Bidder-Optimal Information Structures in Auctions

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Question

what is the optimal information structure to maximize the buyer surplus in an auction if the seller is choosing the optimal mechanism?

Motivation

- in many allocation problems the choice of information disclosure interacts strongly with the mechanism that guides the allocation
- recommender system and item/menu pricing in digital marketplaces
- bidding algorithms in digital advertising

Example: Digital Advertising I

- advertisers are bidding for display or sponsored product advertisements
- match between advertisers and viewers on websites of publishers are made through intermediaries
- a demand side platform seeks to make bids across websites to maximize surplus of advertisers (to be as attractive as possible for the advertisers)
- a supply side platform seeks to design the auction to maximize the revenue for the publishers
- demand and supply platform receive a commission of the surplus

Example: Digital Advertising II

- demand-side platform (DSP)
  - advertisers and agencies use DSP to set the parameters of “real-time bids” for ad impressions based on relevant attributes of the ad space and the viewer navigating to it; these attributes may be demographic, behavioral, contextual, or location-based
- supply-side platform (SSP)
  - a supply-side platform, or SSP, enables publisher to sell their ad space to multiple ad exchanges, it controls how their ad space is sold by setting price floors, what types of advertisers can bid

Example: Digital Advertising III

- a prominent tool of the demand side platform is to manage the match information between advertiser and viewer through the design of bidding categories and characteristics.
- the demand side platform seeks to influence the information regarding values and bids
- the supply side platform chooses the optimal auction format

Question

what is the optimal information structure to maximize the buyer (i.e., DSP) surplus in an auction if the seller (i.e., SSP) is choosing the optimal selling mechanism?

Result 1: Optimality of Positive Regular Distributions

- optimal information structure will always generate a positive regular distribution
- an implication of regularity is that (in our symmetric setting) the optimal auction can always be implemented by a second price auction with a reserve price
- thus bidders induce/force seller to adopt second price auction (with reserve price)
- excluded bidders

Result 2: Linear Revenue Function

- complete characterization of optimal information structure
- information structure generates a distribution of posterior expectations of the bidders with three segments:
    1. lower segment: bidders excluded, might as well receive complete information in this segment that agrees with their prior distribution
    2. intermediate segment: bidders pooled with truncated generalized Pareto distribution inducing decreasing linear revenue
    3. upper segment: each bidder again has complete information

Model

Model

- seller offers an indivisible good to N bidders
- bidder i has value $v \in \mathbb{R}_+$ for probability $q_i \in [0,1]$ against transfer $p_i \in \mathbb{R}_+$

$$u(v_i, q_i, p_i) = v_i q_i - p_i$$

- value v_i has common prior distribution F, independent and identical across all i:

$$F \in \Delta([\underline{v}, \bar{v}]), \qquad 0 \leq \underline{v} < \bar{v} \leq \infty$$

- seller maximizes revenue:

$$\sum_{i=1}^{N} p_i$$

Information Design

- bidder may not observe their ex post value v_i but rather observe a signal $s(v)$ about their value
- posterior distribution of values

$$w = \mathbb{E}\left[v \,|\, s\right]$$

is denoted by

$$G \in \Delta([\underline{v}, \bar{v}])$$

- there exists an information signal that induces a distribution G of expected values if and only if G is a mean-preserving contraction of F, i.e.,

$$\int_v^{\overline{v}} F(t)dt \leq \int_v^{\overline{v}} G(t)dt, \, \forall v \in [\underline{v}, \bar{v}], \quad \Leftrightarrow \; F \prec G$$

with equality for $v = \underline{v}$, "G **majorizes** F"

Mechanism

- direct and symmetric (interim) mechanism:

$$Q, P : [\underline{v}, \bar{v}] \to [0,1] \times \mathbb{R}_+,$$

  where $Q(v)$ is probability of winning, and $P(v)$ is payment
- incentive compatibility and participation constraints:

$$\begin{aligned} wQ(w) - P(w) &\geq wQ(w') - P(w'); \\ wQ(w) - P(w) &\geq 0; \qquad \forall w, w' \in \mathbb{R}_+ \end{aligned}$$

- feasibility of symmetric allocation rule $Q(w)$
- sequential game: first information, then mechanism
- NOT simultaneous move game

Related Literature I

- single buyer chooses information and seller chooses optimal mechanism [Roesler/Szentes AER 2017]
- an optimal information structure is a truncated Pareto distribution with:
  1. zero virtual utility, constant revenue
  2. no exclusion at the bottom, efficiency
  3. multiple solutions at the upper tail, one solution is always a truncated Pareto distribution
- Condorelli/Szentes (JPE 2020) consider the case where distribution of values chosen without majorization constraint
- this paper: **many** buyers

Related Literature II

- many buyers (bidders) but efficient mechanism (absolute second price auction): [BHM/Sorokin/Winter AER:I 2022]
- buyer optimal information structure is:
  - complete information disclosure at upper interval of support
  - single pooling interval at lower interval of support
  - quantile threshold between two regimes is independent of prior distribution, dependent only on the number of bidders
- this paper: optimal information structure given **revenue maximizing** mechanism

Related Literature III

- information structure and mechanism jointly maximize revenue
  - many buyers; Bergemann and Pesendorfer (JET 2008)
  - one buyer, but variable quality (BHM wp 2023)
- this paper: information structure maximizes **buyer surplus**

Bidder Surplus in Quantile Space

Quantile Space

- quantile space $t \in [0,1]$ rather than value space $v \in \mathbb{R}_+$:

$$v = F^{-1}(t)$$

- denote inverses:

$$V(t) \triangleq F^{-1}(t) \quad \text{and} \quad W(t) \triangleq G^{-1}(t).$$

- majorization reverses:

$$F \prec G \Leftrightarrow V \succ W$$

- revenue of single bidder is

$$v(1 - F(v)) \text{ or } F^{-1}(t)(1-t) \text{ or } V(t)(1-t)$$

- we write

$$\pi_v(t) \triangleq V(t)(1-t) \text{ and } \pi_w(t) \triangleq W(t)(1-t)$$

Feasible Mechanism

- efficient allocation rule q assigns object with probability one to bidder in highest quantile:

$$q(t) \triangleq t^{N-1}$$

- a quantile allocation rule r is feasible if and only if q weakly majorizes, i.e.,

$$\int_t^1 r(s)ds \leq \int_t^1 q(s)ds, \text{ for all } t \in [0,1]$$

- **weakly majorizes** because we do not impose equality at $t = 0$
- we write

$$r \prec_w q$$

- symmetric allocation and mechanism

Revenue in Quantile Space

- allocation rule $r(t)$ determines payments via envelope theorem:

$$\Pi \triangleq N\left(\int_0^1 (1-t)W(t)dr(t) + r(0)W(0)\right)$$

- revenue (=profit) selling from quantile t up

$$\pi_w(t) = (1-t)W(t)$$

- marginal revenue is virtual utility:

$$-\frac{d\pi_w}{dt} = -\left[(1-t)W'(t) - W(t)\right] = W(t) - \frac{1-G(W(t))}{g(W(t))}$$

since quantile

$$t = G(W(t)), \text{ so } W'(t) = \frac{1}{g(W(t))}$$

Bidder Surplus in Quantile Space

bidder surplus of an individual bidder is:

$$
\begin{aligned}
U &\triangleq \int_0^1 r(t)W(t)dt - \frac{\Pi}{N} \\
&= \int_0^1 r(t)W(t)dt - \left(\int_0^1 W(t)(1-t)dr(t) + r(0)W(0)\right)
\end{aligned}
$$

Bidders' Surplus Maximization

maximum bidder surplus U^* is given by

$$\max_{\substack{\{W:W\prec V\}\\ \{r:r\prec_w q\}}} \int_0^1 r(t)W(t)dt - \int_0^1 (1-t)W(t)dr(t) - r(0)W(0)$$

$$\text{s.t. } r \in \arg\max_{\{\hat{r}:\hat{r}\prec_w q\}} \int_0^1 (1-t)W(t)d\hat{r}(t) + \hat{r}(0)W(0)$$

- three constraints:
  1. majorization
     $$\{W : W \prec V\}$$
  2. feasibility
     $$\{r : r \prec_w q\}$$
  3. optimal mechanism
     $$r \in \arg\max_{\{\hat{r}:\hat{r}\prec_w q\}} \int_0^1 (1-t)W(t)d\hat{r}(t) + \hat{r}(0)W(0)$$

Three Steps

1. replace majorization constraint...

$$\{W : W \prec V\}$$

   with support constraint

$$w \in [0, 1]$$

2. replace support constraint

$$w \in [0, 1]$$

   with support constraint

$$w \in [m, 1]$$

3. restore majorization constraint...

$$\{W : W \prec V\}$$

1. Optimality without Value Majorization Constraint

Optimal Revenue of Seller I

- denote by $\mathrm{cav}[\pi]$ the concavification of π

Proposition (Seller's Revenue)

For any given information structure W, the seller's revenue is:

$$\max_{\{r:r\prec_w q\}} \int_0^1 \pi_w(t)dr(t) = q(t_x)\pi_w(t_x) + \int_{t_x}^1 \mathrm{cav}[\pi_w](t)dq(t),$$

with

$$t_x \in \arg\max_t \pi_w(t).$$

Optimal Revenue of Seller II

- exclusion below zero virtual utility t_x where

$$\frac{d\pi_w(t)}{dt} = 0 \Leftrightarrow w - \frac{1 - G(w)}{g(w)} = 0$$

- efficient allocation beyond t_x:

$$r(t) \Rightarrow q(t)$$

- Myerson (1981), Kleiner et al. (2021)

Positive Regular Information Structure

- consider the following set of information structures:

  $\mathcal{W}_+ \triangleq \{W \in \Delta[0,1] : W(t)(1-t) \text{ is decreasing and concave}\}$

- virtual values:

$$W(t) - \frac{1 - G(W(t))}{g(W(t))}, \forall t$$

  are nonnegative and increasing

- positive regular information structure

Positive Regular Information Structure is Optimal

- optimal choice is in smaller class of information structures

Proposition (Positive Regular is Optimal)

An information structure W^* is optimal only if $W^* \in \mathcal{W}_+$.

- suppose W were not positive regular so either $\pi^*_w(t) < \text{cav}[\pi^*_w](t)$ or $t_x > 0$
- can generate more bidder surplus with

$$\widehat{W}(t) = \begin{cases} \frac{\text{cav}[\pi^*_w](t)}{1-t} & \text{if } t \geq t_x \\ \frac{\text{cav}[\pi^*_w](t_x)}{1-t} & \text{if } t < t_x. \end{cases}$$

- $\widehat{W}$ first-order stochastically dominates W, yet generates the same revenue

(Absolute) Second Price Auction is Optimal

- so optimal mechanism is simple

Corollary (Second Price Auction is Optimal)

An absolute second price auction is an optimal auction for all $W \in \mathcal{W}_+$, and in particular $W^* \in \mathcal{W}_+$.

- zero exclusion, or $t_x = 0$
- efficient allocation everywhere:

$$r(t) \Rightarrow q(t)$$

A Simpler Problem

But knowing that allocation takes this form, we can give more concise description of our optimization problem:

$$W^* \in \arg\max_{W \in \mathcal{W}_+} \int_0^1 W(t) \frac{ds(t)}{dt} dt$$

where

$$s(t) \triangleq -q(t)(1-t)$$

Reflecting Second Price Auction

- $s(t)$ represents the difference between the first and second order statistic
- this is information rent
- writing $w_{(1)}$ and $w_{(2)}$ for first and second-order statistics
- for any $t \in [0,1]$:

$$\begin{aligned}\mathbb{P}\{w_{(1)} &\leq W(t)\} = t^N \\ \mathbb{P}\{w_{(2)} &\leq W(t)\} = Nt^{N-1} - (N-1)t^N.\end{aligned}$$

- difference of order statistics at quantile t is $s(t)$:

$$\begin{aligned}s(t) &= -q(t)(1-t) \\ &= \left(\mathbb{P}\{w_{(1)} \leq W(t)\} - \mathbb{P}\{w_{(2)} \leq W(t)\}\right)/N\end{aligned}$$

Optimal Information Structure and Shape of s(t)

- since

$$s\,(t) = -q(t)(1-t) = t^{N-1}(1-t),$$

- s is quasiconvex, first decreasing, then increasing
- minimum and inflection point

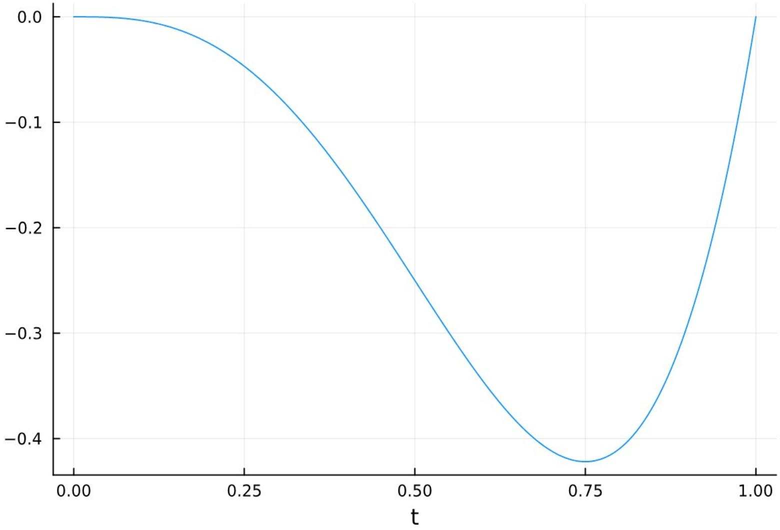
$$t_s \triangleq \arg\min_{t\in[0,1]}\ s(t); \quad t_i \triangleq \arg\min_{t\in[0,1]}\ \frac{ds(t)}{dt}.$$

relate only to N, number of bidders

$$t_i = \frac{N-2}{N} < \frac{N-1}{N} = t_s;$$

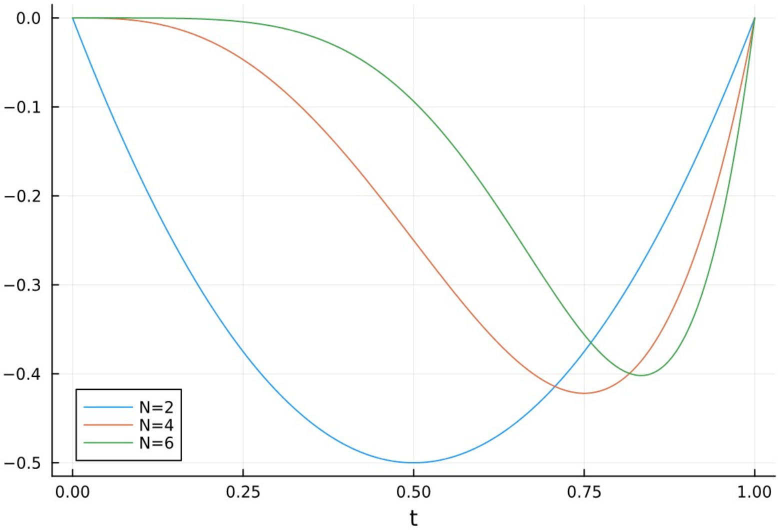
Shape of s(t)

depends on N only: $N = 4$



Shape of s(t)

depends on N only:



Optimal Control Problem

- we have

$$W^* \in \underset{W \in \mathcal{W}_+}{\arg\max} \int_0^1 W(t) \frac{ds(t)}{dt} dt$$

- choose $W(t)$ to maximize the integral
- weight is given by differential

$$\frac{ds(t)}{dt}$$

  changes sign once from negative to positive
- $W(t)$ is monotone $\Rightarrow$ suggests $0, 1$ step function as optimal solution
- $W \in \mathcal{W}_+ \Leftrightarrow W(t)(1-t)$ is decreasing and concave is a second constraint

Truncated Pareto Distribution

- Pareto distribution with mass point at $w = 1$ with probability $1 - t_z$...

$$t_z = \lim_{w \to 1} G(w)$$

  where

$$G(w) = \begin{cases} 1 - \frac{1-t_z}{w} & \text{if } 1 - t_z \leq w < 1; \\ 1 & \text{if } w = 1. \end{cases}$$

- probability t_z of mass point set lower bound of support

Proposition (Optimal Information Structure)

An optimal information structure $G^*(w)$ has:

$$G(w) = \begin{cases} 1 - \frac{1-t_z}{w} & \textit{if } 1 - t_z \leq w < 1; \\ 1 & \textit{if } w = 1. \end{cases}$$

Truncated Pareto Distribution

- restate the Pareto distribution in quantile space

Corollary (Value W^* and Revenue π^*)

An optimal information structure W^* has:

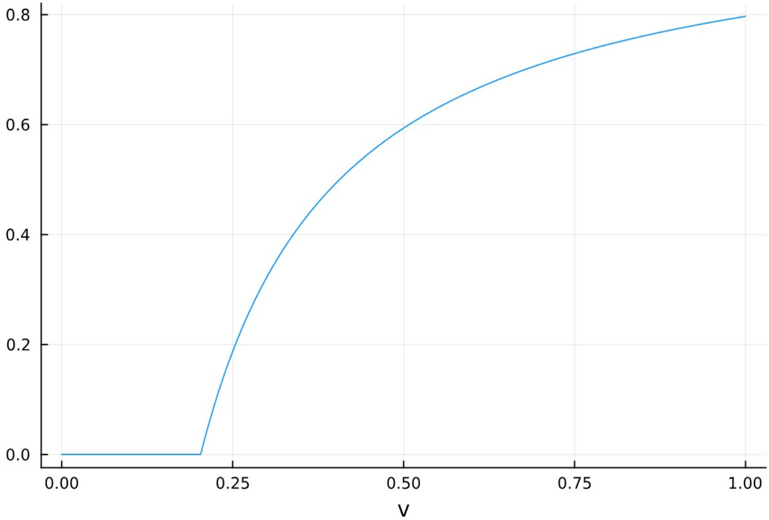
1. $W^*(t) = (1 - t_z) \,/\, (1 - t)$ for some $t \in [0, 1 - t_z]$;
2. The revenue function is

$$\pi^*(t) = \begin{cases} 1 - t_z & \text{if } t < t_z; \\ 1 - t & \text{if } t \geq t_z. \end{cases}$$

3. The virtual utility is zero (except at 1)

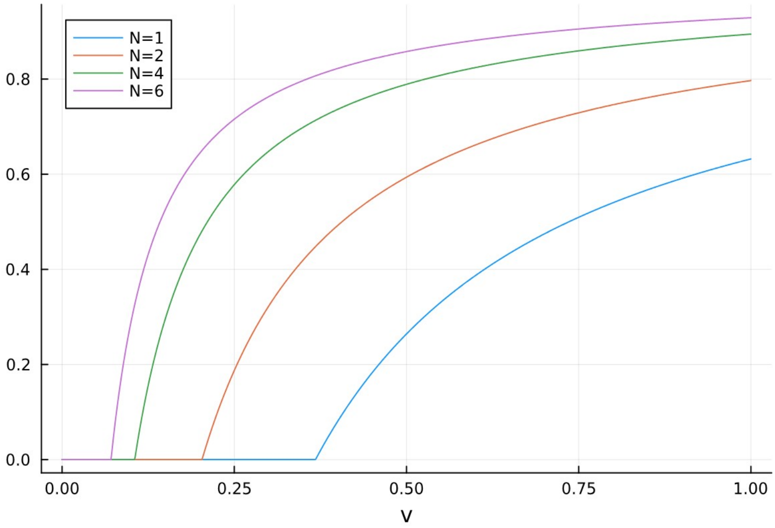
Pareto Distribution

truncated Pareto distribution: $N = 2$



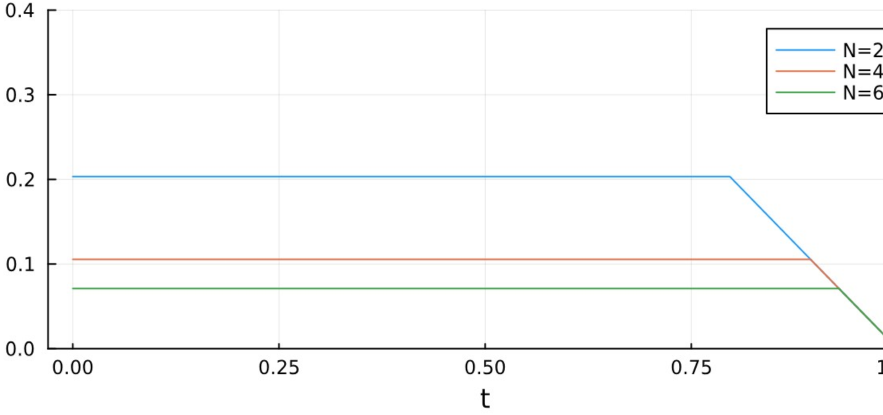
Pareto Distribution

truncated Pareto distribution varies with N



Revenue Function

revenue function with $N = 2, 4, 6$



2. Optimality with tight support constraintt

Tighten Support Restrictions

- suppose

$$w \in [m, 1], \quad m > 0$$

- range of private information becomes smaller
- now maintaining zero virtual utility is becoming impossible
- still, the best way to generate information rent is to keep the seller at a constant virtual utility
- zero virtual utility is replaced by constant but positive virtual utility

Decreasing Demand Elasticity

- remember $w \in [m, 1]\,, m > 0$

Proposition (Optimal Information Structure)

Every optimal information structure $G^*(w)$ has

$$G^*(w) = \begin{cases} 1 - \frac{(1-t_z)(\alpha+1)}{\alpha+w} & \text{if } m < w < 1; \\ 1 & \text{if } w = 1; \end{cases}$$

where α is:

$$\alpha = \frac{(1-t_z) - m}{t_z}.$$

- generalized Pareto distribution with location and scale parameter
- linear decreasing rather than constant revenue function

Linear Revenue Function

Corollary (Value W^* and Revenue π^*)

A information structure W^* is optimal iff:

1. Value $W^*(t)$ is given by

$$W(t) = (1+\alpha)\frac{1-t_z}{1-t} - \alpha;$$

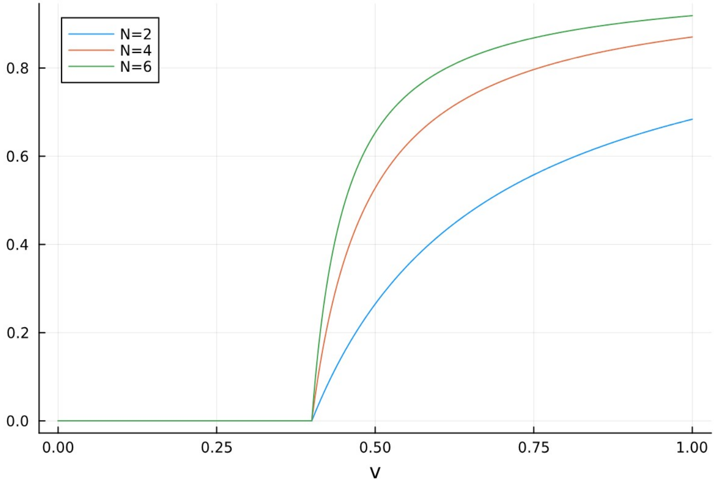
2. Revenue function is

$$\pi^*(t) = \begin{cases} (1-t_z) - \alpha(t_z - t) & \text{if } t \leq t_z; \\ (1-t) & \text{if } t \geq t_z. \end{cases}$$

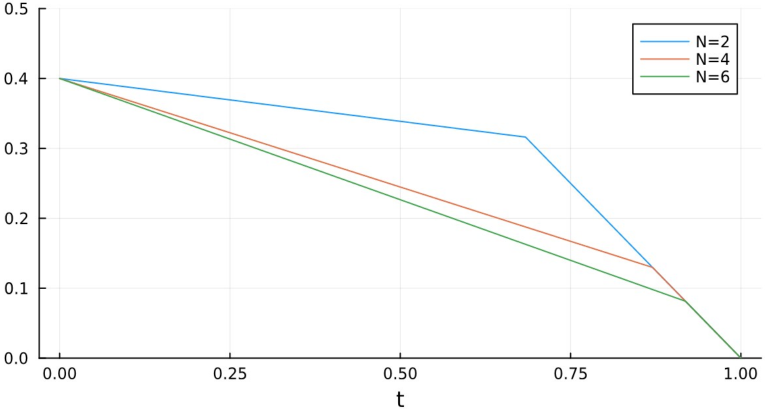
- revenue function is linear in quantile
- slope of revenue function is α

Distributions with Support Constraint

- support constraint $m = 0.4$



- support constraint $m = 0.4$



3. Optimality with tight support constraintt

Include Majorization Constraint

- adding back majorization constraint

$$\{W : W \prec V\}$$

- in particular, restoring Bayes plausibility
- where might problems arise:

1. sufficient low to justify collusion may exist
2. high values may not exist (e.g., mass point at top of distribution)l

Optimal Revenue

- optimal revenue for any fixed information structure
- denote by $\text{cav}[\pi_w]$ the concavification of π_w
- denote by t_x a critical quantile below which the seller excludes bidder

$$t_x \in \arg\max_t \pi_w(t)$$

Proposition (Seller's Revenue)

For any given distribution G, the seller's revenue is given by:

$$\max_{\{r : r \prec_w q\}} \int_0^1 \pi_w(t) dr(t) = q(t_x)\pi_w(t_x) + \int_{t_x}^1 \text{cav}[\pi_w](t) dq(t).$$

Almost Positive Regular Distribution

- almost positive regular distribution $\mathcal{W}_+(t_x) \triangleq$

$$\left\{ \begin{array}{c} W \prec V : \\ \forall t < t_x : W(t) = V(t), V(t)(1-t) \leq W(t_x)(1-t_x); \\ \forall t > t_x : W(t)(1-t) \text{ decreasing and concave} \end{array} \right.$$

any distribution in $\mathcal{W}(t_x)$ has the following properties:

1. revenue-maximizing quantile is t_x;
2. any information structure in $\mathcal{W}(t_x)$ is decreasing and concave for $t > t_x$
3. information structure is complete information for $t < t_x$.

Implication of Regular Distribution

- what is the optimal mechanism for regular distribution

Proposition (Implication of Regularity)

If $W \in \mathcal{W}(t_x)$ *then*

1. *a second-price auction with reserve price* $W(t_x)$ *maximizes revenue;*
2. *bidders whose expected value is below the reserve price know their ex post value;*
3. *any* W *satisfying these two properties also satisfies* $W \in \mathcal{W}(t_x)$*, for some* $t_x \in [0, 1]$*.*

- allocates is interim efficient above the reserve price
- winning bidder $\neq$ bidder with highest *ex post* value

Optimality of Regular Distribution

- bidder-optimal information structure is indeed regular

Theorem (Regular is Optimal)

An information structure W^* *solves* U^* *only if* $W^* \in \mathcal{W}(t_x)$ *for some* $t_x \in [0, 1]$.

- if seller pools allocation, it decreases total surplus and increases profit; hence, it is detrimental for bidder surplus
- bidder-optimal information structure induces seller to allocate good interim efficiently (if expected value above the reserve price)

Maximizing Bidders' Surplus

- we aim to solve U^* :

$$\max_{\substack{\{W:W\prec V\}\\ \{r:r\prec_w q\}}} \int_0^1 r(t)W(t)dt - \left(\int_0^1 W(t)(1-t)dr(t) + r(0)W(0)\right)$$

subject to: $r \in \arg\max_{\{\hat{r}:\hat{r}\prec_w q\}} \int_0^1 (1-t)W(t)d\hat{r}(t) + \hat{r}(0)W(0)$

- we replace the inner optimization problem by an efficient allocation and a reserve price:

$$\max_{\substack{\{W:W\prec V\}\\ \{r:r\prec_w q\}}} \int_0^1 r(t)W(t)dt - \left(\int_0^1 W(t)(1-t)dr(t) + r(0)W(0)\right)$$

Bidder Optimal Information Structure

Bidder Surplus Expressed in Order Statistics

- bidder surplus using this function

Corollary (Computing Bidder-Optimal W^*)

An information structures W^* is optimal iff it solves:

$$W^* \in \underset{t_x \in [0,1], W \in \mathcal{W}(t_x)}{\arg\max} \left(\int_{t_x}^{1} W(t) \frac{ds(t)}{dt} dt - (1 - t_x) q(t_x) W(t_x) \right)$$

Bidder Optimal Information Structure

Theorem (Bidder Optimal Information Structure)

A bidder-optimal information structure $G^*(w)$ *has*

$$G^*(w) = \begin{cases} 1 - \frac{(1-t_z)(\alpha+1)}{\alpha+w} & \text{if } w \in [W(t_x), W(t_z)]; \\ F(w) & \text{if } w \notin [W(t_x), W(t_z)]. \end{cases}$$

for some parameters (α, t_x, t_z). *Furthermore,* $\alpha \geq 0$ *and, if* $t_x > 0$*, then* $\alpha = 0$.

Bidder Surplus and Revenue Function

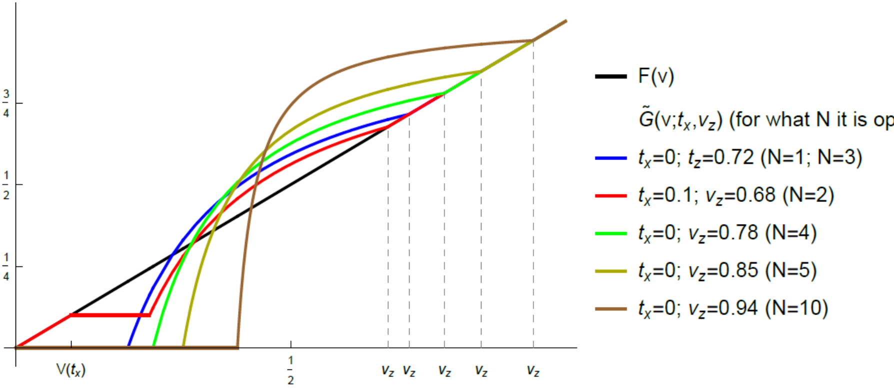
Proposition (Bidder Optimal Revenue Function)

Any bidder-optimal revenue function π_w^* is given by:

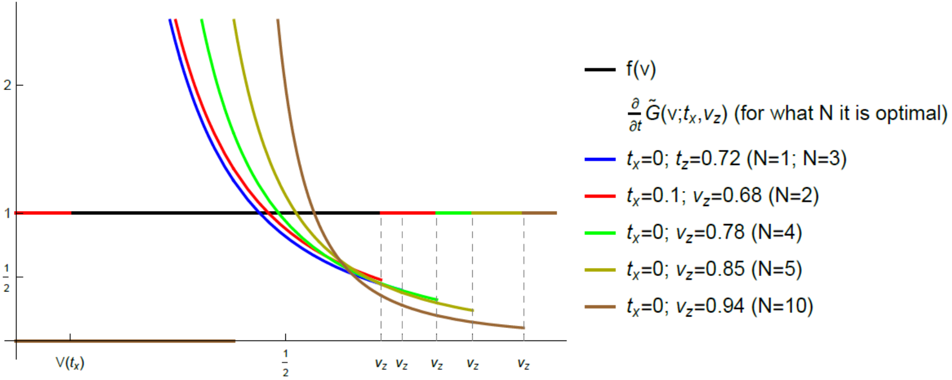
$$\pi_w^*(t) = \begin{cases} \pi(t_z) - \alpha(t - t_z) & \textit{if } t \in [t_x, t_z]; \\ \pi(t) & \textit{if } t \notin [t_x, t_z]. \end{cases}$$

for some parameters (t_x, t_z). Furthermore, $\alpha \geq 0$ and, if $t_x > 0$, then $\alpha = 0$.

Nature of Solution: Distribution



Nature of Solution: Density



Nature of Solution: Revenue

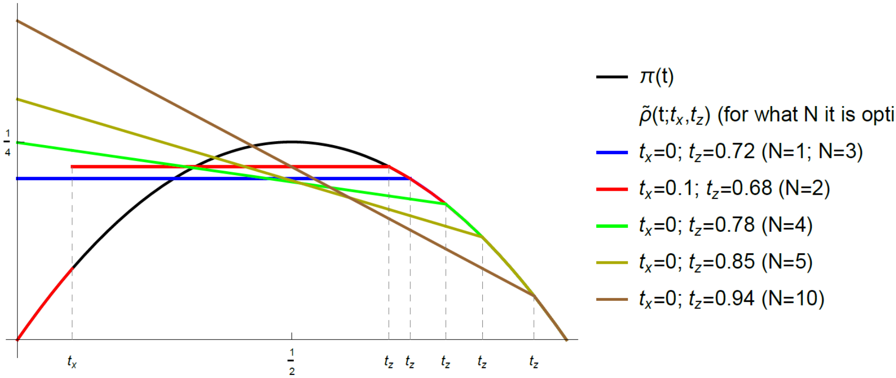


Figure: Profit Functions in Quantile Space for Uniform Distribution $F(v) = v$.

Conclusion

- interaction of information design and mechanism design
- bidder optimal information structure conflates values to generate information rents
- complete disclosure at the lower and upper tail
- demand management in the intermediate rangeλ

Extensions

1. maximize total surplus - λ.profits
   - our case $\lambda = 1$, also $\lambda = 0$ (efficiency), $\lambda = \infty$ (minmax)
2. multiunit
3. large market
4. asymmetric / correlated