)\right) + \sum_{\substack{t=T_1+1,\dots,T_1+\tilde{T}_1: \\ \text{players } -i \text{ communicate}}} \tilde{\pi}_i^{a^0}\left(a_{-i,t}(i-1), \omega_{-i,t}(i-1)\right)$$

as follows. By Lemma 12, for sufficiently large T_1, there exists a function $\tilde{\pi}_i^{\text{cancel}} : A^2 \to \mathbb{R}$ such that

$$\mathbb{E}\left[\tilde{\pi}_i^{\text{cancel}}\left(a_{-i,t}(i-1), \omega_{-i,t}(i-1)\right) | a_{-i,t}, \omega_{-i,t}\right] = \pi_i^{\text{cancel}}\left(a_{-i,t}, \omega_{-i,t}\right)$$

for all $t = T_1 + 1, ..., T_1 + \tilde{T}_1$ and $(a_{-i}, \omega_{-i}) \in A^2$. Similarly, for sufficiently large T_1, there exists a function $\tilde{\pi}_i^{a^0} : A^2 \to \mathbb{R}$ such that

$$\mathbb{E}\left[\tilde{\pi}_i^{a^0}\left(a_{-i,t}(i-1), \omega_{-i,t}(i-1)\right) | a_{-i,t}, \omega_{-i,t}\right] = \pi_i^{a^0}\left(a_{-i,t}, \omega_{-i,t}\right).$$

Recall that π_i^{cancel} and $\pi_i^{a^0}$ are defined in (28) and (29), respectively. Since π_i^{cancel} and $\pi_i^{a^0}$ are bounded, $\tilde{\pi}_i^{\text{cancel}}$ and $\tilde{\pi}_i^{a^0}$ are bounded by Lemma 12, uniformly over sufficiently large T_1. Hence, by Lemma 8, there exist a function $\pi_i(h_{i-1}^{\tilde{T}_2})$, where $h_{i-1}^{\tilde{T}_2} = (a_{i-1,t}, \omega_{i-1,t})_{t=T_1+\tilde{T}_1+1}^{T_1+\tilde{T}_1+\tilde{T}_2}$, such that $\sigma^{**}(x)$ is a BFE in periods $T_1 + \tilde{T}_1 + 1$ to $T_1 + \tilde{T}_1 + \tilde{T}_2$ with payoffs

$$\sum_{t=T_1+\tilde{T}_1+1}^{T_1+\tilde{T}_1+\tilde{T}_2} \hat{u}_i(\mathbf{a}_t) + \pi_i(h_{i-1}^{\tilde{T}_2}) + \left(\begin{array}{c} \sum_{t=T_1+1}^{T_1+\tilde{T}_1} \tilde{\pi}_i^{\text{cancel}}\left(a_{-i,t}(i-1), \omega_{-i,t}(i-1)\right) \\ + \sum_{\substack{t=T_1+1,\dots,T_1+\tilde{T}_1: \\ \text{players } -i \text{ communicate}}} \tilde{\pi}_i^{a^0}\left(a_{-i,t}(i-1), \omega_{-i,t}(i-1)\right) \end{array} \right), \tag{100}$$

and $\left|\pi_i(h_{i-1}^{\tilde{T}_2})\right| = O(\tilde{T}_2)$.

Finally, we define the reward function $\pi_i^{**}\left(x_{i-1}, h_{i-1}^{T_2}\right) = \tilde{\pi}_i^{**}\left(x_{i-1}, h_{i-1}^{T_2}\right) + C_{x_{i-1}}(T_1)$, where $C_{x_{i-1}}(T_1)$ is a constant to be determined and

$$\begin{aligned}\tilde{\pi}_i^{**}\left(x_{i-1}, h_{i-1}^{T_2}\right) &= \tilde{\pi}_i^{*}\left(\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}\right) + \sum_{t=T_1+1}^{T_1+\tilde{T}_1} \tilde{\pi}_i^{\text{cancel}}\left(a_{-i,t}(i-1), \omega_{-i,t}(i-1)\right) \\ &+ \sum_{\substack{t=T_1+1,\ldots,T_1+\tilde{T}_1: \\ \text{players } -i \text{ communicate}}} \tilde{\pi}_i^{a^0}\left(a_{-i,t}(i-1), \omega_{-i,t}(i-1)\right) + \pi_i(h_{i-1}^{\tilde{T}_2}).\end{aligned}$$

It remains to verify the premise of Lemma 14.

[Sequential Rationality]

We verify (45) for all $t = 1, \ldots, T_2$ by backward induction. For $t = T_1 + \tilde{T}_1 + 1$, ..., $T_1 + \tilde{T}_1 + \tilde{T}_2$, all payoffs except for (100) are sunk, so (45) holds by Lemma 8, viewing the last term in (100) as $w_i(h^{\tilde{T}_2})$.

By Claim 2 of Lemma 8, a player's payoff $\sum_{t=T_1+\tilde{T}_1+1}^{T_1+\tilde{T}_1+\tilde{T}_2} \hat{u}_i(\mathbf{a}_t) + \pi_i(h_{i-1}^{\tilde{T}_2})$ is independent of x and $h^{T_1+\tilde{T}_1}$. Hence, for $t = T_1+1, ..., T_1+\tilde{T}_1$, player i maximizes the conditional expectation of

$$\begin{aligned}&\sum_{t=T_1+1}^{T_1+\tilde{T}_1} \hat{u}_i(\mathbf{a}_t) + C_{x_{i-1}}(T_1) + \tilde{\pi}_i^{*}\left(\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}\right) \\ &+ \sum_{t=T_1+1}^{T_1+\tilde{T}_1} \tilde{\pi}_i^{\text{cancel}}\left(a_{-i,t}(i-1), \omega_{-i,t}(i-1)\right) + \sum_{\substack{t=T_1+1,\ldots,T_1+\tilde{T}_1: \\ \text{players } -i \text{ communicate}}} \tilde{\pi}_i^{a^0}\left(a_{-i,t}(i-1), \omega_{-i,t}(i-1)\right).\end{aligned}$$

Given (99) and (96), for sufficiently large T_1, the premise for secure communication with magnitude T^6 for player i is satisfied, for each $x \in \{G, B\}^N$. Moreover, (32) holds for sufficiently large T_0. Hence, Lemma 9 implies (45) for $t = T_1 + 1, ..., T_1 + \tilde{T}_1$.

Finally, by construction, the expected value of

$$\begin{aligned}&\sum_{t=T_1+1}^{T_1+\tilde{T}_1} \hat{u}_i(\mathbf{a}_t) + \sum_{t=T_1+1}^{T_1+\tilde{T}_1} \tilde{\pi}_i^{\text{cancel}}\left(a_{-i,t}(i-1), \omega_{-i,t}(i-1)\right) \\ &+ \sum_{\substack{t=T_1+1,\ldots,T_1+\tilde{T}_1: \\ \text{players } -i \text{ communicate}}} \tilde{\pi}_i^{a^0}\left(a_{-i,t}(i-1), \omega_{-i,t}(i-1)\right) + \pi_i(h_{i-1}^{\tilde{T}_2}) \qquad (101)\end{aligned}$$

does not depend on x or h^{T_1}. Since (97) implies that π_i^* and $\tilde{\pi}_i^*$ are equal in expectation given $\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}$ (assuming players follow σ^{**} in the last $\tilde{T}_1+\tilde{T}_2$ periods, as we have shown to be optimal), (50) implies (45) for $t = 1, ..., T_1$.

[Promise Keeping]

Let

$$\hat{v}_i\left(x_{i-1}\right) := \frac{1}{T_2}\mathbb{E}^{\sigma^*(x)}\left[\sum_{t=1}^{T_2}\hat{u}_i\left(\mathbf{a}_t\right) + \tilde{\pi}_i^{**}\left(x_{-i}, h_{i-1}^{T_2}\right)\right].$$

(As (101) is independent of x, (51) and (97) imply that this expectation does not depend on $x_{-(i-1)}$.) Since π_i^{cancel}, $\pi_i^{a^0}$, and $\pi_i(h_{i-1}^{\tilde{T}_2})$ are bounded and $(\tilde{T}_1 + \tilde{T}_2)/T_1 \to 0$, equation (51) implies

$$\lim_{T_1\to\infty}\hat{v}_i\left(x_{i-1}\right) = v_i\left(x_{i-1}\right) + \text{sign}(x_{i-1})3\varepsilon^*.$$

Hence, there exists $C_{x_{i-1}}\left(T_1\right)$ with

$$\text{sign}(x_{i-1})C_{x_{i-1}}\left(T_1\right) \geq 0 \tag{102}$$

such that, for sufficiently large T_0, (46) holds:

$$v_i\left(x_{i-1}\right) + \text{sign}(x_{i-1})4\varepsilon^* = \frac{1}{T_2}\mathbb{E}^{\sigma^{**}(x)}\left[\sum_{t=1}^{T_2}\hat{u}_i\left(\mathbf{a}_t\right) + \pi_i^{**}\left(x_{-i}, h_{i-1}^{T_2}\right)\right].$$

[Self-Generation]

Recall that (98) holds, and hence $\text{sign}\left(x_{i-1}\right)\tilde{\pi}_i^*\left(\tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}\right) \geq -\frac{7}{2}\varepsilon^* T_2\ \forall \tilde{x}_{-i}, h_{i-1}^{T^*}, \tilde{h}_{-i}^{\mathbb{T}''}$ (since $T_2 \geq T_1$). Moreover, the sum

$$\sum_{t=T_1+1}^{T_1+\tilde{T}_1}\tilde{\pi}_i^{\text{cancel}}\left(a_{-i,t}\left(i-1\right), \omega_{-i,t}\left(i-1\right)\right) + \sum_{\substack{t=T_1+1,\ldots,T_1+\tilde{T}_1: \\ \text{players } -i \text{ communicate}}}\tilde{\pi}_i^{a^0}\left(a_{-i,t}\left(i-1\right), \omega_{-i,t}\left(i-1\right)\right) + \pi_i(h_{i-1}^{\tilde{T}_2})$$

is $O\left(\left(\tilde{T}_1 + \tilde{T}_2\right)\right)$. Hence, (52) and (102) imply (47).

J.7 Proof of Lemma 16

Let $T_1 = T^* + \tilde{T}_1$, where

$$\tilde{T}_1 = 2NL\left(\mathcal{T}\left((T_0)^6, (T^*)^{\frac{1}{2}}\right) + (N-2)\left\lceil \log_2\left(|A|^2+1\right)\right\rceil (T^*)^{\frac{1}{2}}\right). \tag{103}$$

By (6), (16) holds and

$$\lim_{T_0\to\infty} \frac{\tilde{T}_1}{(T^*)^{\frac{1}{2}+\varepsilon}} = 0 \text{ for each } \varepsilon > 0.$$

We construct strategies $(\sigma_i^*(x_i))_{i,x_i}$ and reward functions $\left(\pi_i^{**}\left(x_{-i}, h_{-i}^{\mathbb{T}'}\right)\right)_{i,x_{-i},h_{-i}^{\mathbb{T}'}}$ in the T_1-period game that satisfy the premise of Lemma 15.

Construction of $\sigma_i^{**}(x_i)$

Play within each T^*-period block is given by $(\sigma_i^*(x_i))_i$. After each T^*-period block, players communicate as follows for $\tilde{T}_1$ periods:

Communication for periods $T^*+1, \ldots, T^*+\tilde{T}_1$:

- For $i=1$, player $i-1 \pmod N$ sends $t_{i-1}(1), ..., t_{i-1}(L)$ using the verified communication protocol with repetition $T = (T^*)^{\frac{1}{2}}$ and $\mathcal{I}_{\text{jam}} = -i$. This takes $2L\mathcal{T}\left((T_0)^6, (T^*)^{\frac{1}{2}}\right)$ periods. As a result, each player $n \in I$ infers a message $t_{i-1}(1)(n), ..., t_{i-1}(L)(n)$.

- Sequentially, each player $n \neq i, i-1$ sends $h_{n,t_{i-1}(l)(n)} = \left(a_{n,t_{i-1}(l)(n)}, \omega_{n,t_{i-1}(l)(n)}\right)_{l=1,\ldots,L}$ using the secure communication protocol with repetition $T = (T^*)^{\frac{1}{2}}$ and $\mathcal{I}_{\text{jam}} = \{i-1\}$. This takes $2(N-2)L\left\lceil \log_2\left(|A|^2+1\right)\right\rceil (T^*)^{\frac{1}{2}}$ periods. For each $n \neq i, i-1$, player $i-1$ infers a message $h_{n,t_{i-1}(l)(n)}(i-1)$.

- If (i)-(a) there exists a player $n \neq i$ with $\text{susp}(h_n)$ or (i)-(b) $\theta_i(h_{-i}) = E$ in the verified protocol, or if (ii) player $i-1$ plays JAM during a round where she receives a message via the secure protocol, let $s_{i-1} = 0$ ("communication fails"). Otherwise, $s_{i-1} = 1$ ("communication succeeds"). Note that s_{i-1} is a function of $h_{-i}^{\mathbb{T}'}$. Here, ζ_n is assumed to equal jam for each $n \neq i$ and reg for i, and so is omitted from θ_i.

- Repeat this procedure for $i = 2, \ldots, N$.

Construction of β^{**}

In periods where player n sends a message via the secure protocol, specify trembles as in Lemma 9. In periods where players use the verified protocol, any consistent belief system suffices. For periods $1, ..., T^*$, let $\beta^{**} = \beta^*$.

Construction of $\pi_i^{**}\left(x_{-i}, h_{-i}^{\mathbb{T}'}\right)$

If $s_{i-1} = 1$, we denote player $i-1$'s inference of player n's message during periods $T^*+1, \ldots, T^*+\tilde{T}_1$ by $h_n^{\mathbb{L}_{i-1}}(i-1)$. As in the proof of Lemma 15, define $\tilde{h}_{-i}^{\mathbb{L}_{i-1}} = h_{-i}^{\mathbb{L}_{i-1}}(i-1)$ if $s_{i-1} = 1$ and $\tilde{h}_{-i}^{\mathbb{L}_{i-1}} = 0$ otherwise. Note that (i) inequality (2) for secure communication and Claim 4 of Lemma 6 for verified communication imply, for sufficiently large T_0,

$$\min_{h_{-i}^{\mathbb{L}_{i-1}}} \Pr\left(s_{i-1} = 1 | h_{-i}^{\mathbb{L}_{i-1}}\right) \geq 1 - (T^*)^{-\frac{8}{2}+\frac{1}{2}}, \tag{104}$$

(ii) $s_{i-1} = 1$ implies $h_{-i}^{\mathbb{L}_{i-1}}(i-1) = h_{-i}^{\mathbb{L}_{i-1}}$, and (iii) π_i^* satisfies (57). Hence, in the notation of Lemma 13,

$$\begin{aligned} T &= (T^*)^{\frac{1}{2}}, \\ p_T(m_i) &= 1 - (T^*)^{-\frac{7}{2}} = 1 - T^{-7} \quad \forall m_i \in M(T), \\ F(T) &= \frac{1}{2}(T^*)^3 = \frac{1}{2}T^6, \\ c &= 2\varepsilon^*. \end{aligned} \tag{105}$$

Since $\lim_{T\to\infty}(1 - p_T(m_i)) \max\{F(T), cT\} = 0$, Lemma 13 implies that, for sufficiently large T_0, there exists $\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}\right)$ such that

$$\max_{x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}} \left|\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}\right)\right| \leq T^6, \tag{106}$$

$$\begin{aligned} &\mathbb{E}\left[\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}\right) | x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right] = \pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right), \\ &\text{sign}(x_{i-1})\, \tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}\right) \geq -\frac{5}{2}\varepsilon^* T^* \ \forall x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}, \text{ and} \\ &\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}\right) \text{ is minimized when } s_{i-1} = 0. \end{aligned} \tag{107}$$

We define the reward function $\pi_i^{**}\left(x_{-i}, h_{-i}^{\mathbb{T}''}\right) = \tilde{\pi}_i^{**}\left(x_{-i}, h_{-i}^{\mathbb{T}''}\right) + C_{x_{i-1}}(T^*)$, where $C_{x_{i-1}}(T^*)$

is a constant to be determined and

$$\begin{aligned}\tilde{\pi}_i^{**}\left(x_{-i}, h_{-i}^{\mathbb{T}''}\right) &= \tilde{\pi}_i^{*}\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}\right) \\ &+ \sum_{\substack{t=1,\dots,T_1 \\ t\notin\bigcup_{l=1}^{L}\mathbb{T}(\text{main}(l))}} \tilde{\pi}_i^{\text{cancel}}\left(a_{-i,t},\omega_{-i,t}\right) + \sum_{\substack{t=T^*+1,\dots,T_1: \\ \text{players } -i \text{ communicate}}} \tilde{\pi}_i^{a^0}\left(a_{-i,t},\omega_{-i,t}\right) \\ &+ \sum_{\substack{t=T^*+1,\dots,T_1: \\ t \text{ is in } i\text{-checking round in verified communication}}} \pi_i^{\bar{a}_{i,t}}\left(h_{-i}^{\mathbb{T}''}\right),\end{aligned}$$

where $\pi_i^{\bar{a}_{i,t}}$ is defined as in (34).

We now verify that the premise of Lemma 15 is satisfied. Other than sequential rationality (equation (50)), the verification is parallel to the proof of Lemma 15.

We verify (50) for $t = 1, \dots, T_1$ by backward induction. For $t' = T^* + 1$, ..., T_1, player i maximizes the conditional expectation of

$$\begin{aligned}&\sum_{t=t'}^{T_1} \hat{u}_i\left(\mathbf{a}_t\right) + C_{x_{i-1}} + \tilde{\pi}_i^{*}\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}\right) \\ &+ \sum_{t=t'}^{T_1} \tilde{\pi}_i^{\text{cancel}}\left(a_{-i,t},\omega_{-i,t}\right) + \sum_{\substack{t=t',\dots,T^*+\tilde{T}_1: \\ \text{players } -i \text{ communicate}}} \tilde{\pi}_i^{a^0}\left(a_{-i,t},\omega_{-i,t}\right) \\ &+ \sum_{\substack{t=T^*+1,\dots,T_1: \\ t \text{ is in } i\text{-checking round in verified communication}}} \pi_i^{\bar{a}_{i,t}}\left(h_{-i}^{\mathbb{T}''}\right).\end{aligned}$$

Given (106) and the fact that $\tilde{\pi}_i^{*}\left(x_{-i}, h_{-i}^{\mathbb{T}'}, \tilde{h}_{-i}^{\mathbb{L}_{i-1}}\right)$ is minimized at $s_{i-1} = 0$, for sufficiently large T^*, the premise for secure communication with magnitude $(T^*)^{\frac{7}{2}}$ for player i is satisfied for all $x \in \{G, B\}^N$. In addition, while players communicate to calculate $\tilde{\pi}_i^*$, as $v_i^E(T^*)^{\frac{6}{2}} = v_i^0(T^*)^{\frac{6}{2}} =$ [value of $\tilde{\pi}_i^*$ given $s_{i-1} = 0$], for sufficiently large T^*, the premise for verified communication with magnitude $(T^*)^{\frac{6}{2}}$ for player i is satisfied for all $x \in \{G, B\}^N$, and the continuation payoff of players $-i$ does not depend on the history during this communication. Moreover, (32) holds for sufficiently large T_0. Finally, by (104) and (106), (38) holds for sufficiently large T_0. Hence, Lemma 9 and the second part of Lemma 10 imply (50) for $t = T^* + 1, ..., T_1$. Finally, since π_i^* and $\tilde{\pi}_i^*$ are equal in expectation given $x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}$, (55) implies (50) for $t = 1, \dots, T^*$.

J.8 Proof of Lemma 17

As compared to Lemma 16, we have introduced (58) and replaced (54) with (59) (a more restrictive condition), (55) with (60) (a less restrictive condition) and (56) with (61) (again, a less restrictive condition). We first show that the third replacement is without loss of generality, and then show the same for the second.

Given (60), let

$$\hat{v}_i(x_{-i}) := \frac{1}{T^*}\mathbb{E}^{\sigma^*(x)}\left[\sum_{t\in\bigcup_{l=1}^{L}\mathbb{T}(\text{main}(l))}\hat{u}_i(\mathbf{a}_t) + \pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)\right].$$

Define

$$\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right) = \pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right) - \left(\hat{v}_i(x_{-i}) - (v_i(x_{i-1}) + 2\text{sign}(x_{i-1})\varepsilon^*)\right)T^*.$$

Note that changing the reward function from π_i^* to $\tilde{\pi}_i^*$ only subtracts a constant and thus does not affect sequential rationality. In addition, since

$$\text{sign}(x_{i-1})\left(\hat{v}_i(x_{-i}) - (v_i(x_{i-1}) + \text{sign}(x_{i-1})2\varepsilon^*)\right) \geq 0$$

by (61), (57) implies

$$\text{sign}(x_{i-1})\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right) \geq -2\varepsilon^* T^*.$$

Hence, self-generation also holds with reward function $\tilde{\pi}_i^*$.

Finally, since $(\hat{v}_i(x_{-i}) - (v_i(x_{i-1}) + 2\text{sign}(x_{i-1})\varepsilon^*))T^*$ is $O(T^*)$, (59) implies

$$\sup_{x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}}\left|\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)\right| < \frac{(T^*)^3}{2}.$$

Hence, (54) also holds with reward function $\tilde{\pi}_i^*$. Therefore, the premise of Lemma 16 holds.

We now show that it is also without loss to replace (55) with (60). To this end, let $\chi_n \in \{0,1\}$ be a function of $(x_n, h_n^{T^*})$, where $\chi_n = 1$ if and only if there exists $t = 1, ..., T^*$ such that $a_{n,t} \notin \text{supp}(\sigma_n^*(x_n)|_{h_n^{t-1}})$ (i.e., player n deviated from $\sigma_n^*(x_n)$ in the first T^* periods).

Since χ_n is binary, by the same proof as Lemma 16, we can assume that $\tilde{\pi}_i^*$ depends on $\chi_{-i} = (\chi_n)_{n \neq i}$—that is, players $-i$ "confess" any deviations.

We also assume that, until main phase l is over, player i believes that $t_{i-1}(l) = t$ with probability $(T_0)^{-6}$ for each $t \in \mathbb{T}(\text{main}(l))$. (This belief results whenever trembles in periods $t = 1, ..., T^*$ are independent of $(\mathbb{L}_i, h_i^{t-1})$, and thus is clearly consistent.)

Define

$$\pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) := \pi_i^{\text{cancel}}(a_{-i}, \omega_{-i}) + \text{sign}(x_{i-1}) \max_{\tilde{a}_{-i}, \tilde{\omega}_{-i}} \pi_i^{\text{cancel}}(\tilde{a}_{-i}, \tilde{\omega}_{-i}).$$

Note that

$$\mathbb{E}\left[\hat{u}_i(\mathbf{a}) + \pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) \,|a\right] = \text{sign}(x_{i-1}) \max_{\tilde{a}_{-i}, \tilde{\omega}_{-i}} \pi_i^{\text{cancel}}(\tilde{a}_{-i}, \tilde{\omega}_{-i}) \tag{108}$$

and

$$\text{sign}(x_{i-1})\pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) \geq 0.$$

Since $T^* \in \mathbb{T}'$, we can define

$$\begin{aligned} &\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}, \chi_{-i}\right) \\ : &= \begin{cases} \pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right) & \text{if } \chi_n = 0 \text{ for all } n \neq i, \\ \sum_{t \in \mathbb{T}'} \pi_i^{\text{cancel}}(x_{i-1}, a_{-i}, \omega_{-i}) \\ \quad + (T_0)^6 \sum_{l=1}^{L} \pi_i^{\text{cancel}}\left(x_{i-1}, a_{-i,t_{i-1}(l)}, \omega_{-i,t_{i-1}(l)}\right) & \text{if } \chi_n = 1 \text{ for some } n \neq i. \end{cases} \end{aligned}$$

Note that the $(T_0)^6$ term cancels the probability that $t_{i-1}(l) = t$ for each $t \in \mathbb{T}(\text{main}(l))$, so with this reward function player i is indifferent over all action profiles when $\chi_n = 1$ for some $n \neq i$.

Given reward function $\tilde{\pi}_i^*$, (55) and (57) hold. Moreover, given (59) for $\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$,

$$\sup_{x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}} \left|\tilde{\pi}_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)\right| < \frac{(T^*)^3}{2}.$$

Therefore, the premise of Lemma 16 holds.

Supplementary Appendix 1: Almost-Perfect Monitoring

This appendix proves Theorem 2, which extends the folk theorem to almost-perfect within-match monitoring.

Theorem 2 *Suppose public randomization is available. For all* $\mathbf{v} \in \text{int}(F^*)$*, there exist* $\bar{\delta} < 1$ *and* $\bar{\epsilon} > 0$ *such that* $\mathbf{v} \in E(\delta, q)$ *for all* $\delta > \bar{\delta}$ *and all* ϵ*-perfect within-match monitoring structures* q *with* $\epsilon \leq \bar{\epsilon}$*.*

The logic is similar to that for perfect monitoring. The main differences are as follows:

- There is a key difference in the communication protocols: The jamming players mix over all actions. This guarantees that players attribute unexpected observations to randomization by the jamming players rather than monitoring errors.

- We let the length of the block be random, which introduces a chance that the players have extra time to communicate at the end of the block.

- Other than the possibility of this "long communication phase" at the end of the block, the calendar time structure is the same as with perfect monitoring.

- The reward adjustment lemmas must be modified to account for possible monitoring errors and to accommodate the long communication phase.

- Although players attribute unexpected observations to randomization with probability converging to 1 as $\epsilon \to 0$, for any $\epsilon > 0$ they still assign positive probability to monitoring errors. We show how to use the long communication phase to preserve incentive-compatibility despite this new source of uncertainty.

A Communication Protocols and Modules

A.1 Modifying the Protocols

The basic communication protocol is the same as with perfect monitoring. As in (1), we have

$$\lim_{\epsilon \to 0} \Pr\left(m_i\left(j\right) = m_i\right) \geq 1 - \lceil \log_2 |M_i| \rceil \exp\left(-\bar{\varepsilon} T\right) \;\; \forall j \neq i. \tag{109}$$

The secure and verified protocols are the same as with perfect monitoring, except for jamming players. Jamming players now use the following strategy in each half-interval:

1. With probability $1 - 2T^{-9}$, play a^0 in every period (i.e., play REG).

2. With probability T^{-9}, play a^1 in every period (i.e., play JAM).

3. With probability T^{-9}, play $\alpha^{\text{mix}} = \frac{1}{|A|} \sum_{a \in A} a$ in every period, mixing independently across periods (call this "playing MIX").

As will be seen, a player who observes $\omega \neq a^0, a^1$ attributes this observation to a jamming player playing MIX. For sufficiently small ϵ, (2) holds for the secure protocol, and Claim 4 of Lemma 6 holds for the verified protocol.

The jamming coordination protocol stays the same. Recall that in each period, each player plays a^1 with probability T^{-2} and plays $\frac{1}{|A|-1} \sum_{a \neq a^1} a$ with probability $1 - T^{-2}$. Given a protocol history h_i, we define $\zeta_i(h_i) = \text{jam}$ if there exists $t \in \{1, 2\}$ with $\omega_{i,t} = a^1$. Let

$$p_i(h_i) = \Pr\left(\zeta_j(h_j) = \text{jam} \;\forall j \neq i | h_i\right).$$

For every protocol history h_i, the probability that all players in $I \setminus \{i, \mu_t(i)\}$ play a^1 for $t \in \{1, 2\}$ and $\mu_1(i) \neq \mu_2(i)$ is at least $\bar{\varepsilon} T^{-4(N-2)}$. Conditional on this event, if $\omega_{j,t} = a_{\mu(j),t}$ for all $j \neq i, t \in \{1, 2\}$ then $\zeta_j(h_j) = \text{jam} \;\forall j \neq i$. Hence,

$$p_i(h_i) \geq \bar{\varepsilon} \left(1 - \epsilon\right)^{2(N-1)} T^{-4(N-2)}. \tag{110}$$

A.2 Modifying the Reward Functions

We extend Lemma 8 for almost perfect monitoring:

Lemma 27 *For sufficiently small $\epsilon > 0$, for each $i \in I$, M_i, T, w, and $K > 2\bar{u}/\bar{\varepsilon}$ satisfying the premise for basic communication with magnitude K, there exists a family of functions $\left(\pi_n : H^{\mathbb{T}} \to \mathbb{R}\right)_{n\in I}$ such that the following hold:*

1. *With payoff functions (26), the basic communication protocol is a BFE for each δ.*

2. *For each $n \in I$ and $m_i \in M_i$, $\mathbb{E}\left[\sum_{t\in\mathbb{T}} \delta^{t-1}\hat{u}_n(\mathbf{a}_t) + \pi_n(h_{n-1})\right] = 0$.*

3. *For each $n \in I$ and $t \in \mathbb{T}$, we have*

$$\max_{h_{n-1},\tilde{h}_{n-1}} \left|\pi_n(h_{n-1}) - \pi_n(\tilde{h}_{n-1})\right| < \frac{\bar{u}+K}{2\bar{\varepsilon}}\left|\mathbb{T}\right|.$$

Proof. Let

$$\Delta_{a_i} := q\left(\omega_{i,t} = a^1 | a_i, a_{\mu(i)} = a^1\right) - q\left(\omega_{i,t} = a^1 | a_i, a_{\mu(i)} = a^0\right).$$

For $n \neq i$, define

$$\hat{\pi}_n(h_{n-1}) = \sum_{t\in\mathbb{T}} \frac{2K\mathbf{1}_{\{\omega_{n-1,t}=a^0\}}}{(1-2\epsilon)\,p_{n-1,n}} + \sum_{t\in\mathbb{T}^{1\text{st}}} \frac{\mathbf{1}_{\{\omega_{n-1,t}=a^1\}}\left(1-\delta^T\right)\left(\hat{u}_n(\mathbf{a}^0) - \hat{u}_n(\mathbf{a}^1)\right)}{\Delta_{a_{n-1,t}}p_{n-1,i}}.$$

(Compared to the definition with perfect monitoring, the first term is inflated to account for monitoring errors, and the denominator of the second term is the difference in the probability of $\omega_{n-1,t} = a^1$ between $a_i = a^1$ and $a_i = a^0$.) Similarly, define

$$\hat{\pi}_i(h_{i-1}) := \sum_{t\in\mathbb{T}} \frac{1}{p_{i-1,i}}\left(\delta^{t-1}\mathbf{1}_{\{\omega_{i-1,t}=a^1\}}\frac{\hat{u}_i(\mathbf{a}^1) - \hat{u}_i(\mathbf{a}^0)}{\Delta_{a_{i-1,t}}} + \mathbf{1}_{\{\omega_{i-1,t}\in\{a^0,a^1\}\}}\frac{2\bar{u}}{1-2\epsilon}\right).$$

The rest of the proof is the same as with perfect monitoring. ■

Since with almost-perfect monitoring (a_{-i}, ω_{-i}) statistically identifies (a_i, ω_i), we can also generalize $\pi_i^{\text{cancel}}(a_{-i}, \omega_{-i})$ and $\pi_i^{a^0}(a_{-i}, \omega_{-i})$ such that, for each $\mathbf{a} \in A^I$, we have

$$\begin{aligned} \mathbb{E}\left[\hat{u}_i(\mathbf{a}) + \pi_i^{\text{cancel}}(a_{-i}, \omega_{-i}) \,|\mathbf{a}\right] &= 0, \\ \mathbb{E}\left[\hat{u}_i(\mathbf{a}) + \pi_i^{a^0}(a_{-i}, \omega_{-i}) \,|\mathbf{a}\right] &= \begin{cases} 0 & a_i = a^0 \\ -1 & a_i \neq a^0 \end{cases}. \end{aligned} \tag{111}$$

Moreover, $\pi_i^{\text{cancel}}(a_{-i}, \omega_{-i})$ and $\pi_i^{a^0}(a_{-i}, \omega_{-i})$ converge to the corresponding rewards with perfect monitoring as $\epsilon \to 0$.

Given this modification, since a player who observes $\omega \neq a^0, a^1$ believes a jamming player played MIX, Lemma 9 holds as written. As we will see, it is not necessary to generalize Lemma 10 or Lemma 11.

B Block Structure, Strategies, Equilibrium Conditions

The calendar time structure is unchanged up to what was the end of the block with perfect monitoring (period T^{**} in the main proof). At that point, depending on public randomization, either the block ends or a final, long communication phase is added.

Up to period T^{**}, strategies are the same as with perfect monitoring, with two exceptions:

1. All protocol strategies are now the revised ones just defined.

2. In each main phase, if $\zeta_i(h_i) = \text{jam}$ then with probability $1 - (T_0)^{-9}$ player i follows her perfect monitoring strategy, and with probability $(T_0)^{-9}$ she plays $\frac{1}{|A|}\sum_{a \in A} a$ in every period (mixing independently across periods). (If instead $\zeta_i(h_i) \neq \text{jam}$, player i follows her perfect monitoring strategy for sure.)

In particular, exactly as with perfect monitoring, a receiver j sets $m_i(j) = 0$ (and hence $\text{susp}(h_j) = 1$) if she observes $\omega \neq a^0, a^1$. However, with perfect monitoring such an observation could only arise following a deviation, whereas now it can also arise as a result of a monitoring error or randomization by a jamming player.

It remains to describe the long communication phase. At the end of the final communication phase to cancel discounting, a random variable $z \in \{0\} \cup I$ is drawn by public randomization, with $\Pr(z=0) = 1-(T_0)^{-9}$ and $\Pr(z=i) = (T_0)^{-9}/N\ \forall i \in I$. If $z=0$, the block is over, as with perfect monitoring. If $z=i$, the following *long communication phase for player i* is played: each player $n \notin \{i-1, i\}$ sequentially broadcasts $\left(x_n, h_n^{T^*}\right)$, her entire history within the block up to the end of sub-block L (i.e., period T^*), using the basic communication protocol with repetition T_0. Since the cardinality of the set of a player's histories up to period T^* is $(2|A|)^{2T^*}$, the long communication phase takes $4(N-2)T^*T_0\lceil\log_2 2|A|\rceil$ periods.

Let T^{***} denote the length of the block, which is now a random variable due to the possible addition of a long communication phase. Note that, since $\Pr(z \neq 0) = (T_0)^{-9}$, we have

$$\lim_{T_0\to\infty} \frac{\mathbb{E}\left[T^{***}\right]}{T^*} = 0. \tag{112}$$

Finally, sufficient conditions for the existence of a block belief-free equilibrium with payoff $\mathbf{v} \in \text{int}(F^*)$ are almost the same as with perfect monitoring (i.e., conditions (10)–(13)). The only difference is that self-generation (condition (13)) must now hold for each history $h_{i-1}^{T^{***}}$ and each realization of T^{***}.

C Reward Adjustment Lemma

We next generalize the reward adjustment lemmas to allow more general errors in communication. Given parameters $T \in \mathbb{N}$ and $\epsilon \in \mathbb{R}_+$, let $M(T)$ be a finite set, let $F(T) \in \mathbb{R}_+$ be a constant, let $f_T : M(T) \to [-F(T), F(T)]$ be a function, let $P(T,\epsilon)$ be a non-negative, row-stochastic $|M(T)| \times |M(T)|$ matrix, and, for any matrix Z, let $r_i(Z) = \sum_{j\neq i}|Z_{i,j}|$ be the sum of absolute values of the off-diagonal elements of the i^{th} row of Z. (Applied to the rest of the proof, T is the length of a half-interval, $M(T)$ is a message set, f_T is a reward function bounded by $F(T)$, and $P(T,\epsilon)_{i,j}$ is the probability that message m_j is received when message m_i is sent.)

Lemma 28 *Suppose that*

$$\lim_{T\to\infty}\lim_{\epsilon\to 0}\max_{i\in\{1,\dots,|M(T)|\}} r_i(P(T,\epsilon)) = 0. \tag{113}$$

For all $\varepsilon > 0$, there exist $\bar{T} \in \mathbb{N}$ and a function $\bar{\epsilon} : \mathbb{N} \to \mathbb{R}_+$ such that, for all $T > \bar{T}$ and $\epsilon < \bar{\epsilon}(T)$, there exists a function $g_{T,\epsilon} : M(T) \to [-(1+\varepsilon) F(T), (1+\varepsilon) F(T)]$ such that $\max_{m \in M(T)} |f_T(m) - g_{T,\epsilon}(m)| < \varepsilon F(T)$ and $P(T,\epsilon)\mathbf{g}_{T,\epsilon} = \mathbf{f}_T$, where $\mathbf{g}_{T,\epsilon} = (g_{T,\epsilon}(m))_{m \in M(T)}$ and $\mathbf{f}_T = (f_T(m))_{m \in M(T)}$.

This lemma corresponds to Lemma 12. It can be straightforwardly extended to satisfy the additional conditions of Lemma 13.

Proof. By (113), the matrix $P(T,\epsilon)$ is strictly diagonally dominant for sufficiently large T and small ϵ (choosing first T and then ϵ). Hence, it is invertible (e.g., Horn and Johnson (2013), Theorem 6.1.10). Let $\mathbf{g}_{T,\epsilon} = P^{-1}(T,\epsilon)\mathbf{f}_T$.

It remains to show $\max_{m \in M(T)} |f_T(m) - g_{T,\epsilon}(m)| < \varepsilon F(T)$. For this, it suffices to show

$$\lim_{T\to\infty} \lim_{\epsilon\to 0} \min_{i\in\{1,\dots,|M(T)|\}} P^{-1}(T,\epsilon)_{i,i} = 1 \text{ and } \lim_{T\to\infty} \lim_{\epsilon\to 0} \max_{i\in\{1,\dots,|M(T)|\}} r_i\left(P^{-1}(T,\epsilon)\right) = 0. \quad (114)$$

Note that

$$\begin{aligned} \lim_{T\to\infty} \lim_{\epsilon\to 0} \max_i \sum_j \left|P^{-1}(T,\epsilon)_{i,j}\right| &\leq \lim_{T\to\infty} \lim_{\epsilon\to 0} \frac{1}{\min_i \left(P(T,\epsilon)_{i,i} - r_i(P(T,\epsilon))\right)} \\ &= 1, \end{aligned} \quad (115)$$

where the first line is the Ahlberg–Nilson–Varah bound (Varah, 1975), and the second line follows by (113). Since $P(T,\epsilon)P^{-1}(T,\epsilon) = I$, we have $\sum_j P(T,\epsilon)_{i,j} P^{-1}(T,\epsilon)_{j,i} = 1$, and therefore

$$P(T,\epsilon)_{i,i} P^{-1}(T,\epsilon)_{i,i} + r_i(P(T,\epsilon))\, r_i\left(P^{-1}(T,\epsilon)\right) \geq 1.$$

By (113) and (115), $r_i(P(T,\epsilon))\, r_i(P^{-1}(T,\epsilon)) \leq 1$ for sufficiently large T and small ϵ. Hence,

$$P^{-1}(T,\epsilon)_{i,i} \geq \frac{1 - r_i(P(T,\epsilon))\, r_i(P^{-1}(T,\epsilon))}{P(T,\epsilon)_{i,i}} \geq 1 - r_i(P(T,\epsilon))\, r_i\left(P^{-1}(T,\epsilon)\right). \quad (116)$$

Therefore,

$$\begin{aligned} r_i\left(P^{-1}(T,\epsilon)\right) &= \sum_j \left|P^{-1}(T,\epsilon)_{i,j}\right| - P^{-1}(T,\epsilon)_{i,i} \\ &\leq \sum_j \left|P^{-1}(T,\epsilon)_{i,j}\right| - 1 + r_i\left(P(T,\epsilon)\right) r_i\left(P^{-1}(T,\epsilon)\right). \end{aligned}$$

By (115), for every $\varepsilon > 0$, there exist $\bar{T} \in \mathbb{N}$ and a function $\bar{\epsilon} : \mathbb{N} \to \mathbb{R}_+$ such that, for all $T > \bar{T}$, $\epsilon < \bar{\epsilon}(T)$, and $i \in \{1, \ldots, |M(T)|\}$, we have $r_i(P^{-1}(T,\epsilon)) \leq r_i(P(T,\epsilon))\, r_i(P^{-1}(T,\epsilon)) + \varepsilon$. By (113), this implies $\lim_{T\to\infty} \lim_{\epsilon\to 0} \max_i r_i(P^{-1}(T,\epsilon)) = 0$. Given this, (116) implies $\lim_{T\to\infty} \lim_{\epsilon\to 0} \min_i P^{-1}(T,\epsilon)_{i,i} = 1$. ■

D Reduction Lemma

As with perfect monitoring, we simplify conditions (10)–(13). Fix T_0 (which determines T^*). Define $h_{-i}^{\mathbb{T}'}$ and $h_{-i}^{\mathbb{L}_{i-1}}$ as with perfect monitoring. We show that the following four conditions on strategies $\sigma_i^*(x_i)$ and reward functions $\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$, and $\pi_i^{**}\left(x_{-i}, h_{-i}^{T^*} | \epsilon\right)$ imply (10)–(13):[22]

1. [Reward Bound]

$$\max_{x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}} \left|\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)\right| < \frac{(T^*)^2}{2}, \tag{117}$$

$$\lim_{\epsilon\to 0} \max_{x_{-i}, h_{-i}^{T^*}} \frac{\left|\pi_i^{**}\left(x_{-i}, h_{-i}^{T^*} | \epsilon\right)\right|}{\Pr(z = i)} = 0 \tag{118}$$

2. [Incentive Compatibility] There exists $\bar{\epsilon} > 0$ such that, for all $\epsilon < \bar{\epsilon}$ and all $x \in \{G, B\}^N$,

[22] To clarify the role of imperfect monitoring, we make explicit the dependence of $\pi_i^{**}\left(x_{-i}, h_{-i}^{T^*+1} | \epsilon\right)$ on ϵ. As we will see, $\sigma_i^*(x_i)$ and $\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}}\right)$ do not depend on ϵ.

for each $h_i^{t-1} \in H_i(x_{-i})$,[23]

$$\sigma_i^*(x_i) \in \arg\max_{\sigma_i \in \Sigma_i} \mathbb{E}^{\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right)}\left[\sum_{t \in \bigcup_{l=1}^{L} \mathbb{T}(\text{main}(l))} \hat{u}_i(\mathbf{a}_t) + \pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right) + \pi_i^{**}\left(x_{-i}, h_{-i}^{T^*}|\epsilon\right)|h_i^{t-1}\right].$$

3. [Promise Keeping] There exists $\bar{\epsilon} > 0$ such that, for all $\epsilon < \bar{\epsilon}$ and all $x \in \{G, B\}^N$,

$$\left.\begin{array}{l} v_i(G) - 2\varepsilon^* \leq \\ v_i(B) + 2\varepsilon^* \geq \end{array}\right\} \frac{1}{T^*}\mathbb{E}^{\sigma^*(x)}\left[\sum_{t \in \bigcup_{l=1}^{L} \mathbb{T}(\text{main}(l))} \hat{u}_i(\mathbf{a}_t) + \pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right) + \pi_i^{**}\left(x_{-i}, h_{-i}^{T^*}|\epsilon\right)|h_i^{t-1}\right].$$

4. [Self-Generation] For all x_{-i}, $h_{-i}^{\mathbb{T}'}$, and $h_{-i}^{\mathbb{L}_{i-1}}$,

$$\text{sign}(x_{i-1})\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right) \geq -2\varepsilon^* T^*.$$

Lemma 29 *Suppose that, for all $\bar{T} > 0$, there exist $T_0 > \bar{T}$, strategies $(\sigma_i^*(x_i))_{i,x_i}$ in the $T^*(T_0)$-period repeated game and reward functions $\left(\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)\right)_{i,x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}}$ and $\left(\pi_i^{**}\left(x_{-i}, h_{-i}^{T^*}|\epsilon\right)\right)_{i,x_{-i},h_{-i}^{T^*}}$ such that [Reward Bound]–[Self-Generation] are satisfied. Then there exist $\bar{\delta} < 1$ and $\bar{\epsilon} > 0$ such that $\mathbf{v} \in E(\delta, q)$ for all $\delta > \bar{\delta}$ and all ϵ-perfect monitoring structures q with $\epsilon < \bar{\epsilon}$.*

Proof. We describe how to modify the proofs of Lemmas 14–17 to prove Lemma 29. In what follows, "for sufficiently large T_0 and small ϵ, ..." means "there exist $\bar{T} \in \mathbb{N}$ and a function $\bar{\epsilon} : \mathbb{N} \to \mathbb{R}_+$ such that, for all $T_0 > \bar{T}$ and $\epsilon < \bar{\epsilon}(T_0)$, ...".

Long Communication Phase for Player i

In the long communication phase for player i, each player $j \neq i$ sends her history $(x_j, h_j^{T^*})$ via basic communication. By (109),

$$\begin{aligned} &\lim_{T_0 \to \infty} \lim_{\epsilon \to 0} \Pr\left((x_j, h_j^{T^*})(n) = (x_j, h_j^{T^*}) \text{ for all } j, n \neq i\right) \\ = &\lim_{T_0 \to \infty} 1 - N^2 \log_2\left\lceil \left(2|A|^2\right)^{2T^*} \right\rceil \exp(-\bar{\varepsilon}^2 T_0) = 1. \end{aligned} \tag{119}$$

[23] Recall that we define $H_i(x_{-i})$ as the set of histories that happen with a positive probability given $(\sigma_i, \sigma_{-i}(x_{-i}))$ for some $\sigma_i \in \Sigma_i^T$. Note that the expectation is calculated based on the equilibrium strategy and Bayes' rule.

Hence, by Lemma 28 and (118), for sufficiently large T_0 and small ϵ, there exist reward functions $\left(\tilde{\pi}_i^{**}(x_{i-1}, h_{i-1}^{T^{***}}|\epsilon)\right)_{x_{i-1}, h_{i-1}^{T^{***}}}$ such that

$$\begin{aligned} \mathbb{E}\left[\mathbf{1}_{\{z=i\}}\tilde{\pi}_i^{**}(x_{i-1}, h_{i-1}^{T^{***}}|\epsilon)|h^{T^*}\right] &= \pi_i^{**}\left(x_{-i}, h_{-i}^{T^*}|\epsilon\right), \\ \max_{i, x_{i-1}, h_{i-1}^{T_2}} \frac{\left|\tilde{\pi}_i^{**}(x_{i-1}, h_{i-1}^{T^{***}}|\epsilon)\right|}{\Pr(z=i)} &< 1. \end{aligned}$$

Fix any $K > \max\{\frac{2\bar{u}}{\bar{\varepsilon}}, 1\}$. By Lemma 27, there exist reward functions $(\pi_n(h_{n-1}))_{n, h_{n-1}}$ such that, in the long communication phase, the sum of player n's instantaneous utilities and the reward $\pi_n(h_{n-1})$ is maximized by following the protocol and is independent of the messages $\left((x_j, h_j^{T^*})\right)_{j \neq i}$. Moreover, by (112), the addition of $\pi_n(h_{n-1})$ does not affect equilibrium payoffs when $T_0 \to \infty$.

Hence, by the same proof as for Lemma 14, for sufficiently large T_0 and small ϵ, it suffices to consider the repeated game until the end of final communication phase to cancel discounting, allowing reward functions of the form $\pi_i^{**}\left(x_{-i}, h_{-i}^{T^*}|\epsilon\right)$.

Final Communication Phase to Cancel Discounting

For sufficiently large T_0 and small ϵ, the conclusion of Lemma 14 holds. The proof is the same, replacing Lemma 8 with Lemma 27 and replacing Lemma 12 with Lemma 28,

Final Communication Phase to Share Information from Non-Main Phases

For sufficiently large T_0 and small ϵ, the conclusion of Lemma 15 holds. The proof is the same, replacing Lemma 8 with Lemma 27, recalling that Lemma 9 holds as written, and replacing Lemma 12 with Lemma 28 (in addition, the variable s_{i-1} constructed in the proof must now be set to 0 when player $i-1$ plays MIX, as well as when she plays JAM).

Final Communication Phase to Share Information from Main Phases

For sufficiently large T_0 and small ϵ, the conclusion of Lemma 16 holds. The proof is the same, replacing Lemma 12 with Lemma 28 (again setting $s_{i-1} = 0$ when player $i-1$ plays MIX). Note that, as is clear from the proof of Lemma 10, players other than the initial sender have strict incentives to follow the equilibrium strategy with perfect monitoring, which will be kept in almost perfect monitoring. In addition, the initial sender is indifferent between any messages, when players $-i$ communicate to construct $\tilde{\pi}_i^*$.

Finally, given Lemma 16, the conclusion of Lemma 17 holds by the same argument. ■

E Reward Functions and Equilibrium Verification

To complete the proof of Theorem 2, it remains to construct reward functions $\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$ and $\pi_i^{**}\left(x_{-i}, h_{-i}^{T^*}|\epsilon\right)$ that satisfy [Reward Bound]–[Self-Generation].

E.1 Construction of $\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$ and $\pi_i^{**}\left(x_{-i}, h_{-i}^{T^*}|\epsilon\right)$

With perfect monitoring, recall that $\pi_i^{\geq 3}\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$ denotes the reward function following the jamming coordination phase, and $\pi_i\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right) = \pi_i^{\text{indiff}}\left(x_{-i}, h_{-i}^{\text{jam}}\right) + \pi_i^{\geq 3}\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$ denotes the total reward function. Define $\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$ by modifying $\pi_i^{\geq 3}\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$ by setting $\theta_i(h_{-i}^{<(l,\text{main})}) = E$ if any player other than i plays MIX prior to main phase l (or if any of the conditions for $\theta_i(h_{-i}^{<(l,\text{main})}) = E$ from the perfect monitoring proof are satisfied).

To construct $\pi_i^{**}\left(x_{-i}, h_{-i}^{T^*}|\epsilon\right)$, we first formalize the observation that players attribute monitoring errors to randomization by jamming players. Fix $i \in I$. By (110), with positive probability $\zeta_j = \text{jam}\ \forall j \neq i$ at the end of the jamming coordination phase. Conditional on this event, every opposing action sequence for rest of the block arises with positive probability (independent of ϵ and δ). Hence, for any history $h_i^{T^*}$, we have

$$\lim_{\epsilon \to 0} \Pr\left(\left\{\omega_{j,t} = a_{\mu(j),t}\ \forall j \in I, t \in \mathbb{T}^*\right\} |h_i^{T^*}\right) = 1. \tag{120}$$

Given (120), conditional on each x_{-i}, if player i observes a history that would not be in $H_i(x_{-i})$ with perfect monitoring, she believes that a jamming player played MIX—and hence $\theta_i(h_{-i}^{<(l,\text{main})}) = E$, so any continuation strategy is optimal—with probability converging to 1 as $\epsilon \to 0$. As the equilibrium strategy is optimal under perfect monitoring with reward function $\pi_i\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$, this implies the equilibrium strategy is almost-optimal under almost-perfect monitoring with reward function $\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$: for each $x_{-i} \in \{G, B\}^{N-1}$, history h_i^{t-1}, and $a_i \in \text{supp}\left(\sigma_i^*\left(x_i\right)\left(\mathbb{L}_i, h_i^{t-1}\right)\right)$, there exists $\varepsilon_i^{x_{-i}}(h_i^{t-1}, a_i) \geq 0$ such that[24]

$$\begin{aligned} & \max_{\sigma_i} \mathbb{E}^{\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right)}\left[\sum_{\tau=1}^{T^*} \hat{u}_i\left(\mathbf{a}_\tau\right)+\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right) | h_i^{t-1}, a_{i,t}=a_i\right] \\ & -\max_{\tilde{a}_i, \sigma_i} \mathbb{E}^{\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right)}\left[\sum_{\tau=1}^{T^*} \hat{u}_i\left(\mathbf{a}_\tau\right)+\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right) | h_i^{t-1}, a_{i,t}=\tilde{a}_i\right] \\ \geq \quad & -\varepsilon_i^{x_{-i}}(h_i^{t-1}, a_i) \end{aligned}$$

and

$$\lim_{\epsilon \to 0} \max_{x_{-i}, h_i^{t-1}, a_i \in \text{supp}\left(\sigma_i^*(x_i)(\mathbb{L}_i, h_i^{t-1})\right)} \varepsilon_i^{x_{-i}}(h_i^{t-1}, a_i) = 0.$$

Since (a_{-i}, ω_{-i}) statistically identifies (a_i, ω_i) for sufficiently small $\epsilon > 0$, by a standard application of the theorem of the alternative there exists a reward function $\left(\pi_{i,t}^{\text{monitor}}\left(x_{-i}, h_{-i}^t\right)\right)_{x_{-i}, h_{-i}^t}$ such that, for $t = T^*$, for each $x_{-i} \in \{G, B\}^{N-1}$, history h_i^{t-1}, and $a_i \in \text{supp}\left(\sigma_i^*\left(x_i\right)\left(\mathbb{L}_i, h_i^{t-1}\right)\right)$, we have

$$\begin{aligned} & \max_{\sigma_i} \mathbb{E}^{\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right)}\left[\sum_{\tau=1}^{T^*} \hat{u}_i\left(\mathbf{a}_\tau\right)+\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)+\pi_{i,t}^{\text{monitor}}\left(x_{-i}, h_{-i}^t\right) | h_i^{t-1}, a_{i,t}=a_i\right] \\ & -\max_{\tilde{a}_i, \sigma_i} \mathbb{E}^{\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right)}\left[\sum_{\tau=1}^{T^*} \hat{u}_i\left(\mathbf{a}_\tau\right)+\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)+\pi_{i,t}^{\text{monitor}}\left(x_{-i}, h_{-i}^t\right) | h_i^{t-1}, a_{i,t}=\tilde{a}_i\right] \\ \geq \quad & 0 \end{aligned}$$

and

$$\lim_{\epsilon \to 0} \max_{x_{-i}, h_{-i}^{t-1}} \left|\pi_{i,t}^{\text{monitor}}\left(x_{-i}, h_{-i}^t\right)\right| = 0.$$

Similarly, by backward induction, there exist reward functions $\left(\pi_{i,t}^{\text{monitor}}\left(x_{-i}, h_{-i}^t\right)\right)_{t \in \{1, \ldots, T^*\}, x_{-i}, h_{-i}^t}$ such that, for each $t \in \{1, \ldots, T^*\}$, $x_{-i} \in \{G, B\}^{N-1}$, history h_i^{t-1}, and $a_i \in \text{supp}\left(\sigma_i^*\left(x_i\right)\left(\mathbb{L}_i, h_i^{t-1}\right)\right)$,

[24] Recall that player i's belief about $(\mathbb{L}_{-i}, h_{-i})$ does not depend on $\mathbb{L}_i$ conditional on (x_{-i}, h_i^{t-1}).

we have

$$\max_{\sigma_i} \mathbb{E}^{\left(\sigma_i,\sigma^*_{-i}(x_{-i})\right)}\left[\sum_{\tau=1}^{T^*}\hat{u}_i\left(\mathbf{a}_\tau\right)+\pi_i^*\left(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}\right)+\sum_{\tau\geq t}\pi_{i,\tau}^{\text{monitor}}\left(x_{-i},h_{-i}^{\tau}\right)|h_i^{t-1},a_{i,t}=a_i\right]$$
$$-\max_{\tilde{a}_i,\sigma_i} \mathbb{E}^{\left(\sigma_i,\sigma^*_{-i}(x_{-i})\right)}\left[\sum_{\tau=1}^{T^*}\hat{u}_i\left(\mathbf{a}_\tau\right)+\pi_i^*\left(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}\right)+\sum_{\tau\geq t}\pi_{i,\tau}^{\text{monitor}}\left(x_{-i},h_{-i}^{\tau}\right)|h_i^{t-1},a_{i,t}=\tilde{a}_i\right]$$
$$\geq \quad 0$$

and

$$\lim_{\epsilon\to 0}\max_{x_{-i},h_{-i}^t}\sum_{t=1}^{T^*}\left|\pi_{i,t}^{\text{monitor}}\left(x_{-i},h_{-i}^{t}\right)\right|=0. \tag{121}$$

Now define

$$\pi_i^{**}\left(x_{-i},h_{-i}^{T^*}|\epsilon\right)=\sum_{t=1}^{T^*}\pi_{i,t}^{\text{monitor}}\left(x_{-i},h_{-i}^{t}\right). \tag{122}$$

Since adding $\pi_{i,\tau}^{\text{monitor}}\left(x_{-i},h_{-i}^{\tau}\right)$ does not affect incentives after period $\tau+1$ (i.e., it is sunk), [Incentive Compatibility] with reward function $\pi_i^*\left(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}\right)+\pi_i^{**}\left(x_{-i},h_{-i}^{T^*}|\epsilon\right)$ follows: For each $h_i^{t-1}\in H_i(x_{-i})$,

$$\sigma_i^*\left(x_i\right)\left(\mathbb{L}_i,h_i^{t-1}\right)\in\operatorname*{argmax}_{\sigma_i}\mathbb{E}^{\left(\sigma_i,\sigma^*_{-i}(x_{-i})\right)}\left[\sum_{\tau=1}^{T^*}\hat{u}_i\left(\mathbf{a}_\tau\right)+\pi_i^*\left(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}\right)+\pi_i^{**}\left(x_{-i},h_{-i}^{T^*}|\epsilon\right)|h_i^{t-1}\right]. \tag{123}$$

E.2 Verification of [Reward Bound]–[Self-Generation]

We have already verified [Incentive Compatibility]. Equation (118) follows immediately from (121) and (122).

For equation (117) and [Self-Generation], note that the perfect-monitoring reward function $\pi_i\left(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}\right)$ satisfies these conditions for large enough T_0, and $\pi_i\left(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}\right)$ and $\pi_i^*\left(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}\right)$ differ only if some player other than i played MIX, in which case $\pi_i^*\left(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}\right)$ equals the perfect-monitoring reward with $\theta_i(h_{-i}^{<(l,\text{main})})=E$. Hence, $\pi_i^*\left(x_{-i},h_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}\right)$ also satisfies (117) and [Self-Generation] for large enough T_0.

For [Promise Keeping], note that, conditional on the event that no player plays JAM or MIX, the ex ante distribution of play paths under almost-perfect monitoring converges to that under perfect monitoring as $\epsilon \to 0$. Since $\pi_i\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$ and $\pi_i^*\left(x_{-i}, h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$ coincide conditional on this event and $\pi_i^{**}\left(x_{-i}, h_{-i}^{T^*+1}|\epsilon\right) \to 0$ as $\epsilon \to 0$, equilibrium payoffs conditional on this event also coincide as $\epsilon \to 0$. Moreover, since each player plays JAM or MIX with probability at most $(T_0)^{-9}$ in each sub-interval (or main phase), the probability that no player plays JAM or MIX converges to 1 as $\epsilon \to 0$. Finally, given T_0, payoffs are continuous in ϵ, since all reward functions except $\pi_i^*\left(x_{-i}, h_{-i}^{T^*+1}|\epsilon\right)$ are bounded and independent of ϵ, and (118) holds. Hence, [Promise Keeping] holds for sufficiently large T_0 and small ϵ.

Supplementary Appendix 2: Non-Pairwise Matching

This appendix proves Theorem 3, which extends the folk theorem to non-pairwise matching. Recall the definitions of "symmetric stage games" and "random player-roles" from the text.

Theorem 3 With non-pairwise matching and either symmetric stage games or random player-roles, for all $\mathbf{v} \in \text{int}(F^*)$, there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.

F Identifiability

As with pairwise matching, if players $-i$ successfully aggregate their information (including the sizes of their groups and, if applicable, their roles), they can perfectly identify player i's action and observation.

Lemma 30 *There exists a function $\varphi : A_{-i} \times \Omega_{-i} \to A_i \times \Omega_i$ such that, if $(a_i, \omega_i)_{i \in I}$ is feasible, then $\varphi(a_{-i}, \omega_{-i}) = (a_i, \omega_i)$.*

Proof. Let $\omega_i(a) = |\{j \in \mu(i) : a_j = a\}|$ be the number of player i's opponents who take action a. We must show how to identify four objects on the basis of (a_{-i}, ω_{-i}): (i) the size of i's group, $n^*(i)$, (ii) i's action, a_i, (iii) for symmetric games, the number of i's opponents taking each action, $(\omega_i(a))_{a \in A[n^*(i)]}$, and (iv) for asymmetric games (i.e., random player-roles), i's role, $i^*(i)$, and the actions taken by i's opponents, $\left(a_{i^*(j)}\right)_{j \in \mu(i)}$.

The argument for (i) and (ii) is the same for symmetric and asymmetric games. For (i), for each $n^* \in \{2, ..., M\}$, let $-i(n^*)$ denote the set of players $j \neq i$ with $n^*(j) = n^*$. Note that, if $|-i(n^*)|/n^* \in \mathbb{N}$ for each $n^* \in \{2, ..., M\}$, then $n^*(i) = 1$. Otherwise, there is a unique number $n^* \in \{2, ..., M\}$ with $|-i(n^*)|/n^* \notin \mathbb{N}$. In this case, $n^*(i) = n^*$.

For the rest of the proof, let $n^* = n^*(i)$.

For (ii), consider those players $j \in I$ with $n^*(j) = n^*$ who only observe *only* the same action as they themselves play: that is, players j such that $n^*(j) = n^*$ and $\omega_j(a_j) = n^* - 1$. Clearly, the number of such players (including i herself) must be a multiple of n^*. Hence, if there exists $\bar{a} \in A$ such that the number of players in $-i(n^*)$ with $a_j = \bar{a}$ and $\omega_j(\bar{a}) = n^* - 1$ is not a multiple of n^*, then $a_i = \bar{a}$.

Otherwise, there exists $a \neq a_i$ such that $\omega_i(a) > 0$. Hence, $\omega_i(a_i) < n^* - 1$. Since an action of a player in a size-n^* group is observed by $n^* - 1$ players,

$$(n^* - 1) \times |\{j \in I : n^*(j) = n^* \cap a_j = a\}| = \sum_{j \in I, n^*(j) = n^*} \omega_j(a) \quad \forall a \in A[n^*].$$

Therefore, since $\omega_i(a_i) < n^* - 1$, for $\bar{a} = a_i$ we have

$$(n^* - 1) \times |\{j \in I \setminus \{i\} : n^*(j) = n^* \cap a_j = \bar{a}\}| < \sum_{j \in I \setminus \{i\}, n^*(j) = n^*} \omega_j(\bar{a}),$$

and for each $\bar{a} \neq a_i$ we have

$$(n^* - 1) \times |\{j \in I \setminus \{i\} : n^*(j) = n^* \cap a_j = \bar{a}\}| \geq \sum_{j \in I \setminus \{i\}, n^*(j) = n^*} \omega_j(\bar{a}).$$

Thus, a_i is perfectly identified from (a_{-i}, ω_{-i}).

For (iii), given that a_i (and hence the complete action profile $\mathbf{a}$) is identified from (a_{-i}, ω_{-i}), for each $a \in A[n^*]$, we have

$$\omega_i(a) = (n^* - 1) \times |\{j \in I : n^*(j) = n^* \cap a_j = a\}| - \sum_{j \in I \setminus \{i\}, n^*(j) = n^*} \omega_j(a).$$

For (iv), identifying player i's role on the basis of her opponents' roles is trivial: $i^*(i)$ is the unique role i^* such that $|\{j \neq i : i^*(j) = i^*\}| / n^* \notin \mathbb{N}$. Moreover, the observation of each player $j \in -i(n^*)$ defines a mapping $g_j : \{1, ..., n^*\} \to A_1[n^*] \times \cdots \times A_{n^*}[n^*]$, where $g_j(n)$ is the action of the player in role n in j's match. Note that, for any such mapping g,

$$|\{j \in I : n^*(j) = n^* \cap g_j = g\}|$$

is a multiple of n^*. Hence, g_i is perfectly identified from $(g_j)_{j \neq i}$. In turn, $a_{i^*} = g_i(i^*)$ for each role i^* in the match including i. ■

G Communication Protocols

Recalling that $|A_{i^*}[n^*]| \geq 2$ for each i^* and n^*, fix two distinct actions in each action set $A_{i^*}[n^*]$, and with slight abuse of notation label them a^0 and a^1. In the specification of each communication protocol, we replace $\omega_{j,t} = a^0$ with $\omega_{j,t}(a^0) = n^*(j) - 1$, replace $\omega_{j,t} = a^1$ with $\omega_{j,t}(a^1) \geq 1$, and replace $\omega_{j,t} \notin \{a^0, a^1\}$ with $\omega_{j,t}(a) > 1$ for some $a \notin \{a^0, a^1\}$.

G.1 Basic, Secure, and Verified Protocols

Given this modification, the basic, secure, and verified protocols are the same as with pairwise matching. Inequality (1) holds as written. We now prove the counterpart of Lemma 3:

Lemma 31 *There exists $\tilde{\varepsilon} > 0$ such that, for any player $j \neq i$ with $I_{\text{jam}} \setminus \{j\} \neq \emptyset$ and any sequence of observations $(\omega_{j,t})_{t=1}^{2T\lceil \log_2 |M_i| \rceil}$ that arises with positive probability when players $-j$ follow the secure protocol, one of the following two conditions holds:*

1. *For all $(a_{j,t})_{t=1}^{2T\lceil \log_2 |M_i| \rceil}$, we have*

$$\Pr\left(ALLREG |\, (a_{j,t}, \omega_{j,t})_{t=1}^{2T\lceil \log_2 |M_i| \rceil}\right) \leq T^9 \exp\left(-\frac{1}{4}\tilde{\varepsilon}T\right).$$

2. *The following two conditions hold:*

   (a) *For all $(a_{j,t})_{t=1}^{2T\lceil \log_2 |M_i| \rceil}$, we have*

$$\begin{aligned} &\Pr\left(m_i(j') \in \{m_i, 0\} \ \ \forall j' \notin \{i, j\} \,|\, (a_{j,t}, \omega_{j,t})_{t=1}^{2T\lceil \log_2 |M_i| \rceil}, ALLREG\right) \\ \geq\ & 1 - N \lceil \log_2 |M_i| \rceil \exp\left(-\tilde{\varepsilon}^4 T\right). \end{aligned}$$

   (b) *If $a_{j,t} = a^0$ for all $t \in \{1, \ldots, 2T \lceil \log_2 |M_i| \rceil\}$, then*

$$\begin{aligned} &\Pr\left(m_i(j') = m_i \ \forall j' \notin \{i, j\} \,|\, (a_{j,t}, \omega_{j,t})_{t=1}^{2T\lceil \log_2 |M_i| \rceil}, ALLREG\right) \\ \geq\ & 1 - N \lceil \log_2 |M_i| \rceil \exp\left(-\tilde{\varepsilon}^4 T\right). \end{aligned}$$

The only difference between this lemma and Lemma 3 is it may be necessary to take $\tilde{\varepsilon}$ smaller than $\bar{\varepsilon}$, the lower bound for the matching probability. Given Lemma 31, Lemmas 5–6 hold as written with $\tilde{\varepsilon}$ in place of $\bar{\varepsilon}$.

Proof. Given $\bar{\varepsilon} > 0$, take $\tilde{\varepsilon} > 0$ sufficiently small such that $\tilde{\varepsilon}(1 - 4\log\bar{\varepsilon}) \leq \bar{\varepsilon}(1 - \tilde{\varepsilon})$ and $\tilde{\varepsilon}^4 \leq \bar{\varepsilon}(\bar{\varepsilon}^3 - \tilde{\varepsilon})$. (This also implies $\tilde{\varepsilon} \leq \bar{\varepsilon}$.)

If $\omega_{j,t}(a^1) \geq 2$ for some t, or if $\omega_{j,t}(a^1) \geq 1$ in both half-intervals of some interval, then $ALLREG$ cannot have occurred, so Condition 1 holds. Assuming such observations do not arise, we have two cases:

1. Suppose that there is some half-interval $\mathbb{S}$ where i plays a^1 in which $n_t^*(j) \geq 3$ in at most $\tilde{\varepsilon}T$ periods and $\omega_{j,t}(a^1) = 1$ in at least $(1 - \bar{\varepsilon}^3)T$ periods. Then

$$\frac{\Pr\left((a_{j,t}, \omega_{j,t})_{t=1}^{2T\lceil\log_2|M_i|\rceil} |_{\mathbb{S}}|j'JAMS\right)}{\Pr\left((a_{j,t}, \omega_{j,t})_{t=1}^{2T\lceil\log_2|M_i|\rceil} |_{\mathbb{S}}|ALLREG\right)} \geq \bar{\varepsilon}^{\tilde{\varepsilon}T}\left(\frac{p_{i,j} + p_{j',j}}{p_{i,j}}\right)^{\gamma}\left(\frac{1 - p_{i,j} - p_{j',j}}{1 - p_{i,j}}\right)^{(1-\tilde{\varepsilon})T-\gamma},$$

where γ is the number of periods in which $n_t^*(j) = 2$ and $\omega_{j,t}(a^1) = 1$, and $p_{i,j}$ is the conditional probability that players i and j match given $n_t^*(j) = 2$. By the same argument as in the proof of Lemma 3, we have

$$\begin{aligned}\frac{\Pr\left((a_{j,t}, \omega_{j,t})_{t=1}^{2T\lceil\log_2|M_i|\rceil} |_{\mathbb{S}}|j'JAMS\right)}{\Pr\left((a_{j,t}, \omega_{j,t})_{t=1}^{2T\lceil\log_2|M_i|\rceil} |_{\mathbb{S}}|ALLREG\right)} &\geq \bar{\varepsilon}^{\tilde{\varepsilon}T}\exp\left(\frac{1}{4}\bar{\varepsilon}(1 - \tilde{\varepsilon})T\right) \\ &= \exp\left(\tilde{\varepsilon}T\log(\bar{\varepsilon}) + \frac{1}{4}\bar{\varepsilon}(1 - \tilde{\varepsilon})T\right) \\ &\geq \exp\left(\frac{1}{4}\tilde{\varepsilon}T\right).\end{aligned}$$

Arguing again as in the proof of Lemma 3, this implies Condition 1.

2. Suppose that, for every half-interval where i plays a^1, either $n_t^*(j) \geq 3$ in at least $\tilde{\varepsilon}T$ periods or $\omega_{j,t}(a^1) = 1$ in at most $(1 - \bar{\varepsilon}^3)T$ periods. Fix a half-interval.

   If $n_t^*(j) \geq 3$ in at least $\tilde{\varepsilon}T$ periods in the half-interval, consider two sub-cases:

   (a) If $\omega_{j,t}(a^1) = 1$ in at least $\tilde{\varepsilon}^2T$ periods then, given $ALLREG$, player j believes that she matched with i and another player in at least $\tilde{\varepsilon}^2T$ periods. For any

$n \in I \setminus \{i,j\}$, the probability that player n is not a part of the group including i and j is $\exp\left(-\bar{\varepsilon}\tilde{\varepsilon}^2 T\right)$ by the same calculation as Lemma 2. Since $\tilde{\varepsilon} \leq \bar{\varepsilon}$, player j therefore believes that each player in $I \setminus \{i,j\}$ matched with player i at least once with probability no less than $1 - N\exp\left(-\tilde{\varepsilon}^3 T\right)$.

(b) If $\omega_{j,t}(a^1) = 1$ in at most $\tilde{\varepsilon}^2 T$ periods, player j believes that each other player matched with player i in at least one of the remaining $\left(\tilde{\varepsilon} - \tilde{\varepsilon}^2\right) T$ periods with probability no less than $1 - N\exp\left(-\bar{\varepsilon}\left(\tilde{\varepsilon} - \tilde{\varepsilon}^2\right) T\right) \geq 1 - N\exp\left(-\tilde{\varepsilon}^4 T\right)$.

If instead $n_t^*(j) \geq 3$ in at most $\tilde{\varepsilon}T$ periods and $\omega_{j,t}(a^1) = 1$ in at most $(1 - \bar{\varepsilon}^3) T$ periods, then player j believes that each other player matched with player i at least once in the remaining $(\bar{\varepsilon}^3 - \tilde{\varepsilon}) T$ periods at least once with probability no less than $1 - N\exp\left(-\bar{\varepsilon}\left(\bar{\varepsilon}^3 - \tilde{\varepsilon}\right) T\right) \geq 1 - N\exp\left(-\tilde{\varepsilon}^4 T\right)$.

Therefore, in every case, player j believes that each other player matched with player i at least once with probability no less than $1 - N\exp\left(-\tilde{\varepsilon}^4 T\right)$. Since this holds for every half-interval where i plays a^1, Condition 2 holds as in the proof of Lemma 3.

■

G.2 Jamming Coordination Protocol

We must modify the jamming coordination protocol. We want it to be the case that, if $\zeta_i(h_i) = \text{reg}$, then with positive probability $\zeta_j(h_j) = \text{jam}\ \forall j \neq i$. Suppose we specified that, as in the pairwise matching construction, $\zeta_j(h_j) = \text{jam}$ if and only if player j observes a^1 during the jamming coordination protocol. The problem is that, if player i is always matches in groups of size N or $N-1$, plays $a \neq a^1$, and observes $\omega_i(a^1) = 0$, then she realizes that $\zeta_j(h_j) = \text{reg}$ with probability 1. To address this issue, we repeat the jamming coordination protocol T times, and if this problematic event occurs too often we set $\zeta_i(h_i) = \text{jam}$ even if i has not observed a^1.

Jamming Coordination Protocol with Parameter T:

- In each period $t \in \{1, ..., T\}$, each player i plays a^1 with probability T^{-2} and play each $a \neq a^1$ with probability $\frac{1-T^{-2}}{|A|-1}$.

Given a protocol history h_i, we define $\zeta_i(h_i) = \text{jam}$ if (i) $\omega_{i,t}(a^1) \geq 1$ for some $t \in \{1, \ldots, T\}$ or (ii) $|\{t : n_t^*(i) \geq N-1\}| \geq T - N$.

As with pairwise matching, let

$$P_i(h_i) = \Pr\left(\zeta_j(h_j) = \text{jam} \;\; \forall j \neq i | h_i\right).$$

For every protocol history h_i, either $\zeta_i(h_i) = \text{jam}$ or (25) holds. To see why, note that $\zeta_i(h_i) = \text{reg}$ implies $|\{t : n_t^*(i) \leq N-2\}| \geq N$. Hence, we may denote by $\mathbb{T}_i^{\text{jam}}$ a set of N periods with $|\{t : n_t^*(i) \leq N-2\}|$. Recalling that each partition of the population into groups of size $\leq M$ occurs with probability at least $\bar{\varepsilon}$, the following event has probability at least $\bar{\varepsilon}^N T^{-4(N-2)}$: (i) $n_t^*(j) \geq 2$ for all $j \in I \setminus \{i, \mu_t(i)\}$ and $t \in \mathbb{T}_i^{\text{jam}}$, (ii) $\bigcup_{t \in \mathbb{T}_i^{\text{jam}}} (I \setminus \mu_t(i)) = I \setminus \{i\}$, and (iii) $a_{j,t} = a^1$ for all $j \in I \setminus \{i \cup \mu_t(i)\}$ and $t \in \mathbb{T}_i^{\text{jam}}$. Conditional on this event, $\zeta_j(h_j) = \text{jam}$ $\forall j \neq i$ with probability 1. Hence, (25) holds.

H Communication Modules

For the basic communication module, Lemma 8 holds as written. The only required modification to the proof is that the definitions of $\hat{\pi}_n(h_{n-1})$ and $\hat{\pi}_i(h_{i-1})$ must be changed to

$$\begin{aligned}
\hat{\pi}_n(h_{n-1}) &= \sum_{t \in \mathbb{T}} \frac{2K \mathbf{1}_{\{n^*(n-1)=2 \cap \omega_{n-1,t}(a^0)=1\}}}{\Pr(\mu(n-1) = \{n\})} \\
&\quad + \sum_{t \in \mathbb{T}^{1\text{st}}} \frac{\mathbf{1}_{\{n^*(n-1)=2 \cap \omega_{n-1,t}(a^1)=1\}} \left(1 - \delta^{-T}\right) \left(\hat{u}_n(\mathbf{a}^0) - \hat{u}_n(\mathbf{a}^1)\right)}{\Pr(\mu(n-1) = \{n\})}, \\
\hat{\pi}_i(h_{i-1}) &= \sum_{t \in \mathbb{T}} \frac{1}{\Pr(\mu(i-1) = \{i\})} \begin{pmatrix} \delta^{t-1} \mathbf{1}_{\{n^*(i-1)=2 \cap \omega_{i-1,t}(a^1)=1\}} \left(\hat{u}_i(\mathbf{a}^1) - \hat{u}_i(\mathbf{a}^0)\right) \\ + \mathbf{1}_{\{n^*(i-1)=2 \cap \max\{\omega_{i-1,t}(a^0), \omega_{i-1,t}(a^1)\}=1\}} 2\bar{u} \end{pmatrix},
\end{aligned}$$

thus conditioning on the event that player $n-1$ (or $i-1$) matches in a 2-player group.

The analysis of the secure and verified modules is unchanged. In particular, Lemmas 9 and 10 hold as written.

For the jamming coordination module, Lemma 11 holds as written, except that now $\pi_{i,t}^{\text{indiff}}$ must be defined for $t \in \{1, \ldots, T\}$ rather than $\{1, 2\}$ (a similar change is required in (39)).

The required modifications to the proof are (i) H_i^0 must be defined as the set of protocol histories such that $\omega_{i,t}(a^1) = 0$ for all t and $|\{t : n_t^*(i) \geq N-1\}| < T - N$, and (ii) the construction of $\pi_{i,t}^{\text{indiff}}$ by backwards induction must begin at period $t = T$ rather than $t = 2$.

I Block Structure and Equilibrium Conditions

In the symmetric stage game case, replace the target actions $\mathbf{a}(x)$ with a target mapping from $n^*(i)$ to $A[n^*(i)]$. In the asymmetric (random player-roles) case, replace $\mathbf{a}(x)$ with a mapping from $(n^*(i), i^*(i))$ to $A_{i^*(i)}[n^*(i)]$. Given this modification, the definition of $(v_i(x_{i-1}))_{i\in I, x_{i-1}\in\{G,B\}} \in \mathbb{R}^{2N}$ and the target mappings—which we denote $(\bar{\mathbf{a}}(x))_{x\in\{G,B\}^N}$—is unchanged. Note that, since F^* is defined with the same punishment strategy for all players, the punishment strategy $\bar{\alpha}^{\min}$ is defined independently of the index of the player being punished.

The calendar time structure of a block is also unchanged, except that the cardinality of the set of signals Ω is larger. Since it is still finite (and independent of T_0), (16)–(18) still hold.

Both the reward adjustment lemmas (Lemmas 12 and 13) and the equilibrium conditions and subsequent reduction lemmas (Lemmas 4 and 14–17) hold as written.

J Strategies, Reward Functions, and Verification

Equilibrium strategies are unchanged, except for the following modifications:

1. The jamming coordination protocol is modified as described above.

2. In main phases, if $i \notin I^D\left(h^{<(l,\text{main})}\right)$, player i follows the target mapping $\bar{\mathbf{a}}(x(i))$; if $i \in I^D\left(h^{<(l,\text{main})}\right)$, player i follows the mapping $\bar{\alpha}_i^{\min}$ in every period.

By Lemma 30, $\left(h_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$ perfectly identifies $\left(h_i^{\mathbb{T}'}, h_i^{\mathbb{L}_{i-1}}\right)$. Hence, we may construct reward functions exactly as in the pairwise matching case.

Finally, in verifying the equilibrium conditions (in particular, in the proof of Lemma 20), we must check that $\Pr\left(\mathcal{I}^{\text{jam}} = \emptyset\right)$ still converges to 0 as $T_0 \to \infty$. This is straightforward:

First, since the jamming coordination phase takes T_0 periods, the probability that any player plays a^1 is at most $NT_0/\left(T_0\right)^2 \to 0$. Second, recalling that each partition into groups of size $\leq M$ occurs with positive probability, the probability that $|\{t : n_t^*(i) \geq N-1\}| \geq T_0 - N$ for any player i converges to 0 by the law of large numbers.

Supplementary Appendix 3: Non-I.I.D. Matching

This appendix proves Theorem 4, which extends the folk theorem to non-i.i.d. matching. Recall the definition of F^* and the required full-rank assumptions on the matrices P and Q.

Theorem 4 With non-i.i.d. matching, for all $\mathbf{v} \in \text{int}(F^*)$, there exists $\bar{\delta} < 1$ such that $\mathbf{v} \in E(\delta)$ for all $\delta > \bar{\delta}$.

We actually use a stronger solution concept, which we call *ex post sequential equilibrium (XSE)*. In this appendix, an XSE is a sequential equilibrium in which sequential rationality is satisfied conditional on each possible realization of the initial match μ_1: that is, for each player i, history h_i^{t-1}, and initial match realization μ_1, the continuation strategy $\sigma_i|_{h_i^{t-1}}$ maximizes $\mathbb{E}^{\left(\sigma_i,\sigma^*_{-i}\right)}[\sum_{\tau=t}^{\infty} \hat{u}_i(\mathbf{a}_\tau)|\mu_1, h_i^{t-1}]$. Note that, while sequential rationality is imposed ex post with respect to μ_1, the requirement that an XSE is a sequential equilibrium implies that players' beliefs must be limits of conditional probabilities resulting from completely mixed strategy profiles in which players condition only on their own information $h_i^{t-1} = (a_{i,\tau}, \omega_{i,\tau})_{\tau=1}^{t=1}$ and not on μ_1. Let $E(\mu_1, \delta)$ be the set of ex post sequential equilibrium payoffs with initial match μ_1. We prove the stronger result that Theorem 4 holds with $E(\delta)$ replaced by $E(\mu_1, \delta)$, for any μ_1.

We must also show that $F := \lim_{\delta \to 1} F(\mu_1, \delta)$ is well-defined, independent of μ_1, where $F(\mu_1, \delta)$ is the feasible payoff set with initial match μ_1 and discount factor δ. Recall also that $F^\kappa(\mu_1, \delta)$ is the set of payoffs attainable by the infinite repetition of a strategy in the κ-period finitely repeated game with initial match μ_1 and discount factor δ.

Proposition 1 *For all matches μ_1, μ_1', we have $\lim_{\delta\to 1} F(\mu_1, \delta) = \lim_{\kappa\to\infty} \lim_{\delta\to 1} F^\kappa(\mu_1', \delta)$. In particular, $F = \lim_{\delta\to 1} F(\mu_1, \delta) = \lim_{\kappa\to\infty} \lim_{\delta\to 1} F^\kappa(\mu_1, \delta)$ is well-defined, independent of μ_1.*

We postpone the proof of this proposition until the end of the appendix.

The proof of Theorem 4 follows the same logic as the i.i.d. case. It is structured as follows: Section K summarizes relevant properties of Markov chains and describes how players can identify a match. Section L presents the communication protocols. All communication protocols need some modification. In particular, we add periods of communication to "cancel out" the effect of the initial match. Section M contains the analysis of the communication modules. Except for the basic communication module, this is very similar to the i.i.d. case. Section N describes the block belief-free structure, where continuation payoffs are independent of the initial match. In Sections O and P, we modify the reward adjustment and reduction lemmas. Finally, we construct the reward function and verify the equilibrium conditions in Section Q.

K Facts about Markov Chains

We start with two lemmas showing that the effect of the initial match vanishes exponentially with t. Let $\mathbf{a}_{1:\infty} \in \left(A^N\right)^{\mathbb{N}}$ denote an infinite sequence of action profiles, where $\mathbf{a}_t$ is played in period t. Let $\Pr\left(\mu_t|\mu_1, \mathbf{a}_{1:t}\right)$ denote the probability that the period t match is μ_t given initial match μ_1 and action sequence $\mathbf{a}_{1:t}$.

Lemma 32 *For any* $\mathbf{a}_{1:\infty} \in \left(A^N\right)^{\mathbb{N}}$ *and* $t \in \mathbb{N}$, *we have* $\max_{\mu_1,\tilde{\mu}_1} \sum_{\mu_t} \left|\Pr\left(\mu_t|\mu_1, \mathbf{a}_{1:t}\right) - \Pr\left(\mu_t|\tilde{\mu}_1, \mathbf{a}_{1:t}\right)\right| \leq (1-\bar{\varepsilon})^t$.

Proof. Fixing $\mathbf{a}_{1:\infty}$, $\Pr_t\left(\mu_t|\mu_{t-1}\right) = \Pr\left(\mu_t|a_{t-1}, \mu_{t-1}\right)$ is a (time-dependent) Markov process with $\Pr_t\left(\mu_t|\mu_{t-1}\right) \geq \bar{\varepsilon}$ for each μ_t, μ_{t-1}. The result now follows from Theorem 4.9 of Seneta (2006). ■

Similarly, let $\Pr^\sigma\left(\mu_t|\mu_1\right)$ denote the probability that the period t match is μ_t given initial match μ_1 and strategy profile σ.

Lemma 33 *Fix* $\kappa \in \mathbb{N}$ *and a strategy* σ^κ *in the* κ*-period finitely repeated game. Let* σ *denote the strategy in the infinitely repeated game that results from repeating* σ^κ*. Then we have*

$$\max_{\mu_1,\tilde{\mu}_1} \sum\nolimits_{\mu_t} \left|\Pr^\sigma\left(\mu_t|\mu_1\right) - \Pr^\sigma\left(\mu_t|\tilde{\mu}_1\right)\right| \leq (1-\bar{\varepsilon})^{\left\lfloor \frac{t}{\kappa} \right\rfloor}. \tag{124}$$

In particular, there is a unique stationary distribution of μ *under strategy* σ*.*

Proof. The proof of (124) is the same as Lemma 32, viewing the repeated game as a repetition of κ-period blocks. The existence of a stationary distribution follows from a fixed point theorem, and uniqueness follows from (124). ■

Lemma 1 holds as it stands, so identification of actions is the same as with i.i.d. matching. We now consider identification of the initial match μ_1. Consider the following finite sequence of action profiles:

1. Each player takes a_0 in every period $t = 1, ..., T$.

2. For the next $N(N-1)/2$ periods, players play $\bar{\sigma}^*$ to identify μ_{T+1}: in each period $t = T+1, ..., T+N(N-1)/2$, players in the pair C_{t-T} (the $(t-T)^{\text{th}}$ element of C) take a^1 and others take a^0.

Suppose players $-i$ communicate the history profile h_{-i} in each of these $T+N(N-1)/2$ periods. By Lemma 1, h_{-i} perfectly identifies h_i (and thus y_C). Since P has full row rank and Q has full rank, by the Sylvester rank inequality, y_C statistically identifies μ_1: that is, $Q^T P$ is invertible. Since $\left(Q^T P\right)^{-1} = P^{-1}\left(Q^{-1}\right)^T$, there exists $\bar{M}$ such that, for each T, we have

$$\left\| \left(Q^T P\right)^{-1} \right\| \leq \bar{M}^T. \tag{125}$$

Note also that Lemma 2 holds as it stands.

L Communication Protocols

Given $S \in \mathbb{N}$, let $\Pr\left(\cdot|\mu_1, a_i^{1:S}, a_{-i}^0\right)$ be the distribution of μ_{S+1} when the initial match is μ_1, player i takes $a_i^{1:S} \in A^S$ from period 1 to S, and players $-i$ always take a_{-i}^0. Let $p_{i,j}\left(\mu_1, a_i^{1:S}, a_{-i}^0\right) = \sum_\mu \Pr\left(\mu|\mu_1, a_i^{1:S}, a_{-i}^0\right) p_{i,j}(\mu)$ be the probability that i and j match in period $S+1$ given μ_1. Fix $S_0 \in \mathbb{N}$ sufficiently large such that $S_0 \geq N(N-1)/2$,

$$(1-\bar{\varepsilon})^{S_0 N(N-1)} < \frac{1}{2}, \tag{126}$$

and for each $i, j, j', \bar{\mu}_1, \mu_1, a_i^{1:S_0}, a_{-i}^0$,

$$\left|\log p_{i,j}\left(\bar{\mu}_1, a_i^{1:S_0}, a_{-i}^0\right) - \log p_{i,j}\left(\mu_1, a_i^{1:S_0}, a_{-i}^0\right)\right| \leq \frac{1}{2}\bar{\varepsilon} \tag{127}$$

and

$$\begin{cases} \left|\log \frac{p_{i,j}\left(\bar{\mu}_1, a_i^{1:S_0}, a_{-i}^0\right)+p_{i,j'}\left(\bar{\mu}_1, a_i^{1:S_0}, a_{-i}^0\right)}{p_{i,j}\left(\bar{\mu}_1, a_i^{1:S_0}, a_{-i}^0\right)} - \log \frac{p_{i,j}\left(\mu_1, a_i^{1:S_0}, a_{-i}^0\right)+p_{i,j'}\left(\mu_1, a_i^{1:S_0}, a_{-i}^0\right)}{p_{i,j}\left(\mu_1, a_i^{1:S_0}, a_{-i}^0\right)}\right| \leq \frac{1}{16}\bar{\varepsilon} \\ \left|\log \frac{1-p_{i,j}\left(\bar{\mu}_1, a_i^{1:S_0}, a_{-i}^0\right)-p_{i,j'}\left(\bar{\mu}_1, a_i^{1:S_0}, a_{-i}^0\right)}{1-p_{i,j}\left(\bar{\mu}_1, a_i^{1:S_0}, a_{-i}^0\right)} - \log \frac{1-p_{i,j}\left(\mu_1, a_i^{1:S_0}, a_{-i}^0\right)-p_{i,j'}\left(\mu_1, a_i^{1:S_0}, a_{-i}^0\right)}{1-p_{i,j}\left(\mu_1, a_i^{1:S_0}, a_{-i}^0\right)}\right| \leq \frac{1}{16}\bar{\varepsilon} \end{cases}. \tag{128}$$

The existence of such S_0 follows from Lemma 32.

L.1 Basic Communication Protocol

Given $S_1 \geq S_0$, we replace each period of the basic communication protocol with the following set of $\tilde{T} = (S_0 + 1 + N(N-1)/2 + 2(S_1+1)S_1N^2(N-1))$ periods, which we refer to as a **unit of basic communication**:[25]

For the first S_0 periods, players take a^0; and then in the $(S_0+1)^{\text{th}}$ period, players take actions as in the basic communication protocol with i.i.d. matching. Intuitively, players take a^0 for S_0 periods so that, by Lemma 32, the effect of μ_1 on the distribution of μ_{S_0+1} is bounded by $(1-\bar{\varepsilon})^{S_0}$ (i.e., "the effect of the initial match is cancelled").

In the next $N(N-1)/2$ periods, players play $\bar{\sigma}^*$ to statistically identify μ_{S_0+1}. Let $\tau(t)$ denote the $(S_0+t+1)^{\text{th}}$ period counting from the beginning of the unit (i.e., the t^{th} period within these $N(N-1)/2$ periods), with $t = 1, ..., N(N-1)/2$. In period $\tau(t)$, players in pair C_t take a^1 and others take a^0. Let $\mathbb{C}$ denote this set of $N(N-1)/2$ periods.

For each player to identify μ_{S_0+1}, players communicate their histories for the set of periods $\mathbb{C}$. Specifically, we view the remaining $2(S_1+1)S_1N^2(N-1)$ periods as $N(N-1)/2$ repetitions of $4N(S_1+1)S_1$-period cycles. In the t^{th} cycle, players in C_t communicate $\omega_{\tau(t)}$ as follows:

The t^{th} cycle is viewed as $2N$ repetitions of $2(S_1+1)S_1$-period "subunit." Intuitively, each player $n \in I$ sends $\omega_{n,\tau(t)}$ twice, once in each subunit. (We explain why players send messages twice when presenting the corresponding module.)

In particular, for each $t = 1, ..., N(N-1)/2$, the t^{th} cycle proceeds as follows:[26]

1. For each $n = 1, ..., N$, player n sends $\omega_{n,\tau(t)}$ as follows:

[25] As in the main text, "play action a in period t" is to read as unconditional on a player's past actions and observations.

[26] To make the following the strategy sequentially rational, we will subsequently slightly modify the off-path behavior. See in Section M.1.

(a) Repeat the following $(S_1 + 1)$-period sequence S_1 times: Players take $\mathbf{a}^0$ for S_1 periods. In the $(S_1 + 1)^{\text{th}}$ period, player n takes a^1 if $\omega_{n,\tau(t)} = a^1$, and takes a^0 otherwise. Other players take a^0.

(b) Then, repeat the following $(S_1 + 1)$-period sequence S_1 times: Players take $\mathbf{a}^0$ for S_1 periods. In the $(S_1 + 1)^{\text{th}}$ period, player n takes a^0 if $\omega_{n,\tau(t)} = a^1$, and takes a^0 otherwise. Other players take a^0.

Call this set of $2(S_1 + 1)S_1$ periods the "$(t, n, 1)$-subunit." Let $\mathbb{S}_1(t, n, 1)$ be the set of $(S_1 + 1)^{\text{th}}$ periods in which player n sends the message. Let $\mathbb{S}_1(t, n, 1, 1)$ denote the first S_1 periods of $\mathbb{S}_1(t, n, 1)$, and let $\mathbb{S}_1(t, n, 1, 2)$ denote the second S_1 periods.

2. For each $n = 1, ..., N$, repeat the $(t, n, 1)$-subunit. Call the set of $2(S_1 + 1)S_1$ periods in which the $(t, n, 1)$ subunit is repeated the "$(t, n, 2)$-subunit." Let $\mathbb{S}_1(t, n, 2)$ be the set of $(S_1 + 1)^{\text{th}}$ periods in which player n sends the message. Define $\mathbb{S}_1(t, n, 2, 1)$ and $\mathbb{S}_1(t, n, 2, 2)$ analogously.

(The reader may wonder why we need to both cancel the effect of the initial match and identify the match. The former makes the effect of the initial match on player i's continuation payoff exponentially small. However, since we need to make player i exactly indifferent between $\sigma_i(G)$ and $\sigma_i(B)$, we still need to identify the match.)

Inference of the message m_i: Inferences of m_i are as in the i.i.d. matching case, except that players use only their observation in the $(S_0 + 1)^{\text{th}}$ period of each unit. By (127) and Lemma 2, we have

$$\Pr\left(m_i\left(j\right) = m_i \ \forall j\right) \geq 1 - N \left\lceil \log_2 \left|M_i\right| \right\rceil \exp\left(-\bar{\varepsilon}T\right). \tag{129}$$

Moreover, if player i follows the basic communication protocol to send message m_i, every player $j \neq i$ plays a^0 in all the $(S_0 + 1)^{\text{th}}$ periods of each unit, and some player $j \neq i$ infers a message $m_i\left(j\right) \in M_i$, then $m_i\left(j\right) = m_i$.

Inference of player n's message $\omega_{n,\tau(t)}$: Inferences of $\omega_{n,\tau(t)}$ are determined by the second subunit in each cycle: each player $n' \neq n$ infers $\omega_{n,\tau(t)}(n') = a^1$ (respectively, $\omega_{n,\tau(t)}(n') = a^0$)

if she observes a^1 at least once in $\mathbb{S}_1(t,n,2,1)$ (resp., $\mathbb{S}_1(t,n,2,2)$) and observes only a^0 in $\mathbb{S}_1(t,n,2,2)$ (resp., $\mathbb{S}_1(t,n,2,1)$). Let $\omega_{n,\tau(t)}(n') = 0$ if player n' always observes a^0 in $\mathbb{S}_1(t,n,2)$. Let $\omega_{n,\tau(t)}(n) = \omega_{n,\tau(t)}$.

Identification of μ_{S_0+1} *in each unit*: By $S_1 \geq S_0$, (126), and Lemma 2, we have

$$\Pr\left(\omega_{n,\tau(t)}(n') = \omega_{n,\tau(t)}\ \forall n, n' \in I, t \in \mathbb{C} | h^{\mathbb{C}}, \mu_{S_0+1}\right) \geq 1 - N^3(N-1)\exp\left(-\bar{\varepsilon} S_1\right),$$

where μ_{S_0+1} is the realized match in the $(S_0+1)^{\text{th}}$ period of the unit. Let $\mathbb{S}_1$ be the set of all $(S_1+1)^{\text{th}}$ periods of a subunit, and let $h_n^{\mathbb{S}_1}$ be player n's history in $\mathbb{S}_1$. Then, $h_n^{\mathbb{S}_1}$ statistically identifies $h^{\mathbb{C}}$ for each μ_{S_0+1} for sufficiently large S_1 and $h^{\mathbb{C}}$ statistically identifies μ_{S_0+1}. Hence, the

$$\prod_{k=0}^{N/2-1} (N-2k-1) \times H_n^{\mathbb{S}_1} \tag{130}$$

matrix $P_n^{\mathbb{S}_1}$ with $\left(\mu_{S_0+1}, h_n^{\mathbb{S}_1}\right)$ element $\Pr\left(h_n^{\mathbb{S}_1} | \mu_{S_0+1}\right)$ has full row rank.

L.2 Secure Communication Protocol

We change the secure communication protocol as follows:

1. For the first $N(N-1)/2$ periods, players identify μ_1: In each period $t = 1, ..., N(N-1)/2$, players play $\bar{\sigma}^*$. Again, let $\mathbb{C}$ denote this set of $N(N-1)/2$ periods.

2. In the next $N^2(N-1)S_0/2$ periods, players send messages to learn (a_t, ω_t) in $\mathbb{C}$.[27] We view this set of $N^2(N-1)S_0/2$ periods as $N(N-1)/2$ repetitions of NS_0-period cycles. The t^{th} cycle consists of N repetitions of S_0-period subunits. In the j^{th} subunit of the t^{th} cycle (the "(t,j)-subunit"), player j takes $a_j = \omega_{j,t}$ and other players take a^0.

3. From period $N(N-1)(1+NS_0)/2+1$ on, players communicate via the (i.i.d. matching) secure communication protocol, where each period of the protocol is replaced with

[27] This is similar to but simpler than the basic communication protocol since, as mentioned after (30), we can rely on future information aggregation via the basic communication protocol after the secure communication protocol is played.

the following set of (S_0+1) periods, which we refer to as a **unit of secure communication**: for the first S_0 periods, players (including the jamming players) take $\mathbf{a}^0$; in the $(S_0+1)^{\text{th}}$ period, players take actions as in the secure communication protocol.

The secure communication protocol with message set M_i now takes $N(N-1)(1+NS_0)/2$ $+2(S_0+1)T\lceil\log_2|M_i|\rceil$ periods. Note that, for sufficiently large T, the length is approximately the same as in the i.i.d. case:[28] for any $\varepsilon>0$,

$$\lim_{T\to\infty}\frac{N(N-1)(1+NS_0)/2+2(S_0+1)T\lceil\log_2|M_i|\rceil}{\lceil\log_2|M_i|\rceil T^{1+\varepsilon}}=0. \tag{131}$$

The history from period $N(N-1)/2+1$ to $N(N-1)/2+N^2(N-1)S_0/2$ statistically identifies the history in the first $N(N-1)/2$ periods (the rank condition follows from (126) and Horn and Johnson (2013), Theorem 6.1.10). Since the history in the first $N(N-1)/2$ periods statistically identifies μ_1, in total the history from period $N(N-1)/2+1$ to $N(N-1)/2+N^2(N-1)S_0/2$ statistically identifies μ_1: the

$$\prod_{k=0}^{N/2-1}(N-2k-1)\times(H_n)^{N^2(N-1)S_0/2} \tag{132}$$

matrix with $\left(\mu_1,(a_{n,t},\omega_{n,t})_{t=N(N-1)/2+1}^{N(N-1)/2+N^2(N-1)S_0/2}\right)$ element $\Pr\left((a_{n,t},\omega_{n,t})_{t=N(N-1)/2+1}^{N(N-1)/2+N^2(N-1)S_0/2}|\mu_1\right)$ has full row rank.

The inference of the messages is the same as in the i.i.d. case, except that players use only their observation in the $(S_0+1)^{\text{th}}$ period of each unit. Since the consecutive S_0 periods of $\mathbf{a}^0$ make the match in the $(S_0+1)^{\text{th}}$ period of each unit almost i.i.d., we have:

Lemma 34 *Let $T'=N(N-1)(1+NS_0)/2+2(S_0+1)T\lceil\log_2|M_i|\rceil$ be the length of the protocol. For any player $j\neq i$ with $I_{\text{jam}}\setminus\{j\}\neq\emptyset$ and any sequence of observations $(\omega_{j,t})_{t=1}^{T'}$ that arises with positive probability when players $-j$ follow the secure protocol, one of the following two conditions holds: For $\hat{\varepsilon}:=\frac{\bar{\varepsilon}}{2}$,*

[28] As will be seen, although $|M_i|$ can depend on T, $|M_i|$ is bounded by a polynominal function of T. Hence, (131) holds.

1. *For all* $(a_{j,t})_{t=1}^{T'}$, *we have*

$$\Pr\left(ALLREG|\left(a_{j,t},\omega_{j,t}\right)_{t=1}^{T'}\right)\leq T^{9}\exp\left(-\frac{1}{4}\hat{\varepsilon}T\right).$$

2. *The following two conditions hold:*

   (a) *For all* $(a_{j,t})_{t=1}^{T'}$, *we have*

$$\begin{aligned}&\Pr\left(m_i\left(j'\right)\in\{m_i,0\}\ \ \forall j'\notin\{i,j\}\,|\left(a_{j,t},\omega_{j,t}\right)_{t=1}^{T'},ALLREG\right)\\ \geq\ &1-N\left\lceil\log_2|M_i|\right\rceil\exp\left(-\hat{\varepsilon}^4T\right).\end{aligned}$$

   (b) *If player* j *follows the protocol, then*

$$\begin{aligned}&\Pr\left(m_i\left(j'\right)=m_i\ \forall j'\notin\{i,j\}\,|\left(a_{j,t},\omega_{j,t}\right)_{t=1}^{T'},ALLREG\right)\\ \geq\ &1-N\left\lceil\log_2|M_i|\right\rceil\exp\left(-\hat{\varepsilon}^4T\right).\end{aligned}$$

Proof. For each unit of communication, each pair (n,n') is matched in the $(S_0+1)^{\text{th}}$ period with probability $p_{n',n}\left(\mu_1,a_i^{1:S_0},a_{-i}^0\right)$, where μ_1 is the realized match in the first period of the unit. For each $\mu_1,\tilde{\mu}_1$, the bounds (127) and (128) hold for the difference between $p_{n',n}\left(\mu_1,a_i^{1:S_0},a_{-i}^0\right)$ and $p_{n,n'}\left(\tilde{\mu}_1,a_i^{1:S_0},a_{-i}^0\right)$. Hence, in the proof of Lemma 3, we can replace $\log\frac{p_{i,j}+p_{j',j}}{p_{i,j}}$ and $\log\frac{1-p_{i,j}-p_{j',j}}{1-p_{i,j}}$ with

$$\begin{aligned}\min_{\mu_1,a_j^{1:S_0}}\log\frac{p_{i,j}\left(\mu_1,a_j^{1:S_0},a_{-j}^0\right)+p_{j',j}\left(\mu_1,a_j^{1:S_0},a_{-j}^0\right)}{p_{i,j}\left(\mu_1,a_j^{1:S_0},a_{-j}^0\right)}&\geq\frac{1}{4}\bar{\varepsilon}\text{ and}\\ \min_{\mu_1,a_j^{1:S_0}}\log\frac{1-p_{i,j}\left(\mu_1,a_j^{1:S_0},a_{-j}^0\right)-p_{j',j}\left(\mu_1,a_j^{1:S_0},a_{-j}^0\right)}{1-p_{i,j}\left(\mu_1,a_j^{1:S_0},a_{-j}^0\right)}&\geq-\frac{1-\frac{1}{2}\bar{\varepsilon}}{\frac{1}{2}\bar{\varepsilon}}\end{aligned}$$

The rest of the proof is unchanged. ■

L.3 Verified Communication Protocol

We change the verified communication protocol as follows: the message round stays the same, and we replace each period of checking rounds with the following $N(N-1)/2+$

$\lceil \log T^{K+2} \rceil + 1$-period unit. As will be seen in the module, we will take K such that the premise of the verified communication module is satisfied with magnitude T^K.

In the τ^{th} unit of the checking round,

1. For the first $N(N-1)/2$ periods, players identify μ by playing $\bar{\sigma}^*$.

2. For the next $\lceil \log T^{K+2} \rceil$ periods, players play $\mathbf{a}^0$ to cancel the effect of the initial match.

3. In the last period, players communicate as in the τ^{th} period in the i.i.d. case.

In total, the verified communication protocol now takes

$$\tilde{\tau}(|M_i|, K, T) = 2\lceil \log_2 |M_i| \rceil + 2N \left\lceil \log_2 A^{4\lceil \log_2 |M_i| \rceil} \right\rceil \left(N(N-1)/2 + \lceil \log T^{K+2} \rceil + 1 \right) T$$

periods. For sufficiently large T, the length is approximately the same as in the i.i.d. case: for any $\varepsilon > 0$ and $K \in \mathbb{N}$, we have

$$\lim_{T \to \infty} \frac{\tilde{\mathcal{T}}(|M_i|, K, T)}{\mathcal{T}(|M_i|, T) T^{\varepsilon}} = 0. \tag{133}$$

The inference of the messages is the same as with i.i.d. matching, except that players use only their observations in the $N(N-1)/2 + \lceil \log T^{K+2} \rceil + 1^{\text{th}}$ period of each unit in the checking round.

We modify the definition of $\text{susp}_j(h_j)$ by letting $\text{susp}_j(h_j) = 1$ if (i) $j \in I$ and $m_i(j) = 0$ or (ii) player j observes $\omega_{j,t} \neq a^0$, where t is in the τ^{th} period of a unit with $\tau \in \{N(N-1)/2 + 1, ..., N(N-1)/2 + \lceil \log T^{K+1} \rceil\}$ where players take $\mathbf{a}^0$.

As will be seen, players conduct private mixture and their continuation play and rewards depend on its realization. Hence, it will be useful to introduce notions of "extended protocol history" in addition to the protocol history. Again let $\mathbb{T}$ be the set of periods comprising a protocol. For an arbitrary collection of random variables $(\chi_{j,t})_{t \in \mathbb{T}}$ with $\chi_{j,t} \in \{0,1\}^N$ for each $t \in \mathbb{T}$, an *extended protocol history* for player j is a vector $\mathfrak{h}_j = (\chi_{j,t}, a_{j,t}, \omega_{j,t})_{t \in \mathbb{T}}$. The random variables $(\chi_{j,t})_{t \in \mathbb{T}}$ will encode different information in different periods, but in all cases $\chi_{j,t}$ will encode the result of a randomization performed by player j in period t, and the cardinality is bounded by 2^N. Since these randomizations are independent across players conditional on protocol history profiles $(h_j)_{j\in I}$, we have

$$\Pr(\mathfrak{h}_{-j}|\mathfrak{h}_j) = \Pr(\mathfrak{h}_{-j}|h_j)\ \forall j, \mathfrak{h}_{-j}, \mathfrak{h}_j, h_j.$$

That is, when calculating probabilities conditional on a player's extended protocol history, it suffices to condition on the protocol history only. Denote the sets of protocol extended protocol histories by $\mathfrak{H}_j$, as opposed to the set of protocol histories H_j.

For $\theta_j(\mathfrak{h}_{-j}, \zeta) \in \{R, E\}$, the following properties are needed to generalize the proof of Lemmas 5–6:

1. If some player $j \neq j'$ fails to match with player j' in a half-interval where player j' takes a^1, then $\theta_j(\mathfrak{h}_{-j}, \zeta, j') = E$.

2. For each $j' \neq j$, the distribution of $\theta_j(\mathfrak{h}_{-j}, \zeta, j')$ is independent of player j''s message given player j's equilibrium strategy.

3. For $j' = j$, the probability of $\theta_j(\mathfrak{h}_{-j'}, \zeta, j') = E$ is maximized when player j' follows $\sigma^*_{j'}$, and this maximized probability is independent of her history in the message round.

Note that Properties 2 and 3 would not be satisfied if we defined $\theta_j(\mathfrak{h}_{-j}, \zeta, j')$ in the same way as in the i.i.d. case, since players can influence the distribution of future matches. To satisfy Properties 2 and 3, we thus introduce a new variable $\tilde{\omega}_j$, which we use to cancel out this effect.

Intuitively, for player $j \neq j'$ and unit τ, suppose player $j-1$ knew μ at the beginning of the unit. Even if player j observes $\omega_j = a^1$ when player j' sends a message in the τ^{th} unit, player $j-1$ sometimes constructs $\theta_j(\mathfrak{h}_{-j}, \zeta, j')$ as if player j did not observe a^1 (that is, player $j-1$ constructs $\tilde{\omega}_j = a^0$ and calculates $\theta_j(\mathfrak{h}_{-j}, \zeta, j')$ as if player j observed $\tilde{\omega}_j$). If player j observes $\omega_j = a^0$, then $\tilde{\omega}_j = a^0$ for sure. We specify the probability of $\tilde{\omega}_j = a^0$ given μ and $a_{j'} = a^1$, so that, for each μ, the conditional probability of $\tilde{\omega}_j = a^0$ is independent of μ, given that $a_{j'} = a^1$. This makes the distribution of $\theta_j(\mathfrak{h}_{-j}, \zeta, j')$ independent of μ, and if player j is not matched with the sender, we have $\theta_j(\mathfrak{h}_{-j}, \zeta, j') = E$ since $\omega_j = a^0$ implies $\tilde{\omega}_j = a^0$. Since player $j-1$ can identify μ from the first $N(N-1)$ periods, we can achieve this statistical property even though player $j-1$ does not directly observe μ.

Formally, for player $j \neq j'$ and $\tau = 1, ..., 2\lceil \log_2 A^{4\lceil \log_2 |M_i| \rceil} \rceil$, player $j-1$ calculates $\tilde{\omega}_{j,t_\tau} \in \{a^0, a^1\}$ from $\mathfrak{h}_{-j}, \zeta$ as follows:

- If $j \in \mathcal{I}_{\text{jam}}$, there is a player in $\mathcal{I}_{\text{jam}} \backslash \{j\}$ who takes JAM, or player j' takes $a_{j',t_\tau+N(N-1)/2+\lceil \log T^{K+2} \rceil+1} \neq a^1$ (that is, player j' does not take a^1 to send a message), then $\tilde{\omega}_{j,t_\tau} = a^0$.
- Otherwise, the definition of $\tilde{\omega}_{j,t_\tau}$ depends on player j's signal $\omega_{j,t_\tau+N(N-1)/2+\lceil \log T^{K+2} \rceil+1}$ (as identified from h_{-j}):
  - If $\omega_{j,t_\tau+N(N-1)/2+\lceil \log T^{K+2} \rceil+1} = a^0$, then $\tilde{\omega}_{j,t_\tau} = a^0$.
  - If $\omega_{j,t_\tau+N(N-1)/2+\lceil \log T^{K+2} \rceil+1} = a^1$, then player $j-1$ draws a private randomization $\tilde{\chi}_{j-1,t_\tau} \in [0,1]$ from Uniform $[0,1]$. Given the history profile $h_{-j}^{t_\tau+1:t_\tau+N(N-1)/2}$ and $\tilde{\chi}_{j-1,t_\tau}$, player $j-1$ sets $\tilde{\omega}_{j,t_\tau} = a^0$ if $\tilde{\chi}_{j-1,t_\tau} \leq X(h_{-j}^{t_\tau+1:t_\tau+N(N-1)/2})$ and $\tilde{\omega}_{j,t_\tau} = a^1$ otherwise, where $X(h_{-j}^{t_\tau+1:t_\tau+N(N-1)/2})$ is a function specified below.

For each $j \neq j'$ and τ, we encode the realized $\tilde{\omega}_{j,t_\tau} \in \{a^0, a^1\}$ as $\chi_{j-1,t_\tau+N(N-1)/2+\lceil \log T^{K+2} \rceil+1}$.

We construct $\tilde{\omega}_{j,t_\tau}$ to ensure that (i) given the equilibrium strategy in periods $t_\tau + 1$, ..., $t_\tau + N(N-1)/2 + \lceil \log T^{K+2} \rceil$, the distribution of $\tilde{\omega}_{j,t_\tau}$ is independent of $\mu_{t_\tau+1}$, (ii) player j cannot manipulate the distribution of $\tilde{\omega}_{j,t_\tau}$ by deviating in periods $t_\tau+1$, ..., $t_\tau+N(N-1)/2$, and (iii) $\omega_{j,t_\tau+N(N-1)/2+\lceil \log T^{K+2} \rceil+1} = a^0$ implies $\tilde{\omega}_{j,t_\tau} = 0$:

Lemma 35 *There exists $\bar{T} \in \mathbb{N}$ such that, for each $T > \bar{T}$, $K \in \mathbb{N}$, $j' \in I$, and $j \neq j'$, there exists $X(h_{-j}^{t_\tau+1:t_\tau+N(N-1)/2}) \in [0,1]$ such that, if we define $\tilde{\omega}_{j,t_\tau}$ given $X(h_{-j}^{t_\tau+1:t_\tau+N(N-1)/2})$ as above, then $\tilde{\omega}_{j,t_\tau}$ satisfies the following properties: For each $a_{j'} \in \{a^0, a^1\}$,*

1. *$\Pr(\tilde{\omega}_{j,t_\tau} = a^0 | \mu_{t_\tau+1}, \bar{\sigma}^*, \mathbf{a}^0, a_{j'}, a^0_{-j'})$ is independent of $\mu_{t_\tau+1}$. Here, $\mathbf{a}^0$ means that players take $\mathbf{a}^0$ from period $t_\tau + N(N-1)/2 + 1$ to $t_\tau + N(N-1)/2 + \lceil \log T^{K+2} \rceil$, and $(a_{j'}, a^0_{-j'})$ is the action profile in period $t_\tau + N(N-1)/2 + \lceil \log T^{K+2} \rceil + 1$.*

2. *For each* $\mu_{t_\tau+1}$, h_j^{t-1} *with* $t \in \{t_\tau+1, ..., t_\tau + N(N-1)/2\}$, σ_j, *and* $a_{j'} \in A$, *we have*

$$\left| \begin{array}{c} \Pr(\tilde{\omega}_{j,t_\tau} = a^0 | \mu_{t_\tau+1}, \bar{\sigma}^*, \mathbf{a}^0, a_{j'}, a^0_{-j'}) \\ -\Pr(\tilde{\omega}_{j,t_\tau} = a^0 | \mu_{t_\tau+1}, h_j^{t-1}, \sigma_j, \bar{\sigma}^*_{-j}, \mathbf{a}^0, a_{j'}, a^0_{-j'}) \end{array} \right| < \frac{1}{T^{K+1}}. \quad (134)$$

Here, we assume that player j *follows* σ_j *until period* $t_\tau + N(N-1)/2$, *takes* a^0 *from* $t_\tau + N(N-1)/2 + 1$ *to* $t_\tau + N(N-1)/2 + \lceil \log T^{K+2} \rceil + 1$.

3. *If* $j \notin \mathcal{I}_{\text{jam}}$ *and* $REG_{j',-j}$ *holds, then* $\omega_{j,t_\tau+N(N-1)/2+\lceil \log T^{K+2}\rceil+1} = a^0$ *implies* $\tilde{\omega}_{j,t_\tau} = 0$.

Proof. The third claim follows from the definition specified above. Moreover, given $a_{j',t_\tau+N(N-1)/2+\lceil \log T^{K+2}\rceil+1} \neq a^1$, we have $\tilde{\omega}_{j,t_\tau} = a^0$. Hence, we are left to specify $X(h_{-j}^{t_\tau+1:t_\tau+N(N-1)/2})$ to satisfy the first two claims given $a_{j',t_\tau+N(N-1)/2+\lceil \log T^{K+2}\rceil+1} = a^1$.

Given $\mu_{t_\tau+1}$, $\bar{\sigma}^*$, and $\mathbf{a}^0$, player $j-1$ calculates the probability that player j observes a^0 when player j' sends the message:

$$\tilde{p}(\mu_{t_\tau+1}) := \Pr\left(\mu_{t_\tau+N(N-1)/2+\lceil \log T^{K+2}\rceil+1}(j) \neq j' | \mu_{t_\tau+1}, \bar{\sigma}^*, \mathbf{a}^0\right).$$

In addition, player $j-1$ also calculates the largest probability with respect to $\mu_{t_\tau+1}$: $q := \max_{\mu_{t_\tau+1}} \tilde{p}(\mu_{t_\tau+1})$. Fix $\hat{q} \in (q, 1)$ arbitrarily.

Let $h^{\mathbb{C}}$ be the history profile in periods $t_\tau+1, ..., t_\tau + N(N-1)/2$ identified from h_{-j}. If $h^{\mathbb{C}}$ is an on-path history, then $h^{\mathbb{C}}$ statistically identifies $\mu_{t_\tau+1}$. Hence, there exist $p(h^{\mathbb{C}})$ such that

$$\mathbb{E}\left[\frac{\hat{q} - p(h^{\mathbb{C}})}{1 - p(h^{\mathbb{C}})} | \mu_{t_\tau+1}, \bar{\sigma}^*, \mathbf{a}^0\right] = \frac{\hat{q} - \tilde{p}(\mu_{t_\tau+1})}{1 - \tilde{p}(\mu_{t_\tau+1})}$$

for each on-path $h^{\mathbb{C}}$. For off-path $h^{\mathbb{C}}$, pick an on-path $\tilde{h}^{\mathbb{C}}$ arbitrarily and define

$$\frac{\hat{q} - p(h^{\mathbb{C}})}{1 - p(h^{\mathbb{C}})} = \frac{\hat{q} - p(\tilde{h}^{\mathbb{C}})}{1 - p(\tilde{h}^{\mathbb{C}})}.$$

We define $X(h_{-j}^{t_\tau+1:t_\tau+N(N-1)/2}) = \dfrac{\hat{q} - p(h^{\mathbb{C}})}{1 - p(h^{\mathbb{C}})}$, that is, player $j-1$ draws a private randomization $\tilde{\chi}_{j,t_\tau} \in [0,1]$ from Uniform $[0,1]$, and if

$$\tilde{\chi}_{j,t_\tau} \leq \frac{\hat{q} - p(h^{\mathbb{C}})}{1 - p(h^{\mathbb{C}})} \tag{135}$$

then player j defines $\tilde{\omega}_{j,t_\tau} = a^0$. Otherwise, $\tilde{\omega}_{j,t_\tau} = a^1$.

We first verify that the right-hand side of (135) lies in $[0,1]$ for sufficiently large T:

$$\frac{\hat{q} - p(h^{\mathbb{C}})}{1 - p(h^{\mathbb{C}})} \in [0,1]. \tag{136}$$

Since players take $\mathbf{a}^0$ from period $t_\tau + N(N-1)/2 + 1$ to $t_\tau + N(N-1)/2 + \lceil \log T^{K+2} \rceil$, we have

$$\begin{aligned} & \max_{\mu_{t_\tau+1}, \mu'_{t_\tau+1}} \sum_{\mu_{t_\tau+N(N-1)/2+\lceil \log T^{K+2} \rceil+1}} \left| \begin{array}{c} \Pr\left(\mu_{t_\tau+N(N-1)/2+\lceil \log T^{K+2} \rceil+1} | \mu_{t_\tau+1}, \bar{\sigma}^*, \mathbf{a}^0\right) \\ - \Pr\left(\mu_{t_\tau+N(N-1)/2+\lceil \log T^{K+2} \rceil+1} | \mu'_{t_\tau+1}, \bar{\sigma}^*, \mathbf{a}^0\right) \end{array} \right| \\ \leq \; & (1-\bar{\varepsilon})^{\lceil \log T^{K+2} \rceil}, \end{aligned} \tag{137}$$

as in Lemma 33. Since $\bar{\varepsilon} \leq \tilde{p}(\mu_{t_\tau+1}) \leq 1 - \bar{\varepsilon}$ by the full-support assumption and

$$1 > \hat{q} > q := \max_{\mu_{t_\tau+1}} \tilde{p}(\mu_{t_\tau+1}) \geq \tilde{p}(\mu_{t_\tau+1}),$$

we have

$$\begin{aligned} \max_{\tilde{\mu}_{t_\tau+1}} \left| \frac{\hat{q} - \tilde{p}(\mu_{t_\tau+1})}{1 - \tilde{p}(\mu_{t_\tau+1})} - \frac{\hat{q} - \tilde{p}(\mu_{t_\tau+1})}{1 - \tilde{p}(\mu_{t_\tau+1})} \right| & \leq \frac{(1-\bar{\varepsilon})^{\lceil \log T^{K+2} \rceil}}{\bar{\varepsilon}}; \\ \frac{\hat{q} - \tilde{p}(\mu_{t_\tau+1})}{1 - \tilde{p}(\mu_{t_\tau+1})} & \in \left[\frac{\hat{q} - q}{1 - q}, \frac{\hat{q} - \bar{\varepsilon}}{1 - \bar{\varepsilon}} \right]. \end{aligned} \tag{138}$$

Given (125), we have (136), as desired.

We now verify the claims of the lemma. Given $j \notin \mathcal{I}_{\text{jam}}$ and $REG_{j',-j}$, for each $\mu_{t_\tau+1}$,

$$\begin{aligned}
& \Pr\left(\tilde{\omega}_{j,t_\tau} = a^0 | \mu_{t_\tau+1}, \bar{\sigma}^*, \mathbf{a}^0, a_j^1, a_{-j}^0\right) \\
= & \Pr\left(\mu_{t_\tau+N(N-1)/2+\lceil \log T^{K+2}\rceil+1}(j) \neq j' | \mu_{t_\tau+1}, \bar{\sigma}^*, \mathbf{a}^0\right) \\
& + \Pr\left(\mu_{t_\tau+N(N-1)/2+\lceil \log T^{K+2}\rceil+1}(j) = j' | \mu_{t_\tau+1}, \bar{\sigma}^*, \mathbf{a}^0\right) \Pr\left(\tilde{\chi}_{j,t_\tau} \leq \frac{\hat{q} - p(h^{\mathbb{C}})}{1 - p(h^{\mathbb{C}})} | \mu_{t_\tau+1}, \bar{\sigma}^*, \mathbf{a}^0\right) \\
= & \tilde{p}(\mu_{t_\tau+1}) + \left(1 - \tilde{p}(\mu_{t_\tau+1})\right) \frac{\hat{q} - \tilde{p}(\mu_{t_\tau+1})}{1 - \tilde{p}(\mu_{t_\tau+1})} = \hat{q}.
\end{aligned}$$

Hence, Claim 1 holds.

For Claim 2, since players take $\mathbf{a}^0$ from period $t_\tau + N(N-1)/2 + 1$ to $t_\tau + N(N-1)/2 + \lceil \log T^{K+2} \rceil$, it suffices to show that

$$\begin{aligned}
& \max_{h^{\mathbb{C}}, \mu_{t_\tau+N(N-1)/2+1}, \tilde{h}^{\mathbb{C}}, \tilde{\mu}_{t_\tau+N(N-1)/2+1}} \left| \begin{array}{c} \Pr(\tilde{\omega}_{j,t_\tau} = a^0 | h^{\mathbb{C}}, \mu_{t_\tau+N(N-1)/2+1}, \mathbf{a}^0, a_{j'}^1, a_{-j'}^0) \\ -\Pr(\tilde{\omega}_{j,t_\tau} = a^0 | \tilde{h}^{\mathbb{C}}, \tilde{\mu}_{t_\tau+N(N-1)/2+1}, \mathbf{a}^0, a_{j'}^1, a_{-j'}^0) \end{array} \right| \\
< & \frac{1}{T^{K+2}}.
\end{aligned}$$

By definition, we have

$$\begin{aligned}
& \Pr\left(\tilde{\omega}_{j,t_\tau} = a^0 | h^{\mathbb{C}}, \mu_{t_\tau+N(N-1)/2+1}, \mathbf{a}^0, a_{j'}^1, a_{-j'}^0\right) \\
= & \Pr\left(\mu_{t_\tau+N(N-1)/2+\lceil \log T^{K+2}\rceil+1}(j) \neq j' | \mu_{t_\tau+N(N-1)/2+1}, \mathbf{a}^0\right) \\
& + \Pr\left(\mu_{t_\tau+N(N-1)/2+\lceil \log T^{K+2}\rceil+1}(j) = j' | \mu_{t_\tau+N(N-1)/2+1}, \mathbf{a}^0\right) \frac{\hat{q} - p(h^{\mathbb{C}})}{1 - p(h^{\mathbb{C}})}. \qquad (139)
\end{aligned}$$

By Lemma 33, the effect of $\mu_{t_\tau+N(N-1)/2}$ is canceled out exponentially:

$$\begin{aligned}
& \max_{\mu_{t_\tau+N(N-1)/2+1}, \tilde{\mu}_{t_\tau+N(N-1)/2+1}} \left| \begin{array}{c} \Pr\left(\mu_{t_\tau+N(N-1)/2+\lceil \log T^{K+2}\rceil+1}(j) = j' | \mu_{t_\tau+N(N-1)/2+1}, \mathbf{a}^0\right) \\ -\Pr\left(\mu_{t_\tau+N(N-1)/2+\lceil \log T^{K+2}\rceil+1}(j) = j' | \tilde{\mu}_{t_\tau+N(N-1)/2+1}, \mathbf{a}^0\right) \end{array} \right| \\
\leq & (1 - \bar{\varepsilon})^{\log T^{K+2}}.
\end{aligned}$$

Moreover, by (125) and (138), we have

$$\left| \frac{\hat{q} - \tilde{p}(\mu_{t_\tau+1})}{1 - \tilde{p}(\mu_{t_\tau+1})} - \frac{\hat{q} - p(h^{\mathbb{C}})}{1 - p(h^{\mathbb{C}})} \right| < \frac{2M (1 - \bar{\varepsilon})^{\log T^{K+2}}}{\bar{\varepsilon}}.$$

In total, the range of (139) with respect to $\mu_{t_\tau+N(N-1)/2+1}$ and $h^{\mathbb{C}}$ is bounded by $2(M+1)(1-\bar{\varepsilon})^{\log T^{K+2}}/\bar{\varepsilon}$, as desired. ■

Similarly, when player j' sends a message, given $a_{j'} \in \{a^0, a^1\}$, player $j'-1$ calculates $\tilde{\omega}_{j,t_\tau}^{j'} \in \{a^0, a^1\}$ for each $j \neq j'$ such that

1. $\Pr(\tilde{\omega}_{j,t_\tau}^{j'} = a^0 | \mu_{t_\tau+1}, \bar{\sigma}^*, \mathbf{a}^0, a_{j'}, a_{-j'}^0)$ with $a_{j'} \in \{a^0, a^1\}$ is independent of $\mu_{t_\tau+1}$.

2. For each $\mu_{t_\tau+1}$, $h_{j'}^{t-1}$ with $t \in \{t_\tau+1, ..., t_\tau+N(N-1)/2\}$, $\sigma_{j'}$, and $a_{j'}$, we have

$$\left| \begin{array}{c} \Pr(\tilde{\omega}_{j,t_\tau}^{j'} = a^0 | \mu_{t_\tau+1}, \bar{\sigma}^*, \mathbf{a}^0, a_{j'}, a_{-j'}^0) \\ -\Pr(\tilde{\omega}_{j,t_\tau}^{j'} = a^0 | \mu_{t_\tau+1}, h_{j'}^{t-1}, \sigma_{j'}, \bar{\sigma}_{-j'}^*, \mathbf{a}^0, a_{j'}, a_{-j'}^0) \end{array} \right| < \frac{1}{T^{K+1}}.$$

3. If $j' \notin \mathcal{I}_{\text{jam}}$ and $REG_{j',-j'}$ holds, then $\omega_{j,t_\tau+N(N-1)/2+\lceil \log T^{K+2} \rceil+1} = a^0$ implies $\tilde{\omega}_{j,t_\tau}^{j'} = 0$.

Again, for each τ, we encode the realized $(\tilde{\omega}_{j,t_\tau}^{j'})_{j\neq j'}$ as $\chi_{j'-1,t_\tau+N(N-1)/2+\lceil \log T^{K+2} \rceil+1}$.

For the j'-checking phase, for each unit, we construct $\tilde{\omega}_{j,t_\tau}$ for $j \neq j'$ and $(\tilde{\omega}_{j,t_\tau}^{j'})_{j\neq j'}$. Then, we define $\theta_j(\mathfrak{h}_{-j}, \zeta, j')$. First, a deviation from $\mathbf{a}^0$ will induce $\theta_j(\mathfrak{h}_{-j}, \zeta, j') = R$: For each $j \in I$, if player j takes $a_{j,t} \neq a^0$, where t is in the τ^{th} period of a unit with $\tau \in \{N(N-1)/2+1, ..., N(N-1)/2+\lceil \log T^{K+1} \rceil\}$ where players take a^0, we define $\theta_j(\mathfrak{h}_{-j}, \zeta, j') = R$ for each $j' \in I$.

Otherwise, we define $\theta_j(\mathfrak{h}_{-j}, \zeta, j') = E$ with $\tilde{\omega}_{j,t}$ replacing $\omega_{j,t}$ for player j and $(\tilde{\omega}_{j,t}^{j'})_{j\neq j'}$ replacing $(\omega_{j,t})_{j\neq j'}$ for player j': for $j, j' \in I$, we define $\theta_j(\mathfrak{h}_{-j}, \zeta, j') = E$ if and only if one or more of the following three conditions holds:

1. $\zeta_j = \text{jam}$.

2. There exists $n \in \mathcal{I}_{\text{jam}} \setminus \{j, j'\}$ who plays JAM in some half-interval.

3. *[Condition FAIL]* $j \neq j'$, and there exists a half-interval in $\mathbb{T}(j')$ such that, with $\mathbb{S}$ being the set of $(N(N-1)/2+\lceil \log T^{K+2} \rceil+1)^{\text{th}}$ period of each unit of the half-interval, there exists $n \neq j'$ such that player j' plays a^1 throughout $\mathbb{S}$ but $\tilde{\omega}_{n,t} = a^0$ for all $t \in \mathbb{S}$.

4. *[Condition FAILj']* $j = j'$, player j' followed σ_j^* in $\mathbb{T}(j')$, but there exist a half-interval in $\mathbb{T}(j')$ such that, with $\mathbb{S}$ being the set of $(N(N-1)/2+\lceil \log T^{K+2} \rceil+1)^{\text{th}}$ period of

each unit of the half-interval, there exists $n \neq j'$ such that player j' plays a^1 throughout $\mathbb{S}$ but $\tilde{\omega}^{j'}_{n,t} = a^0$ for all $t \in \mathbb{S}$.

Lemma 35 guarantees Properties 1-3, given that players take $(\bar{\sigma}^*, \mathbf{a}^0)$ in the first $N(N-1)/2 + \lceil \log T^{K+2} \rceil$ periods of each unit.

Lemma 5 hold as it stands, except that in Claim 1 we require that player j follows $(\bar{\sigma}^*, \mathbf{a}^0)$ in the first $N(N-1)/2 + \lceil \log T^{K+2} \rceil$ periods of each unit.

Lemma 6 holds as it stands for the following reasons: (i) By definition of $\text{susp}_n(h_n)$ and $\theta_j(\mathfrak{h}_{-j}, \zeta, j')$, if player j deviates from $\mathbf{a}^0$, then $\theta_j(\mathfrak{h}_{-j}, \zeta) = R$ and $\text{susp}_n(h_n) = 1$ for some $n \neq j$. Otherwise, (ii) Property 1 holds, and if there exists player $j' \neq j$ and $n \neq j'$ such that player n is not matched with player j' in some half-interval when player j' takes a^1, then $\theta_j(\mathfrak{h}_{-j}, \zeta, j') = E$, and otherwise either player n infers player j''s message correctly or $\text{susp}_n(h_n) = 1$. (iii) In addition, given that she takes $\mathbf{a}^0$, (134) implies that $\Pr(\tilde{\omega}_{j,t_\tau} = a^0 | \mu_{t_\tau+1}, \sigma_j, \bar{\sigma}^*_{-i}, \mathbf{a}^0, a^1_{j'}, a^0_{-j'}) \in (0,1)$ for each $\mu_{t_\tau+1}, \sigma_j$. Hence, by the same proof as Lemma 2, the probability of FAIL is of order $\exp(-T)$. Hence, the probability of $\theta_j(\mathfrak{h}_{-j}, \zeta) = E$ remains bounded by T^{-8}.

Finally, Lemma 7 holds given that player j follows $(\bar{\sigma}^*, \mathbf{a}^0)$ in the first $N(N-1)/2 + \lceil \log T^{K+2} \rceil$ periods of each unit, and $\bar{\varepsilon}$ replaced with

$$\hat{\varepsilon} \min_n \Pr(\tilde{\omega}_{n,t_\tau} = a^1 | \mu_{t_\tau+1}, \bar{\sigma}^*, \mathbf{a}^0, a^1_{j'}, a^0_{-j'}) > 0.$$

Note that we replace $\bar{\varepsilon}$ with $\hat{\varepsilon}$ as in Lemma 34, and then we multiply the probability of $\tilde{\omega}_{n,t_\tau} = a^1$ given $a^1_{j'}$ since this is the lower bound of the probability that player n observes $\tilde{\omega}_{n,t_\tau} = a^1$ conditional on that players j' and n match.

L.4 Jamming Coordination Protocol

We change the jamming coordination protocol as follows:

- For the first $N(N-1)/2$ periods, players play $\bar{\sigma}^*$ to identify μ_1.
- For the next 2 periods, each player i plays a^1 with probability $\left(\frac{1}{T}\right)^2$ and plays $a \neq a^1$ with probability $\frac{1-\left(\frac{1}{T}\right)^2}{|A|-1}$.

Given a protocol history h_i, we define $\zeta_i(h_i) = \text{jam}$ as in the i.i.d. case. Inequality (25) holds as it stands.

M Communication Modules

M.1 Basic Communication Module

For each player $n \in I$, payoff functions in the module again take the form

$$\sum_{t \in \mathbb{T}} \delta^{t-1} \hat{u}_n(\mathbf{a}_t) + \pi_n(h_{n-1}) + w_n(\mu_{T+1}, h). \tag{140}$$

Note that the continuation payoff now depends on μ_{T+1}. We say w *satisfies the premise with magnitude* K if $\max_{\mu,h,\tilde{\mu},\tilde{h}} \left| w(\mu, h) - w(\tilde{\mu}, \tilde{h}) \right| \leq K$.

Sets of periods: Recall that $\mathbb{S}_1$ is the set of all $(S_1+1)^{\text{th}}$ periods of a subunit; for $t \in \mathbb{C}$, $\mathbb{S}_1[t,n,l]$ with $n \in I$ and $l \in \{1,2\}$ is the subset of $\mathbb{S}_1$ where player n sends $\omega_{n,\tau(t)}$ for the l^{th} time; and $\mathbb{S}_1[t,n,l,k]$ with $k \in \{1,2\}$ is the first or second half of $\mathbb{S}_1[t,n,l]$. Let $\mathbb{S}[a^0]$ be the set of the first S_0 periods of the unit and the first S_1 periods of each (S_1+1)-period cycle, where all players take a^0 regardless of the history or m_i. For the rest of this subsection, we use t only for $t \in \mathbb{C}$.

We wish to construct off-path strategies and beliefs (and corresponding tremble sequences) and rewards $\pi_n(h_{n-1})$ so that players will follow the prescribed strategy for any history, initial match μ_1, message m_i, and w_n satisfying the premise. The most difficult part is establishing incentives to truthfully communicate $\omega_{n,\tau(t)}$.

First, let player $n-1$ punish player n if she observes $\omega_{n-1,s} \notin \{a^0, a^1\}$ in any period, or if $\omega_{n-1,s} \neq a^0$ in any period in $\mathbb{S}[a^0]$. This incentivizes players to take $a \in \{a^0, a^1\}$ in any period and take a^0 in $\mathbb{S}[a^0]$. Since players take $\mathbf{a}^0$ for S_1 periods in between sending messages in $\mathbb{S}_1$, for large S_1 players ignore the effect of their actions in $\mathbb{S}_1$ on the match realization in the next period in $\mathbb{S}_1$. Hence, different matches in $\mathbb{S}_1$ are almost independent. Moreover, since players $-n$'s continuation play in $\mathbb{S}_1$ does not depend on their observations in $\mathbb{S}_1$, player n ignores the effect of her action in $\mathbb{S}_1$ on others' continuation play. Second, for $\mathbb{S}_1[t,n',1]$ or

$\mathbb{S}_1[t, n', 2]$ with $n' \neq n$ (i.e., when player n is a receiver), let player $n-1$ incentivize player n to take a^0 by rewarding her when $\omega_{n-1,s} = a^0$.

We now describe how to incentivize players to truthfully communicate $\omega_{n,\tau(t)}$. The idea is to use other players' reports $\omega_{-n,\tau(t)}$ to punish player n for inconsistent reports. However, if players only reported $\omega_{n,\tau(t)}$ once, it would be difficult to specify off-path play after a player deviates when she is a receiver. In particular, what player n should do depends on her belief about other players' inference of $\omega_{-n,\tau(t)}$, and after she deviates this belief may depend on her belief about the initial match μ_1.

By having players report twice, we can make players report truthfully in $\mathbb{S}_1[t, n, 1]$ even after they deviate when receivers, by having them believe that everyone will report truthfully in $(\mathbb{S}_1[t, n', 2])_{n' \neq n}$. In turn, we construct $\pi_{n'}$ so that reporting truthfully in $\mathbb{S}_1[t, n', 2]$ is optimal for a sender who reported truthfully in $\mathbb{S}_1[t, n', 1]$. Finally, we construct trembles such that senders tremble much less than receivers in $\mathbb{S}_1[t, n', 1]$, so it is always consistent for player n to believe that each player n' reported truthfully in $\mathbb{S}_1[t, n', 1]$, and hence will do so again in $\mathbb{S}_1[t, n', 2]$.

Off-path play: We modify $\sigma_n^{m_i}$ after off-path histories to construct strategy $\bar{\sigma}_n^{m_i}$. That is, we recursively define player n's strategy $\bar{\sigma}_n^{m_i}$ as follows (where, if $n \neq i$, $\bar{\sigma}_n^{m_i} = \bar{\sigma}_n^{\tilde{m}_i}$ $\forall m_i, \tilde{m}_i \in M_i$):

1. For $\mathbb{S}[a^0]$ periods, player n takes a^0.

2. In the $(S_0+1)^{\text{th}}$ period, player $n \neq i$ takes a^0 and player i takes $a \in \{a^0, a^1\}$ corresponding to m_i.

3. For period $\tau(t)$ with $t \in \mathbb{C}$, player n follows $\bar{\sigma}_{n,t}^*$. Here, we define $\mathring{\omega}_{n,\tau(t)} \in \left\{\omega_{n,\tau(t)}, a^0\right\}$ to be player n's "interpretation" of $\omega_{n,\tau(t)}$, where $\mathring{\omega}_{n,\tau(t)} = \omega_{n,\tau(t)}$ if $\omega_{n,\tau(t)} \in \{a^0, a^1\}$ and $\mathring{\omega}_{n,\tau(t)} = a^0$ otherwise. Similarly, let $\mathring{a}_{n,\tau(t)} = a_{n,\tau(t)}$ if $a_{n,\tau(t)} \in \{a^0, a^1\}$ and $a_{n,\tau(t)} = a^0$ otherwise. As will be seen, after player n observes (or takes) an off-path play of $a \neq a^0$, she "ignores" this deviation and proceeds as if she observed (or took, respectively) a^0.

4. For each $t \in \mathbb{C}$ and $s \in \mathbb{S}_1[t, j, 1]$, players $-j$ take a^0 and player j takes $a_{j,s} = \sigma_j^{m_i}|_{\mathring{\omega}_{j,\tau(t)}}(h_{j,t-1})$ (here, on equilibrium path, player j's equilibrium action depends

only on $\omega_{j,\tau(t)}$. $\sigma_j^{m_i}|_{\mathring{\omega}_{j,\tau(t)}}(h_{j,t-1})$ means that player j follows the equilibrium strategy given the above signal re-interpretation $\mathring{\omega}_{j,\tau(t)}$ after each $h_{j,t-1}$).

5. For each $t \in \mathbb{C}$ and $s \in \mathbb{S}_1[t, j, 2]$, players $-j$ take a^0. Player j "repeats" her action from the first subunit: For $s \in \mathbb{S}_1[t, j, 2, 1]$, if player j took a^1 at least once in $\mathbb{S}_1[t, j, 1, 1]$, she takes a^1; otherwise, she takes a^0. For $s \in \mathbb{S}_1[t, j, 2, 2]$, if player j took a^1 at least once in $\mathbb{S}_1[t, j, 1, 2]$, she takes a^1; otherwise, she takes a^0.

Trembles: Intuitively, we will construct a tremble sequence such that (i) trembles from equilibrium action a^1 to $a \notin \{a^0, a^1\}$ are least likely and (ii) trembles from equilibrium action a^0 to $a \notin \{a^0, a^1\}$ are second least likely. Given the definition of $\bar{\sigma}_n^{m_i}$, other players proceed after trembling to $\{a^0, a^1\}$ as if they took a^0. Given (i), (ii), and the fact that $\omega = a^0$ is observed with positive probability in every period, in turn, a player who observes $\omega \notin \{a^0, a^1\}$ proceeds as if she observed $\omega = a^0$, as prescribed by $\bar{\sigma}_n^{m_i}$.

For trembles from a^0 to a^1 in $\mathbb{S}[a^0]$, since other players proceed after they tremble in $\mathbb{S}[a^0]$ as if they took a^0, a player who observes $\omega \neq a^0$ in $\mathbb{S}[a^0]$ in turn proceeds as if she observed a^0.

Now, consider a history in which $\omega_{n,s} = a^0$ for each $s \in \mathbb{S}[a^0]$ and $\omega_{n,s} \in \{a^0, a^1\}$ for each s. For each $t \in \mathbb{C}$, we specify that (iii) players tremble less in $\tau(t)$ than in $\mathbb{S}_1[t, j, l]$, for each $j \in I$ and $l \in \{1, 2\}$; (iv) in $\mathbb{S}_1[t, j, 1]$, players $-j$ tremble more than player j, and (v) trembles are history-independent. Given (iii), player n believes that players follow $\bar{\sigma}_t^*$ in period $\tau(t)$ with $t \in \mathbb{C}$. Given (iv), in $\mathbb{S}_1[t, j, 1]$, if player n observes an off-path play of a^1 in $\mathbb{S}_1[t, j, 1]$, she believes her current opponent satisfy $\mu_s(n) \neq j$ and trembled from a^0 to a^1. Given (v), she also believes that, in all other matches, players observe on-path actions.

Formally, we define the tremble sequence $(\bar{\sigma}_i^{m_i,k}, \bar{\sigma}_{-i}^k)_{k=1}^\infty$ as follows:

1. Given $\bar{\sigma}_n^{m_i}|_{h_n} = a^1$, player n trembles to $a \notin \{a^0, a^1\}$ with probability $k^{-k^{k^k}}$.

2. Given $\bar{\sigma}_n^{m_i}|_{h_n} = a^0$, player n trembles to $a \notin \{a^0, a^1\}$ with probability $k^{-k^{k}}$.

3. In $\mathbb{S}[a^0]$ (where $\bar{\sigma}_n^{m_i}|_{h_n} = a^0$ for each h_n), player n trembles to a^1 with probability k^{-k^k} (given that type 2 trembles do not occur).

4. In periods $S_1 + 1$ and $\mathbb{C}$, player n trembles uniformly over $\{a^0, a^1\}$ with probability k^{-k^k} (given that type 1 and 2 trembles do not occur).

5. In $s \in \mathbb{S}_1[t, j, 1]$, player $n \neq j$ (who takes a^0 after any history) trembles to a^1 with probability k^{-k}; and player j trembles uniformly over $\{a^0, a^1\}$ with probability k^{-k^k} (given that type 1 and 2 trembles do not occur).

6. In $s \in \mathbb{S}[t, j, 2]$, each player trembles uniformly over $\{a^0, a^1\}$ with probability k^{-1} (given that type 1 and 2 trembles do not occur).

Player n's belief: Given the tremble sequence, let β_n denote the corresponding limit belief as $k \to \infty$. For each history, β_n satisfies the following properties: For any h_n at the end of the unit (hence, by the law of iterated expectation, the following holds after any history within the unit),

1. Since there always exists a player who takes a^0, given Trembles 1 and 2, player n after observing $\omega_{n,s} \notin \{a^0, a^1\}$ believes that $\bar{\sigma}^{m_i}_{\mu_s(n)}|_{h_{\mu_s(n)}} = a^0$ but $\mu_s(n)$ took $a = \omega_{n,s}$.

2. For period $\tau(t)$ with $t \in \mathbb{C}$, if $\omega_{n,\tau(t)} \in \{a^0, a^1\}$, player n believes that players $-n$ took $\bar{\sigma}^*_{-n,t}$, since (i) any $\omega_{n,\tau(t)} \in \{a^0, a^1\}$ can arise given $\bar{\sigma}^*_{-n}$, (ii) trembles in $\mathbb{S}_1$ are much more likely than trembles in $\mathbb{C}$, and (iii) strategies and trembles in $\mathbb{S}[a^0]$ are history-independent. Again, if $\omega_{n,\tau(t)} \notin \{a^0, a^1\}$, then player n believes that only $\mu_{\tau(t)}(i)$ trembled from a^0 to $\omega_{n,\tau(t)}$.

3. For period $s \in \mathbb{S}_1[t, j, 1]$, suppose player n knew $(\mathring{a}_{\tau(t)}, \mathring{\omega}_{\tau(t)})$. Since $\bar{\sigma}^{m_i}_t(h^{t-1})$ depends only on $(\mathring{a}_{\tau(t)}, \mathring{\omega}_{\tau(t)})$ and the matching has full support, player n can determine which observations $\omega_{n,s}$ are probability-0 event given $\bar{\sigma}^{m_i}_t(h^{t-1})$.

   If she observes on-path $\omega_{n,\tau(t)}$, then she believes all players follow $\bar{\sigma}^{m_i}$, since (i) trembles are history-independent, and (ii) players tremble more often in $\mathbb{S}_1[t, j, 2]$ than $\mathbb{S}_1[t, j, 1]$ (hence, player n does not update her belief about period-s history from inconsistent signals in $\mathbb{S}_1[t, j, 2]$). In addition, for player $n = j$, we have

$$\beta\left(\mu_s(n) = n' | h_n\right) \geq 1/\bar{\varepsilon}^2 \text{ for each } n' \in I \setminus \{j, n\} \tag{141}$$

since, given $\mu_{s-1}(n)$ and $\mu_{s+1}(n)$, by the full-support assumption, any $\mu_s(n)$ arises with probability at least $1/\bar{\varepsilon}^2$.

If she observes off-path $\omega_{n,s}$, then since (i) trembles are history-independent, (ii) players tremble more often in $\mathbb{S}_1[t,j,2]$ than $\mathbb{S}_1[t,j,1]$, and (iii) players $-j$ tremble more than player j, she believes that

$$\begin{cases} \beta(\mu_s(n) = n'|h_n) \geq 1/\bar{\varepsilon}^2 \text{ for each } n' \in I \setminus \{j,n\}, \\ \beta(\mu_s(n) = j|h_n) = 0, \text{ and} \\ \text{player } \mu_s(n) \text{ is the only player who trembled.} \end{cases}$$

Since trembles are more likely in $\mathbb{S}_1[t,j,1]$ and $\mathbb{S}_1[t,j,2]$ than in $\mathbb{C}$, without knowing $(\mathring{a}_{\tau(t)}, \mathring{\omega}_{\tau(t)})$, player j after each history believes (141); and players $n \neq j$ believe

$$\text{player } j \text{ and players } -(n, \mu_s(n)) \text{ followed } \bar{\sigma}^{m_i}. \tag{142}$$

Lemma 36 *For each K, there exist $\hat{\delta} < 1$, S_1, and $\bar{K}$ such that, for each $i \in I$, M_i, T, and w satisfying the premise for communication with magnitude K, there exists a family of functions $\left(\pi_n : H^{\mathbb{T}} \to \mathbb{R}\right)_{n \in I}$ such that the following holds:*

1. *For each $\delta \in [\hat{\delta}, 1]$, with payoff functions (140), the basic communication protocol is a XSE.*[29]

2. *For each $n \in I$, $\mathbb{E}\left[\sum_{s \in \mathbb{T}} \delta^{t-1} \hat{u}_n(\mathbf{a}_s) + \pi_n(h_{n-1}) | \mu_1, m_i\right]$ does not depend on μ_1, m_i.*

3. *For each $n \in I$, we have*

$$\max_{h_{n-1}, \tilde{h}_{n-1}} \left| \pi_n(h_{n-1}) - \pi_n\left(\tilde{h}_{n-1}\right) \right| < \bar{K} |\mathbb{T}|.$$

Note that we now need a large discount factor, since players spend many periods cancelling the effect of the initial match.

[29] See the beginning of this supplemental appendix for the definition of XSE. Here, we require that the equilibrium is sequentially rational conditional on μ_1 and m_i.

Proof. We prove that there exists π_n^{unit} which maps player $n-1$'s history in one unit to $\mathbb{R}$ such that, for each unit and each match realization μ_1 at the beginning of the unit, the following four conditions are satisfied: (In what follows, for a given unit, we say $\bar{\sigma}_i^{m_i}$ equals $\bar{\sigma}_i^0$ or $\bar{\sigma}_i^1$ if $\bar{\sigma}_i^{m_i}$ specifies a^0 or a^1 in the $(S_0+1)^{\text{th}}$ period, respectively. We say player i follows the equilibrium strategy if she takes $\bar{\sigma}_i^0$ or $\bar{\sigma}_i^1$.)

1. For each $n \in I$, m_i, and h_n^{s-1}, player n's deviation from the equilibrium strategy reduces her payoff by at least K compared to her equilibrium payoff

$$\mathbb{E}\left[\sum_{s':unit} \delta^{s'-1}\hat{u}_n(\mathbf{a}_s) + \pi_n^{\text{unit}}(h_{n-1}) \,|\mu_1, m_i, h_n^{s-1}\right]. \tag{143}$$

2. For each $n \neq i$, (143) does not depend on μ_1, given $h_n^{s-1} = \emptyset$. Moreover, the difference

$$\mathbb{E}\left[\sum_{s:unit} \delta^{s-1}\hat{u}_n(\mathbf{a}_s) + \pi_n^{\text{unit}}(h_{n-1}) \,|\mu_1, \bar{\sigma}_i^0\right] - \mathbb{E}\left[\sum_{s:unit} \delta^{s-1}\hat{u}_n(\mathbf{a}_s) + \pi_n^{\text{unit}}(h_{n-1}) \,|\mu_1, \bar{\sigma}_i^1\right]$$

   does not depend on the calendar time at the beginning of the unit.

3. For i and each m_i, (143) does not depend on m_i or μ_1 given $h_n^{s-1} = \emptyset$.

4. For each $n \in I$, we have

$$\max_{h_{n-1},\tilde{h}_{n-1}} \left|\pi_n^{\text{unit}}(h_{n-1}) - \pi_n^{\text{unit}}\left(\tilde{h}_{n-1}\right)\right| < \bar{K}.$$

Given Conditions 2 and 3, the continuation payoff from the next unit is independent of μ_1. Moreover, in the basic communication module, since the sender takes $\bar{\sigma}_i^0$ and $\bar{\sigma}_i^1$ with the same frequency, player $n \in I$ is indifferent between player i sending any message. Hence, the existence of such π_n^{unit} is sufficient for the current lemma.

Let us now construct π_n^{unit}. We first define the reward functions given constants $K_1, K_2, K_3, K_4, K_5, K_6 \geq 2\{1, \max_{a,a' \in A^2}|u(a,a')|\}$ and S_1. Then we fix the constants to satisfy the above conditions. For $n = 1, ..., 6$, let $K_{1:n} = K_1 \times \cdots \times K_n$.

For each period s, we add $-S_1 K_{1:6}\mathbf{1}_{\{\omega_{n-1,s} \neq a^0, a^1\}}$. For periods $\mathbb{S}[a^0]$ where all players take a^0, we add $-\sum_{s \in \mathbb{S}[a^0]} S_1 K_{1:6}\mathbf{1}_{\{\omega_{n-1,s} \neq a^0\}}$ to incentivize a^0. In addition to those rewards, we define the per-period reward as follows:

In period $s \in \mathbb{S}_1[t, n', l]$ with $n' \neq n$, $l \in \{1, 2\}$, and $t \in \mathbb{C}$, we define $\pi_{n,s}(h_{n-1}) = K_{1:5}\mathbf{1}_{\{\omega_{n-1,s}=a^0\}}$ to incentivize a^0.

We consider player n's reward for $\mathbb{S}_1[t, n, l]$. Recall that, in the protocol, we defined player $n-1$'s inference of player n's message in $\mathbb{S}_1[t, n, 2]$ as follows: $\omega_{n,\tau(t)}(n-1) = a^1$ if player $n-1$ observes a^1 at least once in $\mathbb{S}_1[t, n, 2, 1]$ and observes a^0 in all $\mathbb{S}_1[t, n, 2, 2]$; and $\omega_{n,\tau(t)}(n-1) = a^0$ if player $n-1$ observes a^1 at least once in $\mathbb{S}_1[t, n, 2, 2]$ and observes a^0 in all $\mathbb{S}_1[t, n, 2, 1]$ periods. In all the other cases, $\omega_{n,\tau(t)}(n-1) = 0$.

If $\omega_{j',\tau(t)}(n-1) = 0$ for some $j' \in I \setminus \{n, j\}$, then let $\omega^*_{n,\tau(t)}(n-1) = 0$. In other cases, let $\omega_{-n,\tau(t)}(n-1) \in \{a^0, a^1\}^{N-1}$. Given the equilibrium strategy $a_{-n,\tau(t)}$, let $\omega^*_{n,\tau(t)}(n-1) \in \{a^0, a^1\}$ be the signal such that, given $\omega_{-n,\tau(t)}(n-1)$ and $a_{-n,\tau(t)}$, we identify $(a_{n,\tau(t)}, \omega_{n,\tau(t)})$, and let $\omega^*_{n,\tau(t)}(n-1) = \omega_{n,\tau(t)}$.

In period $s \in \mathbb{S}_1[t, n, 1, 1]$, let

$$\pi_{n,s}(h_{n-1}) = \begin{cases} K_{1:3}\mathbf{1}_{\{\omega_{n-1,s}=a^1\}} & \text{if } \omega^*_{n,\tau(t)}(n-1) = a^1, \\ K_{1:3}\mathbf{1}_{\{\omega_{n-1,s}=a^0\}} & \text{if } \omega^*_{n,\tau(t)}(n-1) = a^0. \end{cases} \tag{144}$$

Unless players $-n$ deviate or some pair does not match, player n observes $\omega_{n,\tau(t)} = a^1$ and takes a^1 in period s if and only if $\omega^*_{n,\tau(t)}(n-1) = a^1$. Similarly, in period $s \in \mathbb{S}_1[t, n, 1, 2]$, let

$$\pi_{n,s}(h_{n-1}) = \begin{cases} K_{1:3}\mathbf{1}_{\{\omega_{n-1,s}=a^0\}} & \text{if } \omega^*_{n,\tau(t)}(n-1) = a^1, \\ K_{1:3}\mathbf{1}_{\{\omega_{n-1,s}=a^1\}} & \text{if } \omega^*_{n,\tau(t)}(n-1) = a^0. \end{cases} \tag{145}$$

In period $s \in \mathbb{S}_1[t, n, 2, l]$ with $l \in \{1, 2\}$, let $\omega_{n-1}[t, n, l] = a^1$ if player $n-1$ observes at least once in $\mathbb{S}_1[t, n, 1, l]$; and let $\omega_{n-1}[t, n, l] = a^0$ otherwise.

Let $\overline{\Pr}(\cdot|\mathbf{a}^0)$ be the stationary distribution of μ given $\mathbf{a}^0$. Since players take $\mathbf{a}^0$ for S_1 periods, the match distribution in $\mathbb{S}_1$ is close to $\overline{\Pr}(\cdot|\mathbf{a}^0)$. Let $\mu_{\mathbb{S}_1} \sim \overline{\Pr}$ denote the event that matches in $\mathbb{S}_1$ are always drawn from $\overline{\Pr}(\cdot|\mathbf{a}^0)$.

Given this definition of $\mu_{\mathbb{S}_1} \sim \overline{\Pr}$, let

$$\pi_{n,s}\left(h_{n-1}\right) = \begin{cases} \frac{K_{1:2}}{\bar{\varepsilon}^2}\left(\mathbf{1}_{\{\omega_{n-1,s}=a^1\}} - \Pr\left(\mu_s(n) = n-1 | \mu_{\mathbb{S}_1} \sim \overline{\Pr}\right)\right) & \text{if } \omega_{n-1}[t,n,l] = a^1, \\ K_{1:2}\left(\mathbf{1}_{\{\omega_{n-1,s}=a^0\}} - 1\right) & \text{if } \omega_{n-1}[t,n,l] = a^0. \end{cases} \tag{146}$$

Note that if player n followed the equilibrium strategy in $\mathbb{S}_1[t,n,1]$ and matched with $n-1$ at least once in $\mathbb{S}_1[t,n,1]$, then in $\mathbb{S}_1[t,n,2]$, player n's expected payoff is 0 if she follows the equilibrium strategy and at most $-\bar{\varepsilon}K_{1:2}$ otherwise.

For period s that is the $(S_0+t+1)^{\text{th}}$ period of the unit, if player $n-1$ knew $(a_{-n,s}, \omega_{-n,s})$, she could identify $a_{n,\tau(t)}$ and define

$$\pi_{n,s}\left(a_{-n,s}, \omega_{-n,s}\right) = -\delta^s \hat{u}_n\left(\mathbf{a}_s\right) - \begin{cases} 0 & \text{if } a_{n,\tau(t)} \text{ is the equilibrium action,} \\ K_{1:4} & \text{otherwise.} \end{cases}$$

Let $h_{n-1}^{\mathbb{S}_1[t,-n,2]}$ be player $n-1$'s history in $(\mathbb{S}_1[t,n',2])_{n'\neq n}$, where players $-n$ send the message for the first time. Since

$$\Pr(a_{-n,s}(n-1), \omega_{-n,s}(n-1) | a_{-n,s}, \omega_{-n,s}, \{\mu_{\mathbb{S}_1} \sim \overline{\Pr}\}) \geq \frac{1}{2}$$

given (126), there exists

$$\pi_{n,s}\left(h_{n-1}^{\mathbb{S}_1[t,-n,2]}\right) \in [-4K_{1:4}, 4K_{1:4}] \tag{147}$$

such that

$$\mathbb{E}\left[\delta^s \hat{u}_n\left(\mathbf{a}_s\right) + \pi_{n,s}\left(h_{n-1}^{\mathbb{S}_1[t,-n,2]}\right) | \left\{\mu_{\mathbb{S}_1} \sim \overline{\Pr}\right\}, a_s\right] = \begin{cases} 0 & \text{if } a_{n,\tau(t)} \text{ is the equilibrium action,} \\ -K_{1:4} & \text{otherwise.} \end{cases} \tag{148}$$

(The rank condition again follows from Horn and Johnson (2013), Theorem 6.1.10). We add $\pi_{n,s}\left(h_{n-1}^{\mathbb{S}_1[t,-n,2]}\right)$ as the reward in period s that is the $(S_0+t+1)^{\text{th}}$ period of the unit.

In the $(S_0+1)^{\text{th}}$ period of the unit, let $w_n\left(\mu_{s+1}, m_i\right)$ be the continuation payoff from the next period (within the unit). Since the equilibrium strategy in the continuation play is independent of m_i or history up to $(S_0+1)^{\text{th}}$ period, we can write $w_n\left(\mu_{s+1}\right) := w_n\left(\mu_{s+1}, m_i\right)$.

Suppose player $n-1$ knew μ_s and μ_{s+1}. Then, the following reward would make player n indifferent between player i taking a^0 and a^1: Let

$$\alpha_\delta = \mathbf{1}_{\{\text{the current unit is in the first half of an interval}\}} \times (1-\delta^{\tilde{T}\lceil \log_2 M_i \rceil T})$$

be the effect of discounting when player i takes a^1 in the first half of the interval, rather than the second half. Given

$$\begin{aligned}
\pi_{n,s}\left(\mu_s,\mu_{s+1},\omega_{s,n-1}\right) &= -w_n(\mu_{s+1}) + \frac{K_1 \mathbf{1}_{\{\omega_{n-1,s}=a^0\}}}{p_{n-1,n}\left(\mu_s\right)} + \frac{\mathbf{1}_{\{\omega_{n-1,s}=a^1\}}\alpha_\delta\left(\hat{u}_n\left(\mathbf{a}^0\right)-\hat{u}_n\left(\mathbf{a}^1\right)\right)}{p_{n-1,i}\left(\mu_s\right)} \forall n \neq i, \\
\pi_{i,s}\left(\mu_s,\omega_{s,i-1}\right) &= -w_i\left(\mu_{s+1}\right) + \frac{\mathbf{1}_{\{\omega_{i-1,s}=a^1\}}\alpha_\delta}{p_{i-1,i}\left(\mu_s\right)}\left(\hat{u}_i\left(\mathbf{a}^0\right)-\hat{u}_i\left(\mathbf{a}^1\right)\right),
\end{aligned}$$

the same proof as Lemma 8 ensures that there exist $\bar{\pi}_n^0$ and $\bar{\pi}_n^1$ such that, for each μ_s and $n \in I$, we have

$$\begin{aligned}
\bar{\pi}_n^0 &= \mathbb{E}\left[\delta^{s-1}\hat{u}_n\left(\mathbf{a}_s\right) + \pi_{s,n}\left(\mu_s,\mu_{s+1},\omega_{s,n-1}\right) + w_n\left(\mu_{s+1}\right) | \mu_s, a_{-i}^0, a_i^0\right]; \\
\bar{\pi}_n^1 &= \mathbb{E}\left[\delta^{s-1}\hat{u}_n\left(\mathbf{a}_s\right) + \pi_{s,n}\left(\mu_s,\mu_{s+1},\omega_{s,n-1}\right) + w_n\left(\mu_{s+1}\right) | \mu_s, a_{-i}^0, a_i^1\right].
\end{aligned}$$

Moreover, $\bar{\pi}_i^0 = \bar{\pi}_i^1$ for sender i.

Recall that by (130), $h_{n-1}^{\mathbb{S}_1[t,-n,2]}$ statistically identifies $h^{\mathbb{C}}$, $h^{\mathbb{C}}$ statistically identifies μ_s and μ_{s+1} given $\mathbf{a}_s$, and $\omega_{s,n-1}$ identifies $a_{i,s}$ given μ_s and a_{-i}^0. Hence, there exists $\pi_{s,n}\left(\omega_{s,n-1}, h_{n-1}^{\mathbb{S}_1[t,-n,2]}\right)$ such that, for each n, μ_s, μ_{s+1}, $a_{i,s} \in \{a^0, a^1\}$, and $\omega_{s,n-1}$,

$$\mathbb{E}\left[\pi_{s,n}\left(\omega_{s,n-1}, h_{n-1}^{\mathbb{S}_1[t,-n,2]}\right) | \mu_s, a_i, a_{-i}^0, \omega_{s,n-1}, \mu_{s+1}\right] = \pi_{s,n}\left(\mu_s,\mu_{s+1},\omega_{s,n-1}\right). \tag{149}$$

We add $\pi_{s,n}\left(\omega_{s,n-1}, h_{n-1}^{\mathbb{S}_1[t,-n,2]}\right)$ as a reward for period s.

It will be useful to bound the variation of $\pi_{s,n}\left(\omega_{s,n-1}, h_{n-1}^{\mathbb{S}_1[t,-n,2]}\right)$. Since $K_1 \geq 2\max_{a,a' \in A^2}|u(a,a')|$ and $\alpha_\delta \in [0,1]$, there exists $\bar{S}$ such that for each δ, M_i, $S_1 \geq \bar{S}$, N, and T, given (125) and given that $\omega_{n-1,s} = a_{i,s}$ with probability at least $\bar{\varepsilon}$, we have

$$\left|\pi_{s,n}\left(\omega_{s,n-1}, h_{n-1}^{\mathbb{S}_1[t,-n,2]}\right) - \pi_{s,n}\left(\tilde{\omega}_{s,n-1}, \tilde{h}_{n-1}^{\mathbb{S}_1[t,-n,2]}\right)\right| \leq 2\frac{\bar{M}}{\bar{\varepsilon}}\left(\max_{\mu,\mu'}|w_n(\mu)-w_n(\mu')| + \frac{2K_1}{\bar{\varepsilon}}\right).$$

Here, we need additional slack 2 since $h_{n-1}^{\mathbb{S}_1[t,-n,2]}$ only statistically identifies $h^{\mathbb{C}}$. Since $h_{n-1}^{\mathbb{S}_1[t,-n,2]}$ identifies $h^{\mathbb{C}}$ with probability no less than $1 - N^3 \exp(-\bar{\varepsilon} S_1)$, for sufficiently large S_1, multiplying by 2 is sufficient.

We further bound the term $\max_{\mu,\mu'} |w_n(\mu) - w_n(\mu')|$ as follows. In the last period of $\mathbb{C}$—period $\bar{s} = s + N(N-1)/2 + 1$—the continuation payoff after period $\bar{s}$ can be written as $w_n(\mu_{\bar{s}}, h^{\mathbb{C}})$, since the continuation strategy, (144), (145), and (146) are determined by $h^{\mathbb{C}}$.

For any $h^{\mathbb{C}}$, player n takes a^0 and a^1 with the same frequency in continuation play. Moreover, given S_1, the distribution of $\mu_{\tilde{s}}$ for each $\tilde{s} \in \mathbb{S}_1$ is close to the stationary distribution given a^0: by Lemma 32, for each $\mu_{\bar{s}}$, we have

$$\left| \overline{\Pr}\left(\mu_{\tilde{s}} | \mu_{\bar{s}}, \mathbf{a}^0\right) - \overline{\Pr}\left(\mu_{\tilde{s}} | \mathbf{a}^0\right)\right| \leq (1-\bar{\varepsilon})^{S_1}. \tag{150}$$

In total, we have

$$\max_{\mu, h^{\mathbb{C}}, \tilde{\mu}, \tilde{h}^{\mathbb{C}}} \left| w_n(\mu, h^{\mathbb{C}}) - w_n(\tilde{\mu}, \tilde{h}^{\mathbb{C}})\right| \leq \left(N(1-\bar{\varepsilon})^{S_1} + 1 - \delta^{\tilde{T}}\right) \times \tilde{T} \times (\bar{u} + K_{1:5}).$$

Here, discounting represents the effect of players taking a^0 earlier rather than later in $\mathbb{S}_1$.

We now bound the expected reward during periods $\mathbb{C}$. For each $K_{1:4}$ and $\varepsilon > 0$, for sufficiently large S_1 and δ, for each M_i, T, and $s \in \mathbb{C}$, (148) and (150) imply

$$\mathbb{E}\left[\delta^s \hat{u}_n(\mathbf{a}_s) + \pi_{n,s}\left(\omega_{s,n-1}, h_{n-1}^{\mathbb{S}_1[t,-n,2]}\right) | m_i, \mu_{\tau(t)}\right] \begin{cases} \in [-\varepsilon, \varepsilon] & \text{if } n \notin C_s \text{ and } a_{n,s} = a^0, \\ \in [-\varepsilon, \varepsilon] & \text{if } n \in C_s \text{ and } a_{n,s} = a^1, \\ \leq -(1-\varepsilon) K_{1:4} & \text{otherwise.} \end{cases} \tag{151}$$

In total, for each $K_1, K_2, K_3, K_4, K_4, K_6$, we have

$$\lim_{S_1 \to \infty, \delta \to 1} \sup_{T \in \mathbb{N}, \delta \in [0,1], M_i} \left| \pi_{s,n}\left(\omega_{s,n-1}, h_{n-1}^{\mathbb{S}_1[t,-n,2]}\right) - \pi_{s,n}\left(\tilde{\omega}_{s,n-1}, \tilde{h}_{n-1}^{\mathbb{S}_1[t,-n,2]}\right)\right| \leq 2 \frac{\bar{M}}{\bar{\varepsilon}} \frac{K_1}{\bar{\varepsilon}}. \tag{152}$$

We now determine $K_1, K_2, K_3, K_4, K_5, K_6, S_1$ while verifying incentive compatibility.

1. There exists K_6 such that, for each δ, M_i, T, K_1, K_2, K_3, K_4, K_5 and S_1, each player takes $a \in \{a^0, a^1\}$ for any s and h_n^{s-1}, and takes $a = a^0$ for any $s \in \mathbb{S}[a^0]$ and h_n^{s-1}. In

the last period, this is true. Given this incentive, for s being the last period, we have $-K_{1:6}1_{\{\omega_{n-1,s}\neq a^0,a^1\}}=0$, and so $\delta^s\hat{u}_n\left(\mathbf{a}_{\tau(t)}\right)+\pi_{n,s}\left(h_{n-1}\right)=\delta^s u\left(a^0,a^0\right)$ if $s\in\mathbb{S}[a^0]$. Hence, in the preceding periods, players ignore these payoffs. Since other per-period rewards are bounded by $K_{1:5}$, recursively, we establish the incentive compatibility. Players ignore

$$\sum_s -K_{1:6}1_{\{\omega_{n-1,s}\neq a^0,a^1\}}+\sum_{s\in\mathbb{S}[a^0]}\left(\delta^s\hat{u}_n\left(\mathbf{a}_{\tau(t)}\right)-K_{1:6}\mathbf{1}_{\{\omega_{n-1,s}\neq a^0\}}\right). \tag{153}$$

In what follows, we sequentially fix K_5, K_2, K_3, K_4, and K_1. When we say "there exists K_n such that [statement]" it means that given $K_{n'}$'s that have been already fixed, there exist $\bar{S}$ and $\bar{\delta}<1$ such that, for each remaining $K_{n'}$'s, $S_1\geq\bar{S}$, $\delta\geq\bar{\delta}$, M_i, and T, [statement] holds.

2. There exists K_5 such that, for any period $s\in\mathbb{S}_1[t,n',l]$ with $t\in\mathbb{C}$, $n'\neq n$, and $l\in\{1,2\}$, for any history h_n^{s-1}, taking a^0 is optimal.

   The other rewards are bounded by $K_{1:4}$ for $\tau\notin(\mathbb{S}_1[t,n',l])_{t\in\mathbb{C},n'\neq n,l\in\{1,2\}}$. Since players $-n$'s continuation play does not depend on their observations in period s, we are left to bound the effect of changing the distribution of her match in period $s+(S_1+1)$. Since the per-period payoff $\delta^s\hat{u}_n\left(\mathbf{a}_s\right)+\pi_{n,s}(h_{n-1})$ is bounded by $\bar{u}+\bar{K}_{1:5}$, the value at the beginning of period $s+(S_1+1)$ is bounded by $\tilde{T}\left(\bar{u}+\bar{K}_{1:5}\right)$. Given $\bar{\varepsilon}>0$, since players will take a^0 in $\mathbb{S}[a^0]$ in continuation play regardless of the history, the variation of the continuation payoff in period $s+(S_1+1)$ with respect to $a_{n,s}$ is bounded by $(1-\bar{\varepsilon})^{S_1}\tilde{T}\left(\bar{u}+\bar{K}_{1:5}\right)$, by Lemma 33. For large S_1, the per-period reward $\pi_{n,s}\left(h_{n-1}\right)=K_{1:5}1_{\{\omega_{n-1,s}=a^0\}}$ for the current period s is sufficiently large to make a^0 optimal after any history h_n^{s-1}.

3. There exists K_2 such that, for each period $s\in\mathbb{S}_1[t,n,2,l]$ with $l\in\{1,2\}$, player n follows the equilibrium strategy.

   Note that player n believes that, in each $\tau\in\mathbb{S}_1[t,n,1,l]$, (141) holds after each history. Hence, since (i) (144), (145), and $h_{n-1}^{\mathbb{S}_1[t,-n,2]}$ do not depend on the history in $\mathbb{S}_1[t,n,2,l]$ and (ii) the effect of changing the future match distribution is bounded as in Case 2,

(146) implies that it is optimal to take a^1 if and only if she took a^1 at least once in the same cycle in $\mathbb{S}_1[t, n, 1, l]$.

4. There exists K_3 such that, for each period $s \in \mathbb{S}_1[t, n, 1]$, player n follows the equilibrium strategy.

   Player n believes that (i) $\mathring{a}_{-n,t}, \mathring{\omega}_{-n,t}$ follows $\beta(\mathring{a}_{-n,t}, \mathring{\omega}_{-n,t}|\mu_t, a_{n,t}, \omega_{n,t})$ given μ_t, (ii) players $-n$ in each $\tau \in \mathbb{S}_1[t, n, 2]$ will tell the truth about $\mathring{\omega}_{-n,t}$, (iii) player n will follow the equilibrium strategy in $\mathbb{S}_1[t, n, 2]$, and (iv) each pair matches with each other at least once, and so $\omega^*_{n,\tau(t)}(n-1) = \mathring{\omega}_{n,\tau(t)}$ with probability at least $(1-\bar{\varepsilon})^{S_1}$ in $\mathbb{S}_1[t, n, 2]$. Hence, telling the truth about $\mathring{\omega}_{n,\tau(t)}$ maximizes (144) and (145).

   Since (146) is bounded by $K_{1:2}$, $h_{n-1}^{\mathbb{S}_1[t,-n,2]}$ do not depend on the history in $\mathbb{S}_1[t, n, 1]$, and the effect of changing the future match distribution is bounded as in Case 2, telling the truth about $\mathring{\omega}_{n,\tau(t)}$ is optimal.

5. There exists K_4 such that, for each period $t \in \mathbb{C}$, player n follows the equilibrium strategy.

   Players will follow the equilibrium strategy in $\mathbb{S}_1[t, -n, 2]$. Hence, $h_{n-1}^{\mathbb{S}_1[t,-n,2]}$ identifies $(\mathring{a}_{-n,t}, \mathring{\omega}_{-n,t})$ with probability at least $N^2(1-\bar{\varepsilon})^{S_1}$. Therefore, the deviation costs approximately $-K_{1:4}$.

   Since players $-n$ follow $\bar{\sigma}^*_{-n,t}$ and will tell the truth about $\omega_{-n,t}$ and player n will tell the truth about $\mathring{\omega}_{n,t}$, as long as each pair is matched at least once, the expected payoff of (144), (145), and (146) are independent of $\mathring{a}_{n,t}$. Since all pairs match with probability at least $2N^2(1-\bar{\varepsilon})^{S_1}$, for sufficiently large K_4, (151) implies that $\bar{\sigma}_n^{m_i}$ is optimal for each h_n^{s-1}.

6. For $K_1 \geq 2\bar{u}$, player n follows $\bar{\sigma}_n^{m_i}$ in $t = S_0 + 1$ by (149). Moreover, $\bar{\pi}_n^0$ and $\bar{\pi}_n^1$ imply that players' payoffs do not depend on the initial state.

■

M.2 Secure Communication Module

Premise: We modify the premise as follows:

1. All players but player i^* are indifferent about the result of the communication, and sender i satisfies $i \neq i^*$.

2. If $ALLREG$ does not occur, then $w_{i^*}(\mathfrak{h}) = 0$ for all $\mathfrak{h}$.

3. If $ALLREG$ occurs, then player i^*'s continuation payoff depends only on $m_i(i^* - 1)$ and the first $N(N-1)(1+NS_0)/2$-period history of player $i^* - 1$, denoted by $h_{i^*-1}^{\leq N(N-1)(1+NS_0)/2}$. Denote this continuation payoff by $w_{i^*}(m_i(i^*-1), h_{i^*-1}^{\leq N(N-1)(1+NS_0)/2})$.

   (a) If $m_i(i^*-1) = 0$, then the continuation payoff does not depend on $h_{i^*-1}^{\leq N(N-1)(1+NS_0)/2}$: $w_{i^*}(m_i(i^*-1), h_{i^*-1}^{\leq N(N-1)(1+NS_0)/2}) := w_{i^*}(0)$.

   (b) Otherwise, given the same $m_i(i^*-1)$, the magnitude by which the reward depends on $h_{i^*-1}^{\leq N(N-1)(1+NS_0)/2}$ is small:

$$\max_{m_i \in M_i, h_{i^*-1}, \tilde{h}_{i^*-1}} \left| w_{i^*}(m_i, h_{i^*-1}^{\leq N(N-1)(1+NS_0)/2}) - w_{i^*}(m_i, \tilde{h}_{i^*-1}^{\leq N(N-1)(1+NS_0)/2}) \right| \leq 1.$$

   (c) $w_{i^*}(0) \leq w_{i^*}(m_i(i^*-1), h_{i^*-1}^{\leq N(N-1)(1+NS_0)/2})$ for all $m_i(i^*-1) \in M_i$ and $h_{i^*-1}^{\leq N(N-1)(1+NS_0)/2}$.

Note that these premises imply that the continuation payoff is independent of the realized match at the beginning of the protocol given $h_{i^*-1}^{\leq N(N-1)(1+NS_0)/2}$.

Reward Function: Since now player j's reward can depend on $(a_{-j,t}, \omega_{-j,t})$ and we can identify $(a_{j,t}, \omega_{j,t})$ perfectly, we can define the rewards $\pi_j^{\text{cancel}}(a_{-j}, \omega_{-j})$ and $\pi_j^a(a_{-j}, \omega_{-j})$ for each $a \in A$ such that, conditional on any realized matching and $\mathbf{a} \in A^I$, we have

$$\hat{u}_j(\mathbf{a}) + \pi_j^{\text{cancel}}(a_{-j}, \omega_{-j}) = 0, \tag{154}$$

and

$$\pi_j^a(a_{-j}, \omega_{-j}) = \begin{cases} 0 & a_j = a \\ -1 & a_j \neq a \end{cases}. \tag{155}$$

Given this definition, we add the following rewards: For the first $N(N-1)/2$ periods, we add $\pi_j^{\text{cancel}}(a_{-j}, \omega_{-j}) + \pi_j^{\bar{a}_{j,t}}(a_{-j,t}, \omega_{-j,t})$, where $\bar{a}_{j,t}$ is player j's equilibrium strategy (recall that players' strategies are pure and independent of m_i for the first $N(N-1)/2$ periods). Let $h^{\mathbb{C}}$ be the history profile in periods $\mathbb{C}$ uniquely identified from $h_{-j}^{\mathbb{C}}$. For the next $N^2(N-1)S_0/2$ periods, we add $\pi_j^{\text{cancel}}(a_{-j}, \omega_{-j}) + \pi_j^{\bar{a}_{j,t}|_{h^{\mathbb{C}}}}(a_{-j,t}, \omega_{-j,t})$ where $\bar{a}_{j,t}|_{h^{\mathbb{C}}}$ is player j's equilibrium strategy given $h^{\mathbb{C}}$. For the remaining periods, we add $\pi_j^{\text{cancel}}(a_{-j}, \omega_{-j})$ for players $-i^*$ and $\pi_{i^*}^{\text{cancel}}(a_{-i^*}, \omega_{-i^*}) + \pi_{i^*}^{a^0}(a_{-i^*}, \omega_{-i^*})$ for player i^*, as in the i.i.d. matching case.

We will show that, given this definition of the reward function, the equilibrium strategy is a j-quasi-ex-post belief-free equilibrium:

Definition 3 *A strategy profile σ is a i^**-quasi-ex-post belief-free equilibrium (j-QXBFE) *if (i) for each player $n \neq i^*$, μ_1, and extended history $\mathfrak{h}_n$, the continuation strategy $\sigma_n|_{\mathfrak{h}_n}$ is a best response against $\sigma_{-n}|_{\mathfrak{h}_{-n}}$ for every μ_1 and opposing history profile $\mathfrak{h}_{-n}$, and (ii) for player i^*, there exists a sequence of families of completely mixed strategy profiles $\left((\sigma_i^{m_i,k}, \sigma_{-i}^k)_{m_i \in M_i}\right)_{k=1}^{\infty}$ and a corresponding family of belief systems $\beta(\mathfrak{h}_{-i^*}|\mu_1, m_i, h_{i^*})$ (where $\beta(\mathfrak{h}_{-i^*}|\mu_1, m_i, h_{i^*})$ is the limit of conditional probabilities derived from $\left(\left(\sigma_i^{m_i,k}, \sigma_{-i}^k\right)\right)_{k=1}^{\infty}$) such that, for each μ_1, m_i, and $h_{i^*}^{t-1}$, $\sigma_{i^*}|_{\mathfrak{h}_{i^*}^{t-1}}$ is sequentially rational given β.*

Note that players' per-period payoff and reward are independent of the realized match: for each $j \in I$, $(\bar{a}_{j,t}, a_{-j,t})$, and μ_t,

$$\hat{u}_j(\mathbf{a}_t) + \pi_j^{\text{cancel}}(a_{-j,t}, \omega_{-j,t}) + \pi_j^{\bar{a}_{j,t}}(a_{-j,t}, \omega_{-j,t}) = \begin{cases} 0 & a_{j,t} = \bar{a}_{j,t}, \\ -1 & a_{j,t} \neq \bar{a}_{j,t}. \end{cases}$$

In addition, given Lemma 34, the same proof as Lemma 9 implies that the secure communication protocol is an i^*-QXBFE from period $N(N-1)(1+NS_0)/2$ on. Hence, given 3(a) and 3(b) of the premise, the expected payoff loss from deviating of 1 (due to rewards $\pi_j^{\bar{a}_{j,t}}$ and $\pi_j^{\bar{a}_{j,t}|_{h^{\mathbb{C}}}}$) is sufficient to show that the secure communication protocol is an i^*-QXBFE for the first $N(N-1)(1+NS_0)/2$ periods.

M.3 Verified Communication Module

The premise is as in the i.i.d. case. In addition, assume that

$$\max_{j \in I, \mathfrak{h}, \mathfrak{h}', \zeta, \zeta'} \left| w_j(\mathfrak{h}, \zeta) - w_j(\mathfrak{h}', \zeta') \right| \leq T^{K+\frac{1}{2}}. \tag{156}$$

Suppose player j maximizes $\pi_j\left(h_{-j}, \zeta_j\right) + w_j(\mathfrak{h}, \zeta)$, where

$$\begin{aligned} \pi_j\left(h_{-j}, \zeta_j\right) &= \mathbf{1}_{\{\zeta_j = \text{reg}\}} \sum_{t \in \mathbb{T}_j^{a^0}} \pi_j^{a^0}\left(a_{-j,t}, \omega_{-j,t}\right) + \mathbf{1}_{\{\zeta_j = \text{reg}\}} \sum_{t \in \mathbb{T}} \sum_{\tau = t+1}^{t+N(N-1)/2} \pi_{j,\tau}^{\bar{\sigma}^*}\left(a_{-j,\tau}, \omega_{-j,\tau}\right) \\ &\quad + \mathbf{1}_{\{\zeta_j = \text{reg}\}} \sum_{t \in \mathbb{T}(j)} \pi_j^{\bar{a}_{j,t}}\left(h_{-j}\right). \end{aligned} \tag{157}$$

Here, $\mathbb{T}_j^{a^0}$ is the set of periods in which player j takes a^0 regardless of the history given $\zeta_j = \text{reg}$ (that is, the $N(N-1)/2 + 1^{\text{th}}$ to $N(N-1)/2 + \left\lceil \log T^{K+2} \right\rceil^{\text{th}}$ period of each unit, and the $N(N-1)/2 + \left\lceil \log T^{K+2} \right\rceil + 1^{\text{th}}$ period in each round where player j is not a sender); and $\mathbb{T}$ is the set of first periods of all the units. Moreover, define

$$\pi_{j,t}^{\bar{\sigma}^*}\left(a_{-j,t}, \omega_{-j,t}\right) = \begin{cases} \pi_j^{a^1}\left(a_{-j,t}, \omega_{-j,t}\right) & \text{if player } j \text{ is supposed to take } a^1 \text{ in } \bar{\sigma}^*, \\ \pi_j^{a^0}\left(a_{-j,t}, \omega_{-j,t}\right) & \text{otherwise.} \end{cases} \tag{158}$$

Finally, $\mathbb{T}(j)$ is the set of $N(N-1)/2 + \left\lceil \log T^{K+2} \right\rceil + 1^{\text{th}}$ periods of j-checking round, and $\pi_j^{\bar{a}_{j,t}}\left(h_{-j}\right)$ is defined as (34).

Definition 4 *Given a prior* $\Pr_j$ *on* $(\zeta_n)_{n \in I}$ *for each player* $j \in I$*, we say that the verified communication protocol is an ex post sequential equilibrium if, for each* $j \in I$ *and* μ_1*,* σ_j *is sequentially rational given belief* $\Pr_j$*, realized match* μ_1*, and strategy* σ_{-j}*.*

Note that the reward function does not depend on the realized match, since (34) and (155) hold conditional on the realized match. Given this definition of $\pi_j\left(h_{-j}, \zeta_j\right)$, Lemma 10 holds (with sequential equilibrium replaced with ex post sequential equilibrium). We modify the proof in Section J.3 as follows:

Incentive from the $N(N-1)/2 + 1^{\text{th}}$ *to the* $N(N-1)/2 + \left\lceil \log T^{K+2} \right\rceil$ *period:* Each player $j \in I$ has an incentive to take a^0, since otherwise $\text{susp}(h_n) = 1$ for some $n \neq j$ and

$\theta_j(\mathfrak{h}_{-j},\zeta) = R$, which leads to the worst continuation payoff.

Incentive from the 1st to $N(N-1)/2^{th}$ periods: Given that players take a^0 from the $N(N-1)/2+1^{\text{th}}$ to the $N(N-1)/2+\lceil \log T^{K+2}\rceil^{\text{th}}$ period, the effect of player i's strategy in $N(N-1)^{\text{th}}$ period changes the distribution of $\mu_{t_\tau+N(N-1)/2+\lceil \log T^{K+2}\rceil+1}$ by only $(1-\bar{\varepsilon})^{\lceil \log T^{K+2}\rceil}$, and changes the distribution of $\tilde{\omega}_{j,t_\tau}$ and $\tilde{\omega}^{j'}_{j,t_\tau}$ by $\frac{1}{T^{K+1}}$. Since $\max_{\mathfrak{h},\zeta,\mathfrak{h}',\zeta'}|w_j(\mathfrak{h},\zeta)-w_j(\mathfrak{h}',\zeta')| \leq T^{K+\frac{1}{2}}$, player j has an incentive to take $\bar{\sigma}^*$ given $\pi^{\bar{\sigma}^*}_{j,t}$.

Receiver's incentives in the checking round: Given the above strategy, given that other players have $\tilde{\omega}_{n,t_\tau} = a^1$ at least once in each half-interval when player j' takes a^1, taking a^0 maximizes the equilibrium payoff since this strategy guarantees $\tilde{\omega}_{n,t_\tau} = a^0$ outside of the half-intervals in which player j' takes a^1. Hence, the proof in the i.i.d. case goes through given the modification of Lemma 7.

Sender's incentives in the checking round: Given the definition of $\left(\tilde{\omega}^{j'}_{j,t_\tau}\right)_{j\neq j'}$, the probability of FAILj' is maximized when player j' follows $\sigma^*_{j'}$ (recall Property 3 of Section L.3). Hence, the proof in the i.i.d. case goes through.

Initial Sender's incentives in the message round: Given the definition of $\left(\tilde{\omega}^{j'}_{j,t_\tau}\right)_{j\neq j'}$, the probability of FAILj' is independent of her history in the message round (again follows from Property 3). Hence, the proof in the i.i.d. case goes through.

M.4 Jamming Coordination Module

We now augment the jamming coordination protocol. The premise is as in the i.i.d. case (in particular, the continuation payoff is independent of the realized match). As in Lemma 11, given this premise, we construct $\pi^{\text{indiff}}_{i,t}(h_{-i}|T)$ such that

1. $\lim_{T\to\infty}\max_{h_{-i}}\frac{\sum_{t=1}^{N(N-1)/2+2}\left|\pi^{\text{indiff}}_{i,t}(h_{-i}|T)\right|}{T^6} = 0$.

2. If the premise for jamming coordination is satisfied, then the jamming coordination protocol is a sequential equilibrium conditional on the realization of the initial match.

3. Moreover, we require that the value $\mathbb{E}[\sum_{t=1}^{N(N-1)/2+2}\pi^{\text{indiff}}_{i,t}(h_{-i}|T) + w_i(h|T)|\mu_1]$ is independent of μ_1.

Suppose player $i-1$ knew $\mu_{N(N-1)/2+1}$. Then, the same proof as in the i.i.d. case implies that, given each $\mu_{N(N-1)/2+1}$, there exists $\hat{\pi}_{i,t}^{\text{indiff}}\left(h_{-i}|\mu_{N(N-1)/2+1},T\right)$ such that

$$\sum_{t\geq N(N-1)/2+1}\hat{\pi}_{i,t}^{\text{indiff}}\left(h_{-i}|\mu_{N(N-1)/2+1},T\right)+w_i(h|T)$$

makes player i indifferent among all actions for periods $t\geq N(N-1)/2+1$, and

$$\lim_{T\to\infty}\max_{\mu_{N(N-1)/2+1},h_{-i}}\frac{\sum_{t\geq N(N-1)/2+1}\left|\hat{\pi}_{i,t}^{\text{indiff}}\left(h_{-i}|\mu_{N(N-1)/2+1},T\right)\right|}{T^6}=0.$$

Given $\mu_{N(N-1)/2+1}$, player i's continuation payoff in period $N(N-1)/2+1$,

$$v_i(\mu_{N(N-1)/2+1}):=\mathbb{E}\left[\sum_{t\geq N(N-1)/2+1}\hat{\pi}_{i,t}^{\text{indiff}}\left(h_{-i}|\mu_{N(N-1)/2+1},T\right)+w_i(h|T)|\mu_{N(N-1)/2+1}\right],$$

can be calculated, assuming player i takes a^0 for the rest of the protocol (given the definition of $\hat{\pi}_{i,t}^{\text{indiff}}$, she is indifferent among all $a\in A$). The probability that player i becomes a jamming player (i.e., observes a^1 in period $N(N-1)/2+1$ or $N(N-1)/2+2$) does not depend on $\mu_{N(N-1)/2+1}$ since each player takes a symmetric and i.i.d. strategy in these two periods. Since the range of $w_i(h|T)$ given ζ_i is of order T^5 given the premise, we have

$$\lim_{T\to\infty}\frac{\max_\mu v_i(\mu)-\underline{v}_i}{T^6}=0,$$

where $\underline{v}_i:=\min_\mu v_i(\mu)$.

Since we assume that $h_{-i}^{N(N-1)/2}$ identifies $\mu_{N(N-1)/2+1}$, by (125), for $t\geq N(N-1)/2+1$ there exists $\pi_{i,t}^{*,\text{indiff}}\left(h_{-i}|T\right)$ such that (i) conditional on players' following the equilibrium strategy in periods $1,...,N(N-1)/2$, we have

$$\mathbb{E}\left[\sum_{t\geq N(N-1)/2+1}\pi_{i,t}^{*,\text{indiff}}\left(h_{-i}|T\right)|\mu_1\right]=\sum_t\hat{\pi}_{i,t}^{\text{indiff}}\left(h_{-i}|\mu_{N(N-1)/2+1},T\right)-v_i(\mu_{N(N-1)/2+1})+\underline{v}_i,$$

and

$$\lim_{T\to\infty}\max_{h_{-i}}\frac{\sum_{t\geq N(N-1)/2+1}\left|\pi_{i,t}^{*,\text{indiff}}\left(h_{-i}|T\right)\right|}{T^{6}}=0.$$

Therefore, if we define

$$\pi_{i,t}^{\text{indiff}}\left(h_{-i}|T\right)=\begin{cases}\pi_{i,t}^{*,\text{indiff}}\left(h_{-i}|T\right) & \text{for } t\geq N(N-1)/2+1,\\ K_{T}\pi_{i,t}^{\bar{\sigma}^{*}}\left(a_{-i,t},\omega_{-i,t}\right) & \text{for } t\leq N(N-1)/2,\end{cases}$$

with

$$\pi_{i,t}^{\bar{\sigma}^{*}}\left(a_{-i,t},\omega_{-i,t}\right)=\begin{cases}\pi_{i}^{a^{1}}\left(a_{-i,t},\omega_{-i,t}\right) & \text{if player } i \text{ is supposed to take } a^{1} \text{ in } \bar{\sigma}^{*},\\ \pi_{i}^{a^{0}}\left(a_{-i,t},\omega_{-i,t}\right) & \text{otherwise}\end{cases}$$

and

$$K_{T}=2\max_{h_{-i}}\sum_{t\geq N(N-1)/2+1}\left|\pi_{i,t}^{\text{indiff}}\left(h_{-i}|T\right)\right|, \tag{159}$$

then we have

$$\lim_{T\to\infty}\max_{h_{-i}}\frac{\sum_{t=1}^{N(N-1)/2+2}\left|\pi_{i,t}^{\text{indiff}}\left(h_{-i}|T\right)\right|}{T^{6}}=0.$$

Moreover, (159) ensures that the jamming coordination protocol is an ex post sequential equilibrium for the first $N(N-1)/2$ periods. Finally, player i's payoff does not depend on μ_1: for each μ_1, we have

$$\begin{aligned}&\mathbb{E}\left[\sum_{t=1}^{N(N-1)/2+2}\pi_{i,t}^{\text{indiff}}\left(h_{-i}|T\right)+w_{i}(h|T)|\mu_{1}\right]\\ =\;&\mathbb{E}\left[\sum_{t\geq N(N-1)/2+1}\hat{\pi}_{i,t}^{\text{indiff}}\left(h_{-i}|\mu_{N(N-1)/2+1},T\right)+w_{i}(h|T)-v_{i}(\mu_{N(N-1)/2+1})+\underline{v}_{i}|\mu_{1}\right]=\underline{v}_{i}.\end{aligned}$$

N Block Belief-Free Structure

We impose the block belief-free equilibrium conditions ex post in μ_1 (the match realization in the first period), and require that $v_i(G)$, $v_i(B)$, and the reward function do not depend on μ_1. Hence, now ex post belief system $\beta=(\beta_i)_{i\in I}$ consists of, for each player $i\in I$, initial match realization μ_1, opposing state vector $x_{-i} \in \{G, B\}^{N-1}$, period $t \in \{1, \ldots, T^{**}\}$, and block history $h_i^{t-1} \in H_i^{t-1}$, a probability distribution $\beta_i\left(\cdot|\mu_1, x_{-i}, h_i^{t-1}\right) \in \Delta\left(H_{-i}^{t-1}\right)$, which satisfies Kreps-Wilson consistency.[30]

In addition, we allow player i's reward function to depend on the result of player $i-1$'s private mixture. That is, we consider the following conditions: Let $\mathfrak{h}_{i-1}^{T^{**}} = \{\chi_{i-1,t}, a_{i-1,t}, \omega_{i-1,t}\}_{t=1}^{T^{**}} \in \mathfrak{H}_{i-1}^{T^{**}}$ be player $i-1$'s extended histories with $\chi_{i-1} \in \{0,1\}^N$.

1. [Sequential Rationality] For all μ_1, $x \in \{G, B\}^N$ and $h_i^{t-1} \in H_i^{t-1}$,

$$\sigma_i^*\left(x_i\right) \in \arg\max_{\sigma_i \in \Sigma_i} \mathbb{E}^{\left(\sigma_i, \sigma_{-i}^*(x_{-i})\right)}\left[\sum_{\tau=1}^{T^{**}} \delta^{\tau-1} \hat{u}_i\left(\mathbf{a}_\tau\right) + \pi_i^*(x_{i-1}, \mathfrak{h}_{i-1}^{T^{**}}) | \mu_1, h_i^{t-1}\right].$$

(Recall that $\chi_{i,t}$ is independent of $\mathfrak{h}_{-i}^{t-1}$ conditional on h_i^{t-1}. Hence, we condition only on player i's non-extended history $h_i^{t-1} = (a_{i,\tau}, \omega_{i,\tau})_{\tau=1}^{t-1}$.)

2. [Promise Keeping] For all μ_1 and $x \in \{G, B\}^N$,

$$v_i^*(x_{i-1}) = \mathbb{E}^{\sigma^*(x)}\left[\frac{1-\delta}{1-\delta^{T^{**}}} \sum_{t=1}^{T^{**}} \delta^{t-1} \hat{u}_i\left(\mathbf{a}_t\right) + \pi_i^*(x_{i-1}, \mathfrak{h}_{i-1}^{T^{**}}) | \mu_1\right].$$

3. [Self-Generation] For all $x_{i-1} \in \{G, B\}$ and $\mathfrak{h}_{i-1}^{T^{**}}$,

$$\begin{aligned} \frac{1-\delta}{\delta^{T^{**}}} \pi_i^*(G, \mathfrak{h}_{i-1}^{T^{**}}) &\leq 0, \frac{1-\delta}{\delta^{T^{**}}} \pi_i^*(B, \mathfrak{h}_{i-1}^{T^{**}}) \geq 0, \left|\frac{1-\delta}{\delta^{T^{**}}} \pi_i^*(x_{i-1}, \mathfrak{h}_{i-1}^{T^{**}})\right| \leq v_i^*(G) - v_i^*(B); \\ v_i^*(B) &< v_i < v_i^*(G). \end{aligned}$$

Lemma 37 (Hörner and Olszewski (2006)) *For all* $\mathbf{v} \in \mathbb{R}^N$ *and* $\delta \in [0,1)$*, if there exist* $T^{**} \in \mathbb{N}$, $(\sigma_i^*\left(x_i\right))_{i \in I, x_i \in \{G,B\}}$, $(v_i^*(x_{i-1}))_{i \in I, x_{i-1} \in \{G,B\}}$, *consistent ex post belief system* β^*, *and*

[30] In particular, together with a block strategy profile $(\sigma_i\left(x_i\right))_{i \in I, x_i \in \{G,B\}}$, an ex post belief system is *consistent* if there exists a sequence of completely mixed block strategy profiles $\left(\left(\sigma_i^k\left(x_i\right)\right)_{i \in I, x_i \in \{G,B\}}\right)_{k \in \mathbb{N}}$ converging pointwise to $(\sigma_i\left(x_i\right))_{i \in I, x_i \in \{G,B\}}$ such that, for each $i \in I$, μ_1, $x_{-i} \in \{G, B\}^{N-1}$, $t \in \{1, ..., T^{**}\}$, and $h^{t-1} \in H^{t-1}$, we have

$$\beta(h_{-i}^{t-1} | \mu_1, x_{-i}, h_i^{t-1}) = \lim_{k \to \infty} \Pr^{\left(\sigma_j^k(x_j)\right)_{j \neq i}}\left(h_{-i}^{t-1} | \mu_1, x_{-i}, h_i^{t-1}\right).$$

$\left(\pi_i^*(x_{i-1}, \mathfrak{h}_{i-1}^{T^{**}})\right)_{i\in I, x_{i-1}\in\{G,B\}, \mathfrak{h}_{i-1}^{T^{**}}\in\mathfrak{H}_{i-1}^{T^{**}}}$ *such that [Sequential Rationality]–[Self-Generation] are satisfied, then* $\mathbf{v} \in E(\delta, \mu_1)$ *for each* μ_1.

As the equilibrium conditions are imposed ex post in μ_1, the proof is the same as in the i.i.d. case. Note that it is straightforward to allow the reward to depend on player $i-1$'s private mixture since player $i-1$ can depend her state transition on her own mixture.

Since the feasible payoff set is now determined by finitely repeated game strategies. For each $x \in \{G,B\}^N$ and finite κ_0, let $\sigma^{\kappa_0}(x)$ be a κ_0-period finitely repeated game strategy. Suppose players take $\sigma^{\kappa_0}(x)$ repeatedly, and let $\mathbf{u}^{\kappa_0}(x)$ be the average payoff profile from $\sigma^{\kappa_0}(x)$ under the resulting stationary distribution. As in the i.i.d. case, we can find κ_0, $\sigma^{\kappa_0}(x)$, and $\varepsilon^* > 0$ such that

$$\begin{aligned} v_i(G) \quad &: \quad = \min_{x:x_{i-1}=G} u_i^{\kappa_0}(x), \\ v_i(B) \quad &: \quad = \max_{x:x_{i-1}=B} \max\left\{u_i^{\kappa_0}(x), \underline{u} + 10\varepsilon^*\right\}, \text{ and} \\ v_i(B) + 10\varepsilon^* \quad &< \quad v_i < v_i(G) - 10\varepsilon^*. \end{aligned} \tag{160}$$

Compared to (160), we have taken $\varepsilon^* > 0$ smaller so that we have $10\varepsilon^*$ slack rather than $9\varepsilon^*$. Given κ_0 and ε^*, fix $\kappa_1 \in \mathbb{N}$ such that

$$\kappa_1 \geq \frac{8N(N-1)\kappa_0 \bar{u}}{\varepsilon^*} \max\left\{\frac{\left\|(P)^{-1}\right\|}{\bar{\varepsilon}}, 1\right\}. \tag{161}$$

By viewing the $\kappa_0\kappa_1$-period finitely repeated game as the repetition of κ_0-period finitely repeated games where players take $\sigma^{\kappa_0}(x)$, by Lemma 33, for each μ_1, we have

$$\left|\frac{1}{\kappa_0\kappa_1}\mathbb{E}\left[\sum_{t=1}^{\kappa_0\kappa_1} \hat{u}(\mathbf{a}_t)|\mu_1\right] - u_i^{\kappa_0}(x)\right| \leq \frac{1}{\kappa_0\kappa_1}\frac{\kappa_0\bar{u}}{\bar{\varepsilon}} = \frac{\bar{u}}{\kappa_1\bar{\varepsilon}}. \tag{162}$$

The calendar time structure is unchanged, except that

1. The phases to coordinate on the set of jamming players and x are expanded, according to the modified protocols defined above.

2. We replace each period of the main payoff phase with the following $N(N-1)/2+\kappa_0\kappa_1$ periods, which we call a **unit of the main payoff phase**:

(a) For the first $N(N-1)/2$ periods, players play $\bar{\sigma}^*$ to identify μ.

(b) For next κ_0 periods, player i plays $\sigma_i^{\kappa_0}(x(i))$ if she has not identified a deviation in the previous phases; and plays $\alpha^{\min}$ otherwise. Repeat this κ_0-period cycle κ_1 times.

Let $\mathbb{T}_{\text{initial}}(l) := \{t_l+1, t_l+(N(N-1)/2+\kappa_0\kappa_1)+1..., t_l+(N(N-1)/2+\kappa_0\kappa_1)((T_0)^6-1)+1\}$ be the set of first periods of units given the first period t_l+1 of sub-block l.

Recall that, in i.i.d. case, players take a constant action in each sub-block, but different actions for different sub-blocks. Here, players take a κ_0-period strategy with $\kappa_0 > 0$ in each unit, but take the same strategy in all the sub-blocks. Both specifications ensure that the average payoff from the entire block satisfies (14)/(160).

3. In communication phase l, part 1, player $i-1$ chooses and sends $t_{i-1}(l)$ from $\mathbb{T}_{\text{initial}}(l)$. In part 2, players communicate the sequence of action-signal pairs $(a_{t_{i-1}(l)}, \omega_{t_{i-1}(l)})$, ..., $(a_{t_{i-1}(l)+N(N-1)/2+\kappa_0\kappa_1}, \omega_{t_{i-1}(l)+N(N-1)/2+\kappa_0\kappa_1})$ in the chosen unit. These communication phases are expanded, according to the modified protocols defined above.

4. Similarly, the contagion and final communication phases are expanded.

O Reward Adjustment Lemma

We now modify the reward adjustment lemma. Given a parameter $T \in \mathbb{N}$, let $M(T) \subset \mathbb{N}$ be a finite set, let $F(T) \in \mathbb{R}_+$ be a constant satisfying $\liminf_{T\to\infty} F(T) > 0$, let $f_T : M(T) \to [-F(T), F(T)]$ be a function, and let $\tilde{m}_i \in M(T) \cup \{0\}$ be a random variable such that, for each μ and $m_i \in M(T)$, $\Pr(\tilde{m}_i = m_i | m_i, \mu) = p_T(m_i, \mu)$ and $\Pr(\tilde{m}_i = 0 | m_i, \mu) = 1 - p_T(m_i, \mu)$. Moreover, suppose there exists a finite random variable $y \in Y$ (independent of T or $M(T)$) which statistically identifies μ: $\Pr(y|m_i, \mu) = \Pr(y|\mu)$ and the matrix $P_Y := (\Pr(y|\mu))_{\mu,y}$ has full row rank.

Applied to the remainder of the proof, T will index the length of an interval, μ will be the realized match at the beginning of the communication phase, y will be the history when players are identifying μ, $M(T)$ will be a message set, f_T will be a reward function bounded by $F(T)$, and $p_T(m_i, \mu)$ will be the probability that message m_i is received when μ is the realized initial match and message m_i is sent.

Lemma 38 *Suppose that* $\lim_{T\to\infty} \min_{\mu, m_i \in M(T)} p_T(m_i, \mu) = 1$. *For all* $\varepsilon > 0$, *there exists* $\bar{T} > 0$ *such that, for all* $T > \bar{T}$, *there exists a function* $g_T : Y \times (M(T) \cup \{0\}) \to [-(1+\varepsilon) F(T), (1+\varepsilon) F(T)]$ *such that*

1. $\max_{m_i \in M(T), y \in Y} |f_T(m_i) - g_T(y, m_i)| \leq \varepsilon F(T)$
2. $\mathbb{E}[g_T(y, \tilde{m}_i)|\mu, m_i] = f_T(m_i)$ *for all* μ *and* $m_i \in M(T)$.
3. $g_T(y, 0) = g_T(\tilde{y}, 0)$ *for all* $y, \tilde{y}$.
4. *If* $p_T(m_i, \mu)$ *does not depend on* μ, *then* g_T *does not depend on* y.

Proof. Given μ, the same proof as for Lemma 12 implies that there exists $\tilde{g}_T(\mu, m_i)$ such that Conditions 1 and 2 hold with y replaced by μ. Since P_Y has full row rank, we can solve

$$P_Y \left(g_T(y, m_i)\right)_y = \left(\tilde{g}_T(\mu, m_i)\right)_\mu$$

for $g_T(y, m_i)$. By definition, Condition 4 holds. Since $\mathbb{E}\left[g_T(y, m_i)|\mu, m_i\right] = \tilde{g}_T(\mu, m_i)$, the law of iterated expectation implies Condition 2. Setting $\tilde{g}_T(\mu, 0) = 0$ for each μ, we have Condition 3. Moreover, as in the proof of Lemma 12, $\lim_{T\to\infty} |\tilde{g}_T(\mu, m_i) - f(m_i)| / F(T) = 0$ for each μ_1, m_i. Since P_Y is independent of $M_i(T)$, we can take $\lim_{T\to\infty} |g_T(y, m_i) - f(m_i)| / F(T) = 0$ for each y, m_i. Hence, Condition 1 holds. ■

P Reduction Lemma

Let $\mathbb{T}'$ be the set of all non-main-phase periods, and let $\mathbb{L}_{i-1}$ be the set of periods comprising one randomly chosen *unit* from each main payoff phase. Suppose there exist $\sigma_i(x_i)$ and $\pi_i^*\left(x_{-i}, \mathfrak{h}_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$ from the jamming coordination phase to the end of the main sub-block that satisfy the following condition: There exist L and $\bar{T}$ such that, for each $T_0 \geq \bar{T}$, there exist $v_i(x_{-i})$ and $\pi_i^*\left(x_{-i}, \mathfrak{h}_{-i}^{\mathbb{T}'}, h_{-i}^{\mathbb{L}_{i-1}}\right)$ satisfying

1. [Reward Bound]
$$\sup_{x_{-i},\mathfrak{h}_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}}\left|\pi_i^*\left(x_{-i},\mathfrak{h}_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}\right)\right|<\frac{(T^*)^2}{2}. \tag{163}$$

2. [Incentive Compatibility] For all μ_1, $x\in\{G,B\}^N$, for each $h_i^{t-1}\in H_i(\mu_1,x_{-i})$,[31] we have
$$\sigma_i^*(x_i)\in\arg\max_{\sigma_i\in\Sigma_i}\mathbb{E}^{\left(\sigma_i,\sigma_{-i}^*(x_{-i})\right)}\left[\sum\nolimits_{t\in\bigcup_{l=1}^{L}\mathbb{T}(\text{main}(l))}\hat{u}_i(\mathbf{a}_t)+\pi_i^*\left(x_{-i},\mathfrak{h}_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}\right)|\mu_1,h_i^{t-1}\right]. \tag{164}$$

3. [Promise Keeping] For all $x\in\{G,B\}^N$ and μ_1, we have
$$v_i(x_{-i})=\frac{1}{T^*}\mathbb{E}^{\sigma^*(x)}\left[\sum\nolimits_{t\in\bigcup_{l=1}^{L}\mathbb{T}(\text{main}(l))}\hat{u}_i(\mathbf{a}_t)+\pi_i^*\left(x_{-i},\mathfrak{h}_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}\right)|\mu_1\right],$$
and
$$v_i(x_{-i})\begin{cases}\geq v_i(G)-\varepsilon^* & \text{if } x_{i-1}=G\\ \leq v_i(B)+\varepsilon^* & \text{if } x_{i-1}=B\end{cases}.$$
Recall that $v_i(G)$ and $v_i(B)$ are fixed in (160).

4. [Self-Generation] For all x_{-i}, $\mathfrak{h}_{-i}^{\mathbb{T}'}$, and $h_{-i}^{\mathbb{L}_{i-1}}$,
$$\text{sign}(x_{i-1})\pi_i^*\left(x_{-i},\mathfrak{h}_{-i}^{\mathbb{T}'},h_{-i}^{\mathbb{L}_{i-1}}\right)\geq-2\varepsilon^*T^*.$$

Then the premise of Lemma 37 is satisfied (and so the theorem is proved). The proof follows the same steps as the reduction lemmas in the i.i.d. matching case (Lemmas 14–17).

First, we use the basic communication protocol to cancel the effect of discounting. In the protocol, in the $(S_0+1)^{\text{th}}$ period of each unit, players take different actions for different messages. By (130), for each unit, player $(i-1)$'s history in $\mathbb{S}_1$, $h_{i-1}^{\mathbb{S}_1}$, statistically identifies μ for the $(S_0+1)^{\text{th}}$ period. Given m_n and μ for the $(S_0+1)^{\text{th}}$ period of the first unit, the distribution of $m_n(i-1)$ is determined. Therefore, we can use Lemma 38 (with y being $h_{i-1}^{\mathbb{S}_1}$

[31] We define $H_i(\mu_1,x_{-i})$ as the set of histories that arise with positive probability given μ_1 and $(\sigma_i,\sigma_{-i}(x_{-i}))$ for some $\sigma_i\in\Sigma_i^{T^*}$. The conditional expectation can be calculated from Bayes' rule given $\sigma_{-i}(x_{-i})$.

for the first unit) to create a reward function based on the result of communication to cancel out discounting as in Lemma 14, instead of Lemma 12.

Viewing this reward function as the continuation payoff w, Lemma 36 ensures that (i) players' incentives are satisfied and (ii) the payoff does not depend on m and μ. Hence, the payoff from the basic communication module does not affect incentives before the communication phase, and the equilibrium payoff from the initial period to the end of final communication phase to share information from non-main phases given x is independent of μ given the original promise keeping condition. Hence, we can cancel out the effect of discounting, as in the i.i.d. case.

Note that, in the definition of XSE in Lemma 36, trembles are independent across units and independent of μ, and players $-i$'s trembles are independent of m_i. Hence, since Lemma 36 holds for each m_i and μ, we can proceed by backward induction, just as we considered BFE in the i.i.d. matching case. The same remark is applicable to the secure and verified communication, since trembles are all independent of μ (and m except for the sender).

Second, we use the basic and secure communication protocols to communicate about the history in non-main phases. In particular, in Lemma 15, we use the basic protocol to construct a reward function for the preceding secure protocol. This part of the proof is the same as Lemma 15, with Lemma 12 replaced with Lemma 38 as explained above. In the secure communication protocol, by equation (132), $h_{i-1}^{\leq N(N-1)(1+NS_0)/2}$ statistically identifies the initial match realization μ. Given m_n and μ, the distribution of $m_n(i-1)$ is determined. Hence, viewing $h_{i-1}^{\leq N(N-1)(1+NS_0)/2}$ as y, Lemma 38 allows us to construct a reward function based on the result of communication. We view this reward as the continuation payoff w of the secure communication. Since Lemma 38 implies that the expected continuation payoff depends only on $m_n(i-1)$ and $h_{i-1}^{\leq N(N-1)(1+NS_0)/2}$, the premise of the module is satisfied. Moreover, π_i^{cancel}, $\pi_i^{a^0}$, and $\pi_i^{a^1}$ ensure that player i's payoff in the secure communication module does not depend on μ or m_i. Hence, the equilibrium payoff from the initial period to the end of final communication phase to share information from main phases given x is independent of μ, given the original promise keeping condition. Since the cardinality of $\chi_{j,t} \in \{0,1\}^N$ is bounded independently of T_0, the extra periods needed to communicate $\chi_{j,t}$ does not affect promise keeping and self generation for a large T_0.

Third, we use the secure and verified protocols to communicate about the history in main phases. For the secure communication protocol, the proof is the same as with i.i.d. matching, with Lemma 12 replaced with Lemma 38. In the verified communication protocol, since Lemmas 5–7 and 10 hold as they stand, the distribution of $m_n(i-1)$ does not depend on the initial match μ given m_n. Hence, Claim 4 of Lemma 38 implies that we can construct a reward function based on the result of the communication, which is independent of the realized match μ at the beginning of the module. Viewing this reward as a continuation payoff w of the verified communication, the premise of verified communication is satisfied.

Finally, given $\mathbb{L}_{i-1}$, since players communicate the history in the chosen unit, the communication about the history in $\mathbb{L}_{i-1}$ now takes $\tilde{T}((N(N-1)/2+\kappa_0\kappa_1)|A|^2, 6, (T^*)^{\frac{1}{2}})$ periods.[32] Together with (131) and (133), the length of the final communication phase divided by $(T_0)^6$ (the length of the main phase) converges to 0 as $T_0 \to \infty$. Hence, asymptotically, the final communication phases do not affect promise keeping or self generation.

One may notice we have $2\varepsilon^* T^*$ instead of $\varepsilon^* T^*$ for [Self-Generation]. Recall that we have $10\varepsilon^*$ slack in (160) compared to $9\varepsilon^*$ in (14). Hence, we can add or subtract a constant $\varepsilon^* T$ to satisfy the original promise keeping and self-generation constraints.

Q Reward Function and Payoffs

When players communicate via verified communication in the non-main phases, we define the reward as (157), such that payoffs outside of main payoff phases are independent of the current match realization.

We define the reward for the main payoff phase to ensure that equilibrium payoffs from a unit do not depend on the match realization at the beginning of the unit. Let $\mathbf{1}_t$ be the indicator function for the event that the t^{th} unit is chosen as part of $\mathbb{L}_{i-1}$ by player $i-1$.

We define $\pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i}|\alpha^{\min})$ and $\pi_i^{v_i}(x_{i-1}, a_{-i,\tau}, \omega_{-i,\tau})$ as in i.i.d. match such that player i obtains the per-period expected payoff of $v_i(x_{i-1})$ conditional on the match. In addition, for each τ in the last $\kappa_0\kappa_1$ periods of the unit, given $(a_{-i,\tau}, \omega_{-i,\tau})$, let $a_\tau(x)$ be the

[32] Recall that 6 stands for the fact that the magnitude of the communication result on the continuation payoff is of order $((T^*)^{\frac{1}{2}})^6$, which follows from the same calculation as (105).

equilibrium strategy, and $a_{i,\tau}$ is player i's action identified from $(a_{-i,\tau}, \omega_{-i,\tau})$. The reward

$$\pi_i^{v_i}(a_{-i,\tau}, \omega_{-i,\tau}|x) = \hat{u}_i(a_\tau(x)) - \hat{u}_i(a_i, a_{-i,\tau}) \tag{165}$$

makes player i indifferent and obtain the average payoff of $\hat{u}_i(a_\tau(x))$.

If $\theta_i(l) = E$, the reward is

$$(T_0)^6 \sum_{t:\text{unit}} \mathbf{1}_t \sum_{\tau \text{ in } t^{\text{th}} \text{ unit}} \pi_i^{\text{cancel}} (x_{i-1}, a_{-i,\tau}, \omega_{-i,\tau}),$$

as with i.i.d. matching. Player i obtains average payoff $u_i^{x_{i-1}}$ for each unit, regardless of the realized match.

If $\theta_i(l) = R$ and $I_{-i}^D(l) = -i$, the reward is

$$(T_0)^6 \sum_{t:\text{unit}} \mathbf{1}_t \left(\begin{array}{c} \sum_{\tau \text{ in the first } N(N-1)/2 \text{ periods of } t^{\text{th}} \text{ unit}} \left(\begin{array}{c} v_i(x_{i-1}) + \pi_i^{\text{cancel}} (a_{-i,\tau}, \omega_{-i,\tau}) \\ + \frac{4\kappa_0 \bar{u}}{\bar{\varepsilon}} \left\| (P)^{-1} \right\| \pi_{i,\tau}^{\bar{\sigma}^*} (a_{-i,\tau}, \omega_{-i,\tau}) \end{array} \right) \\ + \sum_{\tau \text{ after the first } N(N-1)/2 \text{ periods of } t^{\text{th}} \text{ unit}} \pi_i^{v_i} (x_{i-1}, a_{-i}, \omega_{-i} | \alpha^{\min}) \\ -\mathbf{1}_{\{x_{i-1}=G\}} (N(N-1)/2 + \kappa_0 \kappa_1) 2\bar{u} \end{array} \right),$$

where

$$\pi_{i,\tau}^{\bar{\sigma}^*} (a_{-i,\tau}, \omega_{-i,\tau}) = \begin{cases} \pi_i^{a^1} (a_{-i,\tau}, \omega_{-i,\tau}) & \text{if player } i \text{ is supposed to take } a^1 \text{ in } \bar{\sigma}^*, \\ \pi_i^{a^0} (a_{-i,\tau}, \omega_{-i,\tau}) & \text{otherwise} \end{cases}$$

For the last $\kappa_0\kappa_1$ periods in each unit, since each player in $I \setminus \{i\}$ takes $\alpha^{\min}$, player i obtains $v_i(x_{i-1})$ regardless of the match or her action, as with i.i.d. matching. For the first $N(N-1)/2$ periods, players obtain $v_i(x_{i-1})$. Hence, the average payoff is $v_i(x_{i-1}) - \mathbf{1}_{\{x_{i-1}=G\}} 2\bar{u}$, independent of the realized match.

If $\theta_i(l) = R$ but $I^D_{-i}(l) \neq \emptyset, -i$, the reward is

$$(T_0)^6 \sum_{t:\text{unit}} \mathbf{1}_t \left(\begin{array}{c} \sum_{\tau \text{ in the first } N(N-1)/2 \text{ periods of } t^{\text{th}} \text{ unit}} \left(\begin{array}{c} v_i(x_{i-1}) + \pi_i^{\text{cancel}}(a_{-i,\tau}, \omega_{-i,\tau}) \\ + \frac{4\kappa_0 \bar{u}}{\bar{\varepsilon}} \left\| (P)^{-1} \right\| \pi_{i,\tau}^{\bar{\sigma}^*}(a_{-i,\tau}, \omega_{-i,\tau}) \end{array} \right) \\ + \sum_{\tau \text{ after the first } N(N-1)/2 \text{ periods of } t^{\text{th}} \text{ unit}} \pi_i^{v_i}(x_{i-1}, a_{-i,\tau}, \omega_{-i,\tau}) \\ -\mathbf{1}_{\{x_{i-1}=G\}} \left(N(N-1)/2 + \kappa_0\kappa_1 \right) 2\bar{u} \end{array} \right).$$

Again, the average payoff is $v_i(x_{i-1}) - \mathbf{1}_{\{x_{i-1}=G\}} 2\bar{u}$, independent of the realized match.

If $\theta_i(l) = R$ and $I^D_{-i}(l) = \emptyset$, the reward is[33]

$$(T_0)^6 \sum_{t:\text{unit}} \mathbf{1}_t \left(\begin{array}{c} \sum_{\tau \text{ in the first } N(N-1)/2 \text{ periods of } t^{\text{th}} \text{ unit}} \left(\begin{array}{c} v_i(x_{i-1}) + \pi_i^{\text{cancel}}(a_{-i,\tau}, \omega_{-i,\tau}) \\ + \frac{4\kappa_0 \bar{u}}{\hat{\varepsilon}} \left\| (P)^{-1} \right\| \pi_{i,\tau}^{\bar{\sigma}^*}(a_{-i,\tau}, \omega_{-i,\tau}) \end{array} \right) \\ +\pi_i^{N(N-1)/2}\left(x(i-1), h_{-i,t}^{1:N(N-1)/2} \right) \\ +\pi_i(a_{-i,t}, \omega_{-i,t} | x(i-1)) \end{array} \right), \tag{166}$$

where $h_{-i,t}^{1:N(N-1)/2}$ is players $-i$'s history in the first $N(N-1)/2$ periods of the t^{th} unit.

We define $\pi_i^{N(N-1)/2}\left(x, h_{-i,t}^{1:N(N-1)/2}\right)$ such that player i obtains per-period payoff of $v_i(x_{i-1})$ regardless of the initial match realization in the unit. Let $\mu_{N(N-1)/2+1}$ be the realized match in the $N(N-1)/2+1^{\text{th}}$ period of the unit. Since players take $\sigma^{\kappa_0}(x)$ for every κ_0 periods after the $N(N-1)/2+1^{\text{th}}$ period, player i's average expected payoff from these $\kappa_1\kappa_0$ periods given $\mu_{N(N-1)/2+1}$ can be written as $u_i^{\kappa_0}\left(x|\mu_{N(N-1)/2+1}\right)$. Given $\bar{\varepsilon} > 0$, since players take cycles of $\sigma^{\kappa_0}(x)$ every κ_0 periods, (162) implies

$$\max_{x,\mu} \left| u_i^{\kappa_0}\left(x|\mu_{N(N-1)/2+1}\right) - u_i^{\kappa_0}(x) \right| \leq \frac{2\bar{u}}{\kappa_1 \bar{\varepsilon}}. \tag{167}$$

Since $(a_{-i,\tau}, \omega_{-i,\tau})_{\tau \text{ in the first } N(N-1)/2 \text{ periods}}$ statistically identify μ_1 in equilibrium, by (125),

[33] Here, we use $\pi_i(a_{-i,t}, \omega_{-i,t}|x(i-1))$ instead of $\pi_i^{v_i}(x_{i-1}, a_{-i,\tau}, \omega_{-i,\tau})$. The reason is, unlike i.i.d. case, $\pi_i^{v_i}(x_{i-1}, a_{-i,\tau}, \omega_{-i,\tau})$ may not satisfy self generation even if players follow $\sigma^{\kappa_0}(x(i-1))$, depending on the realization of the match sequence.

there exists $\tilde{\pi}_i^{N(N-1)/2}\left(x, h_{-i}^{1:N(N-1)/2}\right)$ such that the average payoff

$$\mathbb{E}^{\sigma(x)}\left[u_i^{\kappa_0}\left(x|\mu_{N(N-1)/2+1}\right)+\frac{1}{\kappa_0\kappa_1}\tilde{\pi}_i^{N(N-1)/2}\left(x, h_{-i}^{1:N(N-1)/2}\right)|x,\mu_1\right]=u_i^{\kappa_0}(x)$$

depends only on x, and

$$\max_{x, h_{-i}^{1:N(N-1)/2}, \tilde{h}_{-i}^{1:N(N-1)/2}}\left|\tilde{\pi}_i^{N(N-1)/2}\left(x, h_{-i}^{1:N(N-1)/2}\right)-\tilde{\pi}_i^{N(N-1)/2}\left(x, \tilde{h}_{-i}^{1:N(N-1)/2}\right)\right| \leq \frac{2\kappa_0\bar{u}}{\bar{\varepsilon}}\left\|(P)^{-1}\right\|.$$

Finally, define

$$\pi_i^{N(N-1)/2}\left(x, h_{-i}^{1:N(N-1)/2}\right)=\tilde{\pi}_i^{N(N-1)/2}\left(x, h_{-i}^{1:N(N-1)/2}\right)+\kappa_0\kappa_1\left(v_i(x_{i-1})-u_i^{\kappa_0}(x)\right).$$

Given (160), we have $\text{sign}(x_{i-1})(v_i(x_{i-1})-u_i^{\kappa_0}(x))\geq 0$ for each x. Hence, we have

$$\mathbb{E}^{\sigma(x)}\left[u_i^{\kappa_0}\left(x|\mu_{N(N-1)/2+1}\right)+\frac{1}{\kappa_0\kappa_1}\pi_i^{N(N-1)/2}\left(x, h_{-i}^{1:N(N-1)/2}\right)|x,\mu_1\right]=v_i(x_{i-1}), \tag{168}$$

$$\max_{x, h_{-i}^{1:N(N-1)/2}, \tilde{h}_{-i}^{1:N(N-1)/2}}\left|\pi_i^{N(N-1)/2}\left(x, h_{-i}^{1:N(N-1)/2}\right)-\pi_i^{N(N-1)/2}\left(x, \tilde{h}_{-i}^{1:N(N-1)/2}\right)\right| \leq \frac{2\kappa_0\bar{u}}{\bar{\varepsilon}}\left\|(P)^{-1}\right\|, \tag{169}$$

and

$$\text{sign}(x_{i-1})\pi_i^{N(N-1)/2}\left(x, \tilde{h}_{-i}^{1:N(N-1)/2}\right)\geq -\frac{2\kappa_0\bar{u}}{\bar{\varepsilon}}\left\|(P)^{-1}\right\|. \tag{170}$$

R Verification

Since we have extended Lemmas 6 and 11, Lemma 20 holds as it stands, except that (69) and (70) hold conditional on the realization of the match μ_3 after the jamming coordination phase, for each μ_3. The proof is the same, except that we modify the proof of self generation as follows.

Given (165), $\pi_i^{v_i}(a_{-i,t},\omega_{-i,t}|x(i-1))$ is equal to 0 unless player i's deviation is identified.

In addition, $\pi_i^{\text{cancel}}(x_{i-1}, a_{-i,\tau}, \omega_{-i,\tau})$ and

$$\sum_{\tau \text{ after the first } N(N-1)/2 \text{ periods of } t^{\text{th}} \text{ unit}} \pi_i^{v_i}(x_{i-1}, a_{-i}, \omega_{-i}|\alpha^{\min}) - \mathbf{1}_{\{x_{i-1}=G\}} (N(N-1)/2 + \kappa_0\kappa_1) 2\bar{u}$$

is no less than 0 if $x_{i-1} = B$ or no more than 0 if $x_{i-1} = G$, as in the i.i.d. case. Moreover, the "additional" reward compared to the i.i.d. case satisfies

$$\begin{aligned}
& \max_{\mathfrak{h}} \operatorname{sign}(x_{i-1}) (T_0)^6 \sum_{t:\text{unit}} \mathbf{1}_t \left(\begin{array}{c} \sum_{\tau \text{ in the first } N(N-1)/2 \text{ periods of } t^{\text{th}} \text{ unit}} \left(\begin{array}{c} v_i(x_{i-1}) + \pi_i^{\text{cancel}}(a_{-i,\tau}, \omega_{-i,\tau}) \\ + \frac{4\kappa_0\bar{u}}{\bar{\varepsilon}} \left\| (P)^{-1} \right\| \pi_{i,\tau}^{\bar{\sigma}^*}(a_{-i,\tau}, \omega_{-i,\tau}) \end{array} \right) \\ + \pi_i^{N(N-1)/2}\left(x(i-1), h_{-i,t}^{1:N(N-1)/2}\right) \end{array} \right) \\
\geq \; & -(T_0)^6 \left(N(N-1)/2 \left(2\bar{u} + \frac{4\kappa_0\bar{u}}{\bar{\varepsilon}} \left\| (P)^{-1} \right\| \right) + \frac{2\kappa_0\bar{u}}{\bar{\varepsilon}} \left\| (P)^{-1} \right\| \right) \text{ given (170)} \\
\geq \; & -\varepsilon^* (N(N-1)/2 + \kappa_0\kappa_1)(T_0)^6 \text{ by (161),}
\end{aligned}$$

which is larger than the additional slack of $\varepsilon^* T^*$ in [Self-Generation] (recall that T^* is approximately equal to $L(N(N-1)/2 + \kappa_0\kappa_1)(T_0)^6$). Hence, the same proof as in Lemma 20 establishes self generation.

R.1 Verification of Promise Keeping and Incentive Compatibility

Since the lemmas for verified communication (Lemmas 5–7 and 10) and identification (Lemma 1) hold as they stand, incentives in non-main phases are verified in the same way as in the i.i.d. case.

For the main phase, as mentioned, the per-period payoff from the main payoff phases is equal to $u_i^{x_{i-1}}$ if $\theta_i(l) = E$ and $v_i(x_{i-1}) - 1_{\{x_{i-1}=G\}} 1_{\{I_{-i}^D \neq \emptyset\}} 2\bar{u}$ if $\theta_i(l) = R$. This implies that promise keeping is satisfied. So, we focus on incentive compatibility. In addition, given the definition of the reward function, the payoff from the last $\kappa_0\kappa_1$ periods of the phase is independent of the match realization unless $\theta_i(l) = R$ and $I_{-i}^D(l) = \emptyset$. Moreover, given the equilibrium strategy and match realization, the payoff from the first $N(N-1)/2$ periods satisfies

$$u_i(\mathbf{a}_\tau) + \pi_i^{\text{cancel}}(a_{-i,\tau}, \omega_{-i,\tau}) + \frac{4\kappa_0\bar{u}}{\bar{\varepsilon}} \left\| (P)^{-1} \right\| \pi_{i,\tau}^{\bar{\sigma}^*}(a_{-i,\tau}, \omega_{-i,\tau}) = 0.$$

Hence, incentives coming from the continuation payoff of future phases are the same as in the i.i.d. case unless $\theta_i(l) = R$ and $I_{-i}^D(l) = \emptyset$. Therefore, the proof of incentive compatibility is the same as in the i.i.d. case, except that the reward $\pi_{i,\tau}^{\bar{\sigma}^*}$ strictly incentivizes $\bar{\sigma}^*$ for the first $N(N-1)/2$ periods.

We are left to verify incentives given $\theta_i(l) = R$ and $I_{-i}^D(l) = \emptyset$. It suffices to show that (i) following the equilibrium strategy is optimal within the unit and (ii) the average payoff from the unit does not depend on the realized match at the beginning of the unit, since (ii) ensures that players do not have incentives to deviate in order to manipulate the realized match in the next unit.

For the last $\kappa_0\kappa_1$ periods of each unit, the continuation payoff gives player i an incentive to follow $\sigma_i^{\kappa_0}(x(i))$, since (165) makes player i indifferent among all actions for the current period, and a deviation in $t_{i-1}(l)$ will induce $I_{-i}^D(l+1) \neq 0$ as in Lemma 18. Given this incentive, player i's non-average expected payoff from

$$\begin{aligned} & \sum_{\tau \text{ after the first } N(N-1)/2 \text{ periods of } t^{\text{th}} \text{ unit}} \left(\hat{u}(\mathbf{a}_\tau) + \mathbf{1}_t \left(T_0\right)^6 \pi_i^{v_i}(a_{-i,t}, \omega_{-i,t} | x(i-1))\right) \\ = & \sum_{\tau \text{ after the first } N(N-1)/2 \text{ periods of } t^{\text{th}} \text{ unit}} \hat{u}(\mathbf{a}_\tau(x(i-1))) \end{aligned}$$

depends on the match in $N(N-1)/2^{\text{th}}$ period by magnitude at most $\frac{2\kappa_0 \bar{u}}{\bar{\varepsilon}}$ by (167). Moreover, the potential gain from manipulating $\pi_i^{N(N-1)/2}\left(x(i-1), h_{-i}^{1:N(N-1)/2}\right)$ is bounded by $\frac{2\kappa_0 \bar{u}}{\bar{\varepsilon}} \left\|(P)^{-1}\right\|$, by (169). In total, the deviation gain in the first $N\left(N-1\right)/2$ periods is bounded by $\frac{4\kappa_0 \bar{u}}{\bar{\varepsilon}} \left\|(P)^{-1}\right\|$.

Hence, in the $N\left(N-1\right)/2^{\text{th}}$ period, player i has an incentive to follow the equilibrium strategy given the magnitude $\frac{4\kappa_0 \bar{u}}{\bar{\varepsilon}} \left\|(P)^{-1}\right\|$ of the per-period reward $\pi_{i,\tau}^{\bar{\sigma}^*}$. Since this per-period reward in the $N\left(N-1\right)/2^{\text{th}}$ period does not depend on the realized match in equilibrium, in the preceding period player i again has an incentive to follow the equilibrium strategy. Recursively, player i follows the equilibrium strategy within the unit, as desired. Finally, the per-unit payoff is independent of the initial match realization by (168).

S Proof of Proposition 1

Clearly, $\lim_{\delta\to 1}\bigcup_{\hat{\delta}\geq\delta} F(\mu,\hat{\delta})$ exists as the limit of a monotonic sequence. We show that

$$\lim_{\delta\to 1}\bigcup\nolimits_{\hat{\delta}\geq\delta} F(\mu,\hat{\delta}) = \lim_{\delta\to 1} F(\mu,\delta) = \lim_{\kappa\to\infty}\lim_{\delta\to 1} F^{\kappa}(\mu,\delta),$$

and that these limits are independent of the initial match μ.

We first show that $\lim_{\delta\to 1}\bigcup_{\hat{\delta}\geq\delta} F(\mu,\hat{\delta})$ is independent of μ. Since $F(\mu,\hat{\delta})$ is convex, we can characterize $\lim_{\delta\to 1}\bigcup_{\hat{\delta}\geq\delta} F(\mu,\hat{\delta})$ by supporting hyperplanes. Fix a unit vector $\boldsymbol{\lambda}\in\mathbb{R}^N$ and consider the following two auxiliary games between nature and the players:

1. Nature chooses the worst distribution of the initial match μ, and then players choose a dynamic game strategy profile σ to maximize $\boldsymbol{\lambda}\cdot\mathbf{u}$:

$$\underline{v}^{\delta} := \min_{p\in\Delta(\mathcal{M})}\max_{\sigma\in(\Delta(\Sigma))^N}(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\sum_{\mu}p(\mu)\boldsymbol{\lambda}\cdot\mathbb{E}^{\sigma}\left[\hat{\mathbf{u}}(\mathbf{a}_t)|\mu\right], \tag{171}$$

   where $\mathcal{M}$ is the set of possible matchings. Since $\Delta(\mathcal{M})$ and $(\Delta(\Sigma))^N$ are compact (in the product topology) and convex; $\boldsymbol{\lambda}\cdot\mathbb{E}^{\sigma}\left[\hat{\mathbf{u}}(\mathbf{a}_t)|\mu\right]$ is continuous in p and σ; and discounted payoffs are continuous at infinity, Sion's minimax theorem implies that a minimizer and maximizer in (171) exist, and that

$$\underline{v}^{\delta} = \max_{\sigma\in(\Delta(\Sigma))^N}\min_{p\in\Delta(\mathcal{M})}(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\sum_{\mu}p(\mu)\boldsymbol{\lambda}\cdot\mathbb{E}^{\sigma}\left[\hat{\mathbf{u}}(\mathbf{a}_t)|\mu\right].$$

   Let $\underline{\sigma}^{*,\delta}$ be a maximizer and let $\underline{\mu}^{*,\delta}$ be a minimizer. Without loss, assume $\underline{\mu}^{*,\delta}$ is degenerate.

2. Nature chooses the best initial match μ and players choose σ to maximize $\boldsymbol{\lambda}\cdot\mathbf{u}$:

$$\bar{v}^{\delta} := \max_{\sigma\in(\Delta(\Sigma))^N}\max_{\mu\in\Delta(\mathcal{M})}(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\sum_{\mu}p(\mu)\boldsymbol{\lambda}\cdot\mathbb{E}^{\sigma}\left[\hat{\mathbf{u}}(\mathbf{a}_t)|\mu\right].$$

   Let $\bar{\sigma}^{*,\delta}$ and $\bar{\mu}^{*,\delta}$ be maximizers.

Since $\underline{v}^{\delta}\leq\max_{\sigma\in(\Delta(\Sigma))^N}(1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}\boldsymbol{\lambda}\cdot\mathbb{E}^{\sigma}\left[\hat{\mathbf{u}}(\mathbf{a}_t)|\mu\right]\leq\bar{v}^{\delta}$ for each δ and μ, it suffices to show that $\left|\bar{v}^{\delta}-\underline{v}^{\delta}\right| \leq (1-\delta)\, 2\bar{u}/\bar{\varepsilon}$. (This implies that, for any μ, μ', the difference in score between $F(\mu,\delta)$ and $F(\mu',\delta)$ in any direction is bounded by $(1-\delta)\,2\bar{u}/\bar{\varepsilon}$. Hence, the scores of $\lim_{\delta\to 1}\bigcup_{\hat{\delta}\geq\delta}F(\mu,\hat{\delta})$ and $\lim_{\delta\to 1}\bigcup_{\hat{\delta}\geq\delta}F(\mu',\hat{\delta})$ in every direction coincide, so the sets are equal.) Given $\bar{\sigma}^{*,\delta}$ and $\bar{\mu}^{*,\delta}$, let τ be the (random) earliest time at which $\mu_t = \underline{\mu}^{*,\delta}$. Note that, at period τ, the continuation payoff from $\bar{\sigma}^{*,\delta}$ is no more than $\underline{v}^{\delta}$. Moreover, by the full support assumption, for each t, $\Pr\left(\tau = t | \bar{\sigma}^{*,\delta}, \bar{\mu}^{*,\delta}, \tau > t-1\right) \geq \bar{\varepsilon}$. Hence, we have

$$\begin{aligned}
\bar{v}^{\delta} &\leq (1-\delta)\,\bar{u} + \sum_{t=2}^{\infty}(1-\bar{\varepsilon})^{t-2}\,\bar{\varepsilon}\left(\left(\delta-\delta^{t-1}\right)\bar{u}+\delta^{t-1}\underline{v}^{\delta}\right) \\
&= \frac{1-\delta}{1-\delta\,(1-\bar{\varepsilon})}\bar{u} + \frac{\delta\bar{\varepsilon}}{1-\delta\,(1-\bar{\varepsilon})}\underline{v}^{\delta}.
\end{aligned}$$

Therefore,

$$\bar{v}^{\delta}-\underline{v}^{\delta} \leq \frac{1-\delta}{1-\delta\,(1-\bar{\varepsilon})}\left(\bar{u}-\underline{v}^{\delta}\right) \leq \frac{1-\delta}{\bar{\varepsilon}}2\bar{u}. \tag{172}$$

Hence, $\lim_{\delta\to 1}\bigcup_{\hat{\delta}\geq\delta}F(\mu,\hat{\delta})$ is independent of μ.

We now show that $\lim_{\delta\to 1}\bigcup_{\hat{\delta}\geq\delta}F(\mu,\hat{\delta}) = \lim_{\delta\to 1}F(\mu,\delta) = \lim_{\kappa\to\infty}\lim_{\delta\to 1}F^{\kappa}(\mu,\delta)$ for all μ. Clearly, $\lim_{\delta\to 1}\bigcup_{\hat{\delta}\geq\delta}F(\mu,\hat{\delta}) \supseteq \lim_{\delta\to 1}F(\mu,\delta) \supseteq \lim_{\kappa\to\infty}\lim_{\delta\to 1}F^{\kappa}(\mu,\delta)$. Conversely, we show that, for each $\mathbf{v}\in\lim_{\delta\to 1}\bigcup_{\hat{\delta}\geq\delta}F(\mu,\hat{\delta})$ and direction $\boldsymbol{\lambda}\in\mathbb{R}^n$, $\boldsymbol{\lambda}\cdot\mathbf{v}$ can be approximated by the repetition of finite-period strategies. Specifically, suppose the players repeat the first T periods of $\underline{\sigma}^{*,\delta}$ ad infinitum. Regardless of the initial match, this strategy achieves payoff at least

$$\min_{\mu\in\Delta(\mathcal{M})}\frac{1-\delta}{1-\delta^{T}}\sum_{t=1}^{T}\delta^{t-1}\boldsymbol{\lambda}\cdot\mathbb{E}^{\underline{\sigma}^{*,\delta}}\left[\hat{\mathbf{u}}(\mathbf{a}_t)|\mu\right]. \tag{173}$$

We show that, for each $\eta>0$, there exist T and $\bar{\delta}<1$ such that, for each $\delta\geq\bar{\delta}$, (173) is no less than $\underline{v}^{\delta}-\eta$. Fix any $\mu\in\Delta(\mathcal{M})$. By (172)

$$\begin{aligned}
\underline{v}^{\delta} &\leq (1-\delta)\sum_{t=1}^{T}\delta^{t-1}\boldsymbol{\lambda}\cdot\mathbb{E}^{\underline{\sigma}^{*,\delta}}\left[\hat{\mathbf{u}}(\mathbf{a}_t)|\mu\right]+\delta^{T}\bar{v}^{\delta} \\
&\leq (1-\delta)\sum_{t=1}^{T}\delta^{t-1}\boldsymbol{\lambda}\cdot\mathbb{E}^{\underline{\sigma}^{*,\delta}}\left[\hat{\mathbf{u}}(\mathbf{a}_t)|\mu\right]+\delta^{T}\underline{v}^{\delta}+\delta^{T}\frac{1-\delta}{\bar{\varepsilon}}2\bar{u}.
\end{aligned}$$

Hence,

$$\underline{v}^{\delta} - \delta^{T}\frac{1-\delta}{1-\delta^{T}}\frac{2\bar{u}}{\bar{\varepsilon}} \leq \frac{1-\delta}{1-\delta^{T}}\sum_{t=1}^{T}\delta^{t-1}\boldsymbol{\lambda}\cdot\mathbb{E}^{\underline{\sigma}^{*,\delta}}\left[\hat{\mathbf{u}}(\mathbf{a}_t)|\mu\right].$$

Since $\lim_{\delta\to 1}\delta^{T}\frac{1-\delta}{1-\delta^{T}} = \frac{1}{T}$, for sufficiently large T we have

$$\underline{v}^{\delta} - \eta \leq \frac{1-\delta}{1-\delta^{T}}\sum_{t=1}^{T}\delta^{t-1}\boldsymbol{\lambda}\cdot\mathbb{E}^{\underline{\sigma}^{*,\delta}}\left[\hat{\mathbf{u}}(\mathbf{a}_t)|\mu\right].$$

Since this holds for all $\mu \in \Delta(\mathcal{M})$, we see that (173) is no less than $\underline{v}^{\delta} - \eta$. Hence, repeating the first T periods of $\underline{\sigma}^{*,\delta}$ approximates $\underline{v}^{\delta}$.