A Strategic Topology on Information Structures

Dirk Bergemann (Yale), Stephen Morris (MIT), Rafael Veiel (MIT)

Panorama of Mathematics II at Hausdorff Center

October 2023

a key ingredient of game theory

- the information structure

a key ingredient of game theory

- the information structure
- players' beliefs and higher-order beliefs about the game

a key ingredient of game theory

- the information structure
- players' beliefs and higher-order beliefs about the game
- i.e., what do players believe about the game, what do they believe that others believe, and so on....?

a key ingredient of game theory

- the information structure
- players' beliefs and higher-order beliefs about the game
- i.e., what do players believe about the game, what do they believe that others believe, and so on....?
- the space of all information structures is an interesting mathematical object

topology on information structures

- we ask: what is the coarsest topology on information structures that generates continuity of equilibrium?

topology on information structures

- we ask: what is the coarsest topology on information structures that generates continuity of equilibrium?
- insightful and interesting question in its own right

topology on information structures

- we ask: what is the coarsest topology on information structures that generates continuity of equilibrium?
- insightful and interesting question in its own right
- important for applied economic questions

answer

- an event is approximate common knowledge (Monderer-Samet 89) if (for p close to 1) everyone believes with probability at least p that it is true, everyone believes with probability at least p that everybody believes it with probability at least p

answer

- an event is approximate common knowledge (Monderer-Samet 89) if (for p close to 1) everyone believes with probability at least p that it is true, everyone believes with probability at least p that everybody believes it with probability at least p
- two information structures are close if each assigns high ex ante probability to there being approximate common knowledge (Monderer-Samet 89) that interim (conditional) beliefs are close

talk

1. setting

talk

1. setting
   1.1 base game

talk

1. setting
   1.1 base game
   1.2 space of information structures (Harsanyi (1967/68), Mertens-Zamir (1985))

talk

1. setting
   1.1 base game
   1.2 space of information structures (Harsanyi (1967/68), Mertens-Zamir (1985))
   1.3 equilibrium

talk

1. setting
   1.1 base game
   1.2 space of information structures (Harsanyi (1967/68), Mertens-Zamir (1985))
   1.3 equilibrium
2. main result

talk

1. setting
   1.1 base game
   1.2 space of information structures (Harsanyi (1967/68), Mertens-Zamir (1985))
   1.3 equilibrium
2. main result
3. if time:

talk

1. setting
   1.1 base game
   1.2 space of information structures (Harsanyi (1967/68), Mertens-Zamir (1985))
   1.3 equilibrium
2. main result
3. if time:
   3.1 modelling choices

talk

1. setting
   1.1 base game
   1.2 space of information structures (Harsanyi (1967/68), Mertens-Zamir (1985))
   1.3 equilibrium
2. main result
3. if time:
   3.1 modelling choices
   3.2 applications

talk

1. setting
   1.1 base game
   1.2 space of information structures (Harsanyi (1967/68), Mertens-Zamir (1985))
   1.3 equilibrium
2. main result
3. if time:
   3.1 modelling choices
   3.2 applications
   3.3 context / literature

Setting

setting

- fix a finite set of players I and a finite set of payoff-relevant states Θ

setting

- fix a finite set of players I and a finite set of payoff-relevant states Θ
- a *base game* $\mathcal{G} = (A_i, u_i)_{i \in I}$, where

setting

- fix a finite set of players I and a finite set of payoff-relevant states Θ
- a *base game* $\mathcal{G} = (A_i, u_i)_{i \in I}$, where
    - each A_i is a finite set of actions

setting

- fix a finite set of players I and a finite set of payoff-relevant states Θ
- a *base game* $\mathcal{G} = (A_i, u_i)_{i \in I}$, where
  - each A_i is a finite set of actions
  - each $u_i : A_i \times A_{-i} \times \Theta \to [-M, M]$ is a bounded payoff function

information: hierarchies of beliefs

- define $(\mathcal{T}_i)_{i \in I}$ recursively:

information: hierarchies of beliefs

- define $(\mathcal{T}_i)_{i \in I}$ recursively:
  - for each i, define $\mathcal{T}_i^0 = \{*\}$ and $\mathcal{T}_i^1 = \Delta(\Theta)$

information: hierarchies of beliefs

- define $(\mathcal{T}_i)_{i \in I}$ recursively:
  - for each i, define $\mathcal{T}_i^0 = \{*\}$ and $\mathcal{T}_i^1 = \Delta(\Theta)$
  - given $\left(\mathcal{T}_i^{m-1}\right)_{i \in I}$ for $m > 1$, define $\mathcal{T}_i^m \subseteq \mathcal{T}_i^{m-1} \times \Delta\left(\mathcal{T}_{-i}^{m-1} \times \Theta\right)$ as

$$\left\{ \left(\left(\tau_i^1, .., \tau_i^{m-1}\right), \tau_i^m\right) \middle| \, \mathbf{marg}_{\mathcal{T}_{-i}^{m-2} \times \Theta} \tau_i^m = \tau_i^{m-1} \right\}$$

information: hierarchies of beliefs

- define $(\mathcal{T}_i)_{i \in I}$ recursively:
  - for each i, define $\mathcal{T}_i^0 = \{*\}$ and $\mathcal{T}_i^1 = \Delta(\Theta)$
  - given $\left(\mathcal{T}_i^{m-1}\right)_{i \in I}$ for $m > 1$, define $\mathcal{T}_i^m \subseteq \mathcal{T}_i^{m-1} \times \Delta\left(\mathcal{T}_{-i}^{m-1} \times \Theta\right)$ as

$$\left\{ \left(\left(\tau_i^1, .., \tau_i^{m-1} \right), \tau_i^m \right) \middle| \, \mathbf{marg}_{\mathcal{T}_{-i}^{m-2} \times \Theta} \tau_i^m = \tau_i^{m-1} \right\}$$

- $\mathcal{T}_i$ is the set of sequences $\tau_i = (\tau_i^m)_m$ such that for $\overline{m} \in \mathbb{N}$, the truncated sequence $(\tau_i^m)_{m \leq \overline{m}}$ belongs to $\overline{\mathcal{T}}_i^m$

information: hierarchies of beliefs

- define $(\mathcal{T}_i)_{i\in I}$ recursively:
  - for each i, define $\mathcal{T}_i^0 = \{*\}$ and $\mathcal{T}_i^1 = \Delta(\Theta)$
  - given $\left(\mathcal{T}_i^{m-1}\right)_{i\in I}$ for $m > 1$, define $\mathcal{T}_i^m \subseteq \mathcal{T}_i^{m-1} \times \Delta\left(\mathcal{T}_{-i}^{m-1} \times \Theta\right)$ as

$$\left\{ \left(\left(\tau_i^1, .., \tau_i^{m-1} \right), \tau_i^m \right) \middle| \, \mathbf{marg}_{\mathcal{T}_{-i}^{m-2} \times \Theta} \tau_i^m = \tau_i^{m-1} \right\}$$

- $\mathcal{T}_i$ is the set of sequences $\tau_i = (\tau_i^m)_m$ such that for $\overline{m} \in \mathbb{N}$, the truncated sequence $(\tau_i^m)_{m \leq \overline{m}}$ belongs to $\overline{\mathcal{T}}_i^m$
- $\tau_i \in \mathcal{T}_i$ is a type of player i

information: universal state space

- versal state space $\Omega = \mathcal{T} \times \Theta$

information: universal state space

- versal state space $\Omega = \mathcal{T} \times \Theta$
- consider product topology on each $\mathcal{T}_i$

information: universal state space

- versal state space $\Omega = \mathcal{T} \times \Theta$
- consider product topology on each $\mathcal{T}_i$
- and let d_{Π} be a metric on $\mathcal{T} \times \Theta$ inducing the product topology on $\mathcal{T}$ and discrete topology on Θ

information: universal state space

- versal state space $\Omega = \mathcal{T} \times \Theta$
- consider product topology on each $\mathcal{T}_i$
- and let d_Π be a metric on $\mathcal{T} \times \Theta$ inducing the product topology on $\mathcal{T}$ and discrete topology on Θ
- Mertens-Zamir 85 showed that for each $\tau_i \in \mathcal{T}_i$, there is a unique belief $\tau_i^* \in \Delta\left(\mathcal{T}_{-i} \times \Theta\right)$ so that, for all $m \in \mathbb{N}$,

$$\tau_i^m = \mathbf{marg}_{\mathcal{T}_{-i}^{m-1} \times \Theta}\left(\tau_i^*\right)$$

and $\tau \to \tau^*$

information structures

- common prior assumption: players' beliefs and higher order beliefs are interim (conditional) beliefs given some common prior distribution of the universal type space

information structures

- common prior assumption: players' beliefs and higher order beliefs are interim (conditional) beliefs given some common prior distribution of the universal type space
- so an information structure is a prior $P \in \Delta(\Omega)$ where there is a version of the conditional probability $P_i : \mathcal{T}_i \rightarrow \Delta(\mathcal{T}_{-i} \times \Theta)$, so that for every $m > 1$,

$$\tau_i^m = \mathbf{marg}_{\mathcal{T}_{-i}^{m-1} \times \Theta}(P_i(\tau_i))$$

information structures

- common prior assumption: players' beliefs and higher order beliefs are interim (conditional) beliefs given some common prior distribution of the universal type space
- so an information structure is a prior $P \in \Delta(\Omega)$ where there is a version of the conditional probability $P_i : \mathcal{T}_i \to \Delta(\mathcal{T}_{-i} \times \Theta)$, so that for every $m > 1$,

$$\tau_i^m = \mathbf{marg}_{\mathcal{T}_{-i}^{m-1} \times \Theta}(P_i(\tau_i))$$

- we write $\mathcal{P}$ for the set of information structures

information structures

- common prior assumption: players' beliefs and higher order beliefs are interim (conditional) beliefs given some common prior distribution of the universal type space
- so an information structure is a prior $P \in \Delta(\Omega)$ where there is a version of the conditional probability $P_i : \mathcal{T}_i \to \Delta(\mathcal{T}_{-i} \times \Theta)$, so that for every $m > 1$,

$$\tau_i^m = \mathbf{marg}_{\mathcal{T}_{-i}^{m-1} \times \Theta}\left(P_i\left(\tau_i\right)\right)$$

- we write $\mathcal{P}$ for the set of information structures
- now $(\mathcal{G}, P)$ is a "game of incomplete information"

solution concept

- we will describe action choices by a *decision rule*: a measurable map $\sigma : \mathcal{T} \times \Theta \rightarrow \Delta(A)$

solution concept

- we will describe action choices by a *decision rule*: a measurable map $\sigma : \mathcal{T} \times \Theta \to \Delta(A)$
- a decision rule σ is ε*-obedient* if, for each player i and some regular conditional probability P_i

$$\int_{\mathcal{T}_{-i} \times \Theta} \sum_{a_{-i}} \left(\begin{array}{c} u_i(a_i, a_{-i}, \theta) \\ -u_i(a_i', a_{-i}, \theta) \end{array} \right) d\sigma \circ P_i(a_i, a_{-i}, \tau_{-i}, \theta | \tau_i) > -\varepsilon \text{ a.s.}$$

for all a_i, a_i'

solution concept

- we will describe action choices by a *decision rule*: a measurable map $\sigma : \mathcal{T} \times \Theta \to \Delta(A)$
- a decision rule σ is ε*-obedient* if, for each player i and some regular conditional probability P_i

$$\int_{\mathcal{T}_{-i} \times \Theta} \sum_{a_{-i}} \begin{pmatrix} u_i(a_i, a_{-i}, \theta) \\ -u_i(a_i', a_{-i}, \theta) \end{pmatrix} d\sigma \circ P_i(a_i, a_{-i}, \tau_{-i}, \theta | \tau_i) > -\varepsilon \text{ a.s.}$$

  for all a_i, a_i'
- a decision rule σ is *belief-invariant* if, for each player i and action $a_i \in A_i$, $\sigma(a_i \times A_{-i} | (\tau_i, \tau_{-i}, \theta)) = \sigma(a_i | \tau_i)$ does not depend on (τ_{-i}, θ)

belief-invariant Bayes correlated equilibrium: definition

Definition

a decision rule is an *ε-belief-invariant correlated equilibrium* (ε-BIBCE) if it ε-obedient and belief invariant

- a BIBCE is a 0-BIBCE

belief-invariant Bayes correlated equilibrium: definition

Definition

a decision rule is an *ε-belief-invariant correlated equilibrium* (ε-BIBCE) if it ε-obedient and belief invariant

- a BIBCE is a 0-BIBCE
- subtlety: a belief-invariant decision rule may induce correlation between a and θ, but a_i alone provides i with no additional information about (τ_{-i}, θ)

belief-invariant Bayes correlated equilibrium: definition

Definition

a decision rule is an *ε-belief-invariant correlated equilibrium* (ε-BIBCE) if it ε-obedient and belief invariant

- a BIBCE is a 0-BIBCE
- subtlety: a belief-invariant decision rule may induce correlation between a and θ, but a_i alone provides i with no additional information about (τ_{-i}, θ)
- equivalent to standard "Bayes Nash equilibrium" (BNE) but players can observe correlating devices that are not individually informative about the state and others' beliefs...

belief-invariant Bayes correlated equilibrium: definition

Definition

a decision rule is an *ε-belief-invariant correlated equilibrium* (ε-BIBCE) if it ε-obedient and belief invariant

- a BIBCE is a 0-BIBCE
- subtlety: a belief-invariant decision rule may induce correlation between a and θ, but a_i alone provides i with no additional information about (τ_{-i}, θ)
- equivalent to standard "Bayes Nash equilibrium" (BNE) but players can observe correlating devices that are not individually informative about the state and others' beliefs...
- will postpone full motivation of solution concept but note two relevant properties:

belief-invariant Bayes correlated equilibrium: definition

Definition

a decision rule is an *ε-belief-invariant correlated equilibrium* (ε-BIBCE) if it ε-obedient and belief invariant

- a BIBCE is a 0-BIBCE
- subtlety: a belief-invariant decision rule may induce correlation between a and θ, but a_i alone provides i with no additional information about (τ_{-i}, θ)
- equivalent to standard "Bayes Nash equilibrium" (BNE) but players can observe correlating devices that are not individually informative about the state and others' beliefs...
- will postpone full motivation of solution concept but note two relevant properties:
  1. existence of BIBCE is guaranteed by Stinchcombe (2011) while it is well known that BNE do not without additional restrictions

belief-invariant Bayes correlated equilibrium: definition

Definition

a decision rule is an *ε-belief-invariant correlated equilibrium* (ε-BIBCE) if it ε-obedient and belief invariant

- a BIBCE is a 0-BIBCE
- subtlety: a belief-invariant decision rule may induce correlation between a and θ, but a_i alone provides i with no additional information about (τ_{-i}, θ)
- equivalent to standard "Bayes Nash equilibrium" (BNE) but players can observe correlating devices that are not individually informative about the state and others' beliefs...
- will postpone full motivation of solution concept but note two relevant properties:
  1. existence of BIBCE is guaranteed by Stinchcombe (2011) while it is well known that BNE do not without additional restrictions
  2. allowing information structures with "redundancies" - i.e., multiple types with the same beliefs and higher-order beliefs - makes no difference to the set of equilibrium outcomes

Main Result

approximate common knowledge

- universal state space $\Omega = \mathcal{T} \times \Theta$

p-belief operator: for every $p \in [0, 1]$, event $E \subseteq \Omega$, define

$$B^p(E) = \{(\tau, \theta) \,|\forall i, \tau_i^*(E_{-i}) \geq p\}$$

where E_{-i} is projection of E on $\mathcal{T}_{-i} \times \Theta$

approximate common knowledge

- universal state space $\Omega = \mathcal{T} \times \Theta$

p-belief operator: for every $p \in [0, 1]$, event $E \subseteq \Omega$, define

$$B^p(E) = \{(\tau, \theta) \,|\forall i, \tau_i^*(E_{-i}) \geq p\}$$

where E_{-i} is projection of E on $\mathcal{T}_{-i} \times \Theta$

- for all $m \in \mathbb{N}$, $[B^p]^m(E)$ is the m-fold application of B^p

approximate common knowledge

- universal state space $\Omega = \mathcal{T} \times \Theta$

p-belief operator: for every $p \in [0, 1]$, event $E \subseteq \Omega$, define

$$B^p(E) = \{(\tau, \theta) \,|\forall i, \tau_i^*(E_{-i}) \geq p\}$$

where E_{-i} is projection of E on $\mathcal{T}_{-i} \times \Theta$

- for all $m \in \mathbb{N}$, $[B^p]^m(E)$ is the m-fold application of B^p
- the set of states where the event E is common p-belief is

$$C^p(E) = \bigcap_{m \in \mathbb{N}} [B^p]^m(E)$$

interim beliefs are close

- an event in the universal state space is belief-closed if all players assign probability 1 to that event whenever it is true

interim beliefs are close

- an event in the universal state space is belief-closed if all players assign probability 1 to that event whenever it is true
- an information structure is minimal if there is no non-trivial belief-closed subset

interim beliefs are close

- an event in the universal state space is belief-closed if all players assign probability 1 to that event whenever it is true
- an information structure is minimal if there is no non-trivial belief-closed subset
- observation: any two distinct minimal information structures are disjoint

interim beliefs are close

- an event in the universal state space is belief-closed if all players assign probability 1 to that event whenever it is true
- an information structure is minimal if there is no non-trivial belief-closed subset
- observation: any two distinct minimal information structures are disjoint
- none the less, we want to talk about whether interim (conditional) beliefs are close across perhaps minimal information structures is a little subtle

interim beliefs are close

- neighborhood of state:

$$\mathcal{N}_{\varepsilon}(\omega) = \left\{\omega' \in \Omega | d_{\Pi}\left(\omega, \omega'\right) < \varepsilon\right\}$$

interim beliefs are close

- neighborhood of state:

$$\mathcal{N}_{\varepsilon}(\omega) = \left\{\omega' \in \Omega \middle| d_{\Pi}\left(\omega, \omega'\right) < \varepsilon\right\}$$

- ε-support of information structures:

$$\mathbf{supp}_{\varepsilon}(P) = \bigcup_{\omega \in \Omega : P(\mathcal{N}_{\varepsilon}(\omega)) > 0} \mathcal{N}_{\varepsilon}(\omega)$$

interim beliefs are close

- neighborhood of state:

$$\mathcal{N}_\varepsilon(\omega) = \{\omega' \in \Omega | d_\Pi(\omega, \omega') < \varepsilon\}$$

- ε-support of information structures:

$$\mathbf{supp}_\varepsilon(P) = \bigcup_{\omega \in \Omega : P(\mathcal{N}_\varepsilon(\omega)) > 0} \mathcal{N}_\varepsilon(\omega)$$

- intersection of information structures' ε-supports:

$$\widehat{T}_\varepsilon(P, P') = \mathbf{supp}_\varepsilon(P) \cap \mathbf{supp}_\varepsilon(P')$$

interim beliefs are close

- neighborhood of state:

$$\mathcal{N}_\varepsilon(\omega) = \left\{\omega' \in \Omega \middle| d_\Pi(\omega, \omega') < \varepsilon\right\}$$

- ε-support of information structures:

$$\mathbf{supp}_\varepsilon(P) = \bigcup_{\omega \in \Omega : P(\mathcal{N}_\varepsilon(\omega)) > 0} \mathcal{N}_\varepsilon(\omega)$$

- intersection of information structures' ε-supports:

$$\widehat{T}_\varepsilon(P, P') = \mathbf{supp}_\varepsilon(P) \cap \mathbf{supp}_\varepsilon(P')$$

  - this is the set of states where interim beliefs are close

picture 1

approximate common knowledge topology

- approximate common knowledge distance:

$$d^{ACK}\left(P, P'\right) = \inf \left\{ \varepsilon \geq 0 \,\middle|\, \begin{array}{c} P\left(C^{1-\varepsilon}\left(\widehat{T}_{\varepsilon}(P, P')\right)\right) \geq 1-\varepsilon \\ P'\left(C^{1-\varepsilon}\left(\widehat{T}_{\varepsilon}(P, P')\right)\right) \geq 1-\varepsilon \end{array} \right\}$$

approximate common knowledge topology

- approximate common knowledge distance:

$$d^{ACK}\left(P,P'\right)=\inf\left\{\varepsilon\geq 0\;\middle|\;\begin{array}{c}P\left(C^{1-\varepsilon}\left(\widehat{T}_{\varepsilon}(P,P')\right)\right)\geq 1-\varepsilon\\ P'\left(C^{1-\varepsilon}\left(\widehat{T}_{\varepsilon}(P,P')\right)\right)\geq 1-\varepsilon\end{array}\right\}$$

- if $d^{ACK}\left(P,P'\right)$ is small, there is high probability under both information structures that there is approximate common knowledge that interim beliefs are close

picture 2

approximate common knowledge topology

Definition

the approximate common knowledge topology is generated by open sets

$$\left\{P' | d^{ACK}\left(P, P'\right) \leq \varepsilon\right\}$$

- thus $P^k \rightarrow_{ACK} P$ if $d^{ACK}\left(P^k, P\right) \rightarrow 0$

approximate common knowledge topology

Definition

the approximate common knowledge topology is generated by open sets

$$\left\{P' | d^{ACK}\left(P, P'\right) \leq \varepsilon\right\}$$

- thus $P^k \rightarrow_{ACK} P$ if $d^{ACK}\left(P^k, P\right) \rightarrow 0$
- this is a metric topology (shown by constructing a variant of the distance)

strategic topology: outcomes

- our object of interest is the set of *outcomes* in $\nu \in \Delta(A \times \Theta)$ that can arise in BIBCE.

strategic topology: outcomes

- our object of interest is the set of *outcomes* in $\nu \in \Delta(A \times \Theta)$ that can arise in BIBCE.
- in particular, a BIBCE *decision rule* $\sigma : \mathcal{T} \times \Theta \to \Delta(A)$ and an *information structure* P will induce an *extended outcome* $\sigma \circ P \in \Delta(A \times \mathcal{T} \times \Theta)$

strategic topology: outcomes

- our object of interest is the set of *outcomes* in $\nu \in \Delta(A \times \Theta)$ that can arise in BIBCE.
- in particular, a BIBCE *decision rule* $\sigma : \mathcal{T} \times \Theta \to \Delta(A)$ and an *information structure* P will induce an *extended outcome* $\sigma \circ P \in \Delta(A \times \mathcal{T} \times \Theta)$
- our outcome of interest is the marginal of $\sigma \circ P \in \Delta(A \times \mathcal{T} \times \Theta)$ on $(A \times \Theta)$ which we write as $\nu_\sigma \in \Delta(A \times \Theta)$

strategic topology: outcomes

- our object of interest is the set of *outcomes* in $\nu \in \Delta(A \times \Theta)$ that can arise in BIBCE.
- in particular, a BIBCE *decision rule* $\sigma : \mathcal{T} \times \Theta \to \Delta(A)$ and an *information structure* P will induce an *extended outcome* $\sigma \circ P \in \Delta(A \times \mathcal{T} \times \Theta)$
- our outcome of interest is the marginal of $\sigma \circ P \in \Delta(A \times \mathcal{T} \times \Theta)$ on $(A \times \Theta)$ which we write as $\nu_\sigma \in \Delta(A \times \Theta)$
- writing $BIBCE(\mathcal{G}, P)$ for the set of BIBCE of $(\mathcal{G}, P)$, the set of BIBCE outcomes is

$$\mathcal{O}(\mathcal{G}, P) = \left\{ \nu \in \Delta(A \times \Theta) \,\middle|\, \begin{array}{c} \exists \sigma \in BIBCE(\mathcal{G}, P) \\ \text{such that } \nu = \nu_\sigma \end{array} \right\}$$

strategic topology: outcomes

- our object of interest is the set of *outcomes* in $\nu \in \Delta(A \times \Theta)$ that can arise in BIBCE.
- in particular, a BIBCE *decision rule* $\sigma : \mathcal{T} \times \Theta \to \Delta(A)$ and an *information structure* P will induce an *extended outcome* $\sigma \circ P \in \Delta(A \times \mathcal{T} \times \Theta)$
- our outcome of interest is the marginal of $\sigma \circ P \in \Delta(A \times \mathcal{T} \times \Theta)$ on $(A \times \Theta)$ which we write as $\nu_\sigma \in \Delta(A \times \Theta)$
- writing $BIBCE(\mathcal{G}, P)$ for the set of BIBCE of $(\mathcal{G}, P)$, the set of BIBCE outcomes is

$$\mathcal{O}(\mathcal{G}, P) = \left\{ \nu \in \Delta(A \times \Theta) \,\middle|\, \begin{array}{c} \exists \sigma \in BIBCE(\mathcal{G}, P) \\ \text{such that } \nu = \nu_\sigma \end{array} \right\}$$

- we want to say that if information structures are close their BIBCE outcomes are close in all games

strategic topology: outcomes

- our object of interest is the set of *outcomes* in $\nu \in \Delta(A \times \Theta)$ that can arise in BIBCE.
- in particular, a BIBCE *decision rule* $\sigma : \mathcal{T} \times \Theta \to \Delta(A)$ and an *information structure* P will induce an *extended outcome* $\sigma \circ P \in \Delta(A \times \mathcal{T} \times \Theta)$
- our outcome of interest is the marginal of $\sigma \circ P \in \Delta(A \times \mathcal{T} \times \Theta)$ on $(A \times \Theta)$ which we write as $\nu_\sigma \in \Delta(A \times \Theta)$
- writing $BIBCE(\mathcal{G}, P)$ for the set of BIBCE of $(\mathcal{G}, P)$, the set of BIBCE outcomes is

$$\mathcal{O}(\mathcal{G}, P) = \left\{ \nu \in \Delta(A \times \Theta) \middle| \begin{array}{c} \exists \sigma \in BIBCE(\mathcal{G}, P) \\ \text{such that } \nu = \nu_\sigma \end{array} \right\}$$

- we want to say that if information structures are close their BIBCE outcomes are close in all games
- but what do we mean by BIBCE outcomes being close?

strategic topology: outcomes

- recall that the set of outcomes induced by BIBCE of $(\mathcal{G}, P)$

$$\mathcal{O}(\mathcal{G}, P) = \left\{ \nu \in \Delta(A \times \Theta) \;\middle|\; \begin{array}{c} \exists \sigma \in BIBCE(\mathcal{G}, P) \\ \text{such that } \nu = \nu_\sigma \end{array} \right\}$$

strategic topology: outcomes

- recall that the set of outcomes induced by BIBCE of $(\mathcal{G}, P)$

$$\mathcal{O}(\mathcal{G}, P) = \left\{ \nu \in \Delta(A \times \Theta) \,\middle|\, \begin{array}{c} \exists \sigma \in BIBCE(\mathcal{G}, P) \\ \text{such that } \nu = \nu_\sigma \end{array} \right\}$$

- define $\mathcal{O}_\varepsilon(\mathcal{G}, P)$ as the set of outcomes that are ε-close to $\varepsilon-$BIBCE outcomes of $(\mathcal{G}, P)$:

$$\mathcal{O}_\varepsilon(\mathcal{G}, P) = \left\{ \nu \in \Delta(A \times \Theta) \,\middle|\, \begin{array}{c} \exists \sigma \in BIBCE_\varepsilon(\mathcal{G}, P) \\ \text{such that } \|\nu_\sigma, \nu\| \leq \varepsilon \end{array} \right\}$$

strategic topology: outcomes

- recall that the set of outcomes induced by BIBCE of $(\mathcal{G}, P)$

$$\mathcal{O}(\mathcal{G}, P) = \left\{ \nu \in \Delta(A \times \Theta) \,\middle|\, \begin{array}{c} \exists \sigma \in BIBCE(\mathcal{G}, P) \\ \text{such that } \nu = \nu_\sigma \end{array} \right\}$$

- define $\mathcal{O}_\varepsilon(\mathcal{G}, P)$ as the set of outcomes that are ε-close to $\varepsilon-$BIBCE outcomes of $(\mathcal{G}, P)$:

$$\mathcal{O}_\varepsilon(\mathcal{G}, P) = \left\{ \nu \in \Delta(A \times \Theta) \,\middle|\, \begin{array}{c} \exists \sigma \in BIBCE_\varepsilon(\mathcal{G}, P) \\ \text{such that } \|\nu_\sigma, \nu\| \leq \varepsilon \end{array} \right\}$$

  - note the two forms of approximation

strategic distance

the strategic distance between two information structures P and P' is given by

$$d^*\left(P, P' | \mathcal{G}\right) = \inf \left\{ \varepsilon \geq 0 \;\middle|\; \begin{array}{c} \mathcal{O}\left(\mathcal{G}, P\right) \subseteq \mathcal{O}_{\varepsilon}\left(\mathcal{G}, P'\right) \\ \mathcal{O}\left(\mathcal{G}, P'\right) \subseteq \mathcal{O}_{\varepsilon}\left(\mathcal{G}, P\right) \end{array} \right\}$$

sufficiency

- we first show that if two information structures are close (in the ACK topology), then they nearby equilibrium outcomes in all games.

Proposition 1 (Sufficiency): for every game $\mathcal{G}$ and $\varepsilon > 0$, there exists $\delta > 0$ so that if $d^{ACK}(P, P') < \delta$, then $d^*(P, P'|\mathcal{G}) < \varepsilon$

necessity

- we then show that if two information structures are not close (in the ACK topology), then equilibrium outcomes and not close in some game.

Proposition 2 (Necessity): for every $\varepsilon > 0$, if $d^{ACK}(P, P') \geq \varepsilon$, then there exists a game $\mathcal{G}$ such that $d^{*}(P, P'|\mathcal{G}) \geq \varepsilon$

bottom line

Theorem: The ACK topology is the coarsest topology generating continuity of strategic outcomes.

Proof Sketch

proof sketch of sufficiency

Proposition 1 (Sufficiency): for every game $\mathcal{G}$ and $\varepsilon > 0$, there exists $\delta > 0$ so that if $d^{ACK}(P, P') < \delta$, then $d^{*}(P, P'|\mathcal{G}) < \varepsilon$

- i.e., must show that for all $\mathcal{G}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that, if (i) σ is a BIBCE of $(\mathcal{G}, P)$ and (ii) $d^{ACK}(P, P') \leq \delta$, then there exists σ', a $\varepsilon-$BIBCE of $(\mathcal{G}, P')$, such that $\|\nu_{P,\sigma}, \nu_{P',\sigma'}\| \leq \varepsilon$

extension of decision rule

- let σ be any BIBCE of $(\mathcal{G}, P)$ and suppose $d^{ACK}(P, P') < \delta$

extension of decision rule

- let σ be any BIBCE of $(\mathcal{G}, P)$ and suppose $d^{ACK}(P, P') < \delta$
- we will continuously extend σ from $\mathbf{supp}(P)$ to $\mathbf{supp}_{\delta}(P)$ and thus to $\widehat{T}_{\delta}(P, P')$ and $C^{1-\delta}\left(\widehat{T}_{\delta}(P, P')\right)$

extension of decision rule

- let σ be any BIBCE of $(\mathcal{G}, P)$ and suppose $d^{ACK}(P, P') < \delta$
- we will continuously extend σ from $\mathbf{supp}(P)$ to $\mathbf{supp}_\delta(P)$ and thus to $\widehat{T}_\delta(P, P')$ and $C^{1-\delta}\left(\widehat{T}_\delta(P, P')\right)$
- if $\omega \notin \mathbf{supp}(P)$, let play at ω be an average of play on the overlap of $\mathbf{supp}(P)$ and an δ-ball around ω

extension of decision rule

- let σ be any BIBCE of $(\mathcal{G}, P)$ and suppose $d^{ACK}(P, P') < \delta$
- we will continuously extend σ from $\mathbf{supp}(P)$ to $\mathbf{supp}_\delta(P)$ and thus to $\widehat{T}_\delta(P, P')$ and $C^{1-\delta}\left(\widehat{T}_\delta(P, P')\right)$
- if $\omega \notin \mathbf{supp}(P)$, let play at ω be an average of play on the overlap of $\mathbf{supp}(P)$ and an δ-ball around ω
- write $\widehat{\sigma}$ for that extension of σ to $\mathbf{supp}_\delta(P)$

extension of decision rule

- consider modified version of game $(\mathcal{G}, P')$: force players to follow $\widehat{\sigma}$ on the event $C^{1-\delta}\left(\widehat{T}_\delta(P, P')\right)$ (which will overlap with **supp**(P'))

extension of decision rule

- consider modified version of game $(\mathcal{G}, P')$: force players to follow $\widehat{\sigma}$ on the event $C^{1-\delta}\left(\widehat{T}_{\delta}(P, P')\right)$ (which will overlap with $\mathbf{supp}(P')$)
- find a BIBCE σ' of this modified game

extension of decision rule

- consider modified version of game $(\mathcal{G}, P')$: force players to follow $\widehat{\sigma}$ on the event $C^{1-\delta}\left(\widehat{T}_\delta(P, P')\right)$ (which will overlap with $\mathbf{supp}(P')$)
- find a BIBCE σ' of this modified game
- now because $\widehat{\sigma}$ was mandated on the common $(1-\delta)$-belief event, σ' is an ε-BIBCE of the unmodified game $(\mathcal{G}, P')$, where ε depends on δ and $\mathcal{G}$

extension of decision rule

- consider modified version of game $(\mathcal{G}, P')$: force players to follow $\widehat{\sigma}$ on the event $C^{1-\delta}\left(\widehat{T}_\delta(P, P')\right)$ (which will overlap with $\mathbf{supp}(P')$)
- find a BIBCE σ' of this modified game
- now because $\widehat{\sigma}$ was mandated on the common $(1-\delta)$-belief event, σ' is an ε-BIBCE of the unmodified game $(\mathcal{G}, P')$, where ε depends on δ and $\mathcal{G}$
- because $C^{1-\delta}\left(\widehat{T}_\delta(P, P')\right)$ has probability at least $1-\delta$ under both P and P', and $\widehat{\sigma}$ was a continuous extension of σ, the outcomes induced by σ and σ' are close.

proof sketch of necessity

Proposition 2 (Necessity): for every $\varepsilon > 0$, if $d^{ACK}(P, P') \geq \varepsilon$, then there exists a game $\mathcal{G}$ such that $d^{*}(P, P'|\mathcal{G}) \geq \varepsilon$we will establish contra-positive....

proof sketch of necessity

Proposition 2 (Necessity): for every $\varepsilon > 0$, if $d^{ACK}(P, P') \geq \varepsilon$, then there exists a game $\mathcal{G}$ such that $d^{*}(P, P'|\mathcal{G}) \geq \varepsilon$we will establish contra-positive.........if $d^{ACK}(P, P') > \varepsilon$, we will show the existence of base game $\mathcal{G}$ and a BIBCE σ of $(\mathcal{G}, P)$ generating outcome ν_{σ}, such that every ε-BIBCE of $(\mathcal{G}, P)$ generates an outcome that is far from ν_{σ}

proof sketch of necessity

- now $d^{ACK}(P, P') > \varepsilon$ implies either

  $$P\left(C^{1-\varepsilon}\left(\widehat{T}_{\varepsilon}(P, P')\right)\right) < 1 - \varepsilon$$

  or

  $$P'\left(C^{1-\varepsilon}\left(\widehat{T}_{\varepsilon}(P, P')\right)\right) < 1 - \varepsilon$$

  or both

proof sketch of necessity

- now $d^{ACK}(P, P') > \varepsilon$ implies either

$$P\left(C^{1-\varepsilon}\left(\widehat{T}_{\varepsilon}(P, P')\right)\right) < 1 - \varepsilon$$

  or

$$P'\left(C^{1-\varepsilon}\left(\widehat{T}_{\varepsilon}(P, P')\right)\right) < 1 - \varepsilon$$

  or both
- let's assume

$$P\left(C^{1-\varepsilon}\left(\widehat{T}_{\varepsilon}(P, P')\right)\right) < 1 - \varepsilon$$

proof sketch of necessity

it would be enough to construct a binary action game where

1. action 1 is chosen by all players on the event

$$\mathbf{supp}(P) \setminus C^{1-\varepsilon}\left(\widehat{T}_{\varepsilon}(P, P')\right)$$

   in any ε-BIBCE of $(\mathcal{G}, P)$

proof sketch of necessity

it would be enough to construct a binary action game where

1. action 1 is chosen by all players on the event

$$\mathbf{supp}(P) \setminus C^{1-\varepsilon}\left(\widehat{T}_{\varepsilon}(P, P')\right)$$

   in any ε-BIBCE of $(\mathcal{G}, P)$

2. action 0 is chosen by all players in some BIBCE of $(\mathcal{G}, P')$

proof sketch of necessity

it would be enough to construct a binary action game where

1. action 1 is chosen by all players on the event

$$\mathbf{supp}(P) \backslash C^{1-\varepsilon}\left(\widehat{T}_{\varepsilon}(P, P')\right)$$

   in any ε-BIBCE of $(\mathcal{G}, P)$
2. action 0 is chosen by all players in some BIBCE of $(\mathcal{G}, P')$
   - in this case, action 1 would be played on an event of probability at least ε in $(\mathcal{G}, P)$ and probability 0 in $(\mathcal{G}, P')$

an "email game" argument

- consider a coordination game with two actions, 0 and 1

an "email game" argument

- consider a coordination game with two actions, 0 and 1
- action 1 is the unique ε-best response only if you attach probability at least ε to some other player choosing action 1

an "email game" argument

- consider a coordination game with two actions, 0 and 1
- action 1 is the unique ε-best response only if you attach probability at least ε to some other player choosing action 1
- suppose payoffs are always given by this coordination game except on the event

  $$D_\varepsilon = \mathbf{supp}(P) \setminus \widehat{T}_\varepsilon(P, P')$$

  when players have a dominant strategy to play action 1

figure 3

an "email game" argument

- since D_ε is disjoint from the $\mathbf{supp}(P')$, there is an equilibrium where 0 is always played in $(\mathcal{G}, P')$

an "email game" argument

- since D_ε is disjoint from the $\mathbf{supp}(P')$, there is an equilibrium where 0 is always played in $(\mathcal{G}, P')$
- but in game $(\mathcal{G}, P)$, for action 0 to be played, there must be common $(1 - \varepsilon)$-belief that the state is not in D_ε

an "email game" argument

- since D_ε is disjoint from the $\textbf{supp}(P')$, there is an equilibrium where 0 is always played in $(\mathcal{G}, P')$
- but in game $(\mathcal{G}, P)$, for action 0 to be played, there must be common $(1-\varepsilon)$-belief that the state is not in D_ε
- so 1 must be played on the event

$$\textbf{supp}(P) \setminus C^{1-\varepsilon}\left(\widehat{T}_\varepsilon(P, P')\right)$$

flaw in the argument

- we assumed that we could make it a dominant strategy to play action 1 on the event D_ε

flaw in the argument

- we assumed that we could make it a dominant strategy to play action 1 on the event D_ε
- but we can't do this: payoffs have to be measurable with respect to payoff states

flaw in the argument

- we assumed that we could make it a dominant strategy to play action 1 on the event D_ε
- but we can't do this: payoffs have to be measurable with respect to payoff states
- let's see if we can correct the flaw in the argument....

corrected argument

- make players participate in a second game where they report their mth order beliefs.

corrected argument

- make players participate in a second game where they report their mth order beliefs.
- make it a finite action game by only asking them to report nearest mth order belief on a grid

corrected argument

- make players participate in a second game where they report their mth order beliefs.
- make it a finite action game by only asking them to report nearest mth order belief on a grid
- make it uniquely rationalizable to truthfully report the closest mth order in the grid

corrected argument

- make players participate in a second game where they report their mth order beliefs.
- make it a finite action game by only asking them to report nearest mth order belief on a grid
- make it uniquely rationalizable to truthfully report the closest mth order in the grid
- (is this possible? yes, see iterated scoring rule game in Dekel-Fudenberg-Morris 06)

figure 4

corrected argument

- now choose m and grid of mth order beliefs so that there is a set of reports R sent by players in

$$D_\varepsilon = \mathbf{supp}(P) / \left(\widehat{T}_\varepsilon(P, P')\right)$$

and never sent by players in $\mathbf{supp}(P')$

corrected argument

- now choose m and grid of mth order beliefs so that there is a set of reports R sent by players in

$$D_\varepsilon = \mathbf{supp}(P) / \left(\widehat{T}_\varepsilon(P, P')\right)$$

  and never sent by players in $\mathbf{supp}(P')$
- give players an incentive to choose action 1 whenever they would send a report in R

corrected argument

- now choose m and grid of mth order beliefs so that there is a set of reports R sent by players in

$$D_\varepsilon = \mathbf{supp}(P) / \left(\widehat{T}_\varepsilon(P, P') \right)$$

  and never sent by players in $\mathbf{supp}(P')$
- give players an incentive to choose action 1 whenever they would send a report in R
- symmetric argument ensures there exists a set of reports sent by players in

$$D'_\varepsilon = \mathbf{supp}(P') / \left(\widehat{T}_\varepsilon(P, P') \right)$$

corrected argument

- now choose m and grid of mth order beliefs so that there is a set of reports R sent by players in

$$D_\varepsilon = \mathbf{supp}(P) / \left(\widehat{T}_\varepsilon(P, P') \right)$$

  and never sent by players in $\mathbf{supp}(P')$
- give players an incentive to choose action 1 whenever they would send a report in R
- symmetric argument ensures there exists a set of reports sent by players in

$$D'_\varepsilon = \mathbf{supp}(P') / \left(\widehat{T}_\varepsilon(P, P') \right)$$

- give players an incentrive to choose action 0 whenever they would send a report in R'

corrected argument

- now choose m and grid of mth order beliefs so that there is a set of reports R sent by players in
$$D_\varepsilon = \mathbf{supp}(P) / \left(\widehat{T}_\varepsilon(P, P') \right)$$
and never sent by players in $\mathbf{supp}(P')$
- give players an incentive to choose action 1 whenever they would send a report in R
- symmetric argument ensures there exists a set of reports sent by players in
$$D'_\varepsilon = \mathbf{supp}(P') / \left(\widehat{T}_\varepsilon(P, P') \right)$$
- give players an incentrive to choose action 0 whenever they would send a report in R'
- maintain that 0 is a best response to 0 on the event $C^{1-\varepsilon} \left(\widehat{T}_\varepsilon(P, P') \right)$

corrected argument

- now action 1 chosen in $(\mathcal{G}, P)$ any ε-BIBCE on the event

$$\mathbf{supp}(P) / \left(\widehat{T}_{\varepsilon}(P, P')\right)$$

corrected argument

- now action 1 chosen in $(\mathcal{G}, P)$ any ε-BIBCE on the event

$$\mathbf{supp}(P) / \left(\widehat{T}_{\varepsilon}(P, P')\right)$$

- in any BIBCE of $(\mathcal{G}, P')$, action 0 is always chosen

the game is "minimal"

- we needed (something like) a binary action coordination game and an iterated scoring rule game

the game is "minimal"

- we needed (something like) a binary action coordination game and an iterated scoring rule game
- the binary action coordination game ensures that tails of higher-order beliefs matter

the game is "minimal"

- we needed (something like) a binary action coordination game and an iterated scoring rule game
- the binary action coordination game ensures that tails of higher-order beliefs matter
  - play in the iterated scoring rule game depends only of a finite number of levels of beliefs

the game is "minimal"

- we needed (something like) a binary action coordination game and an iterated scoring rule game
- the binary action coordination game ensures that tails of higher-order beliefs matter
  - play in the iterated scoring rule game depends only of a finite number of levels of beliefs
- the iterated scoring rule game is required to identify when $C^{1-\varepsilon}\left(\widehat{T}_{\varepsilon}(P, P')\right)$ is not true

Literature 1

literature 1

- Monderer-Samet 89 introduced idea of common p-belief and showed its inportance for equilibrium behavior

literature 1

- Monderer-Samet 89 introduced idea of common p-belief and showed its inportance for equilibrium behavior
- Monderer-Samet 96 and Kajii-Morris 97 studied topologies on information structures that are the coarsest generating continuity of equilibrium outcomes

literature 1

- Monderer-Samet 89 introduced idea of common p-belief and showed its inportance for equilibrium behavior
- Monderer-Samet 96 and Kajii-Morris 97 studied topologies on information structures that are the coarsest generating continuity of equilibrium outcomes
- Many (related) minor differences: these papers considered ad hoc, countable (and different) spaces of information structures, BNE, payoff continuity instread of outcome continuity

literature 1

- Monderer-Samet 89 introduced idea of common p-belief and showed its inportance for equilibrium behavior
- Monderer-Samet 96 and Kajii-Morris 97 studied topologies on information structures that are the coarsest generating continuity of equilibrium outcomes
- Many (related) minor differences: these papers considered ad hoc, countable (and different) spaces of information structures, BNE, payoff continuity instread of outcome continuity
- Big difference:

literature 1

- Monderer-Samet 89 introduced idea of common p-belief and showed its inportance for equilibrium behavior
- Monderer-Samet 96 and Kajii-Morris 97 studied topologies on information structures that are the coarsest generating continuity of equilibrium outcomes
- Many (related) minor differences: these papers considered ad hoc, countable (and different) spaces of information structures, BNE, payoff continuity instread of outcome continuity
- Big difference:
  - we look at universal state space and distinguish higher-order beliefs and first order beliefs about payoff states

literature 1

- Monderer-Samet 89 introduced idea of common p-belief and showed its inportance for equilibrium behavior
- Monderer-Samet 96 and Kajii-Morris 97 studied topologies on information structures that are the coarsest generating continuity of equilibrium outcomes
- Many (related) minor differences: these papers considered ad hoc, countable (and different) spaces of information structures, BNE, payoff continuity instread of outcome continuity
- Big difference:
  - we look at universal state space and distinguish higher-order beliefs and first order beliefs about payoff states
  - this makes both directions harder and leads to the need for the continuous extension the decision rule and the mth level scoring rule

literature 2:

- use rationalizability as a solution concept

literature 2:

- use rationalizability as a solution concept
- Dekel-Fudenberg-Morris 06 defined a topology in terms of continuous strategic bahavior and reported some properties

literature 2:

- use rationalizability as a solution concept
- Dekel-Fudenberg-Morris 06 defined a topology in terms of continuous strategic bahavior and reported some properties
- Chen et.al (2017) can a characterization in belief hierarchies, stronger than product topology

literature 2:

- use rationalizability as a solution concept
- Dekel-Fudenberg-Morris 06 defined a topology in terms of continuous strategic bahavior and reported some properties
- Chen et.al (2017) can a characterization in belief hierarchies, stronger than product topology
- interesting connection:

literature 2:

- use rationalizability as a solution concept
- Dekel-Fudenberg-Morris 06 defined a topology in terms of continuous strategic bahavior and reported some properties
- Chen et.al (2017) can a characterization in belief hierarchies, stronger than product topology
- interesting connection:
  - we could have started with their stronger notion of closer hierarchies, and would have got the same topology on information structures

literature 2:

- use rationalizability as a solution concept
- Dekel-Fudenberg-Morris 06 defined a topology in terms of continuous strategic bahavior and reported some properties
- Chen et.al (2017) can a characterization in belief hierarchies, stronger than product topology
- interesting connection:
  - we could have started with their stronger notion of closer hierarchies, and would have got the same topology on information structures
  - but under their stronger (and more complicated) notion of closeness of hierarchies, the common p-belief desideratum would have been for free

Properties

denseness of simple information structure

- an information structure is *finite* if there are a finite set of states

Lemma

finite information structures are dense in the ACK topology

denseness of simple information structure

- an information structure is *finite* if there are a finite set of states
- an information structure is *first order belief* if all types in the support have distinct first order beliefs

Lemma

finite information structures are dense in the ACK topology

denseness of simple information structure

- an information structure is *finite* if there are a finite set of states
- an information structure is *first order belief* if all types in the support have distinct first order beliefs
- an information structure is *simple* if it is finite first order belief

Lemma

finite information structures are dense in the ACK topology

Properties

BNE and embedding correlation devices

- a game is rich if, for every action profile $a \in A$, there exists a state θ_a such that, for all players i,

$$u_i\left(a_i, a_{-i}, \theta_a\right) - u_i\left(a_i', a_{-i}, \theta_a\right)$$

Lemma

Suppose $|\Theta| \geq 2$. For any rich base game $\mathcal{G}$ and any information structure P,

$$\lim_{\varepsilon \downarrow 0} \bigcup_{P' : d^*(P, P') \leq \varepsilon} \mathcal{O}^{BNE}\left(\mathcal{G}, P'\right) = \mathcal{O}\left(\mathcal{G}, P\right)$$

BNE and embedding correlation devices

- a game is rich if, for every action profile $a \in A$, there exists a state θ_a such that, for all players i,

$$u_i\left(a_i, a_{-i}, \theta_a\right) - u_i\left(a_i', a_{-i}, \theta_a\right)$$

Lemma

Suppose $|\Theta| \geq 2$. For any rich base game $\mathcal{G}$ and any information structure P,

$$\lim_{\varepsilon \downarrow 0} \bigcup_{P' : d^*(P,P') \leq \varepsilon} \mathcal{O}^{BNE}\left(\mathcal{G}, P'\right) = \mathcal{O}\left(\mathcal{G}, P\right)$$

- fix any BIBCE σ of $(\mathcal{G}, P)$

BNE and embedding correlation devices

- a game is rich if, for every action profile $a \in A$, there exists a state θ_a such that, for all players i,

$$u_i\left(a_i, a_{-i}, \theta_a\right) - u_i\left(a_i', a_{-i}, \theta_a\right)$$

Lemma

Suppose $|\Theta| \geq 2$. For any rich base game $\mathcal{G}$ and any information structure P,

$$\lim_{\varepsilon \downarrow 0} \bigcup_{P' : d^*(P, P') \leq \varepsilon} \mathcal{O}^{BNE}\left(\mathcal{G}, P'\right) = \mathcal{O}\left(\mathcal{G}, P\right)$$

- fix any BIBCE σ of $(\mathcal{G}, P)$
- find a nearby simple information structure P' (by denseness) and an ε-BIBCE σ' of $(\mathcal{G}, P')$ inducing a nearby outcome.

BNE and embedding correlation devices

- a game is rich if, for every action profile $a \in A$, there exists a state θ_a such that, for all players i,

$$u_i\left(a_i, a_{-i}, \theta_a\right) - u_i\left(a_i', a_{-i}, \theta_a\right)$$

Lemma

Suppose $|\Theta| \geq 2$. For any rich base game $\mathcal{G}$ and any information structure P,

$$\lim_{\varepsilon \downarrow 0} \bigcup_{P' : d^*(P, P') \leq \varepsilon} \mathcal{O}^{BNE}\left(\mathcal{G}, P'\right) = \mathcal{O}\left(\mathcal{G}, P\right)$$

- fix any BIBCE σ of $(\mathcal{G}, P)$
- find a nearby simple information structure P' (by denseness) and an ε-BIBCE σ' of $(\mathcal{G}, P')$ inducing a nearby outcome.
- construct a (non-canonical) information structure that replicates σ' as an ε-BNE.

BNE and embedding correlation devices

- a game is rich if, for every action profile $a \in A$, there exists a state θ_a such that, for all players i,

$$u_i\left(a_i, a_{-i}, \theta_a\right) - u_i\left(a_i', a_{-i}, \theta_a\right)$$

Lemma

Suppose $|\Theta| \geq 2$. *For any rich base game* $\mathcal{G}$ *and any information structure* P,

$$\lim_{\varepsilon \downarrow 0} \bigcup_{P' : d^*(P,P') \leq \varepsilon} \mathcal{O}^{BNE}\left(\mathcal{G}, P'\right) = \mathcal{O}\left(\mathcal{G}, P\right)$$

- fix any BIBCE σ of $(\mathcal{G}, P)$
- find a nearby simple information structure P' (by denseness) and an ε-BIBCE σ' of $(\mathcal{G}, P')$ inducing a nearby outcome.
- construct a (non-canonical) information structure that replicates σ' as an ε-BNE.
- perturb the information structure to make it canonical

BNE and embedding correlation devices

- a game is rich if, for every action profile $a \in A$, there exists a state θ_a such that, for all players i,

$$u_i\left(a_i, a_{-i}, \theta_a\right) - u_i\left(a_i', a_{-i}, \theta_a\right)$$

Lemma

Suppose $|\Theta| \geq 2$. For any rich base game $\mathcal{G}$ and any information structure P,

$$\lim_{\varepsilon \downarrow 0} \bigcup_{P' : d^*(P, P') \leq \varepsilon} \mathcal{O}^{BNE}\left(\mathcal{G}, P'\right) = \mathcal{O}\left(\mathcal{G}, P\right)$$

- fix any BIBCE σ of $\left(\mathcal{G}, P\right)$
- find a nearby simple information structure P' (by denseness) and an ε-BIBCE σ' of $\left(\mathcal{G}, P'\right)$ inducing a nearby outcome.
- construct a (non-canonical) information structure that replicates σ' as an ε-BNE.
- perturb the information structure to make it canonical

information design

- a designer has a continuous (in the Hausdorff topology) objective function

$$V : 2^{\Delta(A \times \Theta)} \backslash \varnothing \to \mathbb{R}$$

Theorem

Now

$$\sup_{P \in \mathcal{P}^{SIMPLE} \cap \mathcal{P}^*} V\left(\mathcal{O}\left(\mathcal{G}, P\right)\right) \leq \sup_{P \in \mathcal{P}^*} V\left(\mathcal{O}\left(\mathcal{G}, P\right)\right)$$

and if $\mathcal{G}$ satisfies strong richness

$$\sup_{P \in \mathcal{P}^{SIMPLE} \cap \mathcal{P}^*} V\left(\mathcal{O}\left(\mathcal{G}, P\right)\right) = \sup_{P \in \mathcal{P}^*} V\left(\mathcal{O}\left(\mathcal{G}, P\right)\right)$$

information design

- a designer has a continuous (in the Hausdorff topology) objective function

$$V : 2^{\Delta(A\times\Theta)}\backslash\varnothing \to \mathbb{R}$$

- for an open subset $\mathcal{P}^* \subseteq \mathcal{P}$ and base game $\mathcal{G}$, the designer chooses P with objective

$$\sup_{P\in\mathcal{P}^*} V\left(\mathcal{O}\left(\mathcal{G},P\right)\right)$$

Theorem

Now

$$\sup_{P\in\mathcal{P}^{SIMPLE}\cap\mathcal{P}^*} V\left(\mathcal{O}\left(\mathcal{G},P\right)\right) \leq \sup_{P\in\mathcal{P}^*} V\left(\mathcal{O}\left(\mathcal{G},P\right)\right)$$

and if $\mathcal{G}$ satisfies strong richness

$$\sup_{P\in\mathcal{P}^{SIMPLE}\cap\mathcal{P}^*} V\left(\mathcal{O}\left(\mathcal{G},P\right)\right) = \sup_{P\in\mathcal{P}^*} V\left(\mathcal{O}\left(\mathcal{G},P\right)\right)$$

information design examples

- suppose the designer has a utility function $u_0 : A \times \Theta \longrightarrow \mathbb{R}$

information design examples

- suppose the designer has a utility function $u_0 : A \times \Theta \longrightarrow \mathbb{R}$
- standard information design

$$V(X) = \max_{\nu \in X} \sum_{a,\theta} \nu(a,\theta)\, u_0(a,\theta)$$

information design examples

- suppose the designer has a utility function $u_0 : A \times \Theta \to \mathbb{R}$
- standard information design

$$V(X) = \max_{\nu \in X} \sum_{a,\theta} \nu(a,\theta)\, u_0(a,\theta)$$

- adversarial information design

$$V(X) = \min_{\nu \in X} \sum_{a,\theta} \nu(a,\theta)\, u_0(a,\theta)$$

take homes

1. canonical information structures are natural and workable

take homes

1. canonical information structures are natural and workable
   - Bergemann-Morris 16 describe a natural "individual sufficiency" ordering over them

take homes

1. canonical information structures are natural and workable
   - Bergemann-Morris 16 describe a natural "individual sufficiency" ordering over them
2. BIBCE needs a better name but there are good reasons for thinking about it

take homes

1. canonical information structures are natural and workable
   - Bergemann-Morris 16 describe a natural "individual sufficiency" ordering over them
2. BIBCE needs a better name but there are good reasons for thinking about it
3. approximate common knowledge matters

take homes

1. canonical information structures are natural and workable
   - Bergemann-Morris 16 describe a natural "individual sufficiency" ordering over them
2. BIBCE needs a better name but there are good reasons for thinking about it
3. approximate common knowledge matters
4. papers on higher-order beliefs are liable to use (i) email game; or (ii) iterated scoring rule game

take homes

1. canonical information structures are natural and workable
   - Bergemann-Morris 16 describe a natural "individual sufficiency" ordering over them
2. BIBCE needs a better name but there are good reasons for thinking about it
3. approximate common knowledge matters
4. papers on higher-order beliefs are liable to use (i) email game; or (ii) iterated scoring rule game
   - we use and need both

Old Slides

equilibrium solution concept

- our main treatment will use "belief-invariant Bayes correlated equilibrium" (BIBCE) solution concept

equilibrium solution concept

- our main treatment will use "belief-invariant Bayes correlated equilibrium" (BIBCE) solution concept
  - the "right" equilibrium solution concept for canonical information structures

equilibrium solution concept

- our main treatment will use "belief-invariant Bayes correlated equilibrium" (BIBCE) solution concept
  - the "right" equilibrium solution concept for canonical information structures
  - equilibrium version of interim correlated rationalizability (ICR) [Liu 15]

equilibrium solution concept

- our main treatment will use "belief-invariant Bayes correlated equilibrium" (BIBCE) solution concept
  - the "right" equilibrium solution concept for canonical information structures
  - equilibrium version of interim correlated rationalizability (ICR) [Liu 15]
  - measurable with respect to the universal type space

equilibrium solution concept

- our main treatment will use "belief-invariant Bayes correlated equilibrium" (BIBCE) solution concept
  - the "right" equilibrium solution concept for canonical information structures
  - equilibrium version of interim correlated rationalizability (ICR) [Liu 15]
  - measurable with respect to the universal type space
  - exists

equilibrium solution concept

- our main treatment will use "belief-invariant Bayes correlated equilibrium" (BIBCE) solution concept
  - the "right" equilibrium solution concept for canonical information structures
  - equilibrium version of interim correlated rationalizability (ICR) [Liu 15]
  - measurable with respect to the universal type space
  - exists
- Bayes Nash equlibrium may not exist and depends on "redundancies" or correlating devices

equilibrium solution concept

- our main treatment will use "belief-invariant Bayes correlated equilibrium" (BIBCE) solution concept
  - the "right" equilibrium solution concept for canonical information structures
  - equilibrium version of interim correlated rationalizability (ICR) [Liu 15]
  - measurable with respect to the universal type space
  - exists
- Bayes Nash equlibrium may not exist and depends on "redundancies" or correlating devices
  - but we will discuss this and argue that our topology remains the relevant one

strategic topology in more detail

so two canonical information structures are ε-close in the strategic topology if, for every BIBCE under one information structure, there is an ε-BIBCE under the other information structure inducing an outcome that is ε-close

approximate common knowledge topology in more detail

- "(interim) beliefs are close" means beliefs are close in the product topology (although the exact topology used here turns out not to be important)

approximate common knowledge topology in more detail

- "(interim) beliefs are close" means beliefs are close in the product topology (although the exact topology used here turns out not to be important)
- an event is "approximate common knowledge" or "common $(1-\varepsilon)$-belief" if everyone believes it with probability at least $1-\varepsilon$, everybody believes with probability $1-\varepsilon$ that everyone believes it with probability at least $1-\varepsilon$, and so on....[Monderer and Samet 89]

approximate common knowledge topology in more detail

- "(interim) beliefs are close" means beliefs are close in the product topology (although the exact topology used here turns out not to be important)
- an event is "approximate common knowledge" or "common $(1-\varepsilon)$-belief" if everyone believes it with probability at least $1-\varepsilon$, everybody believes with probability $1-\varepsilon$ that everyone believes it with probability at least $1-\varepsilon$, and so on....[Monderer and Samet 89]
- so two canonical information structures are ε-close in the ACK topology if each assigns probability at least $1-\varepsilon$ to there being common $(1-\varepsilon)$-belief that belief hierarchies being within ε,,,,,

properties

1. *simple* information structures are dense in the ACK topology

properties

1. *simple* information structures are dense in the ACK topology
   - an information structure is *simple* if each player has finite types and each type has distinct first-order beliefs;

properties

1. *simple* information structures are dense in the ACK topology
   - an information structure is *simple* if each player has finite types and each type has distinct first-order beliefs;
2. therefore without loss of generality to focus simple information structures in information design

properties

1. *simple* information structures are dense in the ACK topology
   - an information structure is *simple* if each player has finite types and each type has distinct first-order beliefs;
2. therefore without loss of generality to focus simple information structures in information design
3. the set of BIBCE outcomes for a given canonical information structure = the set of BNE outcomes of all nearby (general) information structures

literature: (equilibrium) strategic topologies on information structures

- Monderer-Samet 96 and Kajii-Morris 98 obtain similar sounding results

literature: (equilibrium) strategic topologies on information structures

- Monderer-Samet 96 and Kajii-Morris 98 obtain similar sounding results
- but for weird class of information structures where game can depend on higher order beliefs

literature: (equilibrium) strategic topologies on information structures

- Monderer-Samet 96 and Kajii-Morris 98 obtain similar sounding results
- but for weird class of information structures where game can depend on higher order beliefs
    1. an information structure for Monderer-Samet 96 is a profile of (countable) partitions on a fixed probability space

literature: (equilibrium) strategic topologies on information structures

- Monderer-Samet 96 and Kajii-Morris 98 obtain similar sounding results
- but for weird class of information structures where game can depend on higher order beliefs
    1. an information structure for Monderer-Samet 96 is a profile of (countable) partitions on a fixed probability space
    2. an information structure for Kajii-Morris 98 is a probability distribution on a fixed (countable) type/signal space

literature: (equilibrium) strategic topologies on information structures

- Monderer-Samet 96 and Kajii-Morris 98 obtain similar sounding results
- but for weird class of information structures where game can depend on higher order beliefs
  1. an information structure for Monderer-Samet 96 is a profile of (countable) partitions on a fixed probability space
  2. an information structure for Kajii-Morris 98 is a probability distribution on a fixed (countable) type/signal space
- so notion of "interim beliefs are close" not very meaningful

literature: (equilibrium) strategic topologies on information structures

- Monderer-Samet 96 and Kajii-Morris 98 obtain similar sounding results
- but for weird class of information structures where game can depend on higher order beliefs
  1. an information structure for Monderer-Samet 96 is a profile of (countable) partitions on a fixed probability space
  2. an information structure for Kajii-Morris 98 is a probability distribution on a fixed (countable) type/signal space
- so notion of "interim beliefs are close" not very meaningful
- we require novel proof, as I will review

literature: (rationalizability) strategic topology on belief hierarchies

- Dekel-Fudenberg-Morris 06 defined and Chen-di Tillio-Faingold-Xiong 17 characterized

literature: (rationalizability) strategic topology on belief hierarchies

- Dekel-Fudenberg-Morris 06 defined and Chen-di Tillio-Faingold-Xiong 17 characterized
- connection between ex ante (equilibrium) and interim (rationalizability) topologies not previously known

literature: (rationalizability) strategic topology on belief hierarchies

- Dekel-Fudenberg-Morris 06 defined and Chen-di Tillio-Faingold-Xiong 17 characterized
- connection between ex ante (equilibrium) and interim (rationalizability) topologies not previously known
- we will use product topology on hierarchies in constructing our almost common knowledge topology on information structures, but we could have substituted more demanding interim topology, like that of Chen et al

literature: (rationalizability) strategic topology on belief hierarchies

- Dekel-Fudenberg-Morris 06 defined and Chen-di Tillio-Faingold-Xiong 17 characterized
- connection between ex ante (equilibrium) and interim (rationalizability) topologies not previously known
- we will use product topology on hierarchies in constructing our almost common knowledge topology on information structures, but we could have substituted more demanding interim topology, like that of Chen et al
- but then it turns out that approximate common knowledge is for free!

talk outline

- setting
- main result
- proof sketch
- properties

a principled justification for belief-invariant Bayes correlated equilibrium

first consider rationalizability

- in complete information games, we often study correlated rationalizable actions (instead of independent rationalizable actions of Bernheim 82 and Pearce 82).

a principled justification for belief-invariant Bayes correlated equilibrium

first consider rationalizability

- in complete information games, we often study correlated rationalizable actions (instead of independent rationalizable actions of Bernheim 82 and Pearce 82).
    - usual justication: we dont think that a correlation devices is observed; there is just an equivalence between (i) correlated rationalizability; (ii) surviving iterated deletion of strictly dominated strategies; and (iii) consistent with common knowledge of rationality

a principled justification for belief-invariant Bayes correlated equilibrium

first consider rationalizability

- in complete information games, we often study correlated rationalizable actions (instead of independent rationalizable actions of Bernheim 82 and Pearce 82).
  - usual justication: we dont think that a correlation devices is observed; there is just an equivalence between (i) correlated rationalizability; (ii) surviving iterated deletion of strictly dominated strategies; and (iii) consistent with common knowledge of rationality
- in incomplete information games, Dekel-Fudenberg-Morris 07 introduced "interim correlated rationalizability" where a player can believe that there is correlation between an opponent's action and the state even though the player knows nothing about the state.

a principled justification for belief-invariant Bayes correlated equilibrium

first consider rationalizability

- in complete information games, we often study correlated rationalizable actions (instead of independent rationalizable actions of Bernheim 82 and Pearce 82).
  - usual justication: we dont think that a correlation devices is observed; there is just an equivalence between (i) correlated rationalizability; (ii) surviving iterated deletion of strictly dominated strategies; and (iii) consistent with common knowledge of rationality
- in incomplete information games, Dekel-Fudenberg-Morris 07 introduced "interim correlated rationalizability" where a player can believe that there is correlation between an opponent's action and the state even though the player knows nothing about the state.
  - can make same response: I don't need to know source of correlation, we just know that there is an equivalence between (i) ICR; (ii) surviving iterated deletion of (interim) strictly

a principled justification for belief-invariant Bayes correlated equilibrium

now consider equilibrium, i.e., impose common prior assumption and take ex ante perspective

- in complete information games, correlated equilibrium is the equilibrium analogue of correlated rationalizability

a principled justification for belief-invariant Bayes correlated equilibrium

now consider equilibrium, i.e., impose common prior assumption and take ex ante perspective

- in complete information games, correlated equilibrium is the equilibrium analogue of correlated rationalizability
- in incomplete information games, BIBCE is the equilibrium analogue of ICR

pragmatic justifications for belief-invariant Bayes correlated equilibrium

- BIBCE is exactly the solution concept where the set of outcomes induced by BIBCE on a canonical information structure equals the set of outcomes induced by BIBCE on any information structure

pragmatic justifications for belief-invariant Bayes correlated equilibrium

- BIBCE is exactly the solution concept where the set of outcomes induced by BIBCE on a canonical information structure equals the set of outcomes induced by BIBCE on any information structure
- canonical information structures can be separated by BIBCE play in some game, but other information structures cannot

pragmatic justifications for belief-invariant Bayes correlated equilibrium

- BIBCE is exactly the solution concept where the set of outcomes induced by BIBCE on a canonical information structure equals the set of outcomes induced by BIBCE on any information structure
- canonical information structures can be separated by BIBCE play in some game, but other information structures cannot
- we will also argue that if you are interested in Bayes Nash equilibrium (or any solution concept between BNE and BIBCE) you should still be interested in our topology