Information Design

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Bonn Lectures on Information Design 2018

Mechanism Design and Information Design

- "**Basic" Mechanism Design**:
  - Fix an economic environment and information structure
  - Design the rules of the game to get a desirable outcome
- **Information Design**
  - Fix an economic environment and rules of the game
  - Design an information structure to get a desirable outcome

Mechanism Design and Information Design

- **"Basic" Mechanism Design**:
  - Can compare particular mechanisms..
    - e.g., first price auctions versus second price auctions
  - Can work with space of all mechanisms...
    - without loss of generality, let each agent's action space be his set of types...revelation principle
    - e.g., optimal mechanism
- **Information Design**
  - Can compare particular information structures
    - Linkage Principle: Milgrom-Weber 82
    - Information Sharing in Oligopoly: Novshek-Sonnenschein 82
  - Can work with space of all information structures
    - without loss of generality, let each agent's type space be his set of actions......revelation principle

Information Design: Some Leading Cases

1. Uninformed information designer (or "mediator"):
   - Myerson: "Bayesian games with communication" (one piece of expanded mechanism design: also "collective choice problems")
   - Incomplete Information Correlated Equilibrium literature of the 1980s and 1990s (Forges 93)
2. One player (a "receiver") and an informed information designer (or "sender")
   - "Bayesian Persuasion": Kamenica-Gentzkow 11 and large and important literature inspired by it
3. One player and uninformed designer: *very boring*
4. Many players *and* an informed information designer

Setup

- Maintained Environment: Fix players 1,...,I; payoff states Θ; prior on states $\psi \in \Delta(\Theta)$
- Basic Game $G : (A_i, u_i)_{i=1..,I}$ where $u_i : A \times \Theta \to \mathbb{R}$
- Information Structure S: $(T_i)_{i=1..,I}$ and $\pi : \Theta \to \Delta(T)$

Information Designer's Problem

- *Decision rule* $\sigma : T \times \Theta \to \Delta(A)$ is *obedient* for (G, S) if, for all i, t_i, a_i and a_i',

$$\begin{aligned} & \sum_{a_{-i}, t_{-i}, \theta} u_i\left(\left(a_i, a_{-i}\right), \theta\right) \sigma\left(a \middle| t, \theta\right) \pi\left(t \middle| \theta\right) \psi\left(\theta\right) \\ \geq & \sum_{a_{-i}, t_{-i}, \theta} u_i\left(\left(a_i', a_{-i}\right), \theta\right) \sigma\left(a \middle| t, \theta\right) \pi\left(t \middle| \theta\right) \psi\left(\theta\right); \end{aligned}$$

  Obedient decision rule σ is a *Bayes correlated equilibrium* (BCE). Characterizes implementability.
- Information designer with payoff $v : A \times \Theta \to \mathbb{R}$ picks a Bayes correlated equilibrium $\sigma \in BCE\left(G, S\right)$ to maximize

$$V_S\left(\sigma\right) \equiv \sum_{a, t, \theta} \psi\left(\theta\right) \pi\left(t \middle| \theta\right) \sigma\left(a \middle| t, \theta\right) v\left(a, \theta\right).$$

Information Design: Three Interpretations

1. Literal: actual information designer with ex ante commitment
2. Informational robustness: family of objectives characterize set of attainable outcomes
3. Metaphorical: e.g., adversarial / worst case

Revelation Principle

- We implicitly restrict the information designer to give players information only action recommendations
- But allowing the information designer to send richer signals would not have allowed the information designer to induce implement extra outcomes
- standard arguments not reviewed here establish this.

One Uninformed Player: Benchmark Investment Example

- a firm is deciding whether to invest or not:
- binary state: $\theta \in \{B, G\}$, bad or good
- binary action: $a \in \{\text{Invest, Not Invest}\}$
- payoffs

	bad state B	good state G
Invest	-1	x
Not Invest	0	0

  with $0 < x < 1$
- prior probability of each state is $\frac{1}{2}$
- firm is uninformed (so one uninformed player)
- information designer (government) seeks to maximize probability of investment (independent of state)
- leading example of Kamenica-Gentzkow 11: will return to KG and concavification

Decision Rule

- p_θ is probability of investment, conditional on being in state θ

	bad state B	good state G
Invest	p_B	p_G
Not Invest	$1 - p_B$	$1 - p_G$

- interpretation: firm observes signal = "action recommendation," drawn according to (p_B, p_G)

Obedience Constraints

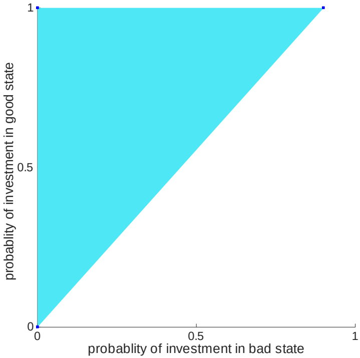
- if "advised" to invest, invest has to be a best response:

$$
\begin{aligned}
-\frac{1}{2}p_B + \frac{1}{2}p_G x &\geq 0 \Leftrightarrow \\
p_G &\geq \frac{p_B}{x}
\end{aligned}
$$

- if "advised" to not invest, not invest has to be a best response
- but because $x < 1$, investment constraint is binding one
- always invest ($p_B = 1$ and $p_G = 1$) is not a BCE
- the "full information equilibrium" has invest only in good state ($p_B = 0$ and $p_G = 1$)

Bayes Correlated Equilibria

equilibrium outcomes (p_B, p_G) for $x = 0.9$



- always invest ($p_B = 1$ and $p_G = 1$) is not a BCE
- the full information equilibrium has invest only in good state ($p_B = 0$ and $p_G = 1$)

Information Design

- recommendation maximizing the probability of investment:

$$p_B = x,\ p_G = 1$$

- best BCE

	B	G
Invest	x	1
Not Invest	$1 - x$	0

- Optimal for government to obfuscate, partially pooling good state and bad state

One Informed Player

- Firm receives a signal which is "correct" with probability $q > 1/2$.
- Formally, the firm observes a signal g or b, with signals g and b being observed with conditionally independent probability q when the true state is G or B respectively:

	bad state B	good state G
bad signal b	q	$1-q$
good signal g	$1-q$	q

- As before, except distribution over states $\theta \in \{B, G\}$ changes in response to firm's signal $t \in \{b, g\}$;
- A decision rule is then a quadruple $\left(p_B^b, p_G^b, p_B^g, p_G^g\right)$.
- For example, if firm's own information is sufficiently noisy, or $q \leq \frac{1}{1+x}$, there is still a binding investment constraint for each signal, e.g.,

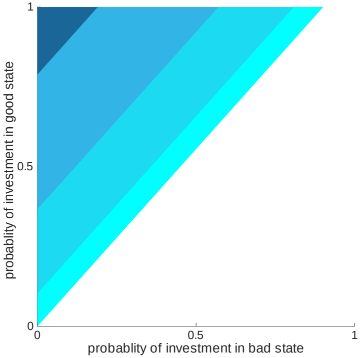
  $$p_G^g \geq \frac{1-q}{q}\frac{p_B^g}{x}.$$

  (if the good signal is good enough, can get investment with probability 1)
- the interesting question: what if we project $\left(p_B^b, p_G^b, p_B^g, p_G^g\right)$ back into ex ante behavior $\left(p_B^b, p_G^b, p_B^g, p_G^g\right)$? e.g.

  $$p_G = qp_G^g + (1-q)\, p_G^b$$

One Informed Player: Bayes Correlated Equilibrium

equilibrium set (for $x = 0.9$ and $q = 0.5$, 0.575, 0.7 and 0.875)



Two Firms

- payoffs almost as before....

$\theta = B$	I	N
I	$-1+\varepsilon$	-1
N	0	0

$\theta = G$	I	N
I	$x+\varepsilon$	x
N	0	0

- ...up to ε term
- assume that information designer (government) wants to maximize the *sum* of probabilities that firms invest....
- if $\varepsilon = 0$, problem is exactly as before firm by firm; doesn't matter if and how firms' signals are correlated
- we will consider what happens when $|\varepsilon| \approx 0$ (so the analysis cannot change very much)
- will now have profile of action recommendations depending on the state

Two Firms: Strategic Complementarities

- If $\varepsilon > 0$, optimal rule is

$\theta = B$	I	N
I	$\frac{x+\varepsilon}{1-\varepsilon}$	0
N	0	$\frac{1-x-2\varepsilon}{1-\varepsilon}$

$\theta = G$	I	N
I	1	0
N	0	0

- the probability of any one firm investing is still about x..
- binding constraints are still investment constraints, slackened by having simultaneous investment...

$$\frac{x+\varepsilon}{1-\varepsilon}(-1+\varepsilon)+x+\varepsilon \geq 0$$

- ....so signals are public

Two Firms: Strategic Substitutes

- If $\varepsilon < 0$, optimal rule is

$\theta = B$	I	N
I	0	$x + \varepsilon$
N	$x + \varepsilon$	$1 - 2x - 2\varepsilon$

$\theta = G$	I	N
I	1	0
N	0	0

- the probability of any one firm investing if the state is bad is still about x....
- binding constraints are still investment constraints, slackened by having minimally correlated investment...

$$(x + \varepsilon)(-1) + x + \varepsilon \geq 0$$

- ....and signals are private
- .....and as "uncorrelated" as possible

Preferences over Correlation of Players Actions

- We assumed information designer had no *intrinsic* preferences over correlation of actions but simply wanted correlation for *instrumental* reasons: with strategic complements, positive correlation relaxed obedience constraint; with strategic substitutes, negative correlation relaxed obedience constraints. (see also Perego, Mathevet and Taneva (2017)).
- Arieli and Babichenko (2016), Meyer (2017) study intrinsic preferences over correlation

Other Objectives and a Benevolent Information Designer

- In one firm case, if government had the same objective as the firm, he would always give them full information...
- But in the two firm case, a benevolent government maximizing the (joint) profits of the two firms might still manipulate information in order to correct for externalities and coordinate behavior
- In game

$\theta = B$	I	N
I	$-1 + \varepsilon + z$	-1
N	z	0

$\theta = G$	I	N
I	$x + \varepsilon + z$	x
N	z	0

benevolent government will behave as an investment maximizing government if z is large enough

Concavification: Example

- recall example

	bad state B	good state G
Invest	-1	x
Not Invest	0	0

  with $0 < x < 1$

- if firm assigns probability q to good state,
  - optimal decision rule is

$$A^*(q) = \begin{cases} \{\text{Invest}\}, \text{ if } q > \frac{1}{1+x} \\ \{\text{Invest, Not Invest}\}, \text{ if } q = \frac{1}{1+x} \\ \{\text{Not Invest}\}, \text{ otherwise} \end{cases}$$

  - the induced probability of investment is

$$v(q) = \begin{cases} 1, \text{ if } q \geq \frac{1}{1+x} \\ 0, \text{ otherwise} \end{cases}$$

Concavification: General One Player Case

- single player $u : A \times \Theta \to \mathbb{R}$
- information designer $v : A \times \Theta \to \mathbb{R}$
- define player's decision rule $A^* : \Delta(\Theta) \to 2^A / \varnothing$

$$a^*(q) = \arg\max_{a \in A} \sum_{\theta} q(\theta)\, u(a, \theta)$$

- define designer's indirect utility over posteriors $\widetilde{v} : \Delta(\Theta) \to \mathbb{R}$

$$\widetilde{v}(q) = \max_{a \in A^*(q)} \sum_{\theta} q(\theta)\, v(a, \theta)$$

Concavification: General

- concavification of f the infinum of the concave functions exceeding f
- define $\widehat{v}$ to be the concavification of the designers indirect utility $\widetilde{v}$

THEOREM:

$$\max_{\sigma \in BCE(G,S)} V_S(\sigma) = \widehat{v}(\psi)$$

Concavification

- We first described two step procedure for solving information design problem (with one or many players):
  1. Characterize all implementable decision rules
  2. Pick the designer's favorite
- Concavification procedure (with one player) [Aumann-Maschler 95 and Kamenica-Gentzkow 11] is in principle a short cut
  - Identify information designer's utility for every belief of the single player
  - Identify utility from optimal design by concavification, identifying information design only implicitly
- Many player generalization: Mathevet al 16
- Always nice interpretation, sometimes (but not always) useful in solving information design problem

Application 1 - Information Sharing: Strategic Substitutes

- Classic Question: are firms better off if they share their information?
- Consider quantity competition when firms uncertain about level of demand (intercept of linear demand curve) with symmetry, normality and linear best response; two effects in conflict:
  1. Individual Choice Effect: Firms would like to be as informed as possible about the state of demand
  2. Strategic Effect: Firms would like to be as uncorrelated with each other as possible

Application 1 - Information Sharing

- Classic Question: are firms better off if they share their information?
- Consider quantity competition when firms uncertain about level of demand: individual and strategic effects in conflict
- Resolution:
  - For large enough price sensitivity (and thus strategic substitutability), strategic effect wins and no information is optimal
  - For low enough price sensitivity (and thus strategic substitutability), individual choice effect wins and full information is optimal
  - For intermediate price sensitivity, there is a non-trivial trade-off and it is optimal to have firms observe noisy signals of demand, but with uncorrelated noise and thus conditionally independent signals, and thus signals which are as uncorrelated as possible conditional on their accuracy

Application 2 - Aggregate Volatility: Wacky Designer Objective

- Classic Question: can informational frictions explain aggregate volatility?
- Consider a setting where each agent sets his output equal to his productivity which has a common component and an idiosyncratic component
- again with symmetry and normality.... common component θ with variance σ^2; idiosyncratic component ε_i with variance τ^2; $\theta_i = \theta + \varepsilon_i$
- Which information structure maximizes variance of average action?

Application 2 - Aggregate Volatility

What information structure maximizes variance of average action?

- suppose that each agent observes a confounding (c.f., Lucas 72) signal $s_i = \lambda\varepsilon_i + (1-\lambda)\theta$ *without noise...*
- optimal action is

$$\frac{(1-\lambda)\sigma^2 + \lambda\tau^2}{(1-\lambda)^2\sigma^2 + \lambda^2\tau^2} s_i$$

- variance of average action is maximized when

$$\lambda = \frac{\sigma}{2\sigma + \sqrt{\sigma^2 + \tau^2}}$$

  and maximum variance of average action is

$$\left(\frac{\sigma + \sqrt{\sigma^2 + \tau^2}}{2}\right)^2$$

Application 2 - Aggregate Volatility

What information structure maximizes variance of average action?

- "optimal" information structure has a confounding (c.f., Lucas 72) signal $s_i = \lambda x_i + (1 - \lambda) y$ *without noise...*
- as $\sigma \to 0$:
  - "optimal" weight on idiosyncratic component goes to 0
  - agents put a lot of weight on their signal in order to put a non-trivial weight on their idiosyncratic component
  - in the limit, the common component becomes a payoff irrelevant but common "sentiments" shock:
- this was actually a non-strategic problem: logic can be extended to strategic setting
- can then be embedded in a richer setting (Angeletos La'O 13)

Application 3 - First Price Auction: Information Shrinking BCE, Adversarial Information Designer, Robust Predictions and Partial Identification

- Example: Two bidders and valuations independently and uniformly distributed on the interval $[0, 1]$
- Plot: (expected bidders' surplus, expected revenue) pairs
- green = feasible pairs, blue = unknown value pairs, red = known value pairs

Application 3 - First Price Auction

1. Known value case (red region) is subset of unknown value case (blue region)
2. Robust Prediction:
    2.1 revenue has lower bound $\approx 1/10$
    2.2 lower bound (w.r.t. first order stochastic dominance) on bids
3. Partial Identification: Winning bid distribution $\Longrightarrow$ Lower bound on Value Distribution (w/o identifying private vs. common values)

Designer Access to Players' Information

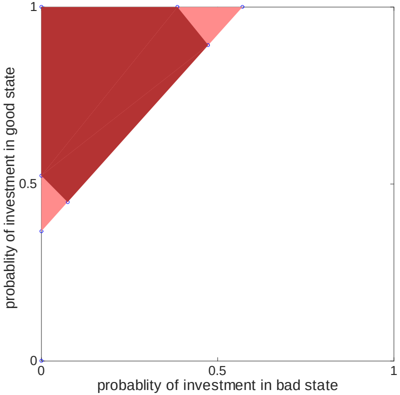
- We want to assume that information designer knows the state θ...
- ...but what should we assume about what information designer knows about players' information? Consider three scenarios:
  1. Omniscient Designer: the designer knows all players' information too...[**maintained assumption so far**]
  2. Communicating Designer: the designer can condition his announcements about the state only on players' reports of their types
  3. Non-Communicating Designer: the designer can tell players about the state but without conditioning on players' information

Back to One Informed Player: Communicating Designer

- as before, firm observes a signal $t \in T$ and government makes a recommendation to invest p_θ^t as a function of reported signal t and state θ
- incentive constraint: add truth-telling to obedience
- to insure truth-telling, differences in recommendations must be bounded across states

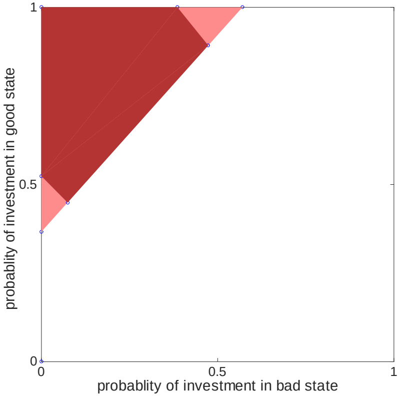
Communicating Designer

- adding truth-telling constraints...($x = 0.9$, $q = 0.7$)



- communicating (red), omniscient (pink)

Communicating Designer



- if there is a large discrepency in recommendations, then firm has an incentive to misreport his signal
- e.g., at maximum investment BCE (top right), firm with good signal is always told to invest; firm with bad signal is given useful information

Non-communicating designer

- firm observes his signal
- government offers a recommendation, independent of the signal, depending on the true state
- In our example, communicating and non-communicating designer can attain the same set of outcomes;
- Kotolin et al show this in a more general - but still restrictive - class of environments
- alternative interpretation: firms can communicate publicly not privately to large audience

Taxonomy

	Single Agent	Many Agent Uninformed Designer	Many Agent Informed Designer
Omniscient	.	Bayesian Solution	BCE
Communicating	Kolotilin et al	Communication Equilibrium	.
Non Communicating	KG informed receiver	Strategic Form Correlated Equilibrium	.

Mechanism Design and Information Design

- Myerson Mechanism Design:
  - Dichotomy in Myerson (1991) textbook
    - Bayesian games with communication (game is fixed)
    - Bayesian collective choice problems (mechanism is chosen by designer)
  - both combined in Myerson (1982, 1987)
- Truth-telling (honesty) and obedience constraints always maintained
- "information design" = "Bayesian games with communication" − truth-telling + informed information designer/mediator

Incomplete Information Correlated Equilibrium

- Decision rule $\sigma : T \times \Theta \to A$ is *incentive compatible* for (G, S) if, for each i, t_i and a_i, we have

$$\begin{aligned} & \sum_{a_{-i}, t_{-i}, \theta} u_i\left(\left(a_i, a_{-i}\right), \theta\right) \sigma\left(a | t, \theta\right) \pi\left(t | \theta\right) \psi\left(\theta\right) \\ \geq & \sum_{a_{-i}, t_{-i}, \theta} u_i\left(\left(\delta\left(a_i\right), a_{-i}\right), \theta\right) \sigma\left(a \middle| \left(t_i', t_{-i}\right), \theta\right) \pi\left(t | \theta\right) \psi\left(\theta\right); \end{aligned} \tag{1}$$

  for all t_i' and $\delta_i : A_i \to A_i$.

- Decision rule $\sigma : T \times \Theta \to A$ is *join feasible* for (G, S) if $\sigma\left(a | t, \theta\right)$ is independent of θ, i.e., $\sigma\left(a | t, \theta\right) = \sigma\left(a | t, \theta'\right)$ for each $t \in T$, $a \in A$,and $\theta, \theta' \in \Theta$.
- Solution Concepts:
  - Bayes correlated equilibrium = obedience
  - Communication equilibrium = incentive compatibility (and thus obedience) and join feasibility
  - Agent normal form correlated equilibrium involves additional feasibility constraints.

Adversarial Equilibrium Selection

- Suppose that an information designer gets to make a communication $\Phi : T \times \Theta \rightarrow \Delta(M)$; new game of incomplete information (G, S, Φ)
- Write $E(G, S, \Phi)$ for the set of Bayes Nash equilibria of (G, S, Φ) and write $V_S^*(\Phi, \beta)$ for the information designer's utility
- We have been studying the maxmax problem

$$\max_{C} \max_{\beta} V_S^*(\Phi, \beta)$$

  using a revelation principle argument to show that this equals

$$\max_{\sigma \in BCE(G,S)} V_S(\sigma)$$

- The maxmin problem

$$\max_{C} \min_{\beta} V^*(S, \Phi, \beta)$$

  does not have a revelation principle characterization

Adversarial Equilibrium Selection

- Starting from (G, S), suppose that an information designer gets to make an expansion $\Phi : T \times \Theta \rightarrow \Delta(M)$; new game of incomplete information (G, S, Φ)
- Write $E(G, S, \Phi)$ for the set of Bayes Nash equilibria of (G, S, Φ) and write $V_S^*(\Phi, \beta)$ for the information designer's utility
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- The maxmin problem

$$\max_C \min_\beta V^*(S, \Phi, \beta)$$

  does not have a revelation principle characterization

Adversarial Equilibrium Selection Example

- two firms with payoffs

$\theta = B$	I	N
I	x, x	$-1, 0$
N	$0, -1$	$0, 0$

$\theta = G$	I	N
I	x, x	$x, 0$
N	$0, -1$	0

- probability of state G is $q < 1 - x$
- player 1 knows the state, $T_1 = \{b, g\}$; $\pi\left(g \middle| G\right) = \pi\left(b \middle| B\right) = 1$
- player 2 knows nothing, $T_2 = \{\varnothing\}$
- information designer wants to induce investment in the worst equilibrium

Adversarial Equilibrium Selection Example

- consider the communication with $M_1 = M_2 = \{0, 1, 2, ...\} \cup \{\infty\}$
- let

$$\begin{aligned}
\pi\left(0, 0 | G, g\right) &= 1 \\
\pi\left(m_1, m_2 | B, b\right) &= \begin{cases} \frac{q}{1-q} x^{m_1+m_2}, & \text{if } m_1 \geq 1 \\ & \text{and } m_2 = m_1 - 1 \text{ or } m_1 \\ 1 - \frac{qx}{1-x}, \text{ if } m_1 = m_2 = \infty \end{cases}
\end{aligned}$$

Adversarial Equilibrium Selection Example

good state	0	1	2	·	k	·	∞
0	1	0	0	·	0	·	0
1	0	0	0	·	0	·	0
2	0	0	0	·	0	·	0
·	·	·	·	·	·	·	·
k	0	0	0	·	0	·	0
·	·	·	·	·	·	·	·
∞	0	0	0	·	0	·	0

Adversarial Equilibrium Selection Example

bad state	0	1	2	$\cdot$	k	$\cdot$	∞
0	0	0	0	$\cdot$	0	$\cdot$	0
1	$\frac{q}{1-q}x$	$\frac{q}{1-q}x^2$	0	$\cdot$	0	$\cdot$	0
2	0	$\frac{q}{1-q}x^3$	0	$\cdot$	0	$\cdot$	0
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
k	0	0	0	$\cdot$	$\frac{q}{1-q}x^{2k}$	$\cdot$	0
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
∞	0	0	0	$\cdot$	0	$\cdot$	$1 - \frac{qx}{1-x}$

Adversarial Equilibrium Selection Example

unconditional distribution	0	1	2	$\cdot$	k	$\cdot$	∞
0	q	0	0	$\cdot$	0	$\cdot$	0
1	qx	qx^2	0	$\cdot$	0	$\cdot$	0
2	0	qx^3	0	$\cdot$	0	$\cdot$	0
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
k	0	0	0	$\cdot$	qx^{2k}	$\cdot$	0
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
∞	0	0	0	$\cdot$	0	$\cdot$	$1 - \frac{q}{1-x}$

Adversarial Equilibrium Selection Example

- now player 1 observing m_1 assigns probability $\frac{1}{1+x}$ to player 2 observing $m_1 - 1$ and probability $\frac{x}{1+x}$ to player 2 observing m_1
- now player 2 observing m_2 assigns probability $\frac{1}{1+x}$ to player 1 observing m_2 and probability $\frac{x}{1+x}$ to player 1 observing $m_2 + 1$
- type 0 of player 1 has dominant strategy to inve
- knowing this, it is a weakly dominant strategy for type 0 of player 2 to invest
- now, by induction, invest is unique action surviving iterated deletion of weakly dominated strategies for all types $k \neq \infty$
- thus not invest survives iterated deletion with probability

$$1 - \frac{q}{1-x}$$

Adversarial Equilibrium Selection Example

- we can get arbitrarily close to this probability replacing weak with strict deletion
- so there is an equilibrium of an information structure where the probability that both players invest can be reduced arbitrarily close to
$$\frac{q}{1-x}$$
- can show that this is best information structure
- if $x > 1$, we could have guaranteed investment everywhere
- bound improves to
$$\frac{q(1+x)}{1-x}$$
if player 1 does not know the state

Adversarial Equilibrium Selection

- this argument: see Kajii-Morris 97; Hoshino 17, Inostroza and Pavan 17,
- other arguments: see Carroll 15, Taneva et al 16,

Multi-Stage Game

- "Information Design in a Multi-Stage Game" by Makris and Renou
- two players and two stages
- player 1 chooses $a_1 \in \{T, B\}$ in the first stage; player 2 is inactive
- player 2 chooses $a_2 \in \{L, R\}$ in the first stage; player 1 is inactive
- payoffs:

	L	R
T	$2, 2$	$0, 1$
B	$3, 0$	$1, 1$

Multi-Stage Game

- what if an information designer can sent information to the players?
- in this complete information case, relevant information is payoff irrelevant and about player 1's action
- with no information, unique Nash equilibrium (B, R)
- if player 2 told player 1's action, Stackleberg outcome (T, L)

Multi-Stage Game

- can also induce $\left(\frac{1}{2}, \frac{1}{2}\right)$ distribution on (B, L) and (T, L):
  - player 1 receives $\left(\frac{1}{2}, \frac{1}{2}\right)$ distribution on signals t or b
  - player 2 receives signal l if and only if (T, t) or (B, b) played in the first period, otherwise signal r
  - equilibrium for players to follow "recommendations"
- in fact, can get exactly convex combination of utility profiles $(1, 1)$, $(2, 2)$,$(2.5, 2)$

Multi-Stage Game

- n players and T stages
- at each stage t, state θ_t is drawn, player i receives signal $s_{it} \in S_{it}$ and chooses an action $a_{it} \in A_{it}$
- probability p_t of state θ_t and signal profile s_t at period t depends on all past actions signals and states
- payoff depends on realized states and actions
- this is base game

Expansions

- Now suppose that players observe addition signals
- At stage t, player i recieves additional signal $m_{it} \in M_{it}$
- The joint probability π_t of state θ_t and signal profile (s_t, m_t) at period t depends on all past actions signals (including additional ones) and states
- expanded game

Admissible Expansion

- no causal effects: conditonal on past actions, signals and states, the probability π_t of current states and signals does not depend on past additional signals, and coincides with p_t
- additional signals at each stage are measurable with respect to past and current states and signals, past actions and past additional signals

Bayes Correlated Equilibria

- Bayes correlated equilibria: set of admissible expansions where additional signals are action recommendations and obedience is satisfied
- The set of outcomes that can be induced by Bayes Nash equilibria in an admissible expansion equals the set of Bayes correlated equilibria

Compare Communication Equilibria

- Traditional study of multi-stage games with communication: add an uninformed mediator, who receives reports from players and makes perhaps correlated recommendations
- Set of possible outcomes: communication equilibria required to satisfy truth-telling as well as obedience
- The set of communication equilibria of expansions will also induce the set of BCE

Refinements

- Traditional to impose sequential rationality in multi-stage games
- Myerson 1986: sequential communication equilibria
- Can define analogous sequential Bayes correlated equilibria and show equivalence between:
  - sequential Bayes correlated equilibria
  - outcomes reached by sequential equilibria in an expansion
  - outcomes reached by sequential communication equilibria in an expansion

Trembling Designer

- Like sequential communication equilibrium, sequential BCE requires the designer to tremble
- Gap between BCE and sequential BCE? Price discrimination example but are there more?

Sequential Information Design

- Doval and Ely (2016) "Sequential Information Design"
- public good contribution game:

	C	NC
C	$v_\theta - c, v_\theta - c$	$v_\theta - c, v_\theta$
NC	$v_\theta, v_\theta - c$	$0, 0$

  where $0 < v_L < c < v_H$
- probability of state H is p
- if $pv_H + (1-p)\,v_L > c$, three Nash equilibria: pure equilibria where one player contributes and mixed equilibrium where each contributes with probability

$$\alpha = \frac{pv_H + (1-p)\,v_L - c}{pv_H + (1-p)\,v_L}$$

- information designer wants to maximize expected total contribution...
- ...so if $\alpha \geq \frac{1}{2}$, the designer's preferred Nash equilibria would be mixed equilibrium

Information Design

- Static information design. In best BCE, both players always contribute in the low state. In the high state both contribute with probability more than $\frac{1}{2}$ with one always contributing.
- Better two stage mechanism:
  - reduce probability that player 1 contributes, while increasing the probability that player 2 contributes in the high state.
  - in static mechanism, this would remove the incentive of the player 1 to contribute
  - but if player 1 moves first, can tell player 2 not to contribute if player 1 does not do so
  - this gives the best two stage mechanism
- Better three stage mechanism:
  - randomize over who is first mover
  - slack in second mover's obedience constraint in the two stage mechanism means there are gains from doing so

Best BCE

- In best BCE, both players always contribute in the low state. In the high state both contribute with probability more than $\frac{1}{2}$ with one always contributing.
- Better sequential mechanism:
  - reduce probability that player 1 contributes, while increasing the probability that player 2 contributes in the high state.
  - in static mechanism, this would remove the incentive of the player 1 to contribute
  - but if player 1 moves first, can tell player 2 not to contribute if player 1 does not do so
  - this gives the best two stage mechanism

General Question

- Fix basic game specifying actions and payoff states
- Construct extensive form with arbitary information and order of moves
- Admissible extensive form: no delegation, know own action, no commitment
- Question: which outcomes (distributions over actions and states) can be induced by some equilibrium and some admissible extensive form?

Canonical Mechanism

- Pick an order to approach players
- Sequentially provide them with information
- The optimal outcome can always be obtained in a canonical mechanism

Optimal Canonical Mechanism in Example

1. Pick a player to move first at random
2. Ask him to contribute with some probability conditional on the state
3. Ask second mover to contribute with some probability conditional on the recommendation and action of the first mover
4. No contribution requested if the first mover does not follow recommendation
5. First mover contributes with probability 1 in low state, second mover always contributes with probability 1
6. First mover contributes with probability

$$\min\left(1, 1 - \frac{2c - pv_H + (1-p)\,v_L}{cp}\right)$$

in the high state. This is the maximum probability consistent with obedience.

Ordering Information

- Fix two information structures $S = \left((T_i)_{i=1}^{I}, \pi \right)$ and $S' = \left((T_i')_{i=1}^{I}, \pi' \right)$
- Information structure $S^* = \left((T_i^*)_{i=1}^{I}, \pi^* \right)$ is a combination of S and S' if
  - $T_i^* = T_i \times T_i'$ for each i
  - $\text{marg}_T \ \pi^* = \pi$ and $\text{marg}_{T'} \ \pi^* = \pi'$
- Information structures S is individually sufficient for S' if

$$\sum_{t'_{-i}} \pi^* \left(\left(t_i', t_{-i}' \right) \mid \left(t_i, t_{-i} \right), \theta \right)$$

is independent of $\left(t_{-i}, \theta \right)$

Key Properties of Individual Sufficiency

- Reduces to Blackwell's order in the one player case.
- But noise must depend on θ
- Information structures S and S' are individually sufficient for each other if and only if they correspond to the same subset of the universal type space.

Ordering Information

- Intuition: more information for the player imposes more constraints on the information designer and reduces the set of outcomes she can induce
- Recall Auction Example
- Say that information structure S "is more incentive constrained than" ($=$ more informed than) S' if it gives rise to a smaller set of BCE outcomes than S' in all games
  - in one player case, this ordering corresponds to Blackwell's sufficiency ordering
  - in many player case, corresponds to "individual sufficiency" ordering
- Bergemann-Morris 16, see also Lehrer et al 10 and 11

Nice Properties of Individual Sufficiency Ordering

- Reduces to Blackwell in one player case
- Transitive
- Neither implies nor implied by Blackwell on join of players' information
- Two information structures are each individually sufficient for each other if and only if they share the same higher order beliefs about Θ
- S is individually sufficient for S' if and only if giving extra signals to S' equals S plus an appropriate correlation device