Welfare Impact of Policy in Incomplete Markets: Theory and Computation

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Contents

List of Tables

List of Figures

ABSTRACT

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Chapter 1 extends the theory of demand to incomplete markets. It starts with smoothness of demand, Slutsky decompositions, and properties of Slutsky matrices. It defines Slutsky perturbations as perturbations of Slutsky matrices that arise from some symmetric perturbation of the Hessian of utility. Finally, it identifies Slutsky perturbations as the solutions to a linear system of equations with budget variables as coefficients.

Chapters 2 and 3 examine the welfare impact of taxation and of financial innovation in incomplete markets. Taking tax policy or financial innovation policy as primitives, it studies the generic existence of Pareto improving policy parameters, their computation, and the size of Pareto improvement. Generic existence obtains if the price adjustment implied by the introduction of tax rates is sufficiently sensitive to the risk aversion of the economy, and if both incompleteness and policy parameters outnumber household heterogeneity. Several known and new tax policies pass this sensitivity test, so does a new financial innovation policy, all therefore supporting Pareto improvements. It is chapter 1's identification of Slutsky perturbations that verifies they pass this test.

Chapter 4 illustrates Pareto improving taxation on current income and asset purchases. The Pareto improvement following taxes is small. This is bounded above by the improve-

ment following the removal of all future uncertainty, also small.

Chapter 5 synthesizes research on the transfer paradox. It reinterprets Samuelson's equivalence of the paradox with instability, as identifying the threshold, the minimum level of trade beyond which the transfer paradox appears. Although the equivalence is false in general, and later research focused on qualifying or debunking it, this reinterpretation generalizes while quantifying the later research.

Chapter 6 documents two Mathematica programs for chapter 4's example, where utility is von Neumann-Morgenstern. In the simpler one the state index is a quadratic transformation of Cobb-Douglas; in the more elaborate one, it is a HARA transformation of CES. To find Pareto improvements from the envelope theorem, the derivative of demand is needed. The former has a closed formula for demand, and computes its derivative symbolically with Mathematica; the latter has not, and computes its derivative instead with chapter 1's Slutsky decompositions.

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Chapter 1

Theory of Demand in Incomplete Markets

1.1 Introduction

I develop the theory of demand for commodities and assets in incomplete markets, given commodity prices, arbitrage-free yield structures, and contingent incomes.

First, the derivative of demand with respect to commodity prices and yield structure decomposes into a substitution effect and an income effect, in terms of a Slutsky matrix.

Next, I identify the properties that every Slutsky matrix must satisfy, and conversely prove that any matrix satisfying these properties must be the Slutsky matrix of some demand.

Then I show that the Slutsky matrix can be perturbed arbitrarily, subject only to preserving these properties, by perturbing the second derivative of the utility generating the original Slutsky matrix, while keeping point demand and marginal utility intact.

Finally, I identify these Slutsky perturbations by explicit linear constraints, defined by prices and the yield structure. Included also is an algorithm that speeds up the computation of Slutsky matrices.

These results for incomplete markets demand mirror exactly those for complete markets derived by Geanakoplos and Polemarchakis (1980). Geanakoplos and Polemarchakis (1986) were the first to apply these results to the study of generic Pareto improvements with incomplete markets. Since they allowed the central planner to decide the agents' asset portfolios, they did not need to go beyond perturbations to the Slutsky matrices of demand in spot markets. To show why weaker interventions may improve welfare, such as anonymous taxes and changes in asset payoffs, it became necessary to take into account how agents' portfolio adjustments cause a further price adjustment. Naturally, this required perturbing demand in asset markets as well as in spot markets. The lack of a theory of Slutsky perturbations in incomplete markets remained an obstacle for over a decade[1], until a breakthrough by Citanna, Kajii, and Villanacci (1998), who circumvented it by analyzing the agents' first order conditions. Researchers have extended the theory of generic Pareto improvements with incomplete markets to many policies by applying this first-order approach; see Cass and Citanna (1998), Citanna, Polemarchakis, and Tirelli (2001), and Bisin et al. (2001).

The results here fill the missing theory of Slutsky perturbations, and allow the study of generic improvements to recover its original, demand based approach. This has certain advantages. First, genericity arguments can target directly the demand function instead of the utility generating it, and the envelope property instead of the first order conditions, budget identities generating it. Second, to compute the welfare impact of interventions, the

[1] The exception is Elul (1995).

policymaker needs to know the derivative of aggregate, not individual, demand. In the first order approach, he needs to know the second derivative of every individual's utility, i.e, he must know the derivative of every individual's demand. Third, to express the economic intuitions the economist can appeal to the familiar language of demand theory, instead of the abstract language of submersions. Fourth, every time the researcher thinks a new result via Slutsky perturbations, he saves himself the work of implicitly identifying them via quadratic utility perturbations.

The paper continues as follows. Section 2 defines demand for commodities and assets in incomplete markets, and lists the basic properties of neoclassical demand. Section 3 defines the Slutsky matrix. Section 4 focuses on a fixed demand, presenting the properties every Slutsky matrix must satisfy, and showing that any matrix satisfying these properties must be the Slutsky matrix of some demand. It also decomposes the derivative of demand into income and substitution effects, notes the envelope property, and speeds up the computation of the Slutsky matrix by a recursion. Section 5 focuses on generic demand, defining Slutsky perturbations and identifying them by linear constraints. Section 6 contains the proofs.

1.2 Demand

The household knows the present state of nature, denoted 0, but is uncertain as to which among $s = 1, ..., S$ nature will reveal in period 1. It consumes commodities $c = 1, ..., C$ in the present and future, and invests in assets $j = 1, ..., J$ in the present only. Markets assign to the household an income $w \in R^{S+1}_{++}$, to commodity c a price $p_{c\cdot} \in R^{S+1}_{++}$, and to asset j a yield $W^j \in R^{S+1}$. We call $(p_c)^C_1 = p = (p_s)$ the spot prices and

$(W^j)_1^J = W = (W_s)$ the yield structure. The set of *budget* variables

$$b \equiv (p, W, w) \in B \equiv R_{++}^{C^*} \times R^{J \times S+1} \times R_{++}^{S+1}$$

has some nonempty, open $B' \subset B$ as a distinguished subset, $C^* = C(S+1)$.

Demand for commodities and assets is a function $\sigma = (x, y) : B' \to R_{+}^{C^*} \times R^J$. It satisfies *Walras' relation* if it makes the following an identity throughout B':

$$p'_s x_s - W'_s y = w_s$$

Alternatively, $[p]' x - W' y = w$ with the useful notation

$$[p] \equiv \begin{bmatrix} \cdot & & & \\ & p_{s-1} & & 0 \\ & & p_s & \\ & 0 & & p_{s+1} \\ & & & & \cdot \end{bmatrix}_{C^* \times S+1}$$

The interpretation is that, faced with spot prices p and yield structure W, the household modifies its income w to $w + W'y(p, W, w)$ by investing in portfolio $y(p, W, w)$, ultimately financing its state contingent consumption $x(p, W, w)$. Here, a yield structure specifies for each asset j that a buyer is to collect, a seller to deliver, a value W_s^j in state s, and a portfolio $y \in R^J$ specifies how much of each asset to buy $(y_j \geq 0)$ or sell $(y_j \leq 0)$, hence yielding $W'y$. For a different emphasis, we may view the assets as having present price $q \equiv -W_0$ and future yield $W_{\mathbf{1}} \equiv (W_s)_{s>0}$.

1.2.1 Neoclassical demand

For $b = (p, W, w) \in B$, the *financeable* bundles are

$$X(b) = \{x \in R_+^{C*} \mid [p]' x - w \in spanW'\}$$

Each $x \in X(b)$ implies a *financing* y, $[p]' x - w = W'y$, which is unique if W has linearly independent rows: $y = y(x, b)$. Given a utility function $u : R_+^{C*} \to R$ and $b \in B'$, suppose the problem

$$\max_{x \in X(b)} \quad u(x)$$

has a unique solution $x(b)$. Then *neoclassical demand* at $b \in B'$ is defined to be $\sigma(b) \equiv (x(b), y(x(b), b))$. The following hinges on $X(b)$ depending on W' only through its span, and on w only through the component that is orthogonal to $spanW'$.

Proposition 1.1. ***Basic properties*** *Suppose B' is X-closed: $b \in B', b \in B, X(b) = X(b') \Rightarrow b' \in B'$.*

- ***Walras' relation*** $[p]' x(p, W, w) - W'y(p, W, w) = w$
- ***Revealed yield preference*** *If $\Delta \in spanW'$ with $w + \Delta \gg 0$, then*

  $$\begin{aligned} &i) \quad x(p, W, w + \Delta) = x(p, W, w) \\ &ii) \quad \lambda(p, W, w + \Delta) = \lambda(p, W, w) \end{aligned}$$

  where $Du'(x(p, W, w)) = [p]\lambda$, should it have a solution, uniquely defines $\lambda(p, W, w) \in R^{S+1}$.

- ***Homogeneity*** $x(p, W, w) = x(p, \tilde{W}, w)$ *if* $spanW' = span\tilde{W}'$.

We now recall a subset $B' \subset B$ for which $x(b)$ exists, is unique and interior. Existence obtains if utility is continuous and $X(b)$ compact; it is well known that $X(p, W, w)$ is compact if and only if W is *arbitrage-free*, $W\lambda = 0$ for some $\lambda \in R^{S+1}_{++}$. Uniqueness and interiority obtain if utility is strictly quasiconcave in R^{C*}_{++} and *boundary averse*, $u(x) > u(\tilde{x})$ whenever $x \in R^{C^*}_{++}, \tilde{x} \in \partial R^{C^*}_{+}$, thanks to the convexity of $X(b)$. In sum, neoclassical demand $\sigma = (x, y) : B' \to R^{C*}_{++} \times R^J$ is defined on

$$B' \equiv \{(p, W, w) \in B \mid W \text{ has linearly independent rows, is arbitrage-free}\}$$

given the hypotheses on utility of continuity, strict quasiconcavity in R^{C*}_{++}, and boundary aversion.

1.3 Slutsky matrices

Assumption 1.1. ***Debreu's setting*** *for* u*:*

$$u \text{ is continuous, } C^{r \geq 2} \text{ in } R^{C*}_{++}$$
$$Du(x) \gg 0 \text{ for } x \gg 0$$
$$D^2u(x) \text{ is negative definite on } Du(x)^{\perp} \text{ for } x \gg 0$$
$$u(x) > u(\tilde{x}) \text{ whenever } x \in R^{C^*}_{++}, \tilde{x} \in \partial R^{C^*}_{+}$$

Debreu's special setting *means the above strengthened to "*$D^2u(x)$ *is negative definite for* $x \gg 0$*."*

All three hypotheses assumed to define interior neoclassical demand are present, save for strict quasi-concavity in $R^{C^*}_{++}$, which is implied by the first and third ones in Debreu's setting.

Proposition 1.2. *Debreu's setting implies* $\sigma = (x, y) : B' \to R^{C*}_{++} \times R^J$ *is* C^{r-1}.

Proof. By definition neoclassical demand is the solution to

$$\max \quad u(x) \quad \text{subject to} \quad x \geq 0, [p]' \, x - W'y = w \tag{max}$$

which exists, is unique, and interior. For now suppose $(x, y) \in R^{C*}_{++} \times R^J$ is neoclassical demand at $b \in B'$ iff there is $\lambda \in R^{S+1}_{++}$ (necessarily unique) such that

$$F(x, y, \lambda; b) \equiv \begin{bmatrix} Du' - [p] \, \lambda \\ W\lambda \\ - [p]' \, x + W'y + w \end{bmatrix} = 0 \tag{F}$$

Then (x, y, λ) is a C^{r-1} implicit function of $b \in B'$[2], if $H \equiv D_{x,y,\lambda}F$ is surjective:

$$H = \begin{bmatrix} D^2u & 0 & -[p] \\ 0 & 0 & W \\ -[p]' & W' & 0 \end{bmatrix} = \begin{bmatrix} M & -\rho \\ -\rho' & 0 \end{bmatrix} \qquad \text{where } M \equiv \begin{bmatrix} D^2u & 0 \\ 0 & 0 \end{bmatrix}, \rho \equiv \begin{bmatrix} [p] \\ -W \end{bmatrix} \tag{H}$$

Invertibility follows easily from (F), Debreu's third condition, and $W's$ linearly independent rows.

We verify the above equivalence for $(x, y) \in R^{C*}_{++} \times R^J$. If it solves (max), the constraint qualification holds given the linear constraints, so there is $\lambda \in R^{S+1}_{+}$ such that (F). (This does not require concavity!) So $\lambda \in R^{S+1}_{++}$ by Debreu's second condition. Conversely, if (F) with $\lambda \gg 0$ then (x, y) solves (max):

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[2] B' is open in B with the product topology. For suppose W has linearly independent rows and $W\lambda = 0, \lambda \in R^{\mathbf{s}+1}_{++}$. Then some open neighborhood O of W preserves the linear independence and admits, by the implicit function theorem, a smooth function $\lambda : O \to R^{\mathbf{s}+1}_{++}$ solving $\tilde{W}\lambda(\tilde{W}) = 0$.

If it did not there would be $\tilde{x}, \tilde{y}$ with $u(\tilde{x}) > u(x)$ (so $\tilde{x} \gg 0$ by boundary aversion and $x \gg 0$) and $[p]' \tilde{x} - W'\tilde{y} = w$. By the strict quasiconcavity in R^{C*}_{++} $u(\tilde{x}(t)) > u(x)$ for all $t \in (0, 1]$, where $\tilde{x}(t) \equiv t\tilde{x} + (1-t)x$, while still $\tilde{x}(t) \gg 0, [p]' \tilde{x}(t) - W'\tilde{y}(t) = w$ with $\tilde{y}(t)$ obviously defined. Writing $\Delta_t \equiv \tilde{x}(t) - x$ in a second order Taylor expansion about $t = 0$,

$$u(\tilde{x}(t)) - u(x) = Du(x)\Delta_t + \frac{1}{2}\Delta_t' D^2 u(x)\Delta_t + o(\|\Delta_t\|^2)$$

The orthogonality $Du(x)\Delta_t = \lambda'[p]'\Delta_t = \lambda' W'(\tilde{y}(t) - y) = 0$ implies $\Delta_t' D^2 u(x)\Delta_t < 0$ by assumption on $D^2 u$, so $u(\tilde{x}(t)) - u(x) < 0$ for all $t \approx 0$, a contradiction. □

Since H is symmetric, so is H^{-1}:

$$\boxed{H^{-1} = \begin{bmatrix} S & -m \\ -m' & -c \end{bmatrix}, \text{ the } \textbf{Slutsky matrices}} \qquad \text{(Slutsky)}$$

To keep track, S, c are symmetric of dimensions $C^* + J, S + 1$, and m is $C^* + J \times S + 1$. We view ρ as playing the role of prices, since $\rho = p_0 = p$ if $J = S = 0$ (sole budget constraint).

Having defined Slutsky matrices, we develop demand theory in two parts. First we treat neoclassical demand for a fixed utility, specifically the Slutsky decomposition, the properties of Slutsky matrices, their computation, and the envelope property. Then we treat neoclassical demand for a generic utility, identifying the range of perturbations of Slutsky matrices that arise from perturbations of the Hessian of utility.

The theory for a fixed utility leads to a general formula for the derivative of equilibrium welfare with respect to the equilibrium parameters, solely in terms of the equilibrium parameters and the Slutsky matrices; it is useful for equilibrium comparative statics. The theory for a generic utility, in conjunction with this formula, allows us to study generic equilibrium welfare; it is useful for the study of generic properties of GEI.

1.4 Fixed neoclassical demand

1.4.1 Slutsky decomposition

We decompose demand into substitution and income effects, generalizing Gottardi and Hens (1999) to multiple commodities and to including the derivative with respect to asset payoffs. Differentiating the identity $F(\sigma(b), \lambda(b); b) \equiv 0$,

$$D_{p,W,w} \begin{bmatrix} \sigma \\ \lambda \end{bmatrix} = -H^{-1} \cdot D_{p,W,w} F = - \begin{bmatrix} S & -m \\ -m' & -c \end{bmatrix} \begin{bmatrix} -L & 0 & 0 \\ 0 & \Lambda & 0 \\ -[x]' & \Psi & I \end{bmatrix}$$

where

$$L \equiv \begin{bmatrix} \cdot & & 0 \\ & \lambda_s I_C & \\ 0 & & \cdot \end{bmatrix}_{C^* \times C^*} \qquad \Lambda \equiv [\lambda_0 I_J : ... : \lambda_s I_J]_{J \times J(S+1)} \qquad \Psi \equiv \begin{bmatrix} y' & & 0 \\ & \cdot & \\ 0 & & y' \end{bmatrix}_{S+1 \times (S+1)J}$$

In differentiating, we vectorized p, W as

$$\begin{bmatrix} \cdot \\ p_s \\ \cdot \end{bmatrix} \qquad \begin{bmatrix} \cdot \\ W_s \\ \cdot \end{bmatrix}$$

Multiplying this out,

$$\boxed{D_p\sigma = S\begin{bmatrix} L \\ 0 \end{bmatrix} - m[x]' \qquad D_W\sigma = -S\begin{bmatrix} 0 \\ \Lambda \end{bmatrix} + m\Psi \qquad \text{with} \ \ D_w\sigma = m} \tag{decomposition}$$

so that m is the marginal propensity to demand; also, $D_w\lambda = c$.

Let us interpret the decomposition in terms of substitution and income effects. The second summands are clearly income effects. For $D_p\sigma$, the value of demanding x is $[x]'p$, so a change in price of $\dot{p}$ implies a change in relative income of $-[x]'\dot{p}$, which implies a change in demand of $-m[x]'\dot{p}$; likewise for $D_W\sigma$, where the value of demanding y is ΨW. The first summands are substitution effects in the following sense. Suppose, given a small change in p, W, that we compensate the household so it can just finance the (x, y) it is demanding. Then its compensated income and demand would be $w(p, W) \equiv [p]'x - W'y, \sigma^C(p, W) \equiv \sigma(p, W, w(p, W))$, and the *substitution effects* be $D_p\sigma^C, D_W\sigma^C$. Computing them,

$$D_p\sigma^C = D_p\sigma + D_w\sigma[x]' = S\begin{bmatrix} L \\ 0 \end{bmatrix} \qquad D_W\sigma^C = D_W\sigma - D_w\sigma\Psi = -S\begin{bmatrix} 0 \\ \Lambda \end{bmatrix}$$

using the chain rule, (decomposition), and $D_w\sigma = m$. Hence the substitution effects are the first summands.

We paraphrase the decomposition to stress the parallel with the traditional one, and to obtain a version that is convenient for general equilibrium analysis. It says about

$D_q = D_{-W_0}$ that

$$D_q\sigma = S\begin{bmatrix} 0 \\ \lambda_0 I_J \end{bmatrix} - m\Psi_0 \qquad \text{where} \qquad \Psi_0 = \begin{bmatrix} y' \\ 0 \end{bmatrix}_{S+1\times J}$$

Concatenating the expressions for $D_p\sigma, D_q\sigma$,

$$D_{p,q}\sigma = SL_+ - m[[x]' : \Psi_0] \qquad \text{where} \qquad L_+ \equiv \begin{bmatrix} L & 0 \\ 0 & \lambda_0 I_J \end{bmatrix}$$

That is,

$$\boxed{D_{p,q}\sigma = SL_+ - m\tilde{\sigma}'} \qquad \text{where} \qquad \tilde{\sigma} \equiv [[x]' : \Psi_0]' \tag{GE}$$

The effect on demand of price changes splits into substitution and income effects, the latter being the product of the marginal propensity to demand with demand itself. (The notation "$\tilde{\sigma}$" expresses that $\sigma, [[x]' : \Psi_0]'$ contain the same information, differing only in its display. $\tilde{\sigma}$ even suggests the absence of asset markets in the future, since Ψ_0 is zero in the coordinates $s > 0$.)

1.4.2 Envelope property

Indirect utility $v : B' \to R, v(b) \equiv u(x(b))$ is derived from demand; inversely says the envelope property, neoclassical demand is derived from indirect utility.

Proposition 1.3. *Indirect utility is C^{r-1} in Debreu's setting, and its gradient $D_b v$ equals*

$$\boxed{D_p v = -\lambda'[x]' \qquad D_W v = \lambda'\Psi \qquad D_w v = \lambda'}$$

Thus $D_{p_s} v = -\lambda_s x_s', D_{W_s} v = \lambda_s y'$.

Proof. v is C^{r-1} since u, x are, in Debreu's setting. By the chain rule and (F) $D_b v = Du \cdot D_b x = \lambda'[p]' \cdot [D_p x : D_W x : D_w x] = *$. Differentiating Walras' relation $[p]'x = W'y + w$ with respect to

$$\begin{aligned} p: &\quad [p]' D_p x + [x]' = W' D_p y \\ W: &\quad [p]' D_W x = W' D_W y + \Psi \\ w: &\quad [p]' D_w x = W' D_w y + I \end{aligned}$$

Inserting this and $\lambda' W' = 0$ from (F), $* = \lambda'[-[x] : \Psi : I]$. □

1.4.3 The Slutsky list of properties

What properties do the Slutsky matrices H^{-1} have? Convenient notations are

$$m = D_w \sigma = \begin{bmatrix} X \\ Y \end{bmatrix} \quad \text{with} \quad \begin{matrix} X_{C^* \times S+1} \\ Y_{J \times S+1} \end{matrix} \qquad \rho \equiv \begin{bmatrix} [p] \\ -W \end{bmatrix}$$

X, Y are the marginal propensities to demand commodities, assets. (H) suggests defining functions

$$H(M) \equiv \begin{bmatrix} M & -\rho \\ -\rho' & 0 \end{bmatrix} \qquad M(D) \equiv \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \tilde{H}(D) \equiv H(M(D)) \tag{functions}$$

Of course, in (H) we have $H = \tilde{H}(D^2 u)$.

Toward the properties of H^{-1}, we take as given some $\mu \in R^{S+1}$ with $W\mu = 0$. In Debreu's setting, we choose the $\mu = \lambda$ in (F); in Debreu's special setting, we choose $\mu = 0$. The point is that μ is unrelated to the second derivative $D = D^2 u$.

Theorem 1.1. *If D is negative definite on $([p]\mu)^{\perp}$ and symmetric, then $\tilde{H}(D)$ is* *invertible, with inverse*

$$Smc \equiv \begin{bmatrix} S & -m \\ -m' & -c \end{bmatrix}$$

for some S, m, c *satisfying the*

$$\boxed{\begin{array}{lc} S & \rho' S = 0, S \ \textit{ is negative definite on } \ \rho^{\perp}, \textit{ symmetric} \\ m & \rho' m = I \qquad XW' = 0 \\ c & cW' = 0, c \ \textit{ is negative definite on } \ \ker X^{\perp} \cap \mu^{\perp}, \textit{ symmetric}^{3} \end{array}} \quad \text{(Slutsky list)}$$

Conversely, if (Slutsky list), then Smc *is invertible, with inverse* $\tilde{H}(D)$*, for some* D *that is negative definite on* $([p]\mu)^{\perp}$ *and symmetric.*

We stress that the Slutsky list of properties is exhaustive, in that it recovers all that we assumed about the one thing (D) defining the Slutsky matrices H^{-1}. Any other Slutsky property must follow from this list; for example, $YW' = -I$ from $\rho' m = I, XW' = 0$.

Note that **revealed yield preference** is manifested infinitesimally in $XW' = 0, cW' = 0$, since this results from differentiating (i,ii) in proposition 1.1 with respect to $\Delta \in spanW'$.

1.4.4 Computation of Slutsky matrices

We can compute Slutsky matrices H^{-1} faster by exploiting the symmetry and sparseness of H.

Express H^{-1} as

$$S = \begin{bmatrix} A & P \\ P' & B \end{bmatrix} \qquad m = \begin{bmatrix} X \\ Y \end{bmatrix} \qquad \Rightarrow \qquad H^{-1} = \begin{bmatrix} A & P & -X \\ P' & B & -Y \\ -X' & -Y' & -c \end{bmatrix} \qquad (*)$$

[3] It is easy to show that $\ker X^{\perp} \cap \mu^{\perp} = W^{\perp} \cap \mu^{\perp}$.

To keep track, the square A, B, c are symmetric of dimensions $C^*, J, S{+}1$, and $P_{C^* \times J}, X_{C^* \times S+1}, Y_{J \times S+1}$.

Algorithm 1.1. *H^{-1} exists if D is negative definite, and is recursively computable if D is symmetric:*

$$\begin{aligned}
& D^{-1} \\
\Phi &\equiv [p]' D^{-1} [p] \\
B &= (W \Phi^{-1} W')^{-1} \\
Y &= -BW\Phi^{-1} \qquad\qquad c = \Phi^{-1} - \Phi^{-1} W' B W \Phi^{-1} \\
P &= -D^{-1}[p] Y' \qquad\qquad X = D^{-1}[p] c \\
A &= (I - X[p]') D^{-1}
\end{aligned}$$

Computing D^{-1} is the most expensive step, which is cheaper with *state separable* utility,

$$u(x) = a(u_0(x_0), ..., u_S(x_S)) \text{ for some } a, (u_s)_s$$

because then D is block diagonal and its inverse too

$$D = \begin{bmatrix} D_0 & & 0 \\ & . & \\ 0 & & D_S \end{bmatrix} \qquad D^{-1} = \begin{bmatrix} D_0^{-1} & & 0 \\ & . & \\ 0 & & D_S^{-1} \end{bmatrix} \qquad \Rightarrow \Phi \text{ is diagonal}$$

The marginal propensity to consume from state s income, $X_s \in R^{C^*}$, may be nonzero in states $t \neq s$. Extra properties of the Slutsky matrices follow as a by-product; by theorem 5.13, they are implied by the Slutsky list.

Corollary 1.1. *Fix D as above. Then B is negative definite, $\ker P = \ker Y'$ and*

$rank Y = J$, $\ker c = \ker X$, c is negative definite on $\left(W\Phi^{-1}\right)^{\perp}$, has rank $S+1-J$, and $c = c\Phi c$. Lastly,

$$\begin{bmatrix} X \\ Y \end{bmatrix} \Phi c = \begin{bmatrix} X \\ 0 \end{bmatrix}$$

meaning that, for marginal income $\dot{w} \in span\Phi c$, marginal demand $m\dot{w}$ is as if asset markets were absent.

1.5 Generic neoclassical demand

1.5.1 Slutsky perturbations

A Slutsky perturbation is a perturbation of the Slutsky matrices arising from a perturbation of the Hessian of utility. We want to identify Slutsky perturbations because they are intrinsic in the study of generic properties of equilibrium in incomplete markets.

More exactly, Slutsky matrices are H^{-1}, where $H = \tilde{H}(D^2u)$. By continuity of $\tilde{H}$, $\tilde{H}(D)$ is invertible for all close enough $D \approx D^2u$. If D is symmetric then the difference $\nabla = \tilde{H}(D)^{-1} - H^{-1}$ is called a **Slutsky perturbation**. Being symmetric, we write

$$\nabla = \begin{bmatrix} \dot{S} & -\dot{m} \\ -\dot{m}' & -\dot{c} \end{bmatrix}$$

and identify a Slutsky perturbation with a triple $\dot{S}, \dot{m}, \dot{c}$. Our main goal is to characterize Slutsky perturbations, without reference to the inversion defining them, in terms of individual constraints on ∇:

$$\boxed{\begin{array}{ll} \text{on } \dot S & \rho'\dot S = 0 \text{ and } \dot S \text{ is symmetric} \\ \text{on } \dot m & \rho'\dot m = 0 \text{ and } \dot X W' = 0 \\ \text{on } \dot c & \dot c W' = 0 \text{ and } \dot c \text{ is symmetric} \end{array}} \tag{constraints}$$

Each of these independent linear constraints is satisfied by zero.

Theorem 1.2. ***Slutsky perturbations characterized*** *Given u in Debreu's setting and b in B', consider the Slutsky matrices H^{-1}. Every small enough Slutsky perturbation ∇ satisfies (constraints). Conversely, every small enough perturbation ∇ that satisfies (constraints) is Slutsky: $H^{-1} + \nabla$ is the inverse of $\tilde H(D)$ for some D that is negative definite on $Du(x(b))^\perp$ and symmetric. (Negative definite given u in Debreu's special setting.)*

Thus Slutsky perturbations are characterized as those that satisfy (constraints), affecting S, m, c simultaneously or separately. Assuming the veracity of theorem 5.13, a proof is trivial, if we appeal to

Lemma 1.1. ***Stability*** *Fix a dimension $0 < d \le C^*$ for the Grassmanian $G_{C^*,d}$. Suppose continuous functions $D : K \to R^{C^* \times C^*}, S : K \to G_{C^*,d}$. If $D(x)$ is negative definite on $S(x)$, then $D(\tilde x)$ is negative definite on $S(\tilde x)$, for all nearby $\tilde x \approx x$.*

Proof. A matrix D is negative definite on a nonzero subspace S iff $\max_{z \in S^*} z'Dz < 0$, by compactness of $S^* \equiv \{z \in S \mid z'z = 1\}$. By hypothesis, $\epsilon(x) \equiv \max_{z \in S^*(x)} z'D(x)z < 0$, and by the maximum principle $\epsilon(\cdot)$ is continuous, so $\epsilon(\tilde x) < 0$ is an open neighborhood of x. (To apply the principle, note $S^*(\cdot)$ is a continuous, nonempty, compact valued correspondence and $(x, z) \mapsto z'D(x)z$ a continuous function.) □

Proof. of theorem 1.2. Clearly $\nabla = \tilde{H}(D)^{-1} - H^{-1}$ satisfies (constraints) if both $H^{-1}, \tilde{H}(D)^{-1}$ satisfy (Slutsky list). This hypothesis in turn holds, by the first part of theorem 5.13, if D^2u, D are (1) symmetric and (2) negative definite on $([p]\mu)^{\perp}$. These conditions hold for D^2u in Debreu's setting; by definition of a Slutsky perturbation (1) holds for D, and by stability ($S \equiv ([p]\mu)^{\perp}$) so does (2), if it is small enough.

Conversely, suppose ∇ satisfies (constraints). By the first part of theorem 5.13 H^{-1} satisfies (Slutsky list), so clearly $H^{-1} + \nabla$ satisfies (Slutsky list), save perhaps for the definiteness statements, which by stability ($S = \rho^{\perp}, \ker(X + \dot{X})^{\perp} \cap \mu^{\perp}$) *do* hold if ∇ is small enough. By the converse part of theorem 5.13, $H^{-1} + \nabla$ is invertible, with inverse $\tilde{H}(D)$ for some D that is negative definite on $([p]\mu)^{\perp}$ and symmetric. Thus $\tilde{H}(D)^{-1} = H^{-1} + \nabla$ and ∇ is a Slutsky perturbation. □

The study of generic properties of GEI entails Slutsky perturbations, that is, it entails generic Slutsky matrices. It is useful to identify the generic Slutsky matrices, because they determine the generic comparative statics of market variables and welfare, through their appearance in the Slutsky decomposition of demand. So what are the generic Slutsky matrices? What is the range of Slutsky perturbations $\nabla = \tilde{H}(D)^{-1} - H^{-1}$? We answered this: Any small ∇ that meets explicit linear constraints can be rationalized as a Slutsky perturbation.

The relevance of this for the study of generic properties of GEI is that genericity can be argued directly in terms of Slutsky matrices and Slutsky perturbations.

1.5.2 Rationalizing Slutsky perturbations

Tacit in the previous subsection is that D may be rationalized as $D^2\tilde{u}(x(b))$ for some nearby $\tilde{u} \approx u$ preserving Debreu's setting and $x(b)$ as neoclassical demand, a well-known "fact" we now recall.

Definition 1.1. *A* quadratic perturbation of utility at $\overline{x} \in R^{C^*}_{++}$ *is a pair* (ω, Δ) *consisting of a* $C^{r\geq 2}$ *weight function* $\omega : R^{C^*}_{+} \to [0,1]$ *that equals unity in a neighborhood of* $\overline{x}$ *and has compact support in* $R^{C^*}_{++}$, *and of a symmetric matrix* Δ *of dimension* C^*. *It operates on functions* $R^{C^*}_{+} \to R$ *as* $u \mapsto u_{(\omega,\Delta)}(x) \equiv u(x) + \frac{\omega(x)}{2}(x-\overline{x})'\Delta(x-\overline{x})$.

Proposition 1.4. ***Rationalizability*** *of a small symmetric perturbation* Δ *of* $D^2u(x)$. *If* u *is in Debreu's setting, so is* $u_{(\omega,\Delta t)}$ *for all small enough support(*ω*),t, and then* $\overline{x}$ *is the* $u-$*demand at* b *iff it is the* $u_{(\omega,\Delta t)}-$*demand at* b. *Last but not least,* $D^2u_{(\omega,\Delta t)}(\overline{x}) = D^2u(\overline{x}) + \Delta t$, *so that* $\frac{\partial}{\partial t}|_{t=0}\, D^2u_{(\omega,\Delta t)}(\overline{x}) = \Delta$.

Conclusion 1.1. *Suppose* u *belongs in Debreu's setting and* b *in* B', *and consider the Slutsky matrices* S, m, c *at* $x(b)$. *Then any small enough perturbation to them that satisfies (constraints), and none other, we can rationalize by a quadratic perturbation* $u_{(\omega,\Delta)}$ *of* u *such that* $u_{(\omega,\Delta)}$ *preserves Debreu's setting and demand* $\sigma_{u_{(\omega,\Delta)}}(b) = \sigma_u(b)$ *at* b, *and has the perturbed* S, m, c *for its Slutsky matrices.*

1.6 Proofs

1.6.1 The Slutsky properties

We tie the Slutsky properties to each of three increasingly stringent descriptions of H in (H):

$$H = \begin{bmatrix} M & -\rho \\ -\rho' & 0 \end{bmatrix}$$

(I) the relationship between M, ρ

(II) $M = M(D)$ for some D

(III) D is negative definite on $([p]\mu)^{\perp}$[4]

Equivalence 1.1. *Fix a matrix* ρ.[5] *Suppose*

$$M \text{ is negative definite on } \rho^{\perp} \text{ and symmetric, and } \rho \text{ has no kernel} \qquad \text{(I)}$$

Then

$$\begin{bmatrix} M & -\rho \\ -\rho' & 0 \end{bmatrix} \qquad (1)$$

is invertible, with inverse

$$\begin{bmatrix} S & -m \\ -m' & -c \end{bmatrix} \qquad (1')$$

for some S, m, c *satisfying*

$$\begin{gathered} \rho' S = 0, S \text{ is negative definite on } \rho^{\perp}\text{, symmetric} \\ \rho' m = I \\ c \text{ is symmetric} \end{gathered} \qquad (I')$$

Conversely, suppose (I'). Then (5.4) is invertible, with inverse (1), for some M *satisfying*

[4] We will take $\mu = \lambda$ or 0, according as we are in Debreu's setting or Debreu's special setting. (III) says "D is negative definite" if $\mu = 0$.

[5] This does not have to be the particular one in (H).

(I).

We use the convenient notation

$$\rho \equiv \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \quad \text{where} \quad \begin{array}{r} \rho_1 = \text{first } \ C^* \ \text{ rows of } \ \rho \\ \rho_2 = \text{last } \ J \ \text{ rows of } \ \rho \end{array}$$

Equivalence 1.2. *Fix* ρ *with no kernel. Suppose (I) and consider the* m, c *implied by Equivalence 1.1. If*

$$M(D)'s \ \textit{ last } \ J \ \textit{ rows and columns are zero} \tag{II}$$

then

$$X\rho_2' = 0 \qquad Y\rho_2' = I \qquad c\rho_2' = 0 \tag{II'}$$

Conversely, suppose (I') and consider the M *implied by Equivalence 1.1. If (II') then (II) for some* D*. Lastly,* $Y\rho_2' = I$ *is redundant in (II') if* ρ_2 *has linearly independent rows.*

Equivalence 1.3. *Fix* ρ *with no kernel and* $\rho_2\mu = 0$*. Suppose (I) and consider the* m, c *implied by Equivalence 1.1; suppose (II). If*

$$D \ \textit{ is negative definite on } \ (\rho_1\mu)^\perp \tag{III}$$

then

$$c \ \textit{ is negative definite on } \ \ker X^\perp \cap \mu^\perp \tag{III'}$$

Conversely, suppose (I') and consider the M *implied by Equivalence 1.1; suppose (II') and* *consider the solution to* $M = M(D)$ *implied by Equivalence 1.2. If (III') then (III).*

We now apply the Equivalences to our particular case:

$$M = M(D^2u) \qquad \rho_1 = [p] \qquad \rho_2 = -W \tag{particular}$$

Lemma 1.2. *Suppose* $M = M(D)$ *with* D *negative definite on* $(\rho_1\mu)^\perp$*, where* $\rho_2\mu = 0$ *and* ρ_2 *has linearly independent rows. Then* M *is negative definite on* $\rho^\perp$*.*

Proof.

$$[a' : b']M \begin{bmatrix} a \\ b \end{bmatrix} = a'Da$$

Suppose $[a' : b'] \in \rho^\perp$, that is, $a'\rho_1 = -b'\rho_2$. *Claim:* $a \in (\rho_1\mu)^\perp$. For $a'\rho_1\mu = -b'\rho_2\mu = 0\rfloor$ So $a'Da < 0$ unless $a = 0 \Rightarrow b'\rho_2 = 0 \Rightarrow b = 0$ given the linearly independent rows. □

Proof. of theorem 5.13. By hypothesis and the lemma, $M(D)$ is negative definite on $\rho^\perp$, and ρ has no kernel because $\rho_1 = [p]$ has none. So by Equivalence 1.1 (I') holds for $S, m, c \approxeq \tilde{H}(D)^{-1}$. Obviously $M(D)$ satisfies (II), so by Equivalence 1.2 (II') holds, with $-YW' = I$ redundant since $\rho_2 = -W$ has linearly independent rows. Lastly, by Equivalence 1.3 (III') holds. That is, (Slutsky list) = (I', II', III') hold.

Conversely, if (Slutsky list) = (I', II', III'), then we apply the converse part of the Equivalences. By Equivalence (1.1) (Smc)=(5.4) is invertible, and the symmetric M appearing in (1) must by Equivalence 1.2 be $M = M(D)$ for some (necessarily symmetric) D (recall $Y\rho_2' = I$ is redundant), and by Equivalence (1.3) D must satisfy (III). □

1.6.2 Equivalence lemmas

Equivalence 1.1

Proof. Invertibility: Suppose $[x', y']'$ is in the kernel of (1). Then $Mx - \rho y = 0$ and $\rho' x = 0 \Rightarrow x' M x = 0$ and $x \in \rho^{\perp} \Rightarrow x = 0 \Rightarrow \rho y = 0 \Rightarrow y = 0$ since ρ has no kernel, as desired. Since (1) is symmetric, so is its inverse, with S, c symmetric. By definition of inverse,

$$\begin{array}{ll} MS + \rho m' = I & -Mm + \rho c = 0 \\ \rho' S = 0 & \rho' m = I \end{array}$$

Hence $\rho' S = 0, \rho' m = I$. Turning to $S's$ semidefiniteness, fix γ and consider $\gamma' S \gamma$. Solve

$$\begin{bmatrix} M & -\rho \\ -\rho' & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \gamma \\ 0 \end{bmatrix} \qquad \equiv \qquad \begin{array}{c} Ma - \rho b = \gamma \\ \rho' a = 0 \end{array}$$

which is possible by invertibility. Then

$$\begin{aligned} \gamma' S \gamma &= \\ (a'M - b'\rho') S \gamma &= a' M S \gamma = a'(I - \rho m')\gamma = \\ a'\gamma &= a'(Ma - \rho b) = a' M a \end{aligned}$$

Since $a \in \rho^{\perp}$, by hypothesis on M $\gamma' S \gamma = a' M a < 0$ unless $a = 0 \Rightarrow -\rho b = \gamma$ or $\gamma \in span \rho$. So if $\gamma \in \rho^{\perp}$, then $\gamma = 0$. That is, S is negative definite on $\rho^{\perp}$.

Conversely, suppose (I'). Then the invertibility of (5.4) is established similarly as above.

Since (5.4) is symmetric, so is its inverse

$$\begin{bmatrix} M & -\alpha \\ -\alpha' & \beta \end{bmatrix}$$

with M, β symmetric. Claim: $\alpha = \rho, \beta = 0$. By definition of inverse, $MS + \alpha m' = I$ and $\alpha' S + \beta m' = 0$; postmultiplying by ρ and invoking (I') establishes the claim.⌋ Clearly $\rho' m = I$ implies ρ has no kernel. Lastly, M is negative definite on $\rho^{\perp}$: Fix γ and consider $\gamma' M \gamma$. Solve

$$\begin{bmatrix} S & -m \\ -m' & -c \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \gamma \\ 0 \end{bmatrix} \qquad \equiv \qquad \begin{matrix} Sa - mb = \gamma \\ m'a + cb = 0 \end{matrix}$$

and suppose $\gamma \in \rho^{\perp} \equiv 0 = \rho'(Sa - mb) = -b$. That is, $Sa = \gamma$ and $m'a = 0$. Since $M\gamma = MSa = (I - \rho m')a = a, \gamma' M \gamma = a' Sa$. Invoking (I'), we see $\gamma' M \gamma < 0$ unless $a = \rho\alpha$ for some $\alpha \Rightarrow 0 = m'a = m'\rho\alpha = \alpha \Rightarrow a = 0 \Rightarrow \gamma = Sa = 0$. Hence M is negative definite on $\rho^{\perp}$. □

Equivalence 1.2

Proof. By hypothesis, write

$$M = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

Focusing on the bottom part of $MS + \rho m' = I$,

$$0 + \rho_2 m' = [0 : I]$$

which says $X\rho_2 = 0, Y\rho_2 = I$. As for $c\rho_2' = 0$: Using $-Mm + \rho c = 0$, $0 = M[0 : I]' =$

$Mm\rho_2' = \rho c\rho_2'$. Since ρ has no kernel, $c\rho_2' = 0$.

Conversely, applying (II') to $-Mm + \rho c = 0$:

$$M\begin{bmatrix} 0 \\ I \end{bmatrix} = Mm\rho_2' = \rho c\rho_2' = 0$$

This and the symmetry of M imply that M is zero off the northwestern corner.

Lastly, $I = \rho' m = \rho_1' X + \rho_2' Y$, so $X\rho_2' = 0$ implies $\rho_2' = 0 + \rho_2' Y\rho_2'$ or $\rho_2'(I - Y\rho_2') = 0$. If ρ_2 has linearly independent rows, $I - Y\rho_2' = 0$. □

Equivalence 1.3

Expressing H^{-1} as in (5.5), by definition of inverse we have:

$$\begin{bmatrix} A & P & -X \\ P' & B & -Y \\ -X' & -Y' & -c \end{bmatrix}\begin{bmatrix} D & 0 & -\rho_1 \\ 0 & 0 & -\rho_2 \\ -\rho_1' & -\rho_2' & 0 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$\begin{array}{lll} AD + X\rho_1' = I & X\rho_2' = 0 & A\rho_1 + P\rho_2 = 0 \\ P'D + Y\rho_1' = 0 & Y\rho_2' = I & P'\rho_1 + B\rho_2 = 0 \\ -X'D + c\rho_1' = 0 & c\rho_2' = 0 & X'\rho_1 + Y'\rho_2 = I \end{array}$$

Lemma 1.3. *Every* $z \in R^{C^*}$ *can be expressed as* $z = Aa + Xb$ *for some* $a \in X^{\perp}, b \in \ker X^{\perp}$.

Proof. Set $b = \rho_1' z, a = Dz - \rho_1 cb$. Then $Aa + Xb = A(Dz - \rho_1 cb) + Xb = (AD)z - (A\rho_1)cb + Xb = (I - X\rho_1')z - (-P\rho_2)cb + Xb = z - X(\rho_1' z - b) + P(\rho_2 c)b = z$ since from the equations $\rho_2 c = 0$. Now $a \in X^{\perp} : X'a = X'(Dz - \rho_1 cb) = c\rho_1' z - (I - Y'\rho_2)cb = c(\rho_1' z - b) = 0$. To get $b \in \ker X^{\perp}$, redefine $b = (\rho_1' z)^*$ where "$*$" denotes the orthogonal projection to

$\ker X^\perp$, but keep a as before. □

Lemma 1.4. *If* $S\rho = 0, S$ *is negative definite on* $\rho^\perp$*, symmetric, then* A *is negative definite on* $X^\perp$*.*

Proof. Fix a and write

$$\begin{bmatrix} a \\ 0 \end{bmatrix} = x + y \in \rho^\perp + span\rho$$

Since $\rho' S = 0, S\rho = 0$,

$$a'Aa = [a' : 0]S \begin{bmatrix} a \\ 0 \end{bmatrix} = x'Sx$$

By hypothesis on S, $a'Aa < 0$ unless $x = 0 \Rightarrow [a' : 0]' = y = \rho\gamma$ some $\gamma \Rightarrow 0 = \rho_2\gamma$. If $a \in X^\perp$ then $0 = X'a = X'\rho_1\gamma = (I - Y'\rho_2)\gamma = \gamma \Rightarrow y = 0 \Rightarrow a = 0$. That is, A is negative definite on $X^\perp$. □

Proof. of Equivalence 1.3. Suppose throughout $\rho_2\mu = 0$. We will appeal twice to the string

$$(X\delta)'(\rho_1\mu) = \delta'(X'\rho_1)\mu = \delta'(I - Y'\rho_2)\mu = \delta'\mu \qquad \text{(string)}$$

The third row implies $X'DX = c$: $X'DX = c\rho_1'X = c(I - \rho_2'Y) = c$. For every δ, $\delta' c\delta = (X\delta)'D(X\delta) = *$, and $X\delta \in (\rho_1\mu)^\perp$ if $\delta \in \mu^\perp$ by the (string), so $* < 0$ by hypothesis on D, unless $X\delta = 0$ or $\delta \in \ker X$. If $\delta \in \ker X^\perp$ then $\delta = 0$. That is, c is negative definite on $\delta \in \ker X^\perp \cap \mu^\perp$.

Conversely, fix $z \in R^{C^*}$ and by lemma 1.3 write $z = Aa + Xb$ with $a \in X^\perp, b \in \ker X^\perp$. *Claim*: $z'Dz = a'Aa + b'cb$. $Dz = D(Aa + Xb) = (I - \rho_1 X')a + \rho_1 cb = a + \rho_1 cb$. Thus $z'Dz = (a'A + b'X')(a + \rho_1 cb) = a'Aa + a'A\rho_1 cb + b'X'a + b'X'\rho_1 cb = *$. The second term is zero, since the equations say $A\rho_1 c = -P\rho_2 c$ and $\rho_2 c = 0$, and so is the third one, since $X'a = 0$. So $* = a'Aa + b'(I - Y'\rho_2)cb = a'Aa + b'cb.\rfloor$

By lemma 1.4 and $a \in X^\perp, a'Aa < 0$ unless $a = 0$. By hypothesis on c and $b \in \ker X^\perp, b'cb < 0$ unless $b = 0$–so long as $b \in \mu^\perp$. So to show D is negative definite on $(\rho_1\mu)^\perp$, it suffices that $z \in (\rho_1\mu)^\perp \Leftrightarrow b \in \mu^\perp$. To see this implication, we take the particular $b = (\rho_1' z)^*$ from the proof of lemma 1.3, and apply (string) twice, with $\delta = \rho_1' z$ and $\tilde{\delta} = b$: $z'\rho_1\mu = \delta'\mu = (X\delta)'(\rho_1\mu) = (Xb)'(\rho_1\mu) = b'\mu$ (the definition of $b \Rightarrow \delta - b \in \ker X \Rightarrow X\delta = Xb$). $\square$

1.6.3 Rationalizability

Lemma 1.5. *If* u *is in Debreu's setting, then* $u(x) > \sup_{\partial R_+^{C*}} u(\tilde{x})$ *for* $x \in R_{++}^{C*}$.

Proof. Debreu's last condition implies this holds with " $\geq$" instead. But if "=" at some $x \in R_{++}^{C*}$, then by Debreu's second condition $u(tx) < \sup_{\partial R_+^{C*}} u(\tilde{x})$ for all large enough $t \in (0,1)$, in which case $u(tx) < u(\tilde{x})$ for some $\tilde{x} \in \partial R_+^{C*}$, contrary to " $\geq$". $\square$

Proof. of proposition 1.4. Assuming that $u_{(\omega,\Delta)}$ is also in Debreu's setting, the remainder is easy:

Given its interiority, $\overline{x}$ is the $u-$neoclassical demand at (p, W, w) iff (F) holds at $\overline{x}$ and u iff (F) holds at $\overline{x}$ and $u_{(\omega,\Delta)}$ iff $\overline{x}$ is the $u_{(\omega,\Delta)}-$neoclassical demand at (p, W, w). The first and last equivalences hold because $u, u_{(\omega,\Delta)}$ belong in Debreu's setting, and the middle one because $Du(\overline{x}) = Du_{(\omega,\Delta)}(\overline{x})$.

Last but not least, $\omega \equiv 1$ in a neighborhood $x \approx \overline{x}$, where $u_{(\omega,\Delta t)}(x) \equiv u(x) + \frac{1}{2}(x - \overline{x})'\Delta t(x - \overline{x})$ and $D^2 u_{(\omega,\Delta)}(\overline{x}) = D^2 u(\overline{x}) + \frac{1}{2}(\Delta + \Delta')t = D^2 u(\overline{x}) + \Delta t$, the last equality by $\Delta's$ symmetry.

To verify for $u_{(\omega,\Delta)}$ the four conditions in Debreu's setting, fix ω and write $K \equiv$ support(ω).

First condition. Obvious.

Second and third. These hold with the proviso $x \in R_{++}^{C*} \backslash K$, since $R_{++}^{C*} \backslash K$ is open and $u_{(\omega,\Delta)} \mid_{R_{++}^{C*} \backslash K} = u \mid_{R_{++}^{C*} \backslash K}$, so we turn to $x \in K$. Both $\sup_K \left\| Du_{(\omega,\Delta)}(x) - Du(\overline{x}) \right\|$, $\sup_K \left\| D^2 u_{(\omega,\Delta)}(x) - D^2 u(\overline{x}) \right\|$ are bounded since $Du_{(\omega,\Delta)}(x), D^2 u_{(\omega,\Delta)}(x)$ are continuous in x and K compact, and homogeneous of degree one in t, hence may be chosen smaller than any given $\delta > 0$ by replacing Δ with Δt for all small enough $t > 0$. Choosing δ small enough to make true the implications $\left\| Du_{(\omega,\Delta)}(x) - Du(\overline{x}) \right\| < \delta \Rightarrow Du_{(\omega,\Delta)}(x) \gg 0, \left\| D^2 u_{(\omega,\Delta)}(x) - D^2 u(\overline{x}) \right\| < \delta \Rightarrow D^2 u_{(\omega,\Delta)}(x)$ is negative definite on $Du(x)^{\perp}$ (appealing to lemma 1.1 with $D(x) \equiv D^2 u_{(\omega,\Delta)}(x), S(x) \equiv Du(x)^{\perp}$), these conditions also hold at $x \in K$.

Fourth. This holds with the proviso $x \in R_{++}^{C*} \backslash K$ since $u_{(\omega,\Delta)} \mid_{R_{++}^{C*} \backslash K} = u \mid_{R_{++}^{C*} \backslash K}$, so we turn to $x \in K$. By lemma 1.5 $\epsilon \equiv u(\overline{x}) - \sup_{\partial R_{+}^{C*}} u(\tilde{x}) > 0$. Now suppose that K is small enough, in that $|u(x) - u(\overline{x})| < \frac{\epsilon}{2}$ for $x \in K$ (possible by $u's$ continuity), and that the rescaling of Δ is too, in that $|(x - \overline{x})' \Delta (x - \overline{x})| < \epsilon$ for $x \in K$. So for $x \in K$ $u_{(\omega,\Delta)}(x) = u(x) + \frac{\omega(x)}{2}(x - \overline{x})' \Delta (x - \overline{x}) > u(\overline{x}) - \frac{\epsilon}{2} + \frac{\omega(x)}{2}(-\epsilon) \geq u(\overline{x}) - \epsilon = \sup_{\partial R_{+}^{C*}} u(\tilde{x}) = \sup_{\partial R_{+}^{C*}} u_{(\omega,\Delta)}(\tilde{x})$, the latter since $u = u_{(\omega,\Delta)}$ on $\partial R_{+}^{C*} \rfloor$ □

1.6.4 Computation of Slutsky matrices

As in the proof of Equivalence 1.3, but substituting $\rho_1 = [p], \rho_2 = -W$,

$$\begin{array}{lll} AD + X[p]' = I & XW' = 0 & A[p] - PW = 0 \\ P'D + Y[p]' = 0 & -YW' = I & P'[p] - BW = 0 \\ -X'D + c[p]' = 0 & cW' = 0 & X'[p] - Y'W = I \end{array} \qquad \text{(system)}$$

Proof. of algorithm 1.1. Invertibility is easy. We deduce formulas for A, B, c, P, X, Y recursively, while imposing $A, B, c's$ symmetry, which we verify last, and refer to equation ij as that appearing in row i, column j of the (system). Note $\Phi \equiv [p]'D^{-1}[p]$ is symmetric, negative definite since $[p]$ has no kernel.

Equation 21 holds iff $P \equiv -D^{-1}[p]Y'$; equation 31 iff $X \equiv D^{-1}[p]c$; equation 11 iff $A \equiv [(I - X[p]')D^{-1}]'$. With this definition of X, 12 holds if 32 holds. So far P, X, A are in terms of Y, c, which we describe in terms of B.

Given this formula for P, 23 holds iff $-Y\Phi - BW = 0$ iff $Y \equiv -BW\Phi^{-1}$. Given the formulas for X, Y, 33 holds iff $c\Phi + \Phi^{-1}W'BW = I$ iff $c \equiv \Phi^{-1} - \Phi^{-1}W'BW\Phi^{-1}$.

Claim: A, P as defined make 13 true. $A[p] - PW = D^{-1}(I - [p]X')[p] + D^{-1}[p]Y'W = *$. Since 33 holds by definition of c, $* = D^{-1}(I - [p]X')[p] + D^{-1}[p](X'[p] - I) = 0$.

Now define $B \equiv (W\Phi^{-1}W')^{-1}$. Note, $W\Phi^{-1}W'$ is invertible if negative definite, which it is since Φ^{-1} is (as the inverse of a negative definite matrix) and W' has no kernel.

Claim: B as defined makes $22, 32$ true. 22: $-YW' = BW\Phi^{-1}W' = I$. 32: $cW' = (\Phi^{-1} - \Phi^{-1}W'BW\Phi^{-1})W' = \Phi^{-1}W'(I - B \cdot W\Phi^{-1}W') = \Phi^{-1}W'(0) = 0$.

These definitions solve the system modulo $A, B, c's$ symmetry, which does exist: B

is symmetric indeed, which implies c is, which implies $A = D^{-1}(I - [p]X') = D^{-1} - D^{-1}[p]c[p]'D^{-1}$ is. □

Chapter 2

Welfare Impact of Taxation in Incomplete Markets

2.1 Introduction

When asset markets are incomplete, there are almost always many Pareto improving policy interventions, if there are multiple commodities and households. Remarkably, these policies do not involve adding any new markets.

Focusing on tax policy, I create a framework for proving the existence of Pareto improving taxes, for computing them, and for estimating the size of the improvement.

The protagonist is the price adjustment following an intervention. Its role is to improve on asset insurance by redistributing endowment wealth across states, as anticipated by Stiglitz (1982). The price adjustment is determined by how taxes and prices affect aggregate, not individual, demand.

If taxes targeting current incomes are Pareto improving, then they must cause an equilibrium price adjustment, Grossman (1975). Conversely, I prove that if the price adjustment is sufficiently sensitive to risk aversion, then for almost all risk aversions and endowments, Pareto improving taxes exist. I show how to verify this sensitivity test with standard demand theory, which Turner (2003a) extends from complete to incomplete markets.

To numerically identify the Pareto improving taxes, I give a formula for the welfare impact of taxes. It requires information on the individual marginal utilities and net trades, and on the derivative of aggregate, but not individual, demand with respect to taxes and prices.

To bound the rate of Pareto improvement, I define an equilibrium's insurance deficit. Pareto optimality obtains exactly when the insurance deficit is zero. If the tax policy targets only current incomes, then the implied price adjustment determines the best rate, by integration against the covariance of insurance deficit and net trades across agents. The equilibrium's insurance deficit arises from the agents' component of marginal utility for contingent income standing orthogonally to the asset span.

Many different tax policies generically support a Pareto improvement, because they all pass this one sensitivity test. These policies include (a) taxes on asset purchases, as in Citanna, Polemarchakis, and Tirelli (2001), (b) lump-sum taxes on current income plus one flat tax on asset purchases, similar to Citanna, Kajii, and Villanacci (1998) and to Mandler (2003), (c) asset measurable taxes on capital gains, and (d) excise taxes on current commodities, similar to Geanakoplos and Polemarchakis (2002), who emphasize consumption externality over asset incompleteness.

Some policies fail the sensitivity test and never improve everyone's welfare. For example, reallocate current incomes lump-sum and force households to keep original asset demands.

If utilities are time separable, they keep future commodity demands, inducing utilities for current consumption. The First Welfare Theorem implies this tax policy is not Pareto improving. The example flunks the sensitivity test because the future price adjustment is zero, independently of risk aversion. For another example, for each asset tax purchases and subsidize sales at the same rate. Then each asset's price adjusts to offset the tax, and the final cost of holding a portfolio of assets stays the same. Demand and welfare stay the same. The example flunks the sensitivity test because the price adjustment is the negative of the tax, independently of risk aversion.

To ultimately decide whether a tax policy generically supports a Pareto improvement, I give primitives for the sensitivity of price adjustment. This requires information about the derivatives of aggregate demand with respect to policy and prices. The price adjustment is sensitive to risk aversion if there is (1) Full Reaction of Demand to Policy, and (2) Sufficient Independence of the Reactions of Demand (to Policy and to Prices). That is, if (1) there is high enough rank in the derivative of aggregate demand with respect to policy, and (2) it is possible to affect the derivative of aggregate demand with respect to prices while preserving the derivative with respect to policy, by perturbations to risk aversion. The first example violates (1); the rank is below the number of households by budget balance. The second example violates (2); the derivatives are each other's inverses, whatever the risk aversion.

The existence result for a tax policy, that it supports a Pareto improvement at any equilibrium, speaks not of every economy but only of a generic economy. At some economies the endowments are Pareto optimal, so that no price adjustment could lead to a Pareto improvement; at equilibria of other economies, everyone has the same marginal propensity to demand, so that no price adjustment exists.

In turn, to decide whether a tax policy meets primitives (1), (2), I invoke an extension of Slutsky theory from complete to incomplete markets.

Turner (2003a) develops the Slutsky theory of demand for commodities and assets in incomplete markets. First, it decomposes the derivative of demand with respect to commodity prices, asset prices, and asset payoffs into an income effect and a Slutsky substitution effect. Next, it identifies the properties that every Slutsky matrix must satisfy, and conversely proves that any matrix satisfying these properties is the Slutsky matrix of some demand. Finally, it shows that the Slutsky matrix can be perturbed arbitrarily, subject only to maintaining these properties, by perturbing the second derivative (risk aversion) of the utility generating the original Slutsky matrix, while preserving demand and the income effect matrix. These results for incomplete markets mirror exactly those for complete markets derived by Geanakoplos and Polemarchakis (1980).

For some economies, the price adjustment function does not admit any Pareto improving interventions, even though the equilibrium allocation is not Pareto optimal. By taking Slutsky perturbations of demand, I show that for almost all nearby economies the price adjustment function does admit them. Slutsky perturbations are thus the key to why there exist almost always Pareto improving taxes.

Geanakoplos and Polemarchakis (1986) began the study of generic improvements with incomplete markets, and introduced the idea of Slutsky perturbations from quadratic utility perturbations. Since they allowed the central planner to decide the agents' asset portfolios, they did not need to go beyond perturbing the Slutsky matrices of commodity demand. To show why weaker interventions may improve welfare, such as anonymous taxes and changes in asset payoffs, it became necessary to take into account how agents' portfolio adjustments caused a further price adjustment. Naturally, this required perturbing asset demand as well as commodity demand. The lack of a Slutsky theory for incomplete markets blocked contributions for over ten years[1], until a breakthrough by Citanna, Kajii, and Villanacci (1998), who analyzed first order conditions instead of Slutsky matrices. Researchers have extended the theory of generic improvements with incomplete markets to many policies by applying this first order approach; Cass and Citanna (1998), Citanna, Polemarchakis, and Tirelli (2001), Bisin et al. (2001), and Mandler (2003).

The Slutsky approach has certain advantages. First, to compute the Pareto improving interventions from my formula the policymaker needs to know the derivative of aggregate, but not individual, demand. In the first order approach the policymaker needs to know the second derivative of every individual's utility, i.e., the derivative of every individual's demand function. Second, to express the economic intuitions the economist can keep to the familiar language of demand theory, as in (1), (2), instead of the abstract language of submersions. Third, every time the researcher thinks a new result via Slutsky perturbations, he saves himself the work of implicitly reworking demand theory anew via quadratic utility perturbations.

Turner (2003b) adds to the result on the generic existence of Pareto improving financial innovation, by Elul (1995) and Cass and Citanna (1998). It argues that if the price adjustment to financial innovation passes the test of sufficient sensitivity to risk aversion, then

[1] The sole one is Elul (1995).

generically Pareto improving financial innovation exists. Then Slutsky perturbations reveal that substitution free financial innovation in an existing asset passes this test indeed.

These results suggest that the reason any policy would generically admit Pareto improving parameter values, be it fiscal, financial or otherwise, is precisely the passing of the sensitivity test. They also suggest that Slutsky perturbations are useful in discovering which other policies pass this test.

The paper continues as follows. Section 2 presents a general model of tax policy, and details several examples of tax policy. Section 3 has the formula for the welfare impact of taxes. Section 4 obtains the generic existence of Pareto improving taxes from the sensitivity condition on price adjustment, which it then reinterprets in terms of the Reaction of Demand to Prices and to Policy. Section 5 summarizes the demand theory in incomplete markets necessary to check the sensitivity in terms of the Reactions, then section 6 checks it for the several tax policies. Section 7 estimates the rate of Pareto improvement. Section 8 derives the welfare impact formula, and spells out the notation and the parameterization of economies.

2.2 GEIT model

Households $h = 1, ..., H$ know the present state of nature, denoted 0, but are uncertain as to which among $s = 1, ..., S$ nature will reveal in period 1. They consume commodities $c = 1, ..., C$ in the present and future, and invest in assets $j = 1, ..., J$ in the present only. Each state has commodity C as unit of account, in terms of which all value is quoted. Markets assign to household h an income $w^h \in R^{S+1}_{++}$, to commodity $c < C$

a price $p_{\cdot c} \in R_{++}^{S+1}$, to asset j a price $q^j \in R$ and future yield $a^j \in R^S$. We call $(p_{\cdot c})_1^C = p = (p_{s\cdot})$ the spot prices, $q = (q^j)$ the asset prices, $(a^j) = a = (a_s)$ the asset structure, and $w = (w^h)$ the income distribution, $\mathbf{P} \equiv R_{++}^{(C-1)(S+1)} \times R^J$.[2] Taxes are $t \in T, T$ some Euclidean space, negative coordinates corresponding to subsidies. The set of **budget variables** is

$$b \equiv (P, a, w, t) \in B \equiv \mathbf{P} \times R^{J \times S} \times R_{++}^{(S+1)H} \times T$$

and has some distinguished nonempty relatively open subset $B' \subset B$. B_0 is B with $T = \{0\}$.

Demand for commodities and assets $d = (x, y) : B' \to R_{++}^{C(S+1)} \times R^J$ is a function on B'. The demand $d^h = (x^h, y^h)$ of household h depends on own income only, $(x^h, y^h)(P, a, w, t) = (x^h, y^h)(P, a, w', t)$ if $w^h = w'^h$. **Tax payment** $\tau : B_0' \times codom(d) \to R^{S+1 \times \dim(T)}$ is a function such that $\tau(b_0, d)t$ is the actual tax payment, if demand and taxes are d, t. **Tax policy** $(\tau^h)_h$ is **anonymous** if τ^h is independent of h, and **tax revenue** τ is $\tau(b_0, (d^h)_h) \equiv \Sigma \tau^h(b_0, d^h)$.

An **economy** (a, e, t, t_*, d) consists of an asset structure a, endowments e, taxes t, distribution rates t_*, and demands d. For each household h, **endowments** specify a certain number $e_{sc}^h > 0$ of each commodity c in each state s, the **distribution rates** specify a fraction $t_*^h > 0$ with $\Sigma t_*^h = 1$, and **demands** specify a demand d^h. Let Ω be the set of (a, e, t, t_*, d).[3]

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[2] The numeraire convention is that unity is the price of $sC, s \geq 0$, which $\mathbf{P}$ therefore omits. The addition to p of the $sC, s \geq 0$ coordinates, bearing value unity, is denoted $\overline{p}$. We use the notation $P = (p, q) \in \mathbf{P}$.

[3] The appendix spells out the parameterization of demand d.

A list $(P, r; a, e, t, t^*, d) \in \mathbf{P} \times R^{S+1} \times \Omega$ is a **GEIT** $\leftrightarrow$

$$\sum(x^h(b) - e^h) = 0 \qquad \sum y^h(b) = 0 \qquad r - \tau(b_0, (d^h(b))_h)t = 0$$
$$\text{and} \quad b \equiv (P, a, (w_s^h = e_s^{h\prime}\overline{p}_s + t_*^h r_s)_s^h, t) \in B'$$

We say $(a, e, t, t^*, d) \in \Omega$ has **equilibrium** $(P, r) \in \mathbf{P} \times R^S$. A **GEI** is a GEIT with $t = 0$.

Under neoclassical assumptions $(a, e, 0, t_*, d) \in \Omega$ has an equilibrium[4], and then the implicit function theorem gives conditions for a neighborhood of $(a, e, 0, t_*, d)$ to have an equilibrium.

2.2.1 Neoclassical demand

Consider the **budget** function $\beta^h : B_0 \times R^{C(S+1)} \times R^J \to R^{S+1}$

$$\beta^h(b, x, y) \equiv (\overline{p}_s' x_s - w_s^h)_{s=0}^S - \begin{bmatrix} -q' \\ a' \end{bmatrix} y$$

Demand $d^h = (x^h, y^h)$ is **neoclassical**$_0$ if $T = \{0\}$ and there is a **utility** function $u : R_+^{C(S+1)} \to R$ with

$$u(x^h(b)) = \max_{X_0^h(b)} u \quad \text{throughout} \quad B'$$

$$X_0^h(b) \equiv \{x \in R_+^{C(S+1)} \mid \beta^h(b, x, y) = 0, \text{ some } \; y \in R^J\}$$

More generally, demand $d^h = (x^h, y^h)$ is **neoclassical** if there is a **utility** function $u : R_+^{C(S+1)} \to R$ with

$$u(x^h(b)) = \max_{X^h(b)} u \quad \text{throughout} \quad B'$$

[4] Geanakoplos and Polemarchakis (1986).

$$X^h(b) \equiv \{x \in R_+^{C(S+1)} \mid \beta^h(b_0, x, y) + \tau^h(b_0, x, y)t_b = 0, \text{ some } \ y \in R^J\}^5$$

If taxes $t_b = 0$ are zero, $X^h(b) = X_0^h(b)$. Thus neoclassical demand restricts to neoclassical$_0$ demand.

Neoclassical welfare is $v : B' \to R^H, v(b) = (v^h(b)) \equiv (u^h(x^h(b)))$.

The interpretation of X is that the cost of consumption x in excess of income w is financed by some portfolio $y \in R^J$ of assets, net of taxes. A **portfolio** specifies how much of each asset to buy or sell ($y_j \gtrless 0$), and a_s^j how much value in state s an asset j buyer is to collect, a seller to deliver.

2.2.2 Four examples of tax policy

We detail T, B', τ^h for four tax policies.[6]

Tax rates on asset purchases $t \in T = R^J$:

$$\tau = \begin{bmatrix} y'_+ \\ 0 \end{bmatrix}$$

$$B' = \{(P, a, w, t) \in B \mid q + t_I \in aR_{++}^S \text{ for all subsets } I \text{ }, a \text{ has linearly independent rows}\}^7$$

Lump-sum taxes on current incomes plus flat tax rate on asset purchases $t' = (l', f') \in$

[5] The functions $b \to b_0, \to t_b$ are $(p, q, a, w, t) \to (p, q, a, w, 0), \to t$. Here y is defined by x, if a is full rank.

[6] For a vector v of reals, v_+ is defined by $(v_+)_m = \max(0, v_m)$.

[7] For a subset $I \subset \{1, ..., J\}$ of assets, t_I is defined by $(t_I)_j$ being t_j or 0 according as $j \in I$ or not.

$T = R^H \times R:$

$$\tau^h = \begin{bmatrix} 1^{h\prime} & 1'y_+ \\ 0 & 0 \end{bmatrix}$$

$$B' = \{(P, a, w, t) \in B \mid q + f1_I \in aR^S_{++} \text{ for all subsets } I, a \text{ has linearly independent rows}\}$$

Asset measurable tax rates on future capital gains $t \in T = a'R^J \subset R^S$. Capital gain is $g^h_s = (\overline{p}'_s x_s - w^h_s)_+$. Measurability has every state's tax rate $t_s = a'_s L$ depending linearly on the asset payoffs:[8]

$$\tau^h = \left[g^h\right]$$

$$B' = \{(P, a, w, t) \in B \mid q \in aR^S_{++}, a \text{ has linearly independent rows, } t_s > -1\}$$

Tax rates on net purchases of current commodities $t \in T = R^{C-1}$. *(Excise taxes.)* Given endowments[9]

$$\tau^h = \begin{bmatrix} (x_0 - e^h_0)'_+ \\ 0 \end{bmatrix}$$

$$B' = \{(P, a, w, t) \in B \mid q \in aR^S_{++}, a \text{ has linearly independent rows, } p_{0c} + t_c > 0\}$$

Debreu's smooth preferences imply neoclassical demand exists, and is smooth in a neighborhood of b if $y_j, \overline{p}'_s x_s - w_s, x_{0c} - e_{0c} \neq 0$ for all j, s, c. We term **active** a GEI if it satisfies these inequalities for every household, in the context of these four examples, or if all demands are locally smooth, in a general context.

[8] Occasionally we view g, t as in R^{S+1} with $g_0, t_0 = 0$. For a point $g \in R^{(S+1)k}, [g] \in R^{(S+1)k \times S+1}$ denotes the matrix whose s^{th} column is $g_{s\cdot} \in R^k$ in the s^{th} block and zero in all the other k-blocks. If $k = 1$, as here, this is a diagonal matrix with g along the diagonal. See "aggregate notation" in the appendix.

[9] Occasionally we view t as in R^C with $t_C = 0$.

2.3 Welfare impact of taxes

We think of a smooth path $t = t(\xi)$ of taxes through $t = 0$, and of *infinitesimal taxes* as its initial velocity $\dot{t} = \dot{t}(0)$. Suppose the active GEI $(P, r; a, e, 0, t_*, d)$ is **regular** in that such a path lifts locally to a unique path $(P, r; a, e, t, t_*, d) = (P(\xi), r(\xi); a, e, t(\xi), t_*, d)$ of GEIT through the GEI. Then welfare is $v(b(\xi))$ with $b(\xi) = (P(\xi), r(\xi); a, (w^h_s = e^{h\prime}_s \overline{p}_s(\xi) + t^h_* r_s(\xi))^h_s, t(\xi))$. Thus taxes impact welfare only via the budget variables they imply. By the fundamental theorem of calculus the welfare impact is the integral of $D_b v^h \cdot \dot{b}$, which by abuse we call the *welfare impact*. We compute this product in the appendix, using the envelope theorem for $D_b v^h$ and the chain rule for $\dot{b}$, where the details of the notation appear.

Proposition 2.1 (Envelope). *The **welfare impact** $\dot{v} \in R^H$ of infinitesimal taxes $\dot{t}$ at a regular GEI is*

$$\dot{v} = (\lambda)' \dot{m} \qquad \dot{m} = \underbrace{(t^h_* \dot{r} - \tau^h \dot{t})_h}_{PRIVATE} \quad \underbrace{-\underline{z}\dot{P}}_{PUBLIC}$$

Here $(\lambda)'$ collects the households' marginal utilities of income across states, and $\dot{m}$ the impact on their incomes, private and public. The private one is the impact $\dot{r}$ on revenue distributed at rate $t_* \in R^H$ net of the impact $\tau^h \dot{t}$ on tax payments, and the public one is the impact on the value of their excess demands $\underline{z}$ in all nonnumeraire markets, that implied by the impact $\dot{P}$ on prices.

Policy targeting welfare must account for the equilibrium price adjustment it causes. The equilibrium price adjustment undoes the excess aggregate demand that policy causes, and depends on the reactions of aggregate demand to both policy and prices.

Proposition 2.2 (Revenue Impact). *At a regular GEI* $\dot{r} = \tau\dot{t}$.

This follows from $r = \tau t$, the chain rule, and $t = 0$ at a GEI. At a regular GEI there is a **price adjustment** matrix dP, smooth in a neighborhood of it, such that $\dot{P} = dP\dot{t}$. Thus the welfare impact is

$$\boxed{dv = (\lambda)'\left((t_*^h\tau - \tau^h)_h - \underline{z}dP\right)}$$

A policy targeting current incomes is (first order) Pareto improving only if taxes cause a price adjustment. For if $\tau^h_{s\geq 1}\dot{t} = 0, dP\dot{t} = 0$ then $\Sigma\frac{1}{\lambda_0^h}\dot{v}^h = \Sigma\frac{1}{\lambda_0^h}\lambda^{h\prime}\dot{m}^h = \Sigma\dot{m}_0^h = \Sigma(t_*^h\tau_0 - \tau_0^h) = 0$ so $\dot{v} \gg 0$ is impossible. Next we prove a converse.

2.4 Framework for generic existence of Pareto improving taxes

We prove the generic existence of Pareto improving taxes, stressing the role of changing commodity prices over the role of the particular tax policy. Existence follows directly from a hypothesis on price adjustment. Thus the tax policy is relevant only insofar as it meets the hypothesis on price adjustment. Then we reinterpret this hypothesis on dP in terms of primitives, the Reaction of Demand to Prices and the Reaction of Demand to Policy.

Pareto improving taxes exist if there exists a solution to $dv\dot{t} \gg 0$. In turn this exists if $dv \in R^{H\times \dim T}$ has rank H, which in turn implies that tax parameters outnumber household types $\dim T \geq H$. The key idea is that if $dv = (\lambda)'(t_*^h\tau - \tau^h)_h - (\lambda)'\underline{z}dP$ is rank deficient, then a perturbation of the economy would restore full rank by preserving the first summand but affecting the second one. Namely, if some economy's dP is not appropriate, then almost every nearby economy's dP is.

We have in mind a perturbation of the households' **risk aversion** $(D^2 u^h)_h$, which affects nothing but dP in the welfare impact dv. Now, to restore the rank the risk aversion must map into $(\lambda)'\underline{\overline{z}}dP$ richly enough. Since this map keeps $(\lambda)'\underline{\overline{z}}$ fixed, we require that $(\lambda)'\underline{\overline{z}}$ have rank H and that dP be sufficiently sensitive to risk aversion. Cass and Citanna (1998) gift us the first requirement:

Fact 2.1 (Full Externality of Price Adjustment on Welfare). *Suppose asset incompleteness exceeds household heterogeneity $S - J \geq H > 1$. Then generically in endowments every GEI has $(\lambda_s^h z_{s1}^h)_{s \leq H-1}^{h \leq H}$ invertible.*

Fact 2.2. *At a regular GEI, dP is locally a smooth function of risk aversion; the marginal utilities λ^i, tax payments τ^i, and excess demands z^i are locally constant in risk aversion.*

For $k \in R^{(S+1)(C-1)+J}$ we say that a *commodity coordinate* is one of the first $(S+1)(C-1)$.

Definition 2.1. *At a regular GEI, dP is k-**Sensitive to risk aversion** if for every $\alpha \in R^{\dim(T)}$ there is a path of risk aversion that solves $k'd\dot{P} = \alpha'$.*[10] *It is **Sensitive to risk aversion** if it is k-Sensitive to risk aversion for all k with a nonzero commodity coordinate.*

Figure 1

[10] The appendix spells out a path of risk aversion. Here the dot denotes differentiation with respect to the path's parameter.

Assumption 2.1 (Generic Sensitivity of dP). *If $H > 1$, then generically in endowments and utilities, at every GEI dP is Sensitive to risk aversion.*

Figures 2, 3

This assumption banishes the particulars of the tax policy, leaving only its imprint on dP. Of course, dP is defined only at regular GEI, so implicitly assumed is that regular GEI are generic in endowments.

Theorem 2.1 (Logic of Pareto Improvement). *Fix the tax policy and the desired welfare impact $\dot{v} \in R^H$. Grant the Generic Sensitivity of dP under $\dim(T), S - J \geq H > 1, C > 1$. Then generically in utilities and endowments, at every GEI $\dot{v}$ is the welfare impact of some $\dot{t} \in T$. Hence a nearby Pareto superior GEIT exists.*

Proof. We fix generic endowments, utilities from the fact, assumption, and apply transversality to

1	nonnumeraire excess demand equations
2	$\gamma'(\lambda)'\left((t^h_*\tau - \tau^h)_h - \underline{\overline{z}}dP\right) = 0$
3	$r - \tau t = 0$
4	$\gamma'\gamma - 1 = 0$

Suppose this is transverse to zero and the natural projection is proper. By the transversality theorem, for generic endowments and utilities, this system of $(\dim p + \dim q) + \dim(T) + \dim r + 1$ equations is transverse to zero in the remaining endogenous variables, which number $\dim p + \dim q + \dim r + H$. By hypothesis $\dim(T) \geq H$, so for these endowments and utilities the preimage theorem implies that no endogenous variables solve this system– every GEI has dv with rank H.

This is transverse to zero. As is well known, we can control the first equations by perturbing one household's endowment. For a moment, say that we can control the second equations and preserve the top ones. We then perturb the third equations and preserve the top two, by perturbing r as well as numeraire endowments–to preserve incomes $w_s^h = e_s^{h\prime}\overline{p}_s + t_*^h r_s$. We control the fourth equation and preserve the top three, by scalar multiples of γ. So transversality obtains if our momentary supposition on $\gamma' dv$ holds:

Write $k' \equiv \gamma'(\lambda)'\underline{z}$. Differentiating $\gamma' dv$ with respect to the parameter of a path of risk aversion,

$$\alpha' =_{def} \frac{d}{d\xi}\gamma'(\lambda)'\left((t_*^h\tau - \tau^h)_h - \underline{z}dP\right) = -\gamma'(\lambda)'\underline{z}\frac{d}{d\xi}(dP) = -k'd\dot{P}$$

since λ, τ^i, z are locally constant. We want to make α arbitrary, and we can if dP is k-sensitive, which holds by assumption if k has a nonzero commodity coordinate. It has: Full Externality of Price Adjustment on Welfare, $C > 1, \gamma \neq 0$ imply $\gamma'(\lambda)'\underline{z}$ is nonzero in the coordinate $m = s1$ for some $s \leq H - 1$.

That the natural projection is proper we omit. (The numeraire asset structure is fixed.)

□

We have seen that tax policy targeting current incomes, such as taxes on asset purchases, on net purchases of current commodities, or lump-sum taxes on current incomes, supports a Pareto improvement only if there is a price adjustment. Conversely, tax policy generically supports a Pareto improvement if the price adjustment is sufficiently sensitive to risk aversion. Therefore price adjustment is pivotal.

2.4.1 Expression for Price Adjustment

Before we can check whether a particular policy meets the Sensitivity of dP to Risk Aversion, we need an expression for dP. We express dP in terms of the Reaction of Demand to Prices and the Reaction of Demand to Policy, notions which are well defined at an active GEI.

Let an underbar connote the omission of the numeraire in each state, define

$$d : B' \to R_{++}^{(C-1)(S+1)} \times R^J \qquad d = \Sigma \underline{d}^h$$

and the **aggregate demand** of $(a, e, t, t_*) \in \Omega$

$$d_{a,e,t,t_*}(p, q, r) \equiv d(p, q, a, (w_s^h = e_s^{h\prime}\overline{p}_s + t_*^h r)_s^h, t)$$

with domain $\mathbf{P}_{a,e,t,t_*} \equiv \{(p, q, r) \in \mathbf{P} \times R^{S+1} \mid (p, q, a, (w_s^h = e_s^{h\prime}\overline{p}_s + t_*^h r_s)_s^h, t) \in B'\}$.[11]

Now define

$$\begin{aligned} \nabla &\equiv D_{p,q} d_{a,e,t,t_*} && \text{the } \textbf{Reaction of Demand to Prices} \\ \Delta &\equiv D_r d_{a,e,t,t_*} \cdot \tau + D_t d_{a,e,t,t_*} && \text{the } \textbf{Reaction of Demand to Policy}^{12} \end{aligned} \tag{2.1}$$

Suppose a path of GEIT $(P(\xi), r(\xi), a, (e_s^{h\prime}\overline{p}_s(\xi) + t_*^h r_s(\xi))_s^h, t(\xi))$ through an active GEI. Then

$$d_{a,e,t,t_*}(P, r) = \begin{bmatrix} \sum \underline{e}^h \\ 0 \end{bmatrix}$$

---

[11] P_{a,e,t,t_*} is open, as the preimage by a continuous function of the open B'. Recall the notation $P' = (p', q')$.

[12] Clearly $D_r d_{a,e,t,t_*} = \Sigma D_{w^h} \underline{d}^h t_*^h$.

is an identity in the path's parameter ξ. Differentiating with respect to it,

$$\nabla \dot{P} + D_r d_{a,e,t,t_*} \cdot \dot{r} + D_t d_{a,e,t,t_*} \cdot \dot{t} = 0$$

Substituting for $\dot{r} = \tau \dot{t}$ from the Revenue Impact proposition,

$$\nabla \dot{P} + \Delta \dot{t} = 0$$

An active GEI is **regular** if ∇ is invertible. By the implicit function theorem, a regular GEI lifts a local policy through $t = 0$ to a path of GEIT through itself, such as the one just above.

Proposition 2.3 (Price Adjustment). *At a regular GEI the Price Adjustment to infinitesimal taxes is*

$$dP = -\nabla^{-1} \Delta \tag{dP}$$

where the Reactions ∇, Δ *are defined in (2.1).*

2.4.2 Primitives for the Sensitivity of Price Adjustment to Risk Aversion

Given the Logic of Pareto improvement, we want to check whether a policy meets the Generic Sensitivity of dP. We provide primitives for the Sensitivity of dP, thanks to expression (dP)[13]:

$$d\dot{P} = -\nabla^{-1} \dot{\Delta} + \nabla^{-1} \dot{\nabla} \nabla^{-1} \Delta$$

[13] Applying the chain rule to $JJ^{-1} = I$ gives $\frac{d}{d\xi} J^{-1} = -J^{-1} (\frac{d}{d\xi} J) J^{-1}$.

Recall equation $k' d\dot{P} = \alpha'$ from definition 2.1. If $\dot{\Delta} = 0$ and $\tilde{k}' \equiv_{def} k'\nabla^{-1}$ then the equation reads $\tilde{k}'\dot{\nabla}\nabla^{-1}\Delta = \alpha'$. If Δ has rank $\dim(T)$ then there is a solution β to $\beta'\nabla^{-1}\Delta = \alpha'$ so it suffices to solve $\tilde{k}'\dot{\nabla} = \beta'$. Thus dP is k-Sensitive if (1) Δ has rank $\dim(T)$, (2) $\tilde{k}$ is nonzero everywhere, (3) whenever $\tilde{K}$ is nonzero everywhere and $\beta \in R^{(S+1)(C-1)+J}$, there is a path of risk aversion that solves $\dot{\Delta} = 0, \tilde{K}'\dot{\nabla} = \beta'$. (Take $\tilde{k} = \tilde{K}$.) Thus Generic Sensitivity of dP follows from the following (independently of the $\tilde{k}$ defined):

Lemma 2.1 (Activity). *If $H > 1$, generically in endowments every GEI is active and regular.*[14]

Assumption 2.2 (Full Reaction of Demand to Policy). *If $C > 1$, generically in utilities and endowments, at every GEI Δ has rank* $\dim(T)$.

Lemma 2.2 (Mean Externality of Price Adjustment on Welfare is Regular). *Generically in utilities, at every regular GEI, whenever k is nonzero in some commodity coordinate, $\tilde{k}' \equiv k'\nabla^{-1}$ is nonzero everywhere.*

Assumption 2.3 (Sufficient Independence of Reactions). *If $H > 1$, then generically in endowments and utilities, whenever $\tilde{k} \in R^{(S+1)(C-1)+J}$ is nonzero everywhere and $\beta \in R^{(S+1)(C-1)+J}$, at every GEI there is a path of risk aversion that solves* $\dot{\Delta} = 0, \tilde{k}'\dot{\nabla} = \beta'$.

These primitives for the Generic Sensitivity of dP and the Logic of Pareto Improvement yield

[14] We do not argue this relatively simple statement. For these endowments, both Δ and dP are defined.

Theorem 2.2 (Test for Pareto Improvement). *Fix the tax policy and the desired welfare impact* $\dot{v} \in R^H$*. Say the policy passes the Full Reaction of Demand to Policy and the Sufficient Independence of Reactions under* $\dim(T), S - J \geq H > 1, C > 1$*. Then generically in utilities and endowments, at every GEI* $\dot{v}$ *is the welfare impact of some* $\dot{t} \in T$*. Hence there is a nearby Pareto superior GEIT.*

Next we illustrate how to check whether a tax policy passes this test via demand theory in incomplete markets, as developed by Turner (2003a). We show that the four tax policies in the introduction pass this test, and therefore generically admit Pareto improving taxes, owing to the unifying logic of a sensitive price adjustment. At a GEI ∇ will turn out to be independent of the policy, so we will verify the lemma on the Mean for one and all policies.

2.5 Summary of demand theory in incomplete markets

We must check whether each policy meets the Full Reaction of Demand to Policy and the Sufficient Independence of Reactions. For this we report the theory of demand in incomplete markets as developed by Turner (2003a). The basic idea is to use decompositions of Δ, ∇ in terms of Slutsky matrices, and then to perturb these Slutsky matrices by perturbing risk aversion, while preserving neoclassical demand at the budget variables under consideration. We stress that this theory is applied to, but independent of, equilibrium.

2.5.1 Slutsky perturbations

Define $H : R^{C^* \times C^*} \to R^{C^*+J+(S+1) \times C^*+J+(S+1)}$ as

$$H(D) = \begin{bmatrix} D & 0 & -[\overline{p}] \\ 0 & 0 & W \\ -[\overline{p}] & W' & 0 \end{bmatrix}$$

where $p, W = [-q : a] \in R^{J \times S+1}$ of rank J are given, and $C^* = C(S+1)$. In other notation,

$$H(D) = \begin{bmatrix} M(D) & -\rho \\ -\rho' & 0 \end{bmatrix} \qquad \text{where } M(D) = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \rho = \begin{bmatrix} [\overline{p}] \\ -W \end{bmatrix}$$

In showing the differentiability of demand, the key step is the invertibility of $H(D^2u)$. **Slutsky matrices** are $H(D^2u)^{-1}$. If D is symmetric, so are $H(D), H(D)^{-1}$ when defined. Thus we write

$$H(D)^{-1} = \begin{bmatrix} S & -m \\ -m' & -c \end{bmatrix}$$

where S, c are symmetric of dimensions $C^*+J, S+1$ and $m = (m_x, m_y)$ is $C^*+J \times S+1$. A **Slutsky perturbation** is $\nabla = H(D)^{-1} - H(D^2u)^{-1}$, for some symmetric $D \approx D^2u$ that is close enough for the inverse to exist. A Slutsky perturbation is a perturbation of Slutsky matrices rationalizable by some perturbation of the Hessian of utility. Being symmetric, we write

$$\nabla = \begin{bmatrix} \dot{S} & -\dot{m} \\ -\dot{m}' & -\dot{c} \end{bmatrix}$$

and view a Slutsky perturbation as a triple $\dot{S}, \dot{m}, \dot{c}$. We identify Slutsky perturbations, without reference to the inversion defining them, in terms of independent linear constraints on ∇ :

on $\dot{S}$	$\rho'\dot{S} = 0$ and $\dot{S}$ is symmetric	
on $\dot{m}$	$\rho'\dot{m} = 0$ and $\dot{m}_x W' = 0$	(constraints)
on $\dot{c}$	$\dot{c}W' = 0$ and $\dot{c}$ is symmetric	

Theorem 2.3 (Identification of Slutsky perturbations, Turner 2003a). *Given u smooth in Debreu's sense and b in B' with $t = 0$, consider the Slutsky matrices $H(D^2u)^{-1}$. Every small enough Slutsky perturbation ∇ satisfies (constraints). Conversely, every small enough perturbation ∇ that satisfies (constraints) is Slutsky: $H(D^2u)^{-1} + \nabla$ is the inverse of $H(D)$ for some D that is negative definite and symmetric.*

We use only Slutsky perturbations with $\dot{m}, \dot{c} = 0$ by choosing $\dot{S}$ as follows. A matrix $\underline{\dot{S}} \in R^{(C-1)(S+1)+J \times (C-1)(S+1)+J}$ is extendable in a unique way to a matrix $\dot{S} \in R^{C^*+J \times C^*+J}$ satisfying $\rho'\dot{S} = 0$; we call $\dot{S}$ the **extension** of $\underline{\dot{S}}$. It is easy to verify that if $\underline{\dot{S}}$ is symmetric, so is its extension. In sum, any symmetric $\underline{\dot{S}}$ defines a unique Slutsky perturbation with $\dot{m}, \dot{c} = 0$.

2.5.2 Decomposition of demand

The relevance of Slutsky perturbations is that they allow us to perturb demand functions directly, while preserving their neoclassical nature, without having to think about utility. This is because Slutsky matrices appear in the **decomposition** of demand $D_{p,q}\underline{d}$ at b with $t = 0$:

$$D_{p,q}\underline{d}^h = \underline{S}^h L_+^h - \underline{m}^h \cdot ([\underline{x}^h]' : \overline{y}_0^h) \qquad \text{(dec)}$$

Here L^h_+ a diagonal matrix displaying the marginal utility of contingent income

$$L^h_+ \equiv \begin{bmatrix} L^h & 0 \\ 0 & \lambda^h_0 I_J \end{bmatrix} \qquad L^h \equiv \begin{bmatrix} \cdot & & 0 \\ & \lambda^h_s I_{C-1} & \\ 0 & & \cdot \end{bmatrix}$$

$m^h = D_{w^h} d^h$, and $([\underline{x}^h]' : \overline{y}^h_0)$ is the transpose of $\underline{d}^h$:[15]

$$[\underline{x}^h]' = \begin{bmatrix} \cdot & 0 & 0 \\ 0 & \underline{x}^{h\prime}_s & 0 \\ 0 & 0 & \cdot \end{bmatrix}_{(S+1)\times(C-1)(S+1)} \qquad \overline{y}^h_0 = \begin{bmatrix} y^{h\prime} \\ 0 \end{bmatrix}_{S+1\times J}$$

Writing $(e^{h\prime}_s \overline{p}_s)_s$ as $[e^h]'\overline{p}$, we have $D_{p,q}[e^h]'\overline{p} = ([\underline{e}^h]' : 0)$, so from (2.1) we have

$$\nabla = \Sigma D_{p,q}\underline{d}^h + D_{w^h}\underline{d}^h \cdot ([\underline{e}^h]' : 0)$$

Inserting decomposition (dec),

$$\nabla = \Sigma \underline{S}^h L^h_+ - D_{w^h}\underline{d}^h \cdot ([\underline{x}^h - \underline{e}^h]' : \overline{y}^h_0)$$

Writing $\underline{z}^{h\prime} \equiv ([\underline{x}^h - \underline{e}^h]' : \overline{y}^h_0)$ this reads

$$\boxed{\nabla = \Sigma \underline{S}^h L^h_+ - D_{w^h}\underline{d}^h \cdot \underline{z}^{h\prime}} \qquad (\nabla)$$

This **decomposition** of the aggregate demand of $(a, e, t, t_*) \in \Omega$ generalizes Balasko 3.5.1 (1988) to incomplete markets.

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[15] We view p as one long vector, state by state, and p, q as an even longer one; $(* : \#)$ denotes concatenation of $*, \#$.

One implication of the decomposition is that ∇ is independent of the policy. So let us now provide

Proof that Mean Externality of Price Adjustment on Welfare is Regular. Consider the manifold of regular GEI and a k that is nonzero in some commodity coordinate. Fix a coordinate $n \leq (S+1)(C-1)+J$ and apply transversality to

$$\begin{gathered}\text{nonnumeraire excess demand equations}\\ (k'\nabla^{-1})_n = 0\end{gathered}$$

This is transverse to zero. The burden of the argument is to control the bottom equation independently of the top ones. Consider a Slutsky perturbation with $\dot{m}^1, \dot{c}^1 = 0$ and $\underline{\dot{S}}^1$ symmetric. Then with $\tilde{k}' \equiv k'\nabla^{-1}$

$$\frac{d}{d\xi}(k'\nabla^{-1})_n = -(k'\nabla^{-1}\dot{\nabla}\nabla^{-1})_n = -(\tilde{k}'\dot{\nabla}\nabla^{-1})_n$$

Since ∇^{-1} is invertible, there is α such that $\alpha'\nabla^{-1}$ is the n^{th} basis vector, so it suffices to solve $\tilde{k}'\dot{\nabla} = \alpha'$. From decomposition ($\nabla$) $\dot{\nabla} = \underline{\dot{S}}^1 L^1_+$, so we want to solve $\tilde{k}'\underline{\dot{S}}^1 L^1_+ = \alpha'$ or $\tilde{k}'\underline{\dot{S}}^1 = \alpha'(L^1_+)^{-1} \equiv \beta'$ for symmetric $\underline{\dot{S}}^1$. Since $\tilde{k} \neq 0$, say $\tilde{k}_p \neq 0$. Let column $o \neq p$ of $\underline{\dot{S}}^1$ be $1_p \frac{\beta_o}{\tilde{k}_p}$ so that $(\tilde{k}'\underline{\dot{S}}^1)_o = \beta_o$. To preserve symmetry, let column p of $\underline{\dot{S}}^1$ be $\frac{\beta_o}{\tilde{k}_p}$ in coordinate $o \neq p$ and arbitrary x in coordinate p, so that $(\tilde{k}'\underline{\dot{S}}^1)_{\cdot p} = \Sigma_{o\neq p}\tilde{k}_o \frac{\beta_o}{\tilde{k}_p} + \tilde{k}_p x$. We can set this to β_p and solve for x since $\tilde{k}_p \neq 0$.

By the transversality theorem, for generic utilities in Debreu's setting, the system of $\dim p + \dim q + 1$ equations is transverse in the remaining $\dim p + \dim q$ variables. By the preimage theorem, for these generic utilities every regular GEI with nonzero k has $\tilde{k}_n \neq 0$. Taking the intersection over the finitely many coordinates n, for generic utilities every regular GEI with nonzero k has $\tilde{k}$ nonzero everywhere. □

2.6 Four policies generically admitting Pareto improving taxes

We check for each policy the Full Reaction of Demand to Policy and the Sufficient Independence of Reactions. In computing

$$\Delta = D_t d_{a,e,t,t_*} + (\Sigma D_{w^h} \underline{d}^h t_*^h) \cdot \tau$$

we use the following notation for $\underline{S}^h$, where A^h, B^h are symmetric of dimensions $(C - 1)(S+1), J$:

$$\underline{S}^h = [\underline{S}_p^h : \underline{S}_q^h] = \begin{bmatrix} A^h & P^h \\ P^{h\prime} & B^h \end{bmatrix} \qquad (\underline{S}^h)$$

We can perturb P^h arbitrarily and get a Slutsky perturbation.

Remark 2.1. *In checking the Sufficient Independence of Reactions, the $\dot{\underline{S}}^h$ Slutsky perturbations affect only the Jacobian $\dot{\nabla} = \Sigma \dot{S}^h L_+^h$ in (∇). Also, we solve $\tilde{k}'\dot{\nabla} = \beta'$ piecemeal, solving $\tilde{k}'\dot{\nabla}_p = \beta_p', \tilde{k}'\dot{\nabla}_q = \beta_q'$ by splitting $\beta' = (\beta_p', \beta_q')$, $\dot{\nabla} = [\dot{\nabla}_p : \dot{\nabla}_q]$.*

2.6.1 Tax rates on asset purchases

Corollary 2.1 (Citanna-Polemarchakis-Tirelli 2001). *Fix the desired welfare impact $\dot{v} \in R^H$. Assume $J, S - J \geq H > 1, C > 1$. Then generically in utilities and endowments, at every GEI $\dot{v}$ is the welfare impact of some $\dot{t} \in T$. Hence there is a nearby Pareto superior GEIT with tax rates on asset purchases.*

Proof. The next lemmas, $\dim(T) = J$, and the hypothesis $J \geq H$ enable theorem 2.2. □

The introduction of tax rates on asset purchases amounts to a household specific change in asset prices. The price of asset j changes for household h exactly when $y_j^h > 0$. So $D_t d_{a,e,t,t_*} = \Sigma D_q \underline{d}^h I^h$ where $I^h \in R^{J \times J}$ is a diagonal matrix with entry jj equal to one or zero according as $y_j^h > 0$ or not. Specializing to asset prices, (dec) reads

$$D_q \underline{d}^h = \begin{bmatrix} P^h \\ B^h \end{bmatrix} - D_{w^h} \underline{d}^h \cdot \overline{y}_0^h$$

so that

$$\Delta^q = \Sigma \left(\begin{bmatrix} P^h \\ B^h \end{bmatrix} - D_{w^h} \underline{d}^h \cdot \overline{y}_0^h \right) I^h + D_{w^h} \underline{d}^h t_*^h \cdot \tau \qquad (\Delta^q)$$

Lemma 2.3 (Full Reaction of Demand to Policy). *If* $C > 1$, *generically in utilities and endowments, at every GEI* Δ^q *has rank* $\dim(T)$.

Proof. Fix generic endowments from the Activity lemma and apply transversality to

$$\text{nonnumeraire excess demand equations}$$
$$\hat{\Delta}^q \phi = 0$$
$$\phi' \phi - 1 = 0$$

where the hat omits the last J rows of Δ^q. This is transverse to zero. The burden of the argument is to control the middle equations independently of the top and bottom ones. We perturb only the P^h, so that $\frac{d}{d\xi}\hat{\Delta}^q = \Sigma \dot{P}^h I^h$. Say $\phi_j \neq 0$; we make column j of $\frac{d}{d\xi}\hat{\Delta}^q$ arbitrary and preserve the others. The GEI is active and asset markets clear, so fix h with $y_j^h > 0$; the j^{th} column of $P^h I^h = j^{th}$ column of P^h. So let $\dot{P}^h$ be $(\frac{a_k}{\phi_j})_k$ in column j and zero in the others, and $\dot{P}^{i \neq h} = 0$. Then $(\Sigma \dot{P}^h I^h)\phi = a$ is arbitrary.

By the transversality theorem, generically in endowments and utilities the system of

$\dim p + \dim q + (S+1)(C-1) + 1$ equations is transverse in the remaining $\dim p + \dim q + J$ variables. By the preimage theorem, for these every GEI is active and has $\hat{\Delta}^q$ (a fortiori Δ^q) with linearly independent columns. □

Lemma 2.4 (Sufficient Independence of Reactions). *If $H > 1$, then generically in endowments and utilities, whenever $\tilde{k} \in R^{(S+1)(C-1)+J}$ is nonzero in some commodity coordinate and $\beta \in R^{(S+1)(C-1)+J}$, at every GEI there is a path of risk aversion that solves $\dot{\Delta} = 0, \tilde{k}'\dot{\nabla} = \beta'$.*

Proof. Fix generic endowments from the Activity lemma, a GEI with $\tilde{k}_m \neq 0$ for some commodity coordinate m, and follow remark 2.1. For each asset j fix $h(j)$ with $y_j^{h(j)} < 0$, let $\dot{P}^{h(j)}$ be $1_m \frac{\beta_{(S+1)(C-1)+j}}{\tilde{k}_m}$ in column j and zero in the others, and all $\dot{B}^h = 0$. This keeps $\dot{\Delta} = 0$ and equates $\tilde{k}'\dot{\nabla}$ to $\beta_{(S+1)(C-1)+j}$ in coordinate $(S+1)(C-1)+j$. Having dealt with all asset coordinates $j \leq J$ via the $\dot{P}^h$, we turn to the commodity coordinates $n \leq (S+1)(C-1)$. Let $\gamma' = \Sigma \dot{P}^{h\prime} L^h$. From display ($\underline{S}^h$) it suffices to choose symmetric $\dot{A}^1$ such that $\tilde{k}_p'\dot{A}^1 L^1 + \tilde{k}_q'\gamma' = \beta_p'$ or $\tilde{k}_p'\dot{A}^1 = (\beta_p' - \tilde{k}_q'\gamma')(L^1)^{-1} \equiv \alpha'$. Let column $n \neq m$ of $\dot{A}^1$ be $1_m \frac{\alpha_n}{\tilde{k}_m}$ so that $\tilde{k}_p'\dot{A}^1$ equals α_n in coordinate n. To preserve symmetry, column m of $\dot{A}^1$ must be $\frac{\alpha_n}{\tilde{k}_m}$ in row $n \neq m$ and arbitrary x in row m. Then $\tilde{k}_p'\dot{A}^1$ equals $\Sigma_{n \neq m} \tilde{k}_n \frac{\alpha_n}{\tilde{k}_m} + \tilde{k}_m x$ in coordinate m, which we can equate to α_m by solving for x. Having dealt with all coordinates n, this symmetric $\dot{A}^1$ solves $\tilde{k}'\dot{\nabla} = \beta'$. Since A^1 does not appear in Δ, still $\dot{\Delta} = 0$. □

2.6.2 Lump-sum taxes on current income plus flat tax rate on asset purchases

Corollary 2.2. *Fix the desired welfare impact* $\dot{v} \in R^H$. *Assume* $S - J \geq H > 1, C > 1, J > 0$. *Then generically in utilities and endowments, at every GEI* $\dot{v}$ *is the welfare impact of some* $\dot{t} \in T$. *Hence there is a nearby Pareto superior GEIT with lump-sum taxes on current income plus flat tax rate on asset purchases.*

Proof. The next lemmas and $\dim(T) = H + 1$ enable theorem 2.2. □

The part of $D_t d_{a,e,t,t_*}$ relating to the lump-sum taxes $l \in R^H$ on current income is $-\Sigma D_{w_0^h} \underline{d}^h 1^{h\prime}$, and that relating to the flat tax rate f on asset purchases is $\Delta^q 1$ where $1 \in R^J$. Concatenating,

$$D_t d_{a,e,t,t_*} = \left[-\Sigma D_{w_0^h} \underline{d}^h 1^{h\prime} : \Delta^q 1 \right]$$

Since the first row of τ is $[1' : 1'\Sigma y_+^i]$,

$$(\Sigma D_{w^h} \underline{d}^h t_*^h) \cdot \tau = (\Sigma D_{w_0^h} \underline{d}^h t_*^h) \cdot [1' : 1'\Sigma y_+^i]$$

So

$$\Delta^w = \left[\Sigma D_{w_0^h} \underline{d}^h (t_*^h 1 - 1^h)' : \Delta^q 1 + (\Sigma D_{w_0^h} \underline{d}^h t_*^h)(1'\Sigma y_+^i) \right]$$

For convenience, we reexpress the lump-sum part $\Sigma D_{w_0^h} \underline{d}^h (t_*^h 1 - 1^h)' \dot{l}^h = \Sigma_{h \neq H} \nabla^h (t_*^h 1 - 1^h)' \dot{l}^h$ with $\nabla^h \equiv D_{w_0^h} \underline{d}^h - D_{w_0^H} \underline{d}^H$, to think of only $H - 1$ parameters $(l^h)^{h \neq H}$. Now $\dim(T) = H$ and

$$\Delta^w = \left[\Sigma_{h \neq H} \nabla^h (t_*^h 1 - 1^h)' : \Delta^q 1 + (\Sigma D_{w_0^h} \underline{d}^h t_*^h)(1'\Sigma y_+^i) \right] \qquad (\Delta^w)$$

Lemma 2.5 (Full Reaction of Demand to Policy). *If* $(S+1)(C-1) \geq H > 1$, *generically in utilities and endowments, at every GEI* Δ^w *has rank* $\dim(T)$.

Proof. Fix generic endowments from the Activity lemma and apply transversality to

nonnumeraire excess demand equations

$$\hat{\Delta}^w \phi = 0$$

$$\phi' \phi - 1 = 0$$

where the hat omits the last J rows. This is transverse to zero. The burden of the argument is to control the middle equations independently of the top and bottom ones. Write $\phi = (l, f)$ so that $l^i \neq 0$ for some $i \neq H$ or $f \neq 0$.

If $f \neq 0$, then we want $\frac{d}{d\xi}\hat{\Delta}^q 1 = a$ arbitrary but $\frac{d}{d\xi}\hat{\nabla}^h = 0$, where $\hat{\nabla}^h \equiv D_{w_0^h}\underline{x}^h - D_{w_0^H}\underline{x}^H$, and we can by choosing some h with $y_1^h > 0$ and setting $\dot{P}^h$ to be $\frac{a}{f}$ with $a \in R^{(C-1)(S+1)}$ in column 1 and zero in the others, so that $\frac{d}{d\xi}\hat{\Delta}^q 1 = \dot{P}^h I^h 1 = \frac{a}{f}$ and $\frac{d}{d\xi}(\hat{\Delta}^w \phi) = (\frac{d}{d\xi}\hat{\Delta}^q 1) f = a$.

If $l^i \neq 0$, then we want the i^{th} column of $\Sigma_{h \neq H} \hat{\nabla}^h (t_*^h 1 - 1^h)'$ arbitrary:

$$* = \frac{d}{d\xi} \Sigma_{h \neq H} \hat{\nabla}^h (t_*^h 1 - 1^h)' 1^i = \frac{d}{d\xi}\left[(\Sigma_{h \neq H} \hat{\nabla}^h t_*^h) - \hat{\nabla}^i\right] = \frac{d}{d\xi}\left[(\Sigma_{h \neq i, H} \hat{\nabla}^h t_*^h) - (1 - t_*^i)\hat{\nabla}^i\right] = a$$

but $\diamond = \frac{d}{d\xi}\left(\hat{\Delta}^q 1 + (\Sigma D_{w_0^h}\underline{x}^h t_*^h)(1'\Sigma y_+^i)\right) = 0$. From the identification of Slutsky perturbations, we set $\frac{d}{d\xi} D_{w_0^h}\underline{x}^h = 0$ for all $h \neq i$, and $D_{w_0^i}\underline{x}^i = \frac{a}{1 - t_*^i}$ by setting $\frac{d}{d\xi} D_{w^i}\underline{x}^i = a\frac{\lambda^{i\prime}}{\lambda_0^i}$–a Slutsky perturbation since $\lambda^{i\prime} W' = 0$ from the FOC–so that $* = -(1 - t_*^i)\frac{d}{d\xi} D_{w_0^i}\underline{x}^i = -a$. Any effect on $\diamond$ we can undo, since as just seen we can make $\frac{d}{d\xi}\hat{\Delta}^q 1$ arbitrary while preserving the $D_{w_0^h}\underline{x}^h$.

By the transversality theorem, generically in endowments and utilities the system of

$\dim p + \dim q + (S+1)(C-1)+1$ equations is transverse in the remaining $\dim p + \dim q + H$ variables. By the preimage theorem, for these every GEI is active and has $\hat{\Delta}^w$ (a fortiori Δ^w) with linearly columns. □

Lemma 2.6 (Sufficient Independence of Reactions). *If* $H > 1$, *then generically in endowments and utilities, whenever* $\tilde{k} \in R^{(S+1)(C-1)+J}$ *is nonzero in some commodity coordinate and* $\beta \in R^{(S+1)(C-1)+J}$, *at every GEI there is a path of risk aversion that solves* $\dot{\Delta} = 0, \tilde{k}'\dot{\nabla} = \beta'$.

Proof. The proof of the lemma for Δ^q applies verbatim. □

2.6.3 Asset measurable tax rates on future capital gains

Corollary 2.3. *Fix the desired welfare impact* $\dot{v} \in R^H$. *Assume* $J, S-J \geq H > 1, C > 1$. *Then generically in utilities and endowments, at every GEI* $\dot{v}$ *is the welfare impact of some* $\dot{t} \in T^p$. *Hence there is a nearby Pareto superior GEIT with asset measurable tax rates on future capital gains.*

Proof. The next lemmas, $\dim(T) = J$, and the hypothesis $J \geq H$ enable theorem 2.2. □

Capital gain is $g_s^h = (\overline{p}_s' x_s^h - w_s^h)_+$. State contingent taxes are **asset measurable** if they are a linear function of asset payoffs, $t = a'L$ for some L. The introduction of tax rates on capital gains amounts to a household specific proportional change in commodity prices. The prices of state s commodities change in the same proportion exactly when $\overline{p}_s' x_s^h - w_s^h > 0$, i.e. $p_{\mathbf{1}}^h = [p_{\mathbf{1}}](I + [t^h])$.[16] So $D_t d_{a,e,t,t_*} = \Sigma D_{p_{\mathbf{1}}} \underline{d}^h [p_{\mathbf{1}}] I^h$ where $I^h \in R^{S \times S}$

[16] For $\phi_s = \overline{p}_s' x_s - w_s^h + g_s t - a_s' y$. If $g_s = 0$ then ϕ_s reduces to the GEI ϕ_s. If $g_s \neq 0$ then $g_s = \overline{p}_s' x_s - w_s^h$ so $\phi_s = (1+t_s)(\overline{p}_s' x_s - w_s^h) - a_s' y$. At a GEI $w_s^h = \overline{p}_s' e_s^h + t_*^h r_s$ with $r = 0$, so that $\phi_s = (1+t_s)\overline{p}_s'(x_s - e_s^h) - a_s' y$, as if now prices $p_s(t) = (1+t_s)p_s$. In sum, for every $s \geq 1$ $\phi_s(t) = \overline{p}_s(t)'(x_s - e_s^h) - a_s' y$ with $p_s(t) = (1+t_s^h)p_s$ with $t_s^h = t_s, 0$ according as $\overline{p}_s' x_s^h - w_s^h > 0$ or not.

is a diagonal matrix with entry ss equal to one or zero according as $\overline{p}'_s x^h_s - w^h_s > 0$ or not. Specializing to period 1 commodity prices, (dec) reads

$$D_{p_1}\underline{d}^h = \begin{bmatrix} A^h_{\mathbf{1}} \\ P'^{\,h}_{\mathbf{1}} \end{bmatrix} - D_{w^h_1}\underline{d}^h \cdot [\underline{x}^h_{\mathbf{1}}]'$$

so that with the parameterization $\dot{t} = a'\dot{L}$

$$\Delta^p = \left\{ \Sigma \left(\begin{bmatrix} A^h_{\mathbf{1}} \\ P'^{\,h}_{\mathbf{1}} \end{bmatrix} - D_{w^h_1}\underline{d}^h \cdot [\underline{x}^h_{\mathbf{1}}]' \right) [p_{\mathbf{1}}] I^h + D_{w^h}\underline{d}^h t^h_* \cdot \tau \right\} a' \qquad (\Delta^p)$$

Note that at an active GEI for every s there are h, i with $\overline{p}'_s x^h_s - w^h_s > 0 > \overline{p}'_s x^i_s - w^i_s$. For with $t = 0$ the budget equation is $\overline{p}'_s x^h_s - w^h_s = a'_s y^h$ for all h, so $\Sigma \overline{p}'_s x^h_s - w^h_s = 0$ by asset market clearing.

Lemma 2.7 (Full Reaction of Demand to Policy). *If* $C > 1$, *generically in utilities and endowments, at every GEI* Δ^p *has rank* $\dim(T)$.

Proof. Fix generic endowments from the Activity lemma and apply transversality to

nonnumeraire excess demand equations

$$\{\hat{\cdot}\}\, \phi = 0$$

$$\phi'\phi - 1 = 0$$

where the hat selects the $(s1)_{s \geq 1}$ rows in the bracketed matrix $\{\}$, omitting the $(sc)_{s \leq S, c \neq 1}$ and asset rows. This is transverse to zero. The burden of the argument is to control the middle equations independently of the top and bottom ones. We perturb only the $(A^h_{\mathbf{1}})_{s1} \in R^{S(C-1)}$, so that $(\frac{d}{d\xi}\{\hat{\cdot}\})_s = \Sigma(\dot{A}^h_{\mathbf{1}})_{s1}[p_{\mathbf{1}}]I^h$. Say $\phi_s \neq 0$; the GEI is active so fix $h = h(s)$ with $g^h_s > 0$; for it, the s^{th} column of $[p_{\mathbf{1}}]I^h = s^{th}$ column of $[p_{\mathbf{1}}]$. Now let

$(\dot{A}^h_{\mathbf{1}})_{s1}$ be $(...0 : \frac{a_s}{\phi_s p_{s1}} 1'_{c=1} : 0...)$, so that $(\dot{A}^h_{\mathbf{1}})_{s1}[p_{\mathbf{1}}]I^h\phi = a_s$, and $(\dot{A}^h_{\mathbf{1}})_{t1} = 0$ for $t \neq s$, so that $\dot{A}^h_{\mathbf{1}}[p_{\mathbf{1}}]I^h\phi = 1_s a_s$. Note that $\dot{A}^h$ is symmetric. Finding such $h(s)$ for each s, $\Sigma_s \dot{A}^h_{\mathbf{1}}[p_{\mathbf{1}}]I^h\phi = a$ is arbitrary. Thus let $\dot{A}^i = 0$ for those i distinct from every $h(s)$ to get $\frac{d}{d\xi}\{\hat{:}\}\,\phi = a$.

By the transversality theorem, generically in endowments and utilities the system of $\dim p + \dim q + S + 1$ equations is transverse in the remaining $\dim p + \dim q + S$ variables. By the preimage theorem, for these every GEI is active and has $\{\hat{:}\}$ (a fortiori $\{\}$ and $\{\}\,a'$) with linearly independent columns. □

Lemma 2.8 (Sufficient Independence of Reactions). *If $H > 1$, then generically in endowments and utilities, whenever $\tilde{k} \in R^{(S+1)(C-1)+J}$ is nonzero everywhere and $\beta \in R^{(S+1)(C-1)+J}$, at every GEI there is a path of risk aversion that solves $\dot{\Delta} = 0, \tilde{k}'\dot{\nabla} = \beta'$.*

Proof. Consider generic endowments from the Activity lemma, and follow remark 2.1.

To solve $\tilde{k}'_p\dot{\nabla} = \beta'_p$ we set the $\dot{A}^h = 0$ so that we seek $\tilde{k}'_q\Sigma\dot{P}^{h\prime}L^h = \beta'_p$ or $(\Sigma L^h\dot{P}^h)\tilde{k}_q = \beta_p$. For each s there is $i = i(s)$ with capital loss $0 > \overline{p}'_s x^i_s - w^i_s$, so we can fix $\dot{P}^{i(s)}$ in coordinates $(sc, j)_{c,j}$ and still preserve $\dot{\Delta} = 0$; fix $\dot{P}^{i(s)}$ in coordinates $(sc, J)_c$ to be $\frac{\beta_{sc}}{\tilde{k}_J\lambda^i_s}$ and zero in coordinates $(s'c, J)_c$ for $s' \neq s$, and zero in columns $j < J$. Then $L^{i(s)}\dot{P}^{i(s)}\tilde{k}_q$ equals β_{sc} in coordinates $(sc)_c$ and zero in $(s'c)_{s'\neq s,c}$, so $(\Sigma_s L^{i(s)}\dot{P}^{i(s)})\tilde{k}_q = \beta_p$. We let $\dot{P}^h = 0$ for those h distinct from any $i(s)$, so $(\Sigma L^h\dot{P}^h)\tilde{k}_q = \beta_p$. Recall $\dot{\Delta} = 0$ so far.

To solve $\tilde{k}'_q\dot{\nabla} = \beta'_q$, having fixed the $\dot{P}^h$, we want to solve $\tilde{k}'_q\Sigma\dot{B}^h = \beta'_q - \tilde{k}'_p\Sigma\dot{P}^h \equiv \gamma' \in R^J$ with the $\dot{B}^h$ being symmetric. Since the latter do not figure in Δ, such as solution will complete $\tilde{k}'\dot{\nabla} = \beta'$ with $\dot{\Delta} = 0$. Set $\dot{B}^1$ to be diagonal with j^{th} diagonal element

$\frac{\gamma_j}{\tilde{k}_j}$ and the other $\dot{B}^{h\neq 1} = 0$. □

2.6.4 Excise taxes on current commodities

Corollary 2.4. *Fix the desired welfare impact* $\dot{v} \in R^H$. *Assume* $C-1, S-J \geq H > 1$. *Then generically in utilities and endowments, at every GEI* $\dot{v}$ *is the welfare impact of some* $\dot{t} \in T$. *Hence there is a nearby Pareto superior GEIT with tax rates on net purchases of current commodities.*

Proof. The next lemmas, $\dim(T) = C-1$, and the hypothesis $C-1 \geq H$ enable theorem 2.2. □

The introduction of tax rates on net purchases of commodities, given endowments, amounts to a household specific change in commodity prices. The price of commodity $0c$ changes to $p_{0c} + t_c > 0$ exactly when $x^h_{0c} - e^h_{0c} > 0, c < C$. So $D_t d_{a,e,t,t_*} = \Sigma D_{p_0} \underline{d}^h I^h$ where $I^h \in R^{C-1 \times C-1}$ is a diagonal matrix with coordinate cc one or zero according as $x^h_{0c} - e^h_{0c} > 0$ or not. Specializing to period 0 commodity prices, (dec) reads

$$D_{p_0} \underline{d}^h = \begin{bmatrix} A^h_0 \\ P^{h\prime}_0 \end{bmatrix} \lambda^h_0 - D_{w^h_0} \underline{d}^h \cdot \underline{x}^{h\prime}_0$$

so that

$$\Delta^c = \Sigma \left(\begin{bmatrix} A^h_0 \\ P^{h\prime}_0 \end{bmatrix} \lambda^h_0 - D_{w^h_0} \underline{d}^h \cdot \underline{x}^{h\prime}_0 \right) I^h + D_{w^h} \underline{d}^h t^h_* \cdot \tau \qquad (\Delta^p)$$

Lemma 2.9 (Full Reaction of Demand to Policy). *If* $C > 1$, *generically in utilities and endowments, at every GEI* Δ^c *has rank* $\dim(T)$.

Proof. Fix generic endowments from the Activity lemma and apply transversality to

nonnumeraire excess demand equations

$$\hat{\Delta}^p \phi = 0$$

$$\phi' \phi - 1 = 0$$

where the hat selects the $(Sc)_{c<C}$ rows in (Δ^p). This is transverse to zero. The burden of the argument is to control the middle equations independently of the top and bottom ones. We perturb only the $(\dot{A}_0^h)_{Sc,\cdot} \in R^{C-1}$, so that $(\frac{d}{d\xi}\hat{\Delta}^p)_c = \Sigma \lambda_0^h (\dot{A}_0^h)_{Sc,\cdot} I^h$. Say $\phi_c \neq 0$; since the GEI is active fix $h = h(c)$ with $x_{0c}^h - e_{0c}^h > 0$; we set row $(\dot{A}_0^h)_{Sc,\cdot}$ to be $\frac{\alpha_c 1_c'}{\lambda_0^h}$ so that $\lambda_0^h (\dot{A}_0^h)_{Sc,\cdot} I^h \phi = \alpha_c$. To preserve the symmetry of $\dot{A}^h$, we set $(\dot{A}_S^h)_{\cdot,0c}$ to be $\frac{\alpha_c 1_c}{\lambda_0^h}$ but this does not appear in $\hat{\Delta}^p$. Setting $(\dot{A}_0^h)_{Sc',\cdot} = 0$ for rows $c' \neq c$, we get $\lambda_0^h (\dot{A}_0^h) I^h \phi = 1_c \alpha_c$. Doing so for each $c < C, \Sigma_c \lambda_0^{h(c)} (\dot{A}_0^{h(c)}) I^{h(c)} \phi = \alpha$ is arbitrary. Now set $\dot{A}^i = 0$ for those i distinct from all the $h(c)$. Then $\frac{d}{d\xi}\hat{\Delta}^p \phi = \alpha$ is arbitrary with all $\dot{A}^k$ symmetric.

By the transversality theorem, generically in endowments and utilities the system of $\dim p + \dim q + (C-1) + 1$ equations is transverse in the remaining $\dim p + \dim q + (C-1)$ variables. By the preimage theorem, for these every GEI is active and has $\hat{\Delta}^p$ (a fortiori Δ^p) with linearly independent columns. □

Lemma 2.10 (Sufficient Independence of Reactions). *If $H > 1$, then generically in endowments and utilities, whenever $\tilde{k} \in R^{(S+1)(C-1)+J}$ is nonzero everywhere and $\beta \in R^{(S+1)(C-1)+J}$, at every GEI there is a path of risk aversion that solves $\dot{\Delta} = 0, \tilde{k}'\dot{\nabla} = \beta'$.*

Proof. Fix generic endowments from the Activity lemma, a GEI with $\tilde{k}_m \neq 0$ for every coordinate m, and follow remark 2.1. Fix a commodity coordinate $m = sc$. Pick $h(m)$ with $x_m^{h(m)} - e_m^h < 0$, let $\dot{P}^{h(m)}$ be $1_m \frac{\beta_{(S+1)(C-1)+j}}{\tilde{k}_m}$ in column j, so that $\tilde{k}'\dot{\nabla}$ equals

$\beta_{(S+1)(C-1)+j}$ in coordinate $(S+1)(C-1)+j$, for all $j \leq J$. This $\dot{P}^{h(m)}$ keeps $\dot{\Delta}^p = 0$ because $h(m)$ is a net seller in commodity market m. Having dealt with all asset coordinates via the $\dot{P}^h$, we turn to the commodity coordinates $n \leq (S+1)(C-1)$. Let $\gamma' = \Sigma \dot{P}^{h\prime} L^h$. From display $(\underline{S}^h)$ it suffices to choose symmetric $\dot{A}^h$ such that $\tilde{k}_p' \Sigma \dot{A}^h L^h + \tilde{k}_q' \gamma' = \beta_p'$ or $\tilde{k}_p' \Sigma \dot{A}^h L^h = (\beta_p' - \tilde{k}_q' \gamma') \equiv \alpha'$. For column $n = s'c'$ pick $h(n)$ with $x_n^{h(n)} - e_n^h < 0$ and let $\dot{A}^{h(n)}$ be zero everywhere but $\frac{\alpha_n}{\lambda_{s'}^{h(n)} \tilde{k}_n}$ in coordinate nn, and $\dot{A}^{i \neq h(n)}$ be zero in column n, so that $(\tilde{k}_p' \Sigma \dot{A}^h L^h)_n = \alpha_n$ and still $\dot{\Delta}^p = 0$ because $h(n)$ is a net seller in commodity market n. Doing so simultaneously for all n, we get $\tilde{k}_p' \Sigma \dot{A}^h L^h = \alpha'$. This keeps the symmetry of the $\dot{A}^h$ and $\dot{\Delta}^p = 0$. □

2.7 The insurance deficit bound on the rate of improvement

We bound the rate of Pareto improvement by the equilibrium's *insurance deficit*, which vanishes exactly at Pareto optimality. The bound turns out to be the covariance of the insurance deficit with the marginal purchasing power.

Recall that the welfare impact is $\dot{v}^h = \lambda^{h\prime} dm^h$ where dm^h is marginal purchasing power, for some matrices $\Sigma dm^h = 0$. ($dm^h = (t_*^h \tau - \tau^h) - \underline{z}^h dP$.) Converting marginal welfare from utils to the numeraire at time 0, marginal utility becomes $\frac{\lambda^h}{\lambda_0^h}$, which we rewrite as λ^h with $\lambda_0^h = 1$. In this common unit,

$$dW = \frac{1}{H} \Sigma \lambda^{h\prime} dm^h \qquad \text{the } \textbf{mean welfare impact}$$

Every household's marginal utility of future income projects to a common point in the asset span,

$$\lambda_{\mathbf{1}}^h = \delta^h + c \in a^\perp \oplus a$$

by the first order condition, being unique only in its **insurance deficit** δ^h. If the **mean insurance deficit** is $\overline{\delta} = H^{-1}\Sigma\delta^h$, then the GEI's **insurance deficit** is

$$\Delta = [\delta^1 - \overline{\delta} : ... : \delta^H - \overline{\delta}]_{S\times H}$$

Note that the GEI is Pareto optimal exactly when $\Delta = 0$[17]. Computing the mean welfare impact,

$$\begin{aligned}
H \cdot dW &= \Sigma\lambda_0^h dm_0^h + \Sigma\lambda_{\mathbf{1}}^{h\prime} dm_{\mathbf{1}}^h \\
&= \Sigma dm_0^h + \Sigma(\delta^h + c)' dm_{\mathbf{1}}^h \\
&= 0 + \Sigma\delta^{h\prime} dm_{\mathbf{1}}^h + c'\Sigma dm_{\mathbf{1}}^h \\
&= \Sigma\delta^{h\prime} dm_{\mathbf{1}}^h \\
&= \Sigma(\delta^h - \overline{\delta})' dm_{\mathbf{1}}^h \\
&= H \cdot cov(\Delta, dm_{\mathbf{1}})
\end{aligned}$$

since $\Sigma dm^h = 0$. The **rate of Pareto improvement** is the norm of the functional $dW \mid_{dv\geq 0}$.

Remark 2.2. *At a regular GEI, the mean welfare impact equals the covariance across households of the insurance deficit and the marginal purchasing power,* $dW = cov(\Delta, dm_{\mathbf{1}})$*. So the rate of Pareto improvement is bounded above by the norm of this covariance.*

[17] Also, a household's commodity demand is as though asset markets were complete exactly when $\delta^h = 0$.

If the tax policy targets only current income, i.e. $\tau_{\mathbf{1}}^h, \tau_{\mathbf{1}} = 0$, then $dm_{\mathbf{1}}^h = -\underline{z}_{\mathbf{1}}^h dP_{\mathbf{1}}$ and

$$dW = -cov(\Delta, \underline{z}_{\mathbf{1}}) dP_{\mathbf{1}}$$

The sole control is the future price adjustment, since the GEI sets the insurance deficit and net trade. In a nutshell, the mean welfare impact of the sole control is minus the covariance of insurance deficit and net trade.

2.8 Appendix

2.8.1 Derivation of formula for welfare impact

It is standard how Debreu's smooth preferences, linear constraints, and the implicit function theorem imply the smoothness of neoclassical$_0$ demand. In fact, the implicit function theorem implies smoothness of neoclassical demand in a neighborhood $\tilde{b} \approx b \in B$, if neoclassical$_0$ demand is active at $b \in B_0$. It is standard also that the envelope property follows from the value function's local smoothness, which is the case for v^h as the composition of smooth functions:

$$D_b v^h = D_b L(x, y, \lambda^h) \mid_{(x^h, y^h)(b)}$$

where $b = (p, q, a, w^h, t)$ and

$$L(x, y, \lambda^h) \equiv u^h(x) - \lambda^{h\prime} \left([\overline{p}]' x - w^h - \begin{bmatrix} -q' \\ a' \end{bmatrix} y + \tau^h(b_0, x, y) t \right)$$

Thus

$$D_b v^h = -\lambda^{h\prime}\left([\underline{x}^h]' + D_p\tau^h t : \overline{y}_0^h + D_q\tau^h t : * : -I + D_{w^h}\tau^h t : \tau^h\right) \qquad \text{where } \overline{y}_0^h = \begin{bmatrix} y^{h\prime} \\ 0 \end{bmatrix}$$

If $t = 0$

$$D_b v^h = -\lambda^{h\prime}\left([\underline{x}^h]' : \overline{y}_0^h : * : -I : \tau^h\right)$$

So much for demand theory. Recalling regular GEI from the subsection on the Expression for the Price Adjustment, $dP' = (dp', dq')$ exists and

$$\begin{aligned} w^h &= [\overline{p}]'e^h + t_*^h r \Rightarrow \\ dw^h &= [\underline{e}^h]'dp + t_*^h dr \\ &= ([\underline{e}^h]' : 0)dP + t_*^h \tau \end{aligned}$$

using $dr = \tau$ from the Revenue Impact proposition.

Thus the welfare impact at a regular GEI is

$$\begin{aligned} dv^h &= D_b v^h \cdot db \\ &= -\lambda^{h\prime}\left(([\underline{x}^h]' : \overline{y}_0^h) : * : -I : \tau^h\right) \cdot \left(dP : 0 : ([\underline{e}^h]' : 0)dP + t_*^h\tau : I\right) \\ &= -\lambda^{h\prime}\left(([\underline{x}^h]' : \overline{y}_0^h)dP - ([\underline{e}^h]' : 0)dP - t_*^h\tau + \tau^h\right) \\ &= -\lambda^{h\prime}\left(\underline{z}^{h\prime}dP - t_*^h\tau + \tau^h\right) \end{aligned}$$

where $\underline{z}^{h\prime} \equiv ([\underline{x}^h - \underline{e}^h]' : \overline{y}_0^h)$ by definition. In sum,

$$\boxed{dv^h = \lambda^{h\prime}\left((t_*^h\tau - \tau^h) - \underline{z}^h dP\right)}$$

2.8.2 Aggregate notation

We collect marginal utilities of contingent income, and denote stacking by an upperbar

$$(\lambda)' \equiv \begin{bmatrix} \cdot & & 0 \\ & \lambda^{h\prime} & \\ 0 & & \cdot \end{bmatrix}_{H \times H(S+1)} \qquad \underline{\overline{z}} \equiv \begin{bmatrix} \cdot \\ \underline{z}^{h\prime} \\ \cdot \end{bmatrix}_{H(S+1) \times (S+1)(C-1)+J}$$

Thus

$$dv = (\lambda)' \left((t_*^h \tau - \tau^h)_h - \underline{\overline{z}} dP \right)$$

To visualize the bracket notation $[\cdot]$ defined in footnote 7, it staggers state contingent vectors:

$$[p] \equiv \begin{bmatrix} \cdot & & & \\ & p_{s-1} & 0 & \\ & & p_s & \\ & 0 & p_{s+1} & \\ & & & \cdot \end{bmatrix}_{C(S+1) \times S+1}$$

2.8.3 Transversality

A function $F : M \times \Pi \to N$ defines another one $F_\pi : M \to N$ by $F_\pi(m) = F(m, \pi)$. Given a point $0 \in N$ consider the "equilibrium set" $E = F^{-1}(0)$ and the natural projection $E \to \Pi, (m, \pi) \mapsto \pi$. A function is *proper* if it pulls back sequentially compact sets to sequentially compact sets.

Remark 2.3 (Transversality). *Suppose F is a smooth function between finite dimensional smooth manifolds. If 0 is a regular value of F, then it is a regular value of F_π for almost every $\pi \in \Pi$. The set of such π is open if in addition the natural projection is* *proper.*

A subset of Π is **generic** if its complement is closed and has measure zero. Write $C^* = C(S+1)$. Here the set of parameters is

$$\Pi = O \times O' \times (0, \epsilon)$$

where O, O' are an open neighborhoods of zero in $R^{C^*H}, R^{\frac{C^*(C^*+1)}{2}H}$ relating to endowments and symmetric perturbations of the Hessian of utilities. We have in mind a fixed assignment of utilities, which we perturb by $O' \times (0, \epsilon)$. Specifically, given an equilibrium commodity demand $\overline{x}$ by some household and $\Box \in R^{\frac{C^*(C^*+1)}{2}}, \alpha \in (0, \epsilon)$ we define $u_{\Box,\alpha}$ as

$$u_{\Box,\alpha}(x) \equiv u(x) + \frac{\omega_\alpha(\|x - \overline{x}\|)}{2}(x - \overline{x})'\Box(x - \overline{x})$$

where $\omega_\alpha : R \to R$ is a smooth bump function, $\omega_\alpha \mid_{(-\frac{\alpha}{2},\frac{\alpha}{2})} \equiv 1$ and $\omega_\alpha \mid_{R\setminus(-\alpha,\alpha)} \equiv 0$. In a neighborhood $x \approx \overline{x}$ we have

$$\begin{aligned} u_{\Box,\alpha}(x) &= u(x) + \frac{1}{2}(x - \overline{x})'\Box(x - \overline{x}) \\ Du_{\Box,\alpha}(x) &= Du(x) + (x - \overline{x})'\Box \Rightarrow Du_{\Box,\alpha}(\overline{x}) = Du(x) \\ D^2u_{\Box,\alpha}(x) &= D^2u(x) + \Box \end{aligned}$$

So in an α-neighborhood the Hessian changes, by $\Box$, but the gradient, demand do not. For small enough $\alpha, \Box$ this utility remains in Debreu's setting, so neoclassical demand is defined and smooth when active.

In the Sufficient Independence of Reactions, the path of risk aversion is identified with a linear path $(\Box^h, \alpha^h)(\xi) \equiv (\Box^h \xi, \frac{\|\overline{x}^h\|}{2})$ for each household, so that $\frac{d}{d\xi} D^2 u^h_{\Box,\alpha}(x) = \Box^h$.

Chapter 3

Welfare Impact of Financial Innovation in Incomplete Markets

3.1 Introduction

When asset markets are incomplete, there almost always exist many Pareto improving policy interventions. When they are complete, the First Welfare Theorem implies there never exist any. While the Pareto improvements vanish with the completion of asset markets, the process of completion itself can be Pareto worsening, as shown by Hart (1975) in an example and by Elul (1995) and Cass and Citanna (1998) generically.

I create a framework for proving the existence of Pareto improving financial innovations, and for computing them. The framework requires knowledge of how financial innovation and prices affect aggregate, but not individual, demand.

Financial innovation is Pareto improving only if causes an equilibrium price adjustment, Grossman (1975). The effect of the price adjustment is to redistribute wealth across states, beyond the span of the original assets, according to Stiglitz (1982). Conversely, I prove that if the price adjustment is sufficiently sensitive to risk aversions, then for almost all endowments and risk aversions, Pareto improving financial innovations exist. I show how to verify this sensitivity test with standard demand theory, which Turner (2003a) extends from complete to incomplete markets.

Substitution-free financial innovation in an existing asset passes the sensitivity test, but not financial innovation in a new unwanted asset, as first introduced by Elul (1995).

This result on financial innovation mirrors exactly Turner (2003b) on taxation. There, the sensitivity of price adjustment to risk aversion is sufficient for a tax policy generically to admit Pareto improving taxes. Many different tax policies generically support a Pareto improvement, because they all pass this one sensitivity test. These policies include (a) taxes on asset purchases, as in Citanna, Polemarchakis, and Tirelli (2001), (b) lump-sum taxes on current income plus one flat tax on asset purchases, similar to Citanna, Kajii, and Villanacci (1998) and to Mandler (2003), (c) asset measurable taxes on capital gains, and (d) excise taxes on current commodities, similar to Geanakoplos and Polemarchakis (2002), who emphasize consumption externality over asset incompleteness.

I give a formula for the welfare impact of financial innovation. It requires information about the individual marginal utilities and net trades, and about the derivative of aggregate, but not individual, demand with respect to financial innovation and prices. This information suffices to numerically identify the Pareto improving financial innovations.

To assess the rate of Pareto impairment, I define an agent's equilibrium *insurance deficit*, the marginal utility for contingent income projected to the orthogonal complement of the asset span. This is zero exactly when her commodity demand is as though markets were complete. The rate of Pareto impairment turns out to be quadratic in the insurance deficits, and affine in the level of trade and in the proximity to price crashes.

Elul (1995) and Cass and Citanna (1998) show that if there are multiple commodities and sufficient incompleteness, then typically there is a new asset whose introduction leads to a nearby Pareto worse equilibrium, and one that leads to a nearby Pareto better equilibrium. Their idea is to exploit the same culprit of generic constrained Pareto suboptimality that Geanakoplos and Polemarchakis (1986) identify. The price adjustment that follows creates, through the value of excess demands, a new asset with payoffs beyond the yield span. The main complication is that households are free to demand assets anew after the financial intervention, whereas Geanakoplos and Polemarchakis' (1986) intervention sets asset demands, and the demand theory in Turner (2003a) was developed to address this.

We show that the welfare impact of financial innovation in a new unwanted asset is always rank deficient, in that its rank is smaller than the household heterogeneity. The rank deficiency implies that such financial innovation fails the sensitivity test, and we show exactly how.

3.2 GEI model

Households $h = 1, ..., H$ know the present state of nature, denoted 0, but are uncertain as to which among $s = 1, ..., S$ nature will reveal in period 1. They consume commodities $c = 1, ..., C$ in the present and future, and invest in assets $j = 1, ..., J$ in the present only. Each state has commodity C as unit of account, in terms of which all value is quoted. Markets assign to household h an income $w^h \in R^{S+1}_{++}$, to commodity $c < C$

a price $p_{\cdot c} \in R^{S+1}_{++}$, to asset j a price $q^j \in R$ and future yield $a^j \in R^S$. We call $(p_{\cdot c})_1^C = p = (p_{s\cdot})$ the spot prices, $q = (q^j)$ the asset prices, $(a^j) = a = (a_s)$ the asset structure, and $w = (w^h)$ the income distribution, $\mathbf{P} \equiv R^{(C-1)(S+1)}_{++} \times R^J$.[1] The set of **budget variables** is

$$b \equiv (P, a, w) \in B \equiv \mathbf{P} \times R^{J \times S} \times R^{(S+1)H}_{++}$$

and has some distinguished nonempty relatively open subset $B' \subset B$.

Demand for commodities and assets $d = (x, y) : B' \to R^{C(S+1)}_{++} \times R^J$ is a function on B'. The demand $d^h = (x^h, y^h)$ of household h depends on own income only, $(x^h, y^h)(P, a, w, t) = (x^h, y^h)(P, a, w', t)$ if $w^h = w'^h$.

An **economy** (a, e, d) consists of an asset structure a, endowments e, and demands d. For each household h, **endowments** specify a certain number $e^h_{sc} > 0$ of each commodity c in each state s, and **demands** specify a demand d^h. Let Ω be the set of (a, e, d).[2]

A list $(P; a, e) \in \mathbf{P} \times \Omega$ is a **GEI** $\leftrightarrow$

$$\begin{aligned} &\textstyle\sum(x^h(b) - e^h) = 0 \qquad \sum y^h(b) = 0 \\ &\text{with } \ b \equiv (P, a, (w^h_s = e^{h\prime}_s \overline{p}_s)^h_s) \in B' \end{aligned}$$

We say $(a, e) \in \Omega$ has **equilibrium** $P \in \mathbf{P}$. Under neoclassical assumptions $(a, e) \in \Omega$ has an equilibrium[3].

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[1] The numeraire convention is that unity is the price of $sC, s \geq 0$, which for this reason is omitted from the description of $\mathbf{P}$. The addition of the $sC, s \geq 0$ coordinates, bearing value unity, is denoted $\overline{p}$. We use the notation $P = (p, q) \in \mathbf{P}$.

[2] The appendix spells out the parameterization of demand d.

[3] Geanakoplos and Polemarchakis (1986).

3.2.1 Neoclassical demand

Consider the **budget** function $\beta^h : B \times R^{C(S+1)} \times R^J \to R^{S+1}$

$$\beta^h(b, x, y) \equiv (\overline{p}'_s x_s - w^h_s)_{s=0}^S - \begin{bmatrix} -q' \\ a' \end{bmatrix} y$$

Demand $d^h = (x^h, y^h)$ is **neoclassical** if there is a **utility** function $u : R_+^{C(S+1)} \to R$ with

$$u(x^h(b)) = \max_{X^h(b)} u \text{ throughout } B' \qquad X^h(b) \equiv \{x \in R_+^{C(S+1)} \mid \beta^h(b, x, y) = 0, \text{ some } y \in R^J\}$$

Neoclassical welfare is $v : B' \to R^H, v(b) = (v^h(b)) \equiv (u^h(x^h(b)))$. The **neoclassical domain** is

$$B' = \{(P, a, w) \in B \mid q \in aR^S_{++}, a \text{ has linearly independent rows}\}$$

Debreu's smooth preferences imply neoclassical demand exists and is smooth.

The interpretation of X is that the cost of consumption x in excess of income w is financed by some portfolio $y \in R^J$ of assets. A **portfolio** specifies how much of each asset to buy or sell ($y_j \gtrless 0$), and a^j_s how much value in state s an asset j buyer is to collect, a seller to deliver.

3.3 Welfare impact of financial innovation

Financial innovation in an asset structure a is a smooth path $t = t(\xi)$ in $R^{J\times S}$ through $t(0) = 0$, defining $a(\xi) = a + t(\xi)$ as a new asset structure. We think of *infinitesimal financial innovation* as its initial velocity $\dot{t} = \dot{t}(0)$. Suppose the GEI (P, a, e) is regular in that equilibrium prices are locally a smooth function of the economy, so that financial innovation lifts locally to a unique path $(P(\xi), a + t(\xi), e)$ of nearby GEI. Then welfare is $v(b(\xi))$ with $b(\xi) = (P(\xi), a + t(\xi), (w_s^h = e_s^{h\prime}\overline{p}_s(\xi))_s^h)$. Thus financial innovation impacts welfare only via the budget variables it implies. By the fundamental theorem of calculus the welfare impact is the integral of $D_b v^h \cdot \dot{b}$, which by abuse we call the *welfare impact*. We compute this product in the appendix, using the envelope theorem for $D_b v^h$ and the chain rule for $\dot{b}$, where the details of the notation appear.

Proposition 3.1 (Envelope). *The* ***welfare impact*** $\dot{v} \in R^H$ *of infinitesimal innovation* $\dot{t}$ *at a regular GEI is*

$$\dot{v} = (\lambda)'\dot{m} \qquad \dot{m} = \underbrace{\overline{y}_{\mathbf{1}}\dot{t}}_{PRIVATE} \quad \underbrace{-\underline{z}\dot{P}}_{PUBLIC}$$

Here $(\lambda)'$ *collects the households' marginal utilities of income across states, and* $\dot{m}$ *the impact on their incomes, private and public. The private one is the impact* $\overline{y}_{\mathbf{1}}\dot{a}$ *on portfolio payoffs, and the public one is the impact on the value of their excess demands* $\underline{z}$ *in all nonnumeraire markets, that implied by the impact* $\dot{P}$ *on prices.*

Policy targeting welfare must account for the equilibrium price adjustment it causes.

At a regular GEI there is a **price adjustment** matrix dP, smooth in a neighborhood of it, such that $\dot{P} = dP\dot{t}$. Thus the welfare impact is a differential $\dot{t} \to \dot{v}$,

$$\boxed{dv = (\lambda)'\left(\overline{y}_{\mathbf{1}} - \underline{z}dP\right)} \tag{3.1}$$

Note $dv = dv(b)$ is a function of the budget variables, since v itself is.

We consider two types of financial policy, perturbing an existing asset in a substitution-free way, and perturbing a new unwanted asset, as in Elul (1995) and Cass and Citanna (1998). Aggregate demand is provoked by the income effect of one policy, and by the substitution effect of the other. In either case, financial innovation is parameterized by a vector subspace $\dot{t} \in T = T(b)$ associated with the equilibrium budget variables b :

$$dv : T(b) \to R^H$$

3.4 Framework for generic existence of Pareto improving innovation

We prove the generic existence of Pareto improving innovations, stressing the role of changing commodity prices over the role of the particular financial policy. Existence follows directly from a hypothesis on price adjustment. Thus the financial policy is relevant only insofar as it meets the hypothesis on price adjustment. Then we reinterpret this hypothesis on dP in terms of primitives, the Reaction of Demand to Prices and the Reaction of Demand to Policy.

Pareto improving financial innovation exists if there exists a solution to $dv\dot{t} \gg 0$. In turn this exists if $dv \in R^{H \times \dim T(b)}$ has rank H, which in turn forces us to suppose the innovation parameters outnumber household types $\dim T(b) \geq H$. The key idea is that if $dv = (\lambda)'\overline{y}_{\mathbf{1}} - (\lambda)'\underline{z}dP$ is rank deficient, then a perturbation of the economy would restore full rank by preserving the first summand but affecting the second one. Namely, if some economy's dP is not appropriate, then almost every nearby economy's dP is.

We have in mind a perturbation of the households' **risk aversion** $(D^2 u^h)_h$, which affects nothing but dP in the welfare impact dv. Now, to restore the rank the risk aversion must map into $(\lambda)'\underline{\overline{z}}dP$ richly enough. Since this map keeps $(\lambda)'\underline{\overline{z}}$ fixed, we require that $(\lambda)'\underline{\overline{z}}$ have rank H and that dP be sufficiently sensitive to risk aversion. Cass and Citanna (1998) gift us the first requirement:

Fact 3.1 (Full Externality of Price Adjustment on Welfare). *Suppose asset incompleteness exceeds household heterogeneity* $S - J \geq H > 1$*. Then generically in endowments every GEI has* $(\lambda_s^h z_{s1}^h)_{s \leq H-1}^{h \leq H}$ *invertible.*

Fact 3.2. *At a regular active GEI,* dP *is locally a smooth function of risk aversion; the marginal utilities* λ^i *and excess demands* z^i *are locally constant in risk aversion.*

For $k \in R^{(S+1)(C-1)+J}$ we say that a *commodity coordinate* is one of the first $(S+1)(C-1)$.

Definition 3.1. *At a regular active GEI,* dP *is* k***-Sensitive to risk aversion*** *if for every* $\alpha \in R^{\dim(T)}$ *there is a path of risk aversion that solves* $k' d\dot{P} = \alpha'$.[4] *It is* ***Sensitive to risk aversion*** *if it is* k*-Sensitive to risk aversion for all* k *with a nonzero commodity coordinate.*

Assumption 3.1 (Generic Sensitivity of dP**).** *If* $H > 1$*, then generically in endowments and utilities, at every GEI* dP *is Sensitive to risk aversion.*

This assumption banishes the particulars of the financial innovation policy, leaving only its imprint on dP. Of course, dP is defined only at regular GEI, so implicitly assumed is

[4] The appendix spells out a path of risk aversion. Here the dot denotes differentiation with respect to the path's parameter.

that regular GEI are generic in endowments. Lastly, the requirement that $\dim T(b) \geq H$ with b arising in equilibrium makes sense only with

Assumption 3.2 (Innovation has a dimension). *If $S - J \geq H$, then there is an integer* $\dim$ *such that generically in utilities, at every GEI the vector subspace $\dot{t} \in T = T(b)$ parameterizing financial innovation has dimension* $\dim$. *Call it gen* $\dim$.

Theorem 3.1 (Logic of Pareto Improvement). *Fix a financial policy and the desired welfare impact $\dot{v} \in R^H$. Grant the Generic Sensitivity of dP under gen* $\dim, S - J \geq H > 1, C > 1$. *Then generically in utilities and endowments, at every GEI $\dot{v}$ is the welfare impact of some $\dot{t} \in T$. Hence financial innovation supports a nearby Pareto superior GEI.*

Proof. Fix generic endowments, utilities from the lemma, assumptions, and apply transversality to

1 nonnumeraire excess demand equations

2 $$\gamma'(\lambda)'(\overline{y}_{\mathbf{1}} - \overline{z}dP) = 0$$

3 $$\gamma'\gamma - 1 = 0$$

where $dv : T(b) \to R^H$. Suppose endowments and utilities make this transverse to zero and the natural projection is proper. By the transversality theorem, for generic such, the system of $(\dim p + \dim q) + gen \dim + 1$ equations is transverse to zero in the remaining endogenous variables, which number $\dim p + \dim q + \dim \gamma$. By hypothesis $gen \dim \geq H = \dim \gamma$, so for these endowments and utilities the preimage theorem implies that no endogenous variables solve this system–every GEI has dv with rank H.

This is transverse to zero. As is well known, we can control the first equations by perturbing one household's endowment. For a moment, say that we can control the second equations and preserve the top ones. We then control the third equation and preserve the top two, by scalar multiples of γ. So transversality obtains if our momentary supposition on $\gamma' dv$ holds:

Write $k' \equiv \gamma'(\lambda)'\underline{\overline{z}}$. Differentiating $\gamma' dv$ with respect to the parameter of a path of risk aversion,

$$\alpha' =_{def} \frac{d}{d\xi}\gamma'(\lambda)'\left(\overline{y}_{\mathbf{1}} - \underline{\overline{z}}dP\right) = -\gamma'(\lambda)'\underline{\overline{z}}\frac{d}{d\xi}(dP) = -k'd\dot{P}$$

since λ, z (hence $\overline{y}_{\mathbf{1}}$) are locally constant by fact 3.2. We want to make α arbitrary, and we can if dP is k-sensitive, which holds by assumption if k has a nonzero commodity coordinate. It has: Full Externality of Price Adjustment on Welfare, $C > 1, \gamma \neq 0$ imply $\gamma'(\lambda)'\underline{\overline{z}}$ is nonzero in the coordinate $m = s1$ for some $s \leq H - 1$.

That the natural projection is proper we omit. (The numeraire asset structure is fixed.)

□

Insofar as generically supporting a Pareto improvement, a financial policy need only imply a sensitive price adjustment, and its particulars are irrelevant.

3.4.1 Expression for Price Adjustment

Before we can check whether a particular policy meets the Sensitivity of dP to Risk Aversion, we need an expression for dP. We express dP in terms of the Reaction of Demand to Prices and the Reaction of Demand to Policy.

Let an underbar connote the omission of the numeraire in each state, define

$$d : B' \to R_{++}^{(C-1)(S+1)} \times R^J \qquad d = \Sigma \underline{d}^h$$

and the **aggregate demand** of $(a, e, d) \in \Omega$

$$d_{a,e}(p, q) \equiv d(p, q, a, (w^h_s = e^{h\prime}_s \overline{p}_s)^h_s)$$

with domain $\mathbf{P}_{a,e} \equiv \{(p, q) \in \mathbf{P} \mid (p, q, a, (w^h_s = e^{h\prime}_s \overline{p}_s)^h_s) \in B'\}$.[5]

Now define

$$\begin{aligned} \nabla &\equiv D_{p,q} d_{a,e} \quad \text{the } \textbf{Reaction of Demand to Prices} \\ \Delta &\equiv D_a d_{a,e} \quad \text{the } \textbf{Reaction of Demand to Policy} \end{aligned} \tag{3.2}$$

Suppose a path of GEI $(p(\xi), q(\xi), a + t(\xi), (w^h_s = e^{h\prime}_s \overline{p}_s(\xi))^h_s)$ through a GEI. Then

$$d_{a,e}(P) = \begin{bmatrix} \sum \underline{e}^h \\ 0 \end{bmatrix}$$

is an identity in the path's parameter ξ. Differentiating with respect to it,

$$\nabla \dot{P} + \Delta \dot{t} = 0$$

A GEI is **regular** if ∇ is invertible. By the implicit function theorem, at a regular GEI equilibrium prices P are locally a smooth function of the financial innovation $t(\xi)$.

Proposition 3.2 (Price Adjustment). *At a regular GEI the Price Adjustment to infinitesimal financial innovation exists,*

$$dP = -\nabla^{-1} \Delta \tag{dP}$$

[5] $\mathbf{P}_{a,e}$ is open, as the preimage by a continuous function of the open B'. Recall the notation $P' = (p', q')$.

where the Reactions ∇, Δ *are defined in (3.2).*

3.4.2 Primitives for the Sensitivity of Price Adjustment to Risk Aversion

Given the Logic of Pareto improvement, we want to check whether a policy meets the Generic Sensitivity of dP. We provide primitives for the Sensitivity of dP, thanks to expression $(dP)^6$:

$$d\dot{P} = -\nabla^{-1}\dot{\Delta} + \nabla^{-1}\dot{\nabla}\nabla^{-1}\Delta$$

Recall equation $k'd\dot{P} = \alpha'$ from definition 2.1. If $\dot{\Delta} = 0$ and $\tilde{k}' \equiv_{def} k'\nabla^{-1}$ then the equation reads $\tilde{k}'\dot{\nabla}\nabla^{-1}\Delta = \alpha'$. If Δ has rank $gen \dim$ then there is a solution β to $\beta'\nabla^{-1}\Delta = \alpha'$ so it suffices to solve $\tilde{k}'\dot{\nabla} = \beta'$. Thus dP is k-Sensitive if (1) Δ has rank $gen \dim$, (2) $\tilde{k}$ is nonzero everywhere, (3) whenever $\tilde{K}$ is nonzero everywhere and $\beta \in R^{(S+1)(C-1)+J}$, there is a path of risk aversion that solves $\dot{\Delta} = 0, \tilde{K}'\dot{\nabla} = \beta'$. (Take $\tilde{k} = \tilde{K}$.) Thus Generic Sensitivity of dP obtains (independently of the $\tilde{k}$ defined) if:

Lemma 3.1 (Activity). *If* $H > 1$, *generically in endowments every GEI is regular.*[7]

Assumption 3.3 (Full Reaction of Demand to Policy). *If* $C > 1$, *generically in utilities and endowments, at every GEI* Δ *has rank* $gen \dim$.

Lemma 3.2 (Mean Externality of Price Adjustment on Welfare is Regular). *Generically in utilities, at every regular GEI, whenever* k *is nonzero in some commodity coordinate,* $\tilde{k}' \equiv k'\nabla^{-1}$ *is nonzero everywhere.*

---

[6] Applying the chain rule to $JJ^{-1} = I$ gives $\frac{d}{d\xi}J^{-1} = -J^{-1}(\frac{d}{d\xi}J)J^{-1}$.

[7] We do not argue this standard result. For these endowments, both Δ and dP are defined.

Assumption 3.4 (Sufficient Independence of Reactions). *If* $H > 1$*, then generically in endowments and utilities, whenever* $\tilde{k} \in R^{(S+1)(C-1)+J}$ *is nonzero everywhere and* $\beta \in R^{(S+1)(C-1)+J}$*, at every GEI there is a path of risk aversion that solves* $\dot{\Delta} = 0, \tilde{k}'\dot{\nabla} = \beta'$*.*

These primitives for the Generic Sensitivity of dP and the Logic of Pareto Improvement yield

Theorem 3.2 (Test for Pareto Improvement). *Fix a financial policy and the desired welfare impact* $\dot{v} \in R^H$*. Say the policy passes the Full Reaction of Demand to Policy and the Sufficient Independence of Reactions under* $gen\dim, S - J \geq H > 1, C > 1$*. Then generically in utilities and endowments, at every GEI* $\dot{v}$ *is the welfare impact of some* $\dot{t} \in T$*. Hence financial innovation supports a nearby Pareto superior GEI.*

Next we illustrate how to check whether a financial policy passes this test via demand theory in incomplete markets, as developed by Turner (2003a). We show that substitution free financial innovation passes this test, and so generically supports Pareto improvement, owing to the unifying logic of a sensitive price adjustment. In contrast, financial innovation in a new unwanted asset never passes this test. At a GEI ∇ will turn out to be independent of the policy, so we will verify the lemma on the Mean for one and all policies.

3.5 Summary of demand theory in incomplete markets

We must check whether each policy meets the Full Reaction of Demand to Policy and the Sufficient Independence of Reactions. For this we report the theory of demand in incomplete markets as developed by Turner (2003a). The basic idea is to use decompositions of Δ, ∇ in terms of Slutsky matrices, and then to perturb these Slutsky matrices by perturbing risk aversion, while preserving neoclassical demand at the budget variables under consideration. We stress that this theory is applied to, but independent of, equilibrium.

3.5.1 Slutsky perturbations

Define $H : R^{C^* \times C^*} \to R^{C^*+J+(S+1)\times C^*+J+(S+1)}$ as

$$H(D) = \begin{bmatrix} D & 0 & -[\overline{p}] \\ 0 & 0 & W \\ -[\overline{p}]' & W' & 0 \end{bmatrix}$$

where $p, W = [-q : a] \in R^{J\times S+1}$ of rank J are given, and $C^* = C(S+1)$. In other notation,

$$H(D) = \begin{bmatrix} M(D) & -\rho \\ -\rho' & 0 \end{bmatrix} \qquad \text{where } M(D) = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}, \rho = \begin{bmatrix} [\overline{p}] \\ -W \end{bmatrix}$$

In showing the differentiability of demand, the key step is the invertibility of $H(D^2u)$. **Slutsky matrices** are $H(D^2u)^{-1}$. If D is symmetric, so are $H(D), H(D)^{-1}$ when defined. Thus we write

$$H(D)^{-1} = \begin{bmatrix} S & -m \\ -m' & -c \end{bmatrix}$$

where S, c are symmetric of dimensions $C^*+J, S+1$ and $m = (m_x, m_y)$ is $C^*+J\times S+1$.[8] A **Slutsky perturbation** is of the form $\nabla = H(D)^{-1} - H(D^2u)^{-1}$, for some symmetric $D \approx D^2u$ that is close enough for the inverse to exist. A Slutsky perturbation is a perturbation of Slutsky matrices rationalizable by some perturbation of the Hessian of

[8] It turns out that $m = D_w d$.

utility. Being symmetric, we write

$$\nabla = \begin{bmatrix} \dot{S} & -\dot{m} \\ -\dot{m}' & -\dot{c} \end{bmatrix}$$

and view a Slutsky perturbation as a triple $\dot{S}, \dot{m}, \dot{c}$. We identify Slutsky perturbations, without reference to the inversion defining them, in terms of independent linear constraints on ∇ :

on $\dot{S}$	$\rho'\dot{S} = 0$ and $\dot{S}$ is symmetric	
on $\dot{m}$	$\rho'\dot{m} = 0$ and $\dot{m}_x W' = 0$	(constraints)
on $\dot{c}$	$\dot{c}W' = 0$ and $\dot{c}$ is symmetric	

Theorem 3.3 (Identification of Slutsky perturbations, Turner 2003a). *Given u smooth in Debreu's sense and b in B' with $t = 0$, consider the Slutsky matrices $H(D^2u)^{-1}$. Every small enough Slutsky perturbation ∇ satisfies (constraints). Conversely, every small enough perturbation ∇ that satisfies (constraints) is Slutsky: $H(D^2u)^{-1} + \nabla$ is the inverse of $H(D)$ for some D that is negative definite and symmetric.*

We use only Slutsky perturbations with $\dot{m}, \dot{c} = 0$ by choosing $\dot{S}$ as follows. A matrix $\underline{\dot{S}} \in R^{(C-1)(S+1)+J \times (C-1)(S+1)+J}$ is extendable in a unique way to a matrix $\dot{S} \in R^{C^*+J \times C^*+J}$ satisfying $\rho'\dot{S} = 0$; we call $\dot{S}$ the **extension** of $\underline{\dot{S}}$. It is easy to verify that if $\underline{\dot{S}}$ is symmetric, so is its extension. In sum, any symmetric $\underline{\dot{S}}$ defines a unique Slutsky perturbation with $\dot{m}, \dot{c} = 0$.

Now we turn to decompositions of Δ, ∇ in terms of Slutsky matrices, which in turn make up the inverse the Hessian H matrix. One implication is that knowledge of the Hessian suffices to compute the derivatives of demand functions, enabling numerical comparative statics. Another implication is that with the above identification of Slutsky perturbations, it is possible to perturb demand functions directly, via the Slutsky matrices appearing in the decompositions. The Slutsky approach to genericity is alternative to the first order conditions approach of Cass, Citanna, and Villanacci (1998), and avoids implicitly reworking demand theory at every argument.

3.5.2 Decomposition of demand with respect to prices

The relevance of Slutsky perturbations is that they allow us to perturb demand functions directly, while preserving their neoclassical nature, without having to think about utility. This is because Slutsky matrices appear in the **decomposition** of demand $D_{p,q}\underline{d}$ at b : [9]

$$D_{p,q}\underline{d}^h = \underline{S}^h L_+^h - \underline{m}^h \cdot ([\underline{x}^h]' : \overline{y}_0^h) \tag{dec}$$

Here L_+^h a diagonal matrix displaying the marginal utility of contingent income

$$L_+^h \equiv \begin{bmatrix} L^h & 0 \\ 0 & \lambda_0^h I_J \end{bmatrix} \qquad L^h \equiv \begin{bmatrix} \cdot & & 0 \\ & \lambda_s^h I_{C-1} & \\ 0 & & \cdot \end{bmatrix}$$

[9] Gottardi and Hens (1999) have this in the case $C = 1$. They do not address or define Slutsky perturbations.

$m^h = D_{w^h} d^h$, and $([\underline{x}^h]' : \overline{y}_0^h)$ is the transpose of $\underline{d}^h$:[10]

$$[\underline{x}^h]' = \begin{bmatrix} \cdot & 0 & 0 \\ 0 & \underline{x}_s^{h\prime} & 0 \\ 0 & 0 & \cdot \end{bmatrix}_{(S+1)\times(C-1)(S+1)} \qquad \overline{y}_0^h = \begin{bmatrix} y^{h\prime} \\ 0 \end{bmatrix}_{S+1\times J}$$

Writing $(e_s^{h\prime}\overline{p}_s)_s$ as $[e^h]'\overline{p}$, we have $D_{p,q}[e^h]'\overline{p} = ([\underline{e}^h]' : 0)$, so from (3.2) we have

$$\nabla = \Sigma D_{p,q}\underline{d}^h + D_{w^h}\underline{d}^h \cdot ([\underline{e}^h]' : 0)$$

Inserting decomposition (dec),

$$\nabla = \Sigma \underline{S}^h L_+^h - D_{w^h}\underline{d}^h \cdot ([\underline{x}^h - \underline{e}^h]' : \overline{y}_0^h)$$

Writing $\underline{z}^{h\prime} \equiv ([\underline{x}^h - \underline{e}^h]' : \overline{y}_0^h)$ this reads

$$\boxed{\nabla = \Sigma \underline{S}^h L_+^h - D_{w^h}\underline{d}^h \cdot \underline{z}^{h\prime}} \qquad (\nabla)$$

This **decomposition** of the aggregate demand of $(a, e, t, t_*) \in \Omega$ generalizes Balasko 3.5.1 (1988) to incomplete markets.

One implication of the decomposition is that ∇ is independent of the policy.

Proof that Mean Externality of Price Adjustment on Welfare is Regular. See Turner (2003b).

□

[10] We view p as one long vector, state by state, and p, q as an even longer one; $(* : \#)$ denotes concatenation of $*, \#$.

3.5.3 Decomposition of demand with respect to insurance

There is another **decomposition** of demand $D_a\underline{d}$ at b with $t = 0$:

$$D_a\underline{d}^h = \underline{S}^h \begin{bmatrix} 0 \\ \Lambda_{\mathbf{1}}^h \end{bmatrix} - \underline{m}^h \cdot \overline{y}_{\mathbf{1}}^h$$

Here $\Lambda_{\mathbf{1}}^h$ is a matrix displaying the marginal utility of contingent income

$$\Lambda_{\mathbf{1}}^h \equiv [\lambda_1^h I_J : ... : \lambda_s I_J]_{J\times JS}$$

and $\overline{y}_{\mathbf{1}}^h$ is a repeated display of y^h : [11]

$$\overline{y}_{\mathbf{1}}^h = \begin{bmatrix} 0 & \cdot & 0 \\ y' & \cdot & 0 \\ \cdot & \cdot & \cdot \\ 0 & \cdot & y' \end{bmatrix}_{S+1\times JS}$$

Specializing to a single asset's payoff, this reads

$$\boxed{D_{a^j}d = \Sigma \underline{S}_j^h \lambda_{\mathbf{1}}^{h\prime} - \underline{m}_{\mathbf{1}}^h \cdot y_j^h} \qquad (D_{a^j}d)$$

where $\underline{S}_j^h$ is column $(C-1)(S+1)+ j$ of $\underline{S}^h$.

[11] We view a as one long vector, state by state.

3.5.4 Preparation for genericity

We investigate for each policy the Full Reaction of Demand to Policy and the Sufficient Independence of Reactions. In computing

$$\Delta \equiv D_a d_{a,e}$$

we use the following notation for $\underline{S}^h$, where A^h, B^h are symmetric of dimensions $(C-1)(S+1), J$:

$$\underline{S}^h = [\underline{S}^h_p : \underline{S}^h_q] = \begin{bmatrix} A^h & P^h \\ P^{h\prime} & B^h \end{bmatrix} \qquad (\underline{S}^h)$$

We can perturb P^h arbitrarily and get a Slutsky perturbation.

Remark 3.1. *In checking the Sufficient Independence of Reactions, all marginal utilities* λ^i *and excess demands* z *are automatically fixed by the* $\dot{\underline{S}}^h$ *Slutsky perturbations. Their only effect is on the Jacobian* $\dot{\nabla} = \Sigma \dot{\underline{S}}^h L^h_+$ *in* (∇)*. Also, we solve* $\tilde{k}'\dot{\nabla} = \beta'$ *piecemeal, solving* $\tilde{k}'\dot{\nabla}_p = \beta'_p, \tilde{k}'\dot{\nabla} = \beta'_q$ *by splitting* $\beta' = (\beta'_p, \beta'_q)$, $\dot{\nabla} = [\dot{\nabla}_p : \dot{\nabla}_q]$.

3.6 The insurance deficit

In equilibrium, every household's marginal utility of future income projects to a common point in the asset span[12]

$$\frac{\lambda^h_{\mathbf{1}}}{\lambda^h_0} = u^h + c \in a^{\perp} \oplus span(a)$$

[12] This is the same as the decomposition $\lambda^h_{\mathbf{1}} \in a^{\perp}_{+} \oplus span(a_+)$ by definition of new unwanted asset.

We summarize the **insurance deficit** by

$$U \equiv \begin{bmatrix} \dots & u^h & \dots \end{bmatrix}_{S \times H}$$

Lemma 3.3 (Insurance deficit in general position). *If $S - J \geq H$ then generically in endowments, at every GEI every H rows of the insurance deficit U are linearly independent.*[13]

Proof. Fix $K \subset \{1, ..., S\}$ with cardinality H, and apply transversality to

$$\text{nonnumeraire excess demand equations}$$
$$\pi' U = 0$$
$$\pi'_K \pi_K - 1 = 0$$

where $\pi_{S \backslash K} = 0$. Endowments make this transverse to zero. The burden of the argument is to control the second equations independently of the others. Given $t \in R^H$ we want $\pi' \dot{u}^h = t^h$, where $\dot{u}^h \equiv \frac{d}{d\xi} \Pr_{a^\perp} \left(\frac{\lambda_{\mathbf{1}}^h}{\lambda_0^h} \right)$, via appropriate $\dot{\lambda}^h$, i.e. $\dot{\lambda}^h$ must preserve first order conditions $\dot{\lambda}_0^h q = a \dot{\lambda}_{\mathbf{1}}^h$. Any $\dot{\lambda}^h$ is implementable by an endowment perturbation $\dot{e}^h = \dot{x}^h$ as we show last. If $\dot{\lambda}_0^h = 0$ and $0 = a \dot{\lambda}_{\mathbf{1}}^h$ then first order conditions remain and

$$\frac{\partial}{\partial \cdot} \left(\frac{\lambda_{\mathbf{1}}^h}{\lambda_0^h} \right) = \frac{\dot{\lambda}_{\mathbf{1}}^h}{\lambda_0^h} - \frac{\lambda_{\mathbf{1}}^h}{\lambda_0^{h2}} \dot{\lambda}_0^h = \frac{\dot{\lambda}_{\mathbf{1}}^h}{\lambda_0^h} \text{ so } \dot{u}^h \equiv \frac{\partial}{\partial \cdot} \Pr_{a^\perp} \left(\frac{\lambda_{\mathbf{1}}^h}{\lambda_0^h} \right) = \Pr_{a^\perp} \frac{\dot{\lambda}_{\mathbf{1}}^h}{\lambda_0^h} = \frac{\dot{\lambda}_{\mathbf{1}}^h}{\lambda_0^h}$$

So set $\dot{\lambda}_0^h = 0$ and seek $\dot{\lambda}_{\mathbf{1}}^h$ with $0 = a \dot{\lambda}_{\mathbf{1}}^h, \pi' \frac{\dot{\lambda}_{\mathbf{1}}^h}{\lambda_0^h} = t^h$. To find $\dot{\lambda}_{\mathbf{1}}^h$, say $\pi_s \neq 0, s \in K$ and set $\dot{\lambda}_K^h$ to $\dot{\lambda}_s^h = \frac{\lambda_0^h t^h}{\pi_s}, \dot{\lambda}_{t \neq s}^h = 0$ for $t \in K$ so that, thanks to $\pi_{S \backslash K} = 0$, $\pi' \frac{\dot{\lambda}_{\mathbf{1}}^h}{\lambda_0^h} = t^h$ regardless of $\dot{\lambda}_{S \backslash K}^h$. Having set $\dot{\lambda}_K^h$, define $\dot{\lambda}_{S \backslash K}^h$ as a solution to $0 = a \dot{\lambda}_{\mathbf{1}}^h =$

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[13] This requires that every J columns of a are linearly independent.

$a_K\dot{\lambda}^h_K + a_{S\backslash K}\dot{\lambda}^h_{S\backslash K}$, which exists since these are J equations in $|S\backslash K| = S - H \geq J$ variables and every J columns of a are linearly independent.

To implement this $\dot{\lambda}^h$, solve $D^2u^h \cdot \dot{x}^h = (\overline{p}_s\dot{\lambda}^h_s)_s$ for $\dot{x}^h$, possible by the negative definiteness of D^2u^h and the inverse function theorem. Implement this $\dot{x}^h$ by setting $\dot{e}^h = \dot{x}^h$, while preserving the other equations.

By the transversality theorem, generically in endowments, the system is transverse to zero in the remaining variables. These are $\dim p + \dim q + \dim \pi$ variables and $\dim p + \dim q + H + 1$ equations, with $\dim \pi = H$, so the associated zero set is a submanifold of dimension -1, hence empty. For these endowments E_K, the K rows of U are linearly independent. The intersection of the generic E_K over the finitely many such K is generic still. □

3.7 Substitution free innovation in an existing asset

Substitution free innovation in an existing asset satisfies $\lambda^{h\prime}_{\mathbf{1}}\dot{a}^j = 0$. We parameterize financial innovation by $T(b) = span(a, U)^{\perp}$. Note, $\dot{a}^j \in T(b) \Rightarrow \lambda^{h\prime}_{\mathbf{1}}\dot{a}^j = 0$.

Substitution free innovation provokes only the income effect on demand; formula $(D_{a^j}d)$ implies

$$\Delta \cdot \dot{a}^j = D_{a^j}d \cdot \dot{a}^j = -\Sigma \underline{m}^h_{\mathbf{1}} \cdot y^h_j \dot{a}^j$$

That is,

$$\Delta = -\Sigma \underline{m}^h_{\mathbf{1}} \cdot y^h_j \quad \text{on} \quad T(b)$$

Corollary 3.1. *Fix the desired welfare impact* $\dot{v} \in R^H$. *Assume* $S - J \geq 2H; H, C > 1$. *Then generically in utilities and endowments, at every GEI* $\dot{v}$ *is the welfare impact of* *some* $\dot{t} \in T$*. Hence there is a nearby Pareto superior GEI with substitution free innovation in an existing asset.*

Proof. The next lemmas with $gen\dim = S - J - H$ and the hypothesis $S - J - H \geq H$ enable theorem 3.2. □

Lemma 3.4 (Generic Dimension of Innovation). *If* $S - J \geq H$*, then* $gen\dim = S - J - H$*. That is, generically in endowments, at every GEI the vector subspace* $\dot{t} \in T = T(b)$ *parameterizing financial innovation has dimension* $S - J - H$*.*

Proof. Lemma 3.3 says that generically in endowments U has rank H, and then $span(a, U)'s$ dimension is $J + H$. □

Lemma 3.5 (Full Reaction of Demand to Policy). *If* $C > 1, S - J \geq H > 1$*, generically in utilities and endowments, at every GEI* Δ *has rank* $gen\dim$*.*

Proof. We recall $\Delta = -\Sigma \underline{m}_{\mathbf{1}}^h \cdot y_j^h$ has domain $k \in T(b) = span(a, U)^{\perp}$, and take for granted the very standard result that with $H > 1$ generically in numeraire endowments, at every GEI asset j is traded. Taking generic endowments from this result and the previous lemma's, we apply transversality to

$$\begin{gathered}\text{nonnumeraire excess demand equations}\\ \left(\Sigma \nabla^h \cdot y_j^h\right) k = 0\\ k'k - 1 = 0\end{gathered}$$

where $\nabla_{S \times S}^h$ selects from $\underline{m}_{\mathbf{1}}^h \in R^{S(C-1) \times S}$ only the rows of commodities $(s1)_{s \geq 1}$. Utilities make this transverse to zero. The burden of the argument is to control the middle equations independently of the top and bottom ones. Say $k_{s \geq 1} \neq 0$; we want to perturb arbitrarily column s of the parenthetical sum, as $\frac{d}{d\xi}\left(\Sigma\nabla^h \cdot y_j^h\right) = \frac{a}{k_s}$, and no other. There is h^* with $y_j^{h^*} \neq 0$. From the identification of Slutsky perturbations 3.3, we may perturb arbitrarily any row of $m_x^{h^*}$, hence any row of ∇^{h^*}, subject only to $\dot{m}_x^{h^*}W' = 0$, where $W = [-q : a]$. So perturb it as $\dot{\nabla}_s^{h^*} = [0 : \frac{a_s}{y_j^h}k']$ so that $\frac{d}{d\xi}\left(\Sigma\nabla_s^h \cdot y_j^h\right)k = \frac{d}{d\xi}\left(\dot{\nabla}_s^{h^*} y_j^{h^*}\right)k = a_s(k'k) = a_s$ is arbitrary. Indeed, $\dot{\nabla}_s^{h^*} W' = 0$ since $k \in T(b) \equiv span(a, U)^\perp \subset a^\perp$.

By the transversality theorem, generically in endowments and utilities, this system is transverse to zero in the remaining endogenous variables. These number $\dim p + \dim q + gen\dim$ and there are $\dim p + \dim q + S$ equations, and $gen\dim = S - J - H$, so by the preimage theorem, for these endowments and utilities the associated solution set is empty–every GEI has $\Sigma\nabla^h \cdot y_j^h$ (a fortiori Δ) with linearly independent columns. □

Lemma 3.6 (Sufficient Independence of Reactions). *Generically in endowments and utilities, whenever* $\tilde{k} \in R^{(S+1)(C-1)+J}$ *is nonzero everywhere and* $\beta \in R^{(S+1)(C-1)+J}$*, at every GEI there is a path of risk aversion that solves* $\dot{\Delta} = 0, \tilde{k}'\dot{\nabla} = \beta'$.

Proof. Fix such a $\tilde{k}$, and follow remark 2.1. Since $\Delta = -\Sigma\underline{m}_{\mathbf{1}}^h \cdot y_j^h$ is independent of the substitution matrices $\underline{S}^h$, which is all we perturb, automatically $\dot{\Delta} = 0$ and $\dot{\nabla} = \Sigma\dot{\underline{S}}^h L_+^h$. Left to solve $\tilde{k}'\dot{\nabla} = \Sigma\dot{\underline{S}}^h L_+^h = \beta'$, we set $\dot{\underline{S}}^{h\neq H} = 0$ and so seek to solve $\tilde{k}'\dot{\underline{S}}^H = \beta'\left(L_+^H\right)^{-1} \equiv \tilde{\beta}'$. This is made trivial by a diagonal hence symmetric $\dot{\underline{S}}^H$, with $\dot{\underline{S}}_{mm}^H = \frac{\tilde{\beta}_m}{\tilde{k}_m}$. □

3.8 Innovation in a new unwanted asset

Elul (1995) introduces innovation in a **new unwanted** asset $a^{J+1} \in R^S$. "New" means that it is orthogonal to the existing ones, $aa^{J+1} = 0$, and "unwanted" that it is orthogonal to marginal utilities $\lambda_{\mathbf{1}}^{h\prime} a^{J+1} = 0$ at the GEI. For the economy with this extended asset structure a_{+} Elul then defines a GEI. In this **extended GEI**, the commodities and original assets retain their prices but the new asset is free. It is a GEI because $y_{J+1}^{h} = 0, q^{J+1} = 0$ satisfy the budget identity, and $\lambda_{\mathbf{1}}^{h\prime} a^{J+1} = 0$ the first order conditions, letting households maximize utility by retaining the original demands and ignoring the new asset. Welfare remains the same.

We summarize the **marginal substitutability** of the original assets for the new one by

$$\underline{S}_{*} \equiv \begin{bmatrix} \ldots & \underline{S}_{J+1}^{h} & \ldots \end{bmatrix}_{(C-1)(S+1)+J\times H}$$

and the marginal utilities for current income by

$$\lambda_0 = \begin{bmatrix} \lambda_0^1 & & 0 \\ & \cdot & \\ 0 & & \lambda_0^H \end{bmatrix}_{H\times H}$$

To show that innovation in a new unwanted asset generically supports a Pareto improvement, first we sharpen the welfare impact formula, assuming an extended GEI that is regular.

3.8.1 Welfare impact of innovation in a new unwanted asset

Theorem 3.4 (Welfare impact of innovation in a new unwanted asset). *If the extended GEI is regular, the welfare impact of innovation* $\dot{t}^{J+1} \in R^S$ *is*

$$\dot{v} = (\lambda)' \underline{z} \nabla^{-1} \underline{S}_{*} \lambda_0 U' U \omega$$

where $U\omega = \Pr_U \dot{t}^{J+1}$ *is the orthogonal projection to the insurance deficit* $span(U)$.

Corollary 3.2 (Target insurance deficit). *No achievable welfare impact* $\dot{v}$ *is lost by restricting innovation to the insurance deficit,* $\dot{t}^{J+1} \in T(b) \equiv span(U)$. *Parameterizing* $\dot{t}^{J+1} = U\omega, dv : T(b) \to R^H$ *is*

$$dv = (\lambda)'\underline{z}\nabla^{-1}\underline{S}_*\lambda_0 U'U \tag{3.3}$$

We recognize the **determinants** of the welfare impact. It increases linearly in the excess demands $\underline{z}$, in the proximity ∇^{-1} of the GEI to singularity, and in the marginal substitutability $\underline{S}_*$ of the original assets for the new one, but quadratically in the insurance deficit U. If any determinant is zero, the welfare impact is zero. Also, the welfare impact is zero with equal marginal utilities (say, to λ^*), since $1'dv = -\lambda^{*\prime}1'\underline{z}\nabla^{-1}\underline{S}_*\lambda_0 U'U = 0$ by market clearing $1'\underline{z} = 0$.

Proof. Regularity yields formulas 3.1, dP, $D_{a^j}d$. They would simplify to

$$\begin{gathered}\dot{v} = -(\lambda)'\underline{z}dP = (\lambda)'\underline{z}\nabla^{-1}D_{a^{J+1}}d \cdot \dot{t}^{J+1} \\ D_{a^{J+1}}d = \sum \underline{S}^h_{J+1}\lambda^{h\prime}_{\mathbf{1}}\end{gathered} \tag{*}$$

if $\overline{y}_{\mathbf{1}}\dot{a} = 0, y^h_{J+1} = 0$, equalities which do hold. For the last asset is not demanded in the extended GEI, $y^h_{J+1} = 0$, and financial innovation is only in the last asset, $\dot{a}^j = 0$ for $j \leq J$.

By (*) the welfare impact $\dot{v}$ is additive in the financial innovation $\dot{t}^{J+1}$, so we decompose

$$\dot{t}^{J+1} = U\omega + n_a + n_\perp \in span(U) \oplus span(a_+) \oplus span(a_+, U)^\perp$$

For every household $\lambda_{\mathbf{1}}^h \in span(U) \oplus span(a_+)$, so that $\lambda_{\mathbf{1}}^{h\prime}\dot{t}^{J+1} = \lambda_{\mathbf{1}}^{h\prime}(U\omega + n_a)$ and marginal demand

$$D_{a^{J+1}}d \cdot \dot{t}^{J+1} = D_{a^{J+1}}d \cdot U\omega + D_{a^{J+1}}d \cdot n_a$$

Since the welfare impact $\dot{v}$ is additive in marginal demand, it is the sum of the welfare impacts of the separate financial innovations $U\omega$ and n_a; they are $(\lambda)'\underline{z}\nabla^{-1}\underline{S}_*\lambda_0 U'U\omega$ and 0, as we show next; therefore, $\dot{v} = (\lambda)'\underline{z}\nabla^{-1}\underline{S}_*\lambda_0 U'U\omega + 0$.

Claim on $U\omega$: *The welfare impact of financial innovation* $\dot{t}^{J+1} = U\omega$ *is* $(\lambda)'\underline{z}\nabla^{-1}\underline{S}_*\lambda_0 U'U\omega$.

Given formula (*), it suffices to show $D_{a^{J+1}}d \cdot U\omega = \underline{S}_*\lambda_0 U'U\omega$. Note, $\lambda_{\mathbf{1}}^{h\prime}U\omega = \lambda_0^h(u^h + c)'U\omega = \lambda_0^h u^{h\prime}U\omega$ so that

$$D_{a^{J+1}}d \cdot U\omega = \sum \underline{S}_{J+1}^h \lambda_0^h u^{h\prime}U\omega = \underline{S}_*\lambda_0 U'U\omega$$

Claim on $\mathbf{n}_a$: *The welfare impact of financial innovation* $\dot{t}^{J+1} \in span(a_+)$ *is zero.*

At a regular GEI, by definition $\dot{v}(n_a)$ is the derivative at $t = 0$ of welfare $v(t)$, along any path of GEI $(p, q, a, e)(t)$ with two properties: at $t = 0$ it passes through Elul's extended equilibrium with velocity $\dot{a}^j = 0$ for $j \leq J, \dot{t}^{J+1} = n_a, \dot{e} = 0$. Consider the path $(a(t), e(t))$ with $a^j(t) \equiv a^j$ for $j \leq J, a^{J+1}(t) \equiv a^{J+1} + tn_a, e(t) = e$, where (a, e) is Elul's extended economy. Define $(p(t), q(t))$ in the obvious way to make $(p(t), q(t), a(t), e(t))$ the desired path. Retain $p(t) = p, q^j(t) = q^j$ for $j \leq J$ as in Elul's extended GEI, and set $q^{J+1}(t) = t\Sigma_{j\leq J}q^j\phi^j$ where $n_a = \Sigma_{j\leq J}a^j\phi^j + a^{J+1}\phi^{J+1}$ is the unique expression of $n_a \in span(a_+)$. (The linearity of pricing $\tilde{q}$ at GEI requires $\tilde{q}(a^{J+1}(t)) = \tilde{q}(a^{J+1} + tn_a) = (1 + t\phi^{J+1})\tilde{q}(a^{J+1}) + t\tilde{q}(\Sigma_{j\leq J}a^j\phi^j) = 0 + t\Sigma_{j\leq J}q^j\phi^j$, given that the unwanted asset a^{J+1} is free.) Then commodity prices, yield span, and general equilibrium incomes are the same as in the extended GEI; therefore, budget sets, commodity demands, and welfare $v(t)$ are the same. In particular, commodity markets clear; by Walras' law and the full rank of the extended asset structure, asset markets clear too. So this is not only a GEI, but one with $\dot{v}(t) \equiv 0$ identically and the two properties in the definition of $\dot{v}(n_a)$. □

The obvious and lengthy part is the second claim, which is moot for the corollary.

Application: Symmetric equilibria experience opposite welfare impacts

Is there a financial innovation $\dot{t}^{J+1}$ that is always Pareto improving? Does it have a magical, numerical description? No! Not even if we specify the utilities, equilibrium prices, asset structure, commodity demands. We produce two GEI, which are equal in all these respects, such that essentially *any* innovation $\dot{t}^{J+1}$ is Pareto improving at one if and only if it is Pareto impairing at the other. The trick is to switch the sign of $\underline{z}$ in formula (3.3).

Two GEI_+,GEI_- are **symmetric** if their prices, asset structures, and commodity demands are common, and if their excess demands are the negative of one another, by every household in every of the $(S+1)C+J$ markets. An innovation is **definite** if $\dot{v}^h \neq 0$ for every h.

Lemma 3.7 (Opposite welfare impacts). *Suppose at symmetric extended GEI that the common $S = \Sigma \underline{S}^h L_+^h$ is invertible. If $D_w d\underline{z} \approx 0$ then $\dot{v}_+^h$ and $\dot{v}_-^h$ have opposite signs, given any definite innovation. So it Pareto improves one GEI if and only if it is Pareto impairs the other.*

Proof. Considering formula (3.3), the determinants $\lambda, \underline{S}_*, U$ are functions of the common prices, asset structure, and commodity demands, as are the $S = \Sigma S^h L_+^h$ and $D_w d$ that appear in the decomposition of the Jacobian (∇). So if $D_w d\underline{z} \approx 0$ where $\underline{z}$ is one of the excess demands, then $\nabla_+ \approx S \approx \nabla_-$ and $\nabla_+^{-1} \approx S^{-1} \approx \nabla_-^{-1}$, so that $D_\omega v_- \approx -D_\omega v_-$ since excess demands differ by a sign. □

The lemma makes the nonexistence of an always-Pareto improving financial innovation essentially equivalent to the existence of symmetric extended GEI with $D_w d\underline{z} \approx 0$, a triviality.

Lemma 3.8 (Constructing symmetric GEI). *A no trade GEI and* $z \in R^{[(S+1)(C-1)+J](H-1)} \approx 0$ *define symmetric GEI where the nonnumeraire net trade of* $h < H$ *is* $\underline{z}^h = \pm z^h$.

Proof. At a no trade GEI (p, q, a, e) write x, y for the commodity and asset demands. For $h < H$ define nonumeraire endowments as $\underline{e}_+^h \equiv \underline{x}^h - z_{nonasset}^h, \underline{e}_-^h \equiv \underline{x}^h + z_{nonasset}^h$, and define numeraire endowments uniquely by the requirement that, after liquidation of portfolios $y_+^h = z_{asset}^h, y_-^h = -z_{asset}^h$, the final income equals the cost of consuming x^h. It is easy to check that $(p, q, a, e_+), (p, q, a, e_-)$ are symmetric GEI with the claimed net trades. □

The lemma and proposition reveal a whole neighborhood of no trade GEI that are counterexamples to the existence of a financial innovation that is Pareto improving at both GEI, let alone every GEI.

Proposition 3.3 (Neighborhood of opposing welfare impacts). *Generically in utilities, there is an open neighborhood of no trade GEI, closed under symmetry, such any symmetric pair experiences a welfare impact of opposite sign, household by household, after any definite innovation.*

Proof. A standard argument shows that generically in utilities, any no trade GEI has $S = \Sigma \underline{S}^h L_+^h$ invertible. Constructing symmetric GEI as in lemma 3.8 yields the neighborhood, closed under symmetry, and then the conclusion about any symmetric pair follows from lemma 3.7. □

3.8.2 Deficiency of the welfare impact

The welfare impact of financial innovation in a new unwanted is *always* rank deficient. This is not clear except by an application of formula (3.3) and the Slutsky decomposition of demand.

The reason is that the domain of financial innovation, $T(b) = soan(U)$, has rank H and yet there is one financial innovation which has zero welfare impact. This financial innovation changes no prices other than the new unwanted asset's, and then in equilibrium this price adjusts back to zero, leaving the budget variables hence welfare unchanged.

It follows that financial innovation must not imply a price adjustment generically sensitive to risk aversions. It is easy to see this directly. Indeed, the price adjustment is always *equal to* the "negative" of this financial innovation, and this equality is independent of risk aversions. Further, we can understand this insensitivity as a violation of one of the primitive conditions sufficing for sensitivity, namely, a violation of the independence of the Reactions to Policy and to Prices–one Reaction is the inverse of the other.

Corollary 3.2 states that

$$dv = (\lambda)' \underline{\overline{z}} \nabla^{-1} \underline{S}_* \lambda_0 U' U$$

where by lemma 3.3 U has rank H, generically in endowments when $S - J \geq H$. So there exists $\omega \in R^H$ such that $1 = U'U\omega \in R^H$. We show last that the last column of ∇ is precisely $\underline{S}_*\lambda_0 1$. It follows $\nabla^{-1}\underline{S}_*\lambda_0 U'U\omega = \nabla^{-1}\underline{S}_*\lambda_0 1 = 1_{(S+1)(C-1)+J+1}$ equals a standard unit vector with support in the last coordinate, corresponding to the new unwanted asset. Thus

$$\begin{aligned} dv\omega &= (\lambda)'\underline{\overline{z}}1_{(S+1)(C-1)+J+1} \\ &= (\lambda_0)\begin{bmatrix} y^1_{J+1} & & 0 \\ & \cdot & \\ 0 & & y^H_{J+1} \end{bmatrix} \\ &= (\lambda_0)0 = 0 \end{aligned}$$

using that $y^h_{J+1} = 0$ at Elul's extended GEI.

It would seem that redefining the extended GEI as in Cass and Citanna (1998) would circumvent this rank deficiency of the welfare impact. There, the new asset pays off zero in every state, and the holdings y^h_{J+1} are arbitrary–not zero as above–but for the last agent $h = H$. In our framework, we cannot appeal to this exact trick, because the asset structure becomes singular and the smoothness of demand from the implicit function theorem is at stake. Yet we can borrow the spirit of this trick and start theorem 3.1 with y^h_{J+1} being arbitrary but for one household, by first changing numeraire endowments. Still, an argument identical to the above shows dv is rank deficient on the same domain $T(b) = span(U)$, which by corollary 3.2 is *not* a restriction of the welfare impact.

Finally, we check the last column of ∇ is $\underline{S}_*\lambda_0 1$. Recall decomposition (dec)

$$\nabla = \Sigma \underline{S}^h L^h_+ - D_{w^h}\underline{d}^h \cdot \underline{z}^{h\prime}$$

where $\underline{z}^{h\prime} \equiv ([\underline{x}^h - \underline{e}^h]' : \overline{y}_0^h)$. The last column of L_+^h is the standard basis vector $\lambda_0^h 1_{(S+1)(C-1)+J+1}$ scaled by λ_0^h, so the last column of $\underline{S}^h L_+^h$ is $\underline{S}_{J+1}^h \lambda_0^h$, which added across agents gives $\underline{S}_* \lambda_0 1$, by definition of $\underline{S}_*$. Further, the last column of $\underline{z}^{h\prime}$ is the last column of $\overline{y}_0^h$, which consists of zeros and of the demand for the new unwanted asset, which is zero too at the extended GEI. So the last column of ∇ is $\underline{S}_* \lambda_0 1 + 0 = \underline{S}_* \lambda_0 1$.

3.9 Appendix

3.9.1 Notation

An underbar connotes the omission of the $sC, s \geq 0$ coordinates, as in $\underline{x}^h$; an upperbar on a price $\overline{p}$ connotes the addition of sC coordinates with value $p_{sC} = 1, s \geq 0$.

When differentiating with respect to p, q, a, w, we parameterize these as long vectors:

$$p = \begin{bmatrix} \cdot \\ p_s \\ \cdot \end{bmatrix}_{(C-1)(S+1)\times 1} \qquad q = \begin{bmatrix} \cdot \\ q_j \\ \cdot \end{bmatrix}_{J\times 1} \quad a = \begin{bmatrix} \cdot \\ a_s \\ \cdot \end{bmatrix}_{SJ\times 1} \quad w = \begin{bmatrix} \cdot \\ w^h \\ \cdot \end{bmatrix}_{H(S+1)\times 1}$$

3.9.2 Derivation of formula for welfare impact

It is standard how Debreu's smooth preferences, linear constraints, and the implicit function theorem imply the smoothness of neoclassical demand. It is standard also that the envelope property follows from the value function's local smoothness, which is the case for v^h as the composition of smooth functions:

$$D_b v^h = D_b L(x, y, \lambda^h) \mid_{(x^h, y^h)(b)}$$

where $b = (p, q, a, w^h)$ and

$$L(x, y, \lambda^h) \equiv u^h(x) - \lambda^{h\prime} \left([\overline{p}]' x - w^h - \begin{bmatrix} -q' \\ a' \end{bmatrix} y \right)$$

Thus

$$D_b v^h = -\lambda^{h\prime} \left([\underline{x}^h]' : \overline{y}_0^h : -\overline{y}_{\mathbf{1}}^h : -I \right) \qquad \text{where } \overline{y}_0^h = \begin{bmatrix} y^{h\prime} \\ 0 \end{bmatrix}, \overline{y}_{\mathbf{1}}^h = \begin{bmatrix} 0 & \cdot & 0 \\ y^{h\prime} & & 0 \\ & \cdot & \\ 0 & & y^{h\prime} \end{bmatrix}$$

So much for demand theory. Recalling regular GEI from the subsection on the Expression for the Price Adjustment, $dP' = (dp', dq')$ exists and

$$\begin{aligned} w^h &= [\overline{p}]' e^h \Rightarrow \\ dw^h &= [\underline{e}^h]' dp \\ &= ([\underline{e}^h]' : 0) dP \end{aligned}$$

Thus the welfare impact at a regular GEI is

$$\begin{aligned} dv^h &= D_b v^h \cdot db \\ &= -\lambda^{h\prime} \left(([\underline{x}^h]' : \overline{y}_0^h) : -\overline{y}_{\mathbf{1}}^h : -I \right) \cdot \left(dP : I_{SJ} : ([\underline{e}^h]' : 0) dP \right) \\ &= -\lambda^{h\prime} \left(([\underline{x}^h]' : \overline{y}_0^h) dP - \overline{y}_{\mathbf{1}}^h - ([\underline{e}^h]' : 0) dP \right) \\ &= -\lambda^{h\prime} \left(\underline{z}^{h\prime} dP - \overline{y}_{\mathbf{1}}^h \right) \end{aligned}$$

where $\underline{z}^{h\prime} \equiv ([\underline{x}^h - \underline{e}^h]' : \overline{y}_0^h)$ by definition. In sum,

$$\boxed{dv^h = \lambda^{h\prime}\left(\overline{y}_{\mathbf{1}}^h - \underline{z}^h dP\right)}$$

3.9.3 Aggregate notation

We collect marginal utilities of contingent income, and denote stacking by an upperbar

$$(\lambda)' \equiv \begin{bmatrix} \cdot & & 0 \\ & \lambda^{h\prime} & \\ 0 & & \cdot \end{bmatrix}_{H\times H(S+1)} \qquad \overline{y}_{\mathbf{1}} = \begin{bmatrix} \cdot \\ \overline{y}_{\mathbf{1}}^h \\ \cdot \end{bmatrix}_{H(S+1)\times SJ} \qquad \overline{\underline{z}} \equiv \begin{bmatrix} \cdot \\ \underline{z}^{h\prime} \\ \cdot \end{bmatrix}_{H(S+1)\times(S+1)(C-1)+J}$$

Thus

$$\boxed{dv = (\lambda)'\left(\overline{y}_{\mathbf{1}} - \overline{\underline{z}}dP\right)}$$

To visualize the bracket notation $[\cdot]$ defined in footnote 7, it staggers state contingent vectors:

$$[p] \equiv \begin{bmatrix} \cdot & & & \\ & p_{s-1} & 0 & \\ & & p_s & \\ & 0 & p_{s+1} & \\ & & & \cdot \end{bmatrix}_{C(S+1)\times S+1}$$

3.9.4 Transversality

A function $F : M \times \Pi \to N$ defines another one $F_\pi : M \to N$ by $F_\pi(m) = F(m, \pi)$. Given a point $0 \in N$ consider the "equilibrium set" $E = F^{-1}(0)$ and the natural projection $E \to \Pi, (m, \pi) \mapsto \pi$. A function is *proper* if it pulls back sequentially compact sets to sequentially compact sets.

Remark 3.2 (Transversality). *Suppose F is a smooth function between finite dimensional smooth manifolds. If 0 is a regular value of F, then it is a regular value of F_π for almost every $\pi \in \Pi$. The set of such π is open if in addition the natural projection is proper.*

A subset of Π is **generic** if its complement is closed and has measure zero. Write $C^* = C(S+1)$. Here the set of parameters is

$$\Pi = O \times O' \times (0, \epsilon)$$

where O, O' are an open neighborhoods of zero in $R^{C^*H}, R^{\frac{C^*(C^*+1)}{2}H}$ relating to endowments and symmetric perturbations of the Hessian of utilities. We have in mind a fixed assignment of utilities, which we perturb by $O' \times (0, \epsilon)$. Specifically, given an equilibrium commodity demand $\overline{x}$ by some household and $\Box \in R^{\frac{C^*(C^*+1)}{2}}, \alpha \in (0, \epsilon)$ we define $u_{\Box,\alpha}$ as

$$u_{\Box,\alpha}(x) \equiv u(x) + \frac{\omega_\alpha(\|x - \overline{x}\|)}{2}(x - \overline{x})'\Box(x - \overline{x})$$

where $\omega_\alpha : R \to R$ is a smooth bump function, $\omega_\alpha \mid_{(-\frac{\alpha}{2}, \frac{\alpha}{2})} \equiv 1$ and $\omega_\alpha \mid_{R\setminus(-\alpha,\alpha)} \equiv 0$. In a neighborhood $x \approx \overline{x}$ we have

$$\begin{aligned}
u_{\Box,\alpha}(x) &= u(x) + \frac{1}{2}(x - \overline{x})'\Box(x - \overline{x}) \\
Du_{\Box,\alpha}(x) &= Du(x) + (x - \overline{x})'\Box \Rightarrow Du_{\Box,\alpha}(\overline{x}) = Du(x) \\
D^2u_{\Box,\alpha}(x) &= D^2u(x) + \Box
\end{aligned}$$

So in an α-neighborhood the Hessian changes, by $\Box$, but the gradient, demand do not.

For small enough $\alpha, \Box$ this utility remains in Debreu's setting, so neoclassical demand is defined and smooth when active.

In the Sufficient Independence of Reactions, the path of risk aversions is identified with a linear path $(\Box^h, \alpha^h)(\xi) \equiv (\Box^h \xi, \frac{\|\overline{x}^h\|}{2})$ for each household, so that $\frac{d}{d\xi} D^2 u^h_{\Box,\alpha}(x) = \Box^h$.

Chapter 4

Example of Pareto Improving Taxation

4.1 Introduction

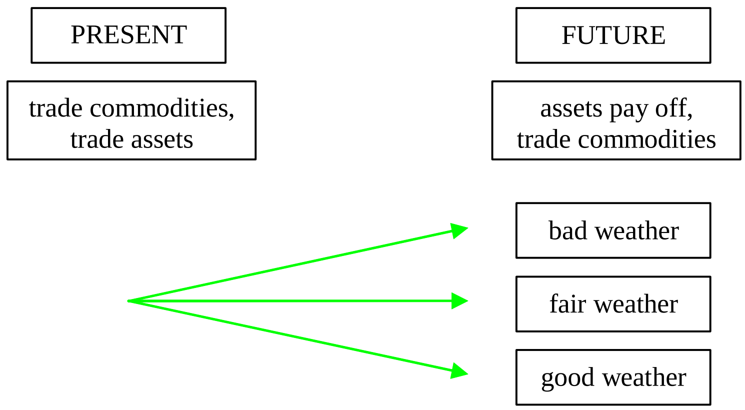
When asset markets are incomplete taxation plays an insurance role. The price adjustment prompted by a tax change provides each agent with marginal purchasing power unspanned by available assets.

In the example the Pareto improving taxes are on current income and on asset purchases. The gist is to engineer each agent's marginal purchasing power to covary negatively with his marginal utility of income.

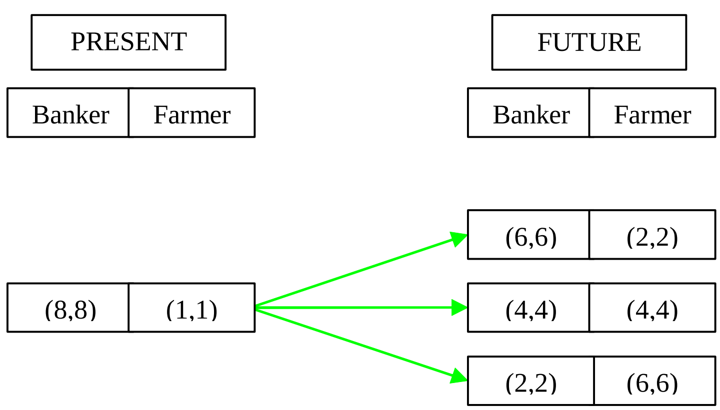
The mean welfare impact of price adjustment equals the covariance between the equilibrium's insurance deficits and net trades. The Pareto improvement following taxes is small, but so is the improvement following the removal of all future uncertainty, i.e. the replacement of future endowments with their expectation.

4.2 Economy

Today a Banker and a Farmer face uncertain future weather, which will turn out to be poor, fair, or good:



They consume corn c and caviar C in both periods, and trade a bond today, which in the future will pay off one unit of caviar, whatever the weather turns out to be. Today the Banker is extremely rich. Future weather will destroy 10% of aggregate supplies, hurting them symmetrically:



Their preferences are von Neumann-Morgenstern, with a quadratic-Cobb Douglas state utility,

$$U(x) = \sum_s \gamma_s Q(u(x_s))$$
$$Q(u) = u - \frac{\beta}{2}u^2$$
$$u(c, C) = c^{\alpha_c} C^{\alpha_C}$$

the Banker being the one more inclined toward caviar and risk, but sharing with the Farmer the same impatience and expectation of weather:

$$\alpha^B = (.25, .75) \qquad \alpha^F = (.75, .25)$$
$$\beta^B = .08 \qquad \beta^F = .15$$
$$\gamma_0 = 1.05 \qquad \gamma_{1,2,3} = \frac{1}{3}$$

where states appear as 0 = today, 1 = poor, 2 = fair, 3 = good.

Each one enjoys corn and caviar equally in every weather, and would be better off by fully insuring. Full insurance would obtain if each one were to donate half his endowment to the other, but in fact they do not trade contingent goods with each other. They trade only income via asset markets, and only in an incomplete way. The only asset is the bond, so they cannot transfer income across states to mitigate the most crushing contingency.

4.3 Equilibrium

If $p_s = (p_{sc}, 1)$ are the prices, $e_s = (e_{sc}, e_{sC})$ the endowments, and $x_s = (x_{sc}, x_{sC})$ the consumptions of corn and caviar in state s, and if each of y units of the bond costs q, then an agent's budgets become

$$p_s' x_s = w_s(y)$$

once he elects his insurance (state contingent income) $w(y)$ by trading the bond:

$$w_s(y) = p_s' e_s - qy \qquad s = 0$$
$$w_s(y) = p_s' e_s + qy \qquad s > 0$$

The problem is to optimize y to then demand as Cobb Douglas and obtain state indirect utility $Q(v_s)$:

$$\begin{aligned} x_i(p_s, w_s) &= \frac{\alpha_i}{p_{si}} w_s \\ v(p_s, w_s) &= \prod_i \left(\frac{\alpha_i}{p_{si}} \right)^{\alpha_i} w_s \end{aligned}$$

The equilibrium variables–corn and bond prices–take the following values (identifying p_s, p_{sc}):

$$p_0 = .626965 \qquad q = 1.10166$$
$$p_1 = .38786 \quad p_2 = .734826 \quad p_3 = 1.3131$$

In the future, the price of corn rises with the endowment wealth of its natural buyer, the Farmer. The economy's future contraction makes the interest rate $1 + r = \frac{1}{q}$ negative. The rich Banker lends by buying the bond $y^B = 2.1214 = -y^F$. So their elected insurance is

B	F	$\longrightarrow$	B	F
			10.4486	.654325
10.6787	3.96403		9.0607	4.81791
			6.7476	11.7572

and net trades in corn are

B	F	$\longrightarrow$	B	F
			.734741	−.734741
−3.74192	3.74192		−.917398	.917398
			−.715331	.715331

4.4 Taxes

The interesting pattern is that the endowment-richer agent is a net demander of corn in extreme weather, poor and good. If the price of corn were to rise in extreme weather, wealth would flow from richer to poorer, netting both better insurance. That is, price adjustment can improve on the incomplete asset insurance. Taxation is one way of engineering such a price adjustment.

4.4.1 Welfare impact of taxes

Write t_1, t_2 for the tax rates on current income and on asset purchases, and distribute tax revenue evenly. Let

$$\tau^i = \begin{bmatrix} w_0^i & y_+^i \\ 0 & 0 \end{bmatrix}_{S+1\times 2} \qquad dP_{S+2\times 2} = ((dp_{sc})_s, dq)$$

be the derivatives of tax payment by agent i and of equilibrium prices, with respect to these tax rates. (Here $y_+ = \max(y, 0)$.) Then the derivative of tax revenue is $\tau = \tau^B + \tau^F$.

Let

$$z^i = \begin{bmatrix} x_{0c}^i - e_{0c}^i & & 0 & y^i \\ & . & & \\ 0 & & x_{Sc}^i - e_{Sc}^i & 0 \end{bmatrix}_{S+1\times S+2}$$

be the net demand for contingent corn and for the bond. Then the derivative of indirect utility is

$$dv^i = \lambda^{i\prime} dm^i$$
$$dm^i = \left(\tfrac{1}{2}\tau - \tau^i\right) - z^i dP$$

where $\lambda^i \in R^{S+1}$ is marginal utility of income and $dm^i \in R^{S+1\times 2}$ the derivative of purchasing power.

4.4.2 Pareto improvement

Normalizing $\lambda^i \rightarrow \frac{\lambda^i}{\lambda^i_0}$ to quote marginal indirect utility in the numeraire today, it turns out that

$$dv = \begin{bmatrix} dv^B \\ dv^F \end{bmatrix} = \begin{bmatrix} -4.49349 & .316879 \\ 4.69088 & -.239745 \end{bmatrix}$$

which is invertible and so admits a solution to $dv^i \cdot \dot{t} = 1$–to Banker and Farmer improving at the same rate. Scaling this solution so that $\dot{t}_2$ equals 1% of the original bond price,

$$\dot{t} = \begin{bmatrix} .000667669 \\ .0110166 \end{bmatrix}$$

Then the price adjustment is

$$\dot{P} = dP \cdot \dot{t} = \begin{bmatrix} .251168 & -.138526 \\ .222179 & .0868223 \\ .277723 & .108528 \\ .370298 & .144704 \\ -.0504975 & -.865367 \end{bmatrix} \dot{t} = \begin{bmatrix} -.00135839 \\ .00110483 \\ .00138104 \\ .00184138 \\ -.00956713 \end{bmatrix} = \begin{bmatrix} \dot{p}_{0c} \\ \dot{p}_{1c} \\ \dot{p}_{2c} \\ \dot{p}_{3c} \\ \dot{q} \end{bmatrix}$$

Indeed, corn prices rise in the future, improving insurance.

The private and public impacts on the Banker's purchasing power are

$$\dot{m}^B_{private} = \left(\frac{1}{2}\tau - \tau^B\right) \cdot \dot{t} = \begin{bmatrix} -5.69438 & -1.0607 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \dot{t} = \begin{bmatrix} -.0154873 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\dot{m}^B_{public} = -z^B \dot{P} = - \begin{bmatrix} -3.74192 \cdot -.00135839 + 2.1214 \cdot -.00956713 \\ .734741 \cdot .00110483 \\ -.917398 \cdot .00138104 \\ -.715331 \cdot .00184138 \end{bmatrix} = \begin{bmatrix} .0151923 \\ -.000808587 \\ .001262 \\ .00131204 \end{bmatrix}$$

so marginal purchasing power is

$$\dot{m}^B = \begin{bmatrix} -.0154873 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} .0151923 \\ -.000808587 \\ .001262 \\ .00131204 \end{bmatrix} = \begin{bmatrix} -000274584 \\ -.000811764 \\ .00126696 \\ .0013172 \end{bmatrix} = \begin{bmatrix} \dot{m}^B_0 \\ \dot{m}^B_1 \\ \dot{m}^B_2 \\ \dot{m}^B_3 \end{bmatrix}$$

Thus the Banker's marginal welfare is

$$\dot{v}^B = \lambda^{B\prime} \dot{m}^B = \begin{bmatrix} 1 & .3133 & .373123 & .415239 \end{bmatrix} \begin{bmatrix} -.000274584 \\ -.000811764 \\ .00126696 \\ .0013172 \end{bmatrix} = .00049078$$

The Farmer's marginal purchasing power is $\dot{m}^F = -\dot{m}^B$ and marginal welfare is

$$\dot{v}^F = \lambda^{F\prime}\dot{m}^F = \begin{bmatrix} 1 & .776987 & .261203 & .0634717 \end{bmatrix} \begin{bmatrix} .000274584 \\ .000811764 \\ -.00126696 \\ -.0013172 \end{bmatrix} = .00049078$$

In sum, we have a Pareto improvement, equal to the agents as intended.

The marginal welfare is positive-sum even though the marginal purchasing power is zero-sum. This is thanks to disperse marginal utilities of income. In the Banker's future, the largest income loss $\dot{m}_1^B = -.000811764$ occurs in the state where his marginal utility of income is lowest $\lambda_1^B = .3133$ and the largest income gain $\dot{m}_3^B = .0013172$ where it is highest $\lambda_3^B = .415239$. Likewise in the Farmer's future. That is, for both, marginal purchasing power covaries negatively with marginal utility of income.

The preceding is a linearization of the Pareto improvement. A calculation of the actual equilibrium with the same tax rates gives the following marginal welfare in terms of the numeraire today

$$\frac{\Delta v^B}{\lambda_0^B} = \frac{.000134}{.304544} = .00044$$
$$\frac{\Delta v^F}{\lambda_0^F} = \frac{.000213}{.440828} = .00048$$

This is a small improvement relative to today's supply of the numeraire, 9. On the other hand, utilities are $v^B = 9.4421, v^F = 4.72425$ and would be only $v^B = 9.60662, v^F = 5.0783$ at an equilibrium of the economy without uncertainty, where future endowments are replaced by their expectation $(4, 4)$ in every state.

4.4.3 Relation of welfare surplus to insurance deficits

Though there is an adjustment of current corn and bond prices, it does not contribute to a Pareto improvement, helping one agent only by hurting another one equally. To emphasize this point, let us compute the derivative of **welfare surplus** $dW = dv^B + dv^F$ in the common unit of account, the numeraire today. From the first order condition, the marginal utility for future income $\lambda_{\mathbf{1}}^i \in R^S$ projects to a common point in the asset span $a = span1$,

$$\lambda_{\mathbf{1}}^i = c + \delta^i \in a + a^{\perp}$$

differing across agents only in the **insurance deficit** δ^i. This sum and the identity $\Sigma dm^i = 0$ imply

$$\begin{aligned} dW &= \Sigma\lambda^{i\prime}dm^i \\ &= \Sigma\lambda_0^i \cdot dm_0^i + \Sigma\lambda_{\mathbf{1}}^{i\prime}dm_{\mathbf{1}} \\ &= \Sigma 1 \cdot dm_0^i + \Sigma(c+\delta^i)'dm_{\mathbf{1}}^i \\ &= 0 + c'(\Sigma dm_{\mathbf{1}}^i) + \Sigma\delta^{i\prime}dm_{\mathbf{1}}^i \\ &= \Sigma\delta^{i\prime}dm_{\mathbf{1}}^i \\ &= \Sigma\delta^{i\prime}\left(-z_{\mathbf{1}}^i dP_{\mathbf{1}}\right) \\ &= -\left(\Sigma\delta^{i\prime}z_{\mathbf{1}}^i\right)dP_{\mathbf{1}} \\ &= -2covariance(\delta, z_{\mathbf{1}})dP_{\mathbf{1}} \end{aligned}$$

using $\Sigma z_{\mathbf{1}}^i = 0$ for the latter. For a Pareto improvement it is necessary that the surplus be positive, $dW \cdot \dot{t} > 0$. Indeed, the surplus is independent of the adjustment in current corn and bond prices dP_0, whatever the tax rates. The only control for Pareto improvement is the future price adjustment $dP_{\mathbf{1}}$, since insurance deficits and net trades are set by the equilibrium. More precisely, the mean surplus from the only control is minus the covariance of insurance deficits with future net trades. Of course, Pareto improvement does lend the current price adjustment dP_0 a role, which is to help allocate the positive surplus.

Chapter 5

How Much Trade for the Transfer Paradox? The threshold computed

5.1 Introduction

Germany's reparations after World War I provoked a controversy about terms of trade. Did the reparations improve or worsen her terms of trade? Did the new terms of trade exacerbate or mitigate her income loss due to reparations? Leontief (1937) showed by example that a donation could so change terms of trade as to erase the income loss and benefit donor–the transfer paradox.

Samuelson (1947) noted the regular equilibria exhibiting the transfer paradox were those unstable with respect to tatonnement. Others confirmed this beautiful characterization of the transfer paradox, at least with two countries and two commodities; Mundell (1968), Balasko (1978).

Theorem 5.1 (Samuelson 1947). *With two countries and two goods, suppose a regular equilibrium. Then the local transfer paradox is present if and only if it is unstable.*

Most deemed instability a theoretical curiosity, the situation where demand increases with prices. By Samuelson's equivalence, the transfer paradox too became a theoretical curiosity, and interest in it waned. Accordingly, Samuelson's equivalence remained the big result on the transfer paradox, and became the wisdom on the topic.

Almost thirty years later, Gale (1974) showed by example that Samuelson's equivalence broke down with a third country.

Theorem 5.2 (Gale 1974). *With three Leontief countries and two goods, there is an example of a stable equilibrium exhibiting the local transfer paradox.*

Yet the example failed to shatter the received wisdom, perhaps because Gale never pointed out its stability, never wrote "transfer paradox."

Chichilnisky (1980) discovered the stability of Gale's example, and further showed its dependence on the preferences of the countries. That it took so long to detect stability evidenced how ingrained Samuelson's wisdom had been–why check, if it must be unstable? Once advertised, this set off a stampede of research in the early eighties, excited by the surprising news, by the renewed plausibility of the transfer paradox, and by the chance to charge at current wisdom.

The stampede mostly split between extending Gale's counterexample and Chichilnisky's analysis, always with two goods. New examples appeared in Polemarchakis (1983), and in Leonard and Manning (1983) with non-Leontief utilities (two Cobb-Douglas, one quasilinear).[1] The analyses (a) relaxed utilities from being Leontief, (b) clarified the role of excess demands, marginal propensities to consume, and elasticities of excess demand, (c) derived

[1] Aumann and Peleg (1974) discarded endowments, instead of reallocating them.

formulas for the welfare impact of small donations in terms of these notions. Yano (1983), Ravallion (1983), Bhagwati et al. (1983), Dixit (1983) singly managed all these extensions. Retaining Leontief utilities, Geanakoplos and Heal (1983), Polemarchakis (1983), and Chichilnisky (1983) gave a priori, equilibrium-independent bounds on endowments and utilities guaranteeing the equilibrium to be unique, globally stable, and consistent with the transfer paradox. Consensus settled on

- the donor's trade level being required large enough,

and on this requisite level being increasing in

- 1) the proximity between the donor's and the recipient's marginal propensities to consume
- 2) the substitution effect, explaining the preponderance of Leontief utilities in examples

In particular, emphasis turned toward the notions in (b) and away from stability.

The remainder focused on the existence question. From Dixit's (1983) formula Safra (1984) obtained

Theorem 5.3 (Safra 1984). *With more than two countries and with two goods, suppose an unstable equilibrium where some trading country's marginal propensity to consume is neither largest nor lowest. Then there is a stable equilibrium exhibiting the transfer paradox, with the same equilibrium prices and incomes but less trade.*

This was another charge, generalizing Gale's example to smooth preferences and multiple countries–curiously, instability did cameo. Earlier, Safra (1983) had argued nonconstructively that for almost any equilibrium prices and incomes, there was a compatible economy exhibiting the transfer paradox. Given any small desired welfare impact and endowment reallocation, Donsimoni and Polemarchakis (1994) showed more constructively that for almost any equilibrium prices and incomes, there was a compatible economy exhibiting the given welfare impact as the de facto welfare impact of the given reallocation.

Altogether, research after Gale and Chichilnisky sidelined Samuelson's equivalence more than the transfer paradox as the theoretical curiosity. If not all targeted what was wrong with Samuelson's equivalence, none looked for what was right with it.

We propose a reinterpretation of Samuelson's equivalence that reaffirms the above consensus. The key idea is that whether an equilibrium is unstable or stable is a precise answer to whether the trade level is or is not large enough relative to

- 1) the proximity between the donor's and the recipient's marginal propensities to consume
- 2) the substitution effect

To see it, we revisit the classical decomposition of the Jacobian J of aggregate demand

$$J = S - \Sigma m^i z^i$$

where S is the sum of the countries' substitution effects, m^i is country $i's$ marginal propensity to consume, and z^i its excess demand for the nonnumeraire commodities. With two countries, it reads

$$J = S - \nabla z^1$$

where $\nabla = m^1 - m^2$ is the difference between their marginal propensities to consume, thanks to market clearing $z^1 + z^2 = 0$. With two goods, an equilibrium is unstable, by definition, if $J > 0$. Thus Samuelson's equivalence is that the transfer paradox is present or absent according as $J > 0$ or $J < 0$. The threshold is $J = 0$, i.e. the **threshold** trade level z^1 is

$$z^1 = \frac{S}{\nabla} \qquad (|)$$

Indeed, this reaffirms the anti consensus, in that the threshold trade level is increasing in the proximity ∇ between marginal propensities to consume, and in the substitution effect S. Samuelson's equivalence encapsulates and quantifies this anti consensus, once reinterpreted.

We show that the threshold reinterpretation generalizes fully to a finite number of countries and commodities. This requires making sense of the ratio $\frac{S}{\nabla}$ with multiple commodities, when ∇ is no longer a scalar. It requires making sense of the trade level $|z|$ with multiple countries, when the equality $|z^1| = |z^2|$ as an unambiguous norm is unavailable.

Fixing the price of C commodities and incomes of H countries, implies the aggregate substitution effect $S = \Sigma S^i$ and the marginal propensities to consume (m^i). Discarding the numeraire, S is negative definite and symmetric, hence defines an inner product on net trades $n \in R^{C-1}$ of nonnumeraire commodities, $(n, n) = n'(-S^{-1})n$, and a norm, $\|n\| = \sqrt{(n, n)}$. If $z = (z^h)$ are the equilibrium net trades at the equilibrium prices and incomes, the **trade level** is $\|z\|^* = \sqrt{\frac{1}{H}\Sigma \|z^h\|^2}$.

What is Samuelson's threshold in this language? Multiplying (|) by $-z^1 S^{-1}$,

$$\begin{aligned} z^1(-S^{-1})z^1 &= \frac{-z^1}{\nabla} = \frac{1}{\nabla(-S^{-1})\nabla} \\ \left\|z^1\right\| &= \frac{1}{\|\nabla\|} \end{aligned}$$

Thanks to market clearing, $\left\|z^2\right\| = \frac{1}{\|\nabla\|}$ and

$$\|z\|^* = \frac{1}{\|\nabla\|} \qquad (|)$$

Theorem 5.4 (Samuelson reinterpreted). *With two countries and two goods, the threshold for the transfer paradox at regular equilibria is* $\frac{1}{\|\nabla\|}$.

One generalization is to multiple goods $C \geq 2$.

Theorem 5.5 (Threshold with multiple goods). *With two countries, the threshold for the transfer paradox at regular equilibria is still* $\frac{1}{\|\nabla\|}$.

With multiple countries, the donor can play the welfare of one recipient against another's, unboundedly. With just two countries, this is impossible because there is a sole recipient. For this reason the threshold is no greater than the above. Specifically, for each country let

$$\nabla^h = m^h - \frac{1}{H-1}\Sigma_{i\neq h} m^i$$

With $H = 2$ clearly $\nabla^1 = \nabla$. Then

Theorem 5.6 (Threshold bounded above). *With* $H, C \geq 2$ *countries and goods, the threshold for* h *to be a protagonist in the transfer paradox at regular equilibria is at* *most* $\frac{1}{\sqrt{H-1}\|\nabla^h\|}$. *So the threshold for the transfer paradox at regular equilibria is at most* $\min_h \frac{1}{\sqrt{H-1}\|\nabla^h\|}$.

Now we compute the threshold for h to be a protagonist.

Definition 5.1. *Fix* $\dot{v} \in R^H$ *with* $\dot{v}^h = 1, 1'\dot{v} = 0$, *to be interpreted as the welfare impact of an infinitesimal donation. Then define the numerator*

$$n(\dot{v}) = \frac{\left(\Sigma_S \dot{v}^i\right)^2}{H - |S|} + \Sigma_S \dot{v}^{i2} \tag{5.1}$$

where $S \subset \{1, ..., H\} \backslash \{h\}$ *is as follows. Ordering* $\dot{v}^{-h} : \dot{v}^{i_1} \leq ... \leq \dot{v}^{i_{H-1}}, S = \{i_1, ..., i_n\}$ *for the largest* n *such that*

$$\text{if } i \in S \qquad \frac{-\Sigma_S \dot{v}^i}{H - |S|} \geq \dot{v}^i$$

Now define

$$\nabla^h(\dot{v}) = m^h + \Sigma_{i \neq h} m^i \dot{v}^i$$

Finally, define[2]

$$\boxed{T^h = \sqrt{\inf \frac{n(\dot{v})}{H\|\nabla^h(\dot{v})\|^2}} \qquad \text{subject to } \dot{v}^h = 1, 1'\dot{v} = 0}$$

Theorem 5.7 (Threshold computed). *With* $H, C \geq 2$ *countries and goods, the threshold for* h *to be a protagonist in the transfer paradox at regular equilibria is* T^h. *So the threshold for the transfer paradox at regular equilibria is* $\min_h T^h$.

It seems impossible to compute T^h in general; after all, the program is the ratio of two

[2] This exists by completeness of the reals, because the objective is bounded below by zero.

convex functions over a noncompact domain. Of course, given particular equilibrium prices and incomes, a computer would.

On the other hand, the upper bound is easily seen to come from $\dot{v}^{-h} = -\frac{1}{H-1}$. For the numerator, note $\dot{v}^{-h} \leq 0$ hence $S = \{1, ..., H\} \backslash \{h\}$, and then compute $n(\dot{v}) = \frac{H}{H-1}$.[3] Noting also $\nabla^h(\dot{v}) = \nabla^h$,

$$T^h \leq \frac{\sqrt{\frac{H}{H-1}}}{\sqrt{H}\left\|\nabla^h\right\|} = \frac{1}{\sqrt{H-1}\left\|\nabla^h\right\|}$$

This gives theorem 5.6. Further, when $H = 2$ the constraint set $\dot{v}^h = 1, 1'\dot{v} = 0$ is a singleton, the above $\dot{v}^{-h} = -\frac{1}{H-1}$, and this upper bound is the $\inf$. This gives theorem 5.5.

Our notion of threshold with multiple commodities is Samuelson's if $H = 2$, but weaker if $H > 2$. It is equal in that no equilibria are paradoxical with trade levels below the threshold. It is different in that not all equilibria with trade levels beyond the threshold need be paradoxical, but there exists a sequence of paradoxical equilibria with trade levels converging from above to the threshold.

5.2 Model

Countries $h = 1, ..., H$ consume commodities $c = 1, ..., C$, C being the unit of account, in terms of which all value is quoted. Markets assign prices $p \in P \equiv R_{++}^{(C-1)}$ to commodities

[3] $n(\dot{v}) = \frac{\left(\Sigma_{i \neq h} - \frac{1}{H-1}\right)^2}{1} + \Sigma_{i \neq h}\left(-\frac{1}{H-1}\right)^2 = 1 + (H-1)\frac{1}{(H-1)^2} = \frac{H}{H-1}$

$c < C$, and incomes $w \in R^H_{++}$ to all countries.[4] The set of **budget variables** is

$$b \equiv (p, w) \in B \equiv P \times R^H_{++}$$

and commodity **demands** $x^h : B \to R^C_{++}$ depend on own income only, $x^h(p, w) = x^h(p, w')$ if $w^h = w'^h$.

The **price-income equilibria** for total **resources** $r \in R^C_{++}$ are

$$B(r) = \left\{ b \in B \mid \Sigma x^h(b) = r \right\}$$

In an **economy**, countries' endowments of commodities make up total resources,

$$\Omega(r) = \left\{ e \in R^{C \times H}_{++} \mid \Sigma e^h = r \right\}$$

The **equilibria** are

$$E(r) = \left\{ (p, e) \in P \times \Omega(r) \mid (p, e'\overline{p}) \in B(r) \right\}$$

There is a natural projection $\pi : E(r) \to B(r), \pi(p, e) = (p, e'\overline{p})$ and a $b-$**equilibrium** is one in $\pi^{-1}(b)$.

Demand is neoclassical if there is a utility $u^h : R^C_+ \to R$ solving $u^h(x(b)) = \max_{\beta^h(b)} u$ throughout $b \in B$, where $\beta^h(b) = \left\{ x \in R^C_+ \mid \overline{p}' x = w^h \right\}$. In this case **welfare** is $v(b) = (v^h(b)) = (u^h(x^h(b)))$. The point of separating budget variables from the economy is that welfare is determined by the budget variables, and in turn these are determined by the

[4] Unity is the price of C, which P omits. The addition to p of the C coordinate with value unity is denoted $\overline{p}$.

economy in equilibrium. We assume Debreu's smooth preferences.

5.3 Welfare impact of reallocation

We think of a smooth path $e(\xi)$ through a given economy $e = e(0)$, and of an **infinitesimal reallocation** as its velocity $\dot{e}$. Suppose the equilibrium (p, e) is regular in that equilibrium prices are locally a smooth function of the economy. Then welfare is $v(b(\xi))$ with $b(\xi) = (p(\xi), e(\xi)'\overline{p}(\xi))$. Thus a reallocation impacts welfare only via the budget variables it implies. By the fundamental theorem of calculus the welfare impact is the integral of $\dot{v} = D_b v \cdot \dot{b}$, which by abuse we call the **welfare impact**. We prefer to quote it not as $\dot{v}^h$, in individual utils, but in the numeraire, as $\dot{v}^{*h} = \frac{\dot{v}^h}{\lambda^h}$, where $\lambda^h = D_{w^h} v^h$ is the marginal utility of the numeraire. Roy's identity gives $D_b v^h$:

Proposition 5.1 (Envelope). *The welfare impact* $\dot{v} \in R^H$ *of* $\dot{e}$ *at a regular equilibrium is*

$$\dot{v}^* = \dot{t} - \underline{z}'\dot{p}$$

where $\dot{t} \equiv \dot{e}'\overline{p}$ *is its value, and* $z \in R^{C \times H}$ *the countries'* ***excess demands***.[5]

As we show next, at a regular equilibrium there is a unique **price adjustment** matrix dp, smooth in a neighborhood of it, such that $\dot{p} = dp\dot{t}$. Thus the welfare impact differential is

$$dv^* = I - \underline{z}'dp \tag{5.2}$$

This implies that a reallocation matters for welfare only through its value, not its identity.

[5]Throughout, an underscore denotes the omission of the numeraire coordinate C.

Remark 5.1. dv^* *is an operator* $\dot{t} \mapsto \dot{v}^*$ *in* $1^\perp \subset R^H$.

Indeed, $1'\dot{t} = 1'\dot{e}'\overline{p} = \dot{r}'\overline{p} = 0$ given that aggregate resources are fixed, and $1'dv^*\dot{t} = (1' - \underline{0}'dp)\dot{t} = 0$ given that total excess demand is zero in equilibrium.

To compute dp, we totally differentiate total **nonnumeraire demand**

$$x^\sigma(b) \equiv \Sigma \underline{x}^h(b)$$

Write

$$J \equiv D_p x^\sigma((p, e'\overline{p}))$$

and suppose a path $(p(\xi), e(\xi))$ of equilibria. Then

$$x^\sigma((p, e'\overline{p})) = \underline{r}$$

is an identity. Differentiating it,

$$J\dot{p} + D_w x^\sigma \dot{t} = 0$$

An equilibrium is **regular** if J is invertible. By the implicit function theorem and Walras' law, at a regular equilibrium (p, e) equilibrium prices are locally a smooth function of the economy.

Proposition 5.2 (Price Adjustment). *At a regular equilibrium the Price Adjustment is*[6]

$$dp = -J^{-1} D_w x^\sigma \qquad (dp)$$

[6]Since demands depend on own income only, $D_w x^\sigma = [D_{w^1}\underline{x}^1 : ... : D_{w^H}\underline{x}^H]$.

This implies that a reallocation matters for prices only through its value, not its identity.

Substituting into (5.2),

$$\boxed{dv^* = I + \underline{z}'J^{-1}D_w x^\sigma} \tag{dv^*}$$

This formula generalizes Dixit (1981) from $C = 2$ and appears in Donsimoni and Polemarchakis (1994). Note, the welfare impact $\dot{v}^*$ of a reallocation equals its value $\dot{t}$ if there is no trade $\underline{z} = 0$ or if all marginal propensities to consume $D_{w^h}\underline{x}^h$ agree. (For then $\dot{t} \in 1^\perp$ implies $D_w x^\sigma \dot{t} = 0$.)

If demand is neoclassical, then the Slutsky decomposition $D_p\underline{x}^h = S^h\lambda^h - D_{w^h}\underline{x}^h \cdot \underline{x}^{h\prime}$ and the equilibrium incomes $e'\overline{p}$ imply that $D_p\underline{x}^h((p, e'\overline{p})) = S^h\lambda^h - D_{w^h}\underline{x}^h \cdot \underline{z}^{h\prime}$. Adding,

$$\boxed{J = S - D_w x^\sigma \cdot \underline{z}'} \tag{5.3}$$

Here the sum $S \equiv \Sigma S^h\lambda^h$ is symmetric and negative definite, since each summand $S^h\lambda^h$ is.

5.4 Definition of threshold

We reinterpret Samuelson's condition for general C, H, in terms of the requisite **trade level** $L \in R$.

Definition 5.2 (Trade levels for a protagonist: Necessary and Sufficient). *Fix $b \in B(r)$ and the associated $S(b) \in R^{C-1\times C-1}$ in (5.3). The* ***norm at*** *b is defined on R^{C-1} as $\|a\| = \sqrt{a \cdot a}$ from the inner product $a \cdot b = a(-S^{-1})b$.*[7] *At a $b-$equilibrium, the* ***trade***

[7] Recall, an inner product is the root of a symmetric, positive definite quadratic form, and indeed $-S^{-1}$ is positive definite and symmetric, according to the consumer theory of Samuelson.

level *is* $\|\underline{z}\| \equiv \sqrt{\frac{1}{H}\Sigma \|\underline{z}^k\|^2}$. L *is* $b-$***necessary for*** h *if every regular* $b-$*equilibrium with* h *a protagonist in the transfer paradox has* $\|\underline{z}\| \geq L$. L *is* $b-$***sufficient for*** h *if for every* $\epsilon > 0$ *there is a regular* $b-$*equilibrium with* h *a protagonist in the transfer paradox and* $\|\underline{z}\| \leq L + \epsilon$.

Whenever L_n is necessary and L_s is sufficient, $L_n \leq L_s$, so there is at most one threshold:

Definition 5.3. *Call* $L^h \in R$ ***the*** $b-$***threshold for*** h *to be a protagonist if it is both* $b-$*sufficient and necessary for* h.

Definition 5.4. *Call* $L \in R$ ***the*** $b-$***threshold*** *for the transfer paradox if* $L = \min_h L^h$.

Remark 5.2. *As shown in the introduction, Samuelson's result with* $C = H = 2$ *means that a threshold exists and equals* $\frac{1}{\|\nabla\|}$ *–for both to be protagonists and for the transfer paradox. To fully generalize this, we need to explicitly compute the inverse of the welfare impact differential.*

5.5 The inverse of the welfare impact differential dv^*

Remarkably, the inverse of dv^* exists and admits an explicit description!

Theorem 5.8 (The inverse of the welfare impact differential dv^*). *Suppose the equilibrium is regular, so that* dv^* *is defined. Then it is invertible, with inverse*

$$\boxed{dv^{*-1} = I - \underline{z}' S^{-1} D_w x^\sigma} \qquad (dv^{*-1})$$

Proof. We use the decomposition $J = S - D_w x^\sigma \cdot \underline{z}'$. By definition, the inverse of dv^*, should it exist, is a solution (necessarily unique) to the equations $dv^* s = I, s dv^* = I$. We show that $I - \underline{z}' S^{-1} D_w x^\sigma$ is such a solution:

$$\begin{aligned}
& dv^* \left(I - \underline{z}' S^{-1} D_w x^\sigma\right) \\
= & \left(I + \underline{z}' J^{-1} D_w x^\sigma\right)\left(I - \underline{z}' S^{-1} D_w x^\sigma\right) \\
= & I - \underline{z}' S^{-1} D_w x^\sigma + \underline{z}' J^{-1} D_w x^\sigma - \underline{z}' J^{-1}\left(D_w x^\sigma \underline{z}'\right) S^{-1} D_w x^\sigma \\
= & I - \underline{z}' S^{-1} D_w x^\sigma + \underline{z}' J^{-1} D_w x^\sigma - \underline{z}' J^{-1} (S - J) S^{-1} D_w x^\sigma \\
= & I - \underline{z}' S^{-1} D_w x^\sigma + \underline{z}' J^{-1} D_w x^\sigma - \underline{z}' \left(J^{-1} - S^{-1}\right) D_w x^\sigma \\
= & I
\end{aligned}$$

Likewise, the equation $\left(I - \underline{z}' S^{-1} D_w x^\sigma\right) dv^* = I$ holds. □

Remark 5.3. dv^{-1*} *is an operator* $\dot{t} \mapsto \dot{v}^*$ *in* $1^\perp \subset R^H$.

This follows from remark 5.1.

5.5.1 A universal example of the arbitrariness of the welfare impact

Donsimoni and Polemarchakis (1994) in the case of general C, H conclude that given any $\dot{t}, \dot{v} \in 1^\perp$ satisfying $\dot{t}^h, \dot{v}^h \neq 0$ for some h, there exist marginal propensities to consume $D_{w^h} \underline{x}^h$ and net trades for which $\dot{v} = dv^* \dot{t}$. Save for Pareto optimality, the welfare impact of reallocations is arbitrary without knowledge of marginal propensities to consume and of net trades. Here we sharpen this result: the welfare impact is arbitrary without knowledge of net trades, even granting knowledge of the marginal propensities to consume. Both in their construction and in ours, equilibrium prices and incomes are known, but endowments may be nonpositive.

Our construction is explicit.

Theorem 5.9 (Universal example of arbitrariness). *Fix* $b \in B(r)$, *plus the desired value* $\dot{t} \in 1^{\perp}$ *of the reallocation and its welfare impact* $\dot{v} \in 1^{\perp}$. *If* $\nabla(b) \equiv D_w \underline{x}^{\sigma} \dot{v} \neq 0$, *then the economy* $\underline{e} \equiv \underline{x}(b) - \underline{z}$ *with excess demands*

$$\underline{z} = \frac{1}{\|\nabla\|^2} \nabla (\lambda \dot{t} - \dot{v})'$$

and numeraire endowments set by b *and the budget identity, defines a regular* $b-$*equilibrium at which* $\dot{v}$ *is the de facto welfare impact of* $\lambda \dot{t}$, *where* λ *is any but finitely many values, unless* $|J(\lambda)|$ *is identically zero.*

Conversely, the trade level $\|\underline{z}\|$ *at any regular* $b-$*equilibrium where* $\dot{v}$ *is the welfare impact of* $\lambda \dot{t}$ *is at least*

$$\frac{\left\|\lambda \dot{t} - \dot{v}\right\|_2}{\sqrt{H}\,\|\nabla\|}$$

with equality only at the latter equilibrium.

Proof. Nonnumeraire markets do clear: $\underline{z}1 = \frac{-1}{\lambda\|\nabla\|^2} \nabla(0 - 0) = 0$. So does the numeraire market: the numeraire endowments imply Walras' law. Suppose a regular $b-$equilibrium. Then $\dot{v}$ is the welfare impact of $\lambda \dot{t}$ iff $dv^{*-1} \dot{v} = \lambda \dot{t}$:

$$\begin{gathered} \left(I - \underline{z}' S^{-1} D_w x^{\sigma}\right) \dot{v} = \lambda \dot{t} \\ -\underline{z}' S^{-1} \nabla = \lambda \dot{t} - \dot{v} \\ -\underline{z}^{k\prime} S^{-1} \nabla = \lambda \dot{t}^k - \dot{v}^k \end{gathered} \tag{5.4}$$

The Cauchy-Schwarz inequality implies $\left\|\underline{z}^k\right\| \|\nabla\| \geq -\underline{z}^{k\prime} S^{-1}\nabla$ hence

$$\left\|\underline{z}^k\right\| \geq \frac{\lambda \dot{t}^k - \dot{v}^k}{\|\nabla\|}$$

with equality only if $\underline{z}^k = \alpha^k \nabla$ for some scalar α^k. Applying definition 5.2 of $\|\underline{z}\|$,

$$\|\underline{z}\| \geq \sqrt{\frac{1}{H}\Sigma\left(\frac{\lambda \dot{t}^k - \dot{v}^k}{\|\nabla\|}\right)^2} = \frac{\left\|\lambda \dot{t} - \dot{v}\right\|_2}{\sqrt{H}\,\|\nabla\|}$$

To find the $\alpha = (\alpha^k)$ achieving equality, substitute $\underline{z}' \equiv \alpha \nabla'$ in (5.4) to get

$$\alpha = \frac{-1}{\nabla' S^{-1} \nabla}(\lambda \dot{t} - \dot{v}) = \frac{1}{\|\nabla\|^2}(\lambda \dot{t} - \dot{v})$$

Thus $dv^{*-1}\dot{v} = \lambda \dot{t}$ follows from

$$\underline{z}' \equiv \frac{1}{\|\nabla\|^2}(\lambda \dot{t} - \dot{v})\nabla'$$

provided this $\underline{z}$ makes the equilibrium regular, i.e., $|J(\lambda)|$ invertible:

Now $|J|$ is polynomial in J, which is linear in λ (writing $\Delta = D_w x^\sigma \dot{t}$) :

$$\begin{aligned} J(\lambda) &= S - D_w x^\sigma \cdot \underline{z}' \qquad (5.5) \\ &= S - \frac{1}{\|\nabla\|^2}(\lambda \Delta - \nabla)\nabla' \end{aligned}$$

So $|J(\lambda)|$ is polynomial in λ, and zero for all but finitely many values, unless it is identically zero. □

5.5.2 A universal example of the transfer paradox

For each price-income equilibrium, we construct a compatible equilibrium with the transfer paradox.

Corollary 5.1 (Universal example of the transfer paradox). *Fix* $b \in B(r)$ *with* $\nabla(b) \equiv D_{w^h}\underline{x}^h - D_{w^i}\underline{x}^i \neq 0$. *Then a donation from* h *to* i, $\dot{t} = \lambda(1^i - 1^h)$, *benefits* h *and hurts* i *and fixes all others' welfare,* $\dot{v}^* = 1^h - 1^i$, *for the economy* $\underline{e} \equiv \underline{x}(b) - \underline{z}$ *with excess demands*

$$\underline{z} = \frac{1+\lambda}{\|\nabla\|^2}\nabla(1^i - 1^h)'$$

and numeraire endowments set by b *and the budget identity, defining a regular* $b-$*equilibrium with the transfer paradox, where* $\lambda > 0$ *is any but finitely many values.*

Proof. This follows from theorem 5.9 since $\dot{t} - \dot{v} = -\lambda\dot{v}^* - \dot{v}^* = (1+\lambda)(1^i - 1^h)$ and $|J(\lambda)|$ is not identically zero. For the latter, substitute $\dot{t} = -\dot{v}$ in (5.5):

$$J(\lambda) = S + \frac{1}{\|\nabla\|^2}(\lambda\nabla + \nabla)\nabla'$$

so that $J(-1) = S$ is negative definite and $|J(-1)| \neq 0$, not the zero polynomial. □

Remark 5.4 (Sufficient level of trade). *Fix* $b \in B(r)$ *where all marginal propensities to consume* $D_{w^h}\underline{x}^h$ *are distinct. Then* $\frac{\sqrt{2}}{\sqrt{H}\|\nabla\|}$ *is a* $b-$*sufficient trade level for the transfer paradox.*

Proof. In example 5.1 $\|\underline{z}\| = \frac{\|\lambda\dot{t} - \dot{v}\|_2}{\sqrt{H}\|\nabla\|}$ and $\lambda\dot{t} - \dot{v} = (1+\lambda)(1^i - 1^h)$, so $\|\underline{z}\| = \frac{\sqrt{2}(1+\lambda)}{\sqrt{H}\|\nabla\|}$.Let $\lambda \searrow 0$. □

Safra (1983) is a predecessor, concluding nonconstructively that ∞ is $b-$sufficient. Note, with $H=2$ this says that $\frac{1}{\|\nabla\|}$ is sufficient, giving half of Samuelson's result. In this example everyone's welfare is fixed other than the donor and the recipient's; in contrast, with $H>2$ there are paradoxical equilibria with even less trade, where the donor affects everyone's welfare. This is not the threshold with $H>2$.

5.6 The threshold for the transfer paradox

Theorem 5.9 states that the trade level at any regular equilibrium where $\dot{v}$ is the welfare impact of $\dot{t}$ is at least $\|\underline{z}\| \geq \frac{\|\dot{t}-\dot{v}\|_2}{\sqrt{H}\|D_w \underline{x}^\sigma \dot{v}\|}$, with equality achieved. To find the threshold, we wish to minimize $\frac{\|\dot{t}-\dot{v}\|_2}{\sqrt{H}\|D_w \underline{x}^\sigma \dot{v}\|}$ subject to the transfer paradox. There is one minimization for each protagonist. Let h be a protagonist, as above normalizing $\dot{t}^h \leq -\lambda < 0, \dot{v}^h = s$ then letting $\lambda \searrow 0$. We wish to solve

$$\min \frac{\left\|\dot{t}-\dot{v}\right\|_2}{\sqrt{H}\left\|D_w \underline{x}^\sigma \dot{v}\right\|} \qquad \text{subject to } \dot{t}^h \leq -\lambda, \dot{v}^h = s, \dot{t}^{-h} \geq 0, 1'\dot{t} = 0 = 1'\dot{v} \qquad (P^h)$$

First, we fix a feasible $\dot{v}$ and solve

$$\min \left\|\dot{t}-\dot{v}\right\|_2^2 \qquad \text{subject to } \dot{t}^h \leq -\lambda, \dot{t}^{-h} \geq 0, 1'\dot{t} = 0 \qquad (5.7)$$

getting a unique minimizer $\dot{t} = \dot{t}(\lambda, \dot{v})$ and value $n(\lambda, \dot{v})$. Since this value is nonincreasing in λ, it is minimized at $\lambda = 0$. In a second step, we wish to solve

$$\min \frac{n(0,\dot{v})}{H\left\|D_w \underline{x}^\sigma \dot{v}\right\|^2} \qquad \text{subject to } \dot{v}^h = s, 0 = 1'\dot{v}$$

This only makes sense if $D_w \underline{x}^\sigma \dot{v} \neq 0$ for some $0 = 1'\dot{v}$, which is equivalent to

Assumption 5.1. *Not all* ***marginal propensities to consume*** $m^i = D_{w^i}\underline{x}^i$ *are equal* $m^1 = ... = m^H$.

We now report $n(0, \dot{v})$.

Lemma 5.1 (Best donation given welfare impact). *Fix* $\dot{v}^h = s, 0 = 1'\dot{v}, \lambda = 0$. *Problem 5.7's value is*

$$n(0, \dot{v}) = \frac{\left(\Sigma_S \dot{v}^i\right)^2}{H - |S|} + \Sigma_S \dot{v}^{i2} \tag{5.8}$$

where, on ordering $\dot{v}^{-h} : \dot{v}^{i_1} \leq ... \leq \dot{v}^{i_{H-1}}, S = \{i_1, ..., i_n\}$ *for the largest* n *such that*

$$\text{if } i \in S \qquad \frac{-\Sigma_S \dot{v}^i}{H - |S|} \geq \dot{v}^i$$

Proof. See appendix. □

For example, if $\dot{v}^{i \neq h} = -\frac{s}{H-1}$ then $S = \{1, ..., H\} \backslash \{h\}$ and $n(0, \dot{v}) = \frac{s^2}{1} + \Sigma_{i \neq h} \left(\frac{s}{H-1}\right)^2 = s^2 \frac{H}{H-1}$.

As above, we normalize $s = 1$ since positive multiples of $(\dot{t}, \dot{v})$ preserve the equation $\dot{t} = dv^{-1}\dot{v}$.

Theorem 5.10 (Protagonist's threshold). *Fix* $b \in B(r)$ *and assumption 5.1. Then the threshold trade level* $T^h(b)$ *for* h *to be a protagonist in the transfer paradox at regular equilibria is*

$$\boxed{T^h(b) = \sqrt{\inf \frac{n(0, \dot{v})}{H \|D_w \underline{x}^\sigma \dot{v}\|^2}} \qquad \text{subject to } \dot{v}^h = 1, 0 = 1'\dot{v}} \tag{*}$$

the numerator being (5.8).

Proof. It is clear that there are no regular equilibria with trade level below this value where h is a protagonist. Conversely, fix $\epsilon > 0$; we want a regular equilibrium with trade level at most $\inf + \epsilon$ where h is a protagonist. Let $\dot{v}_n$ be a sequence that achives the infimum $T^h(b)$. Let $\lambda_n = \frac{1}{n}$ and consider the donation $\dot{t}_n = \dot{t}(\lambda_n, \dot{v}_n)$. Then construction (5.9) gives a $b-$equilibrium, which may be taken regular by slightly increasing $\dot{t}^h_n$, where h is a protagonist and the trade level squared is exactly

$$\frac{n(\lambda_n, \dot{v}_n)}{H \left\| D_w \underline{x}^\sigma \dot{v}_n \right\|^2}$$

which converges to $T^h(b)^2$. □

Corollary 5.2 (Threshold for paradox). *Fix* $b \in B(r)$ *and assumption 5.1. Then the threshold for the transfer paradox is* $\min_h T^h(b)$.

This is clear since the transfer paradox is present if and only if there is a protagonist.

Remark 5.5. *It is hard to make the infimum more explicit, since the objective is the ratio of two convex functions and the constraint set not compact.*

Corollary 5.3 (Protagonist's threshold bounded above). *An explicit upper bound is*

$$T^h(b) \leq \frac{1}{\sqrt{H-1} \left\| \nabla^h \right\|}$$

$\nabla^h = m^h - \frac{1}{H-1} \Sigma_{i \neq h} m^i$ *being the difference from the mean of all others' marginal propensities to consume.*

Proof. We know $\dot{v}^{i \neq h} = -\frac{1}{H-1}$ gives $n(0, \dot{v}) = s^2 \frac{H}{H-1} = \frac{H}{H-1}$, and clearly $D_w \underline{x}^\sigma \dot{v} = \nabla^h$, so

$$T^{h2} \leq \frac{\frac{H}{H-1}}{H \left\| \nabla^h \right\|^2} = \frac{1}{(H-1) \left\| \nabla^h \right\|^2}$$

□

Corollary 5.4 (Appearance of protagonist). *Fix* $b \in B(r)$ *and assumption 5.1. Then* h *is a protagonist in the transfer paradox at some equilibrium with any trade level above* $\frac{1}{\sqrt{H-1}\|\nabla^h\|}$.

Corollary 5.5 (Threshold with multiple goods). *Suppose* $H = 2$. *Fix* $b \in B(r)$ *and assumption 5.1. Then the threshold trade level* $T^h(b)$ *for* h *to be a protagonist in the transfer paradox is exactly* $\frac{1}{\|\nabla^h\|}$.

Proof. When $H = 2$, the constraint set $\dot{v}^h = 1, 0 = 1'\dot{v}$ in (*) is a singleton, namely $\dot{v}^{i \neq h} = -\frac{1}{H-1}$, so the upper bound is the infimum. □

Samuelson (1947) is the special case $H = 2 = C$ of this.

5.7 Appendix

5.7.1 Derivation of formula for welfare impact

By Roy's identity with $\lambda^h = D_{w^h} v^h$,

$$dv^h = \lambda^h(-\underline{x}^h dp + dw^h)$$

In equilibrium $w^h = e^{h\prime}\overline{p}$ so

$$dw^h = \underline{e}^{h\prime} dp + \overline{p}' de^h$$

Letting $dt^h = \overline{p}' de^h$ and substituting,

$$dv^{*h} \equiv \frac{dv^h}{\lambda^h} = -\underline{x}^h dp + \underline{e}^{h\prime} dp + dt^h = dt^h - \underline{z}^{h\prime} dp$$

5.7.2 Minimizing $\|t-v\|_2^2$

Fix $\dot{v}^h = s, 0 = 1'\dot{v}$. We give the value of the problem for small enough $\lambda \geq 0$:

$$\min \left\|\dot{t} - \dot{v}\right\|_2^2 \qquad \text{subject to } \dot{t}^h \leq -\lambda, \dot{t}^{-h} \geq 0, 1'\dot{t} = 0 \tag{5.10}$$

Using the constraints,

$$\begin{aligned} \left\|\dot{t} - \dot{v}\right\|_2^2 &= \left(\dot{t}^h - \dot{v}^h\right)^2 + \left\|\dot{t}^{-h} - \dot{v}^{-h}\right\|_2^2 \\ &= \left(-1'\dot{t}^{-h} - s\right)^2 + \left\|\dot{t}^{-h} - \dot{v}^{-h}\right\|_2^2 \end{aligned}$$

Write x for $\dot{t}^{-h}, y$ for $\dot{v}^{-h}$, so that

$$\|x - y\|_2^2 = \left(1'x + s\right)^2 + (x-y)'(x-y)$$

with remaining constraints being

$$-x \leq 0, 1'x \geq \lambda$$

We solve the problem assuming $1'x \geq \lambda$ is slack, then check that slackness does hold at the candidate solution for small enough $\lambda \geq 0$, and conclude it is a bona fide solution for $\lambda = 0$.

By Kuhn-Tucker (with constraint qualification holding by linearity of the constraint), $x \geq 0$ solves the problem iff there is a nonnegative multiplier $\mu \geq 0$ satisfying complementary slackness such that x minimizes L,

$$L = \left(1'x + s\right)^2 + (x-y)'(x-y) - 2\mu' x$$

This being a convex function, its minimum in R^{H-1} is achieved at $DL = 0$:

$$DL = 2\left(1'x + s\right)1' + 2(x-y)' - 2\mu' = 0$$

That is,

$$x = y + \mu - \left(1'x + s\right)$$

Let $S = \{i \neq h : x^i = 0\}$. By complementary slackness, this says

$$\begin{array}{lll} \text{if } i \notin S & \mu^i = 0 \text{ and } x^i = y^i - (1'x+s) > 0 \\ \text{if } i \in S & x^i = 0 \text{ and } \mu^i = -y^i + (1'x+s) \geq 0 \end{array} \tag{5.11}$$

The above implies

$$\begin{array}{ll} \text{if } i \notin S & y^i > 1'x + s \\ \text{if } i \in S & 1'x + s \geq y^i \end{array}$$

We compute $1'x + s$ now:

$$\begin{aligned} 1'x + s &= \Sigma_S x^i + \Sigma_{\setminus S} x^i + s \\ &= 0 + \Sigma_{\setminus S}\left[y^i - \left(1'x + s\right)\right] + s \\ &= \left(\Sigma_{\setminus S} y^i\right) - (H - 1 - |S|)\left(1'x + s\right) + s \end{aligned}$$

Recalling $y = \dot{v}^{-h}$ satisfies $1'y = -s$, we get $\Sigma_{\setminus S} y^i = -s - \Sigma_S y^i$,

$$1'x + s = -\Sigma_S y^i - (H - 1 - |S|)\left(1'x + s\right) \tag{5.12}$$

$$1'x + s = \frac{-\Sigma_S y^i}{H - |S|}$$

Therefore $S = S(y)$ satisfies

$$\text{if } \; i \notin S \qquad y^i > \frac{-\Sigma_S y^i}{H - |S|} \tag{5.13}$$

$$\text{if } \; i \in S \qquad \frac{-\Sigma_S y^i}{H - |S|} \geq y^i \tag{5.14}$$

Lemma 5.2 (identification of S). *Order $\dot{v}^{-h} : \dot{v}^{i_1} \leq ... \leq \dot{v}^{i_{H-1}}$. The above $S = \{i_1, ..., i_n\} \subset \{1, ..., H\} \setminus \{h\}$ for the largest n such that (5.14). In particular, this description is independent of how ties in v^{-h} are ordered.*

Proof. Since $S \subset \{1, ..., H\} \setminus \{h\}$ must satisfy both (5.13), (5.14) it is clear that it includes the indices corresponding to the $|S|$ smallest elements in v^{-h}, and almost as clear that enlarging S to $S_+ = S \sqcup \{i_+\}$ implies $y^{i_+} > \frac{-\Sigma_{S_+} y^i}{H - |S_+|}$ violating (5.14). In other words, ordering $\dot{v}^{-h} : \dot{v}^{i_1} \leq ... \leq \dot{v}^{i_{H-1}}, S$ is the largest set $\{i_1, ..., i_n\}$ such that (5.14) holds.

Conversely, S is described by its being the largest set $\{i_1, ..., i_n\}$ such that (5.14) holds, and now we show uniqueness. Uniqueness in the description is up to ties in v^{-h} affecting the ordering $\dot{v}^{i_1} \leq ... \leq \dot{v}^{i_{H-1}}$ of indices. It suffices to show S is closed under ties. That is, if $v^{i_n} = v^{i_{n+1}}$ and $i_n \in S$, we want $i_{n+1} \in S$. Clearly this holds if i_n is not the last element of S, so suppose i_n is the last one. With $i_{n+1} \notin S$, it is easy to check that $S_+ = S \cup \{i_{n+1}\}$ satisfies $\frac{-\Sigma_{S'} y^i}{H - |S_+|} \geq y^{i_{n+1}}$ hence (5.14), contradicting that n is the largest such that (5.14). □

Having found $S = S(y)$, the candidate minimizer is described uniquely by substituting (5.12) in (5.11):

$$\begin{array}{ll} \text{if } i \notin S & x^i = y^i + \frac{\Sigma_S y^i}{H-|S|} > 0 \\ \text{if } i \in S & x^i = 0 \end{array}$$

This candidate minimizer is bona fide if $1'x \geq \lambda$ is slack, so for small enough $\lambda \geq 0$ it is bona fide if $1'x > 0$, which holds unless $S = \{1, ..., H\} \backslash \{h\}$, i.e. unless

$$\text{if } i \in S \qquad -s \geq y^i$$

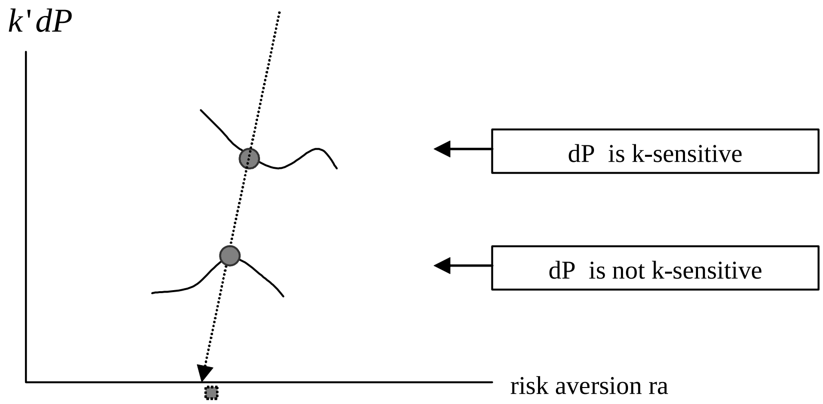
Adding over S, this implies $-s(H-1) \geq -s$, impossible with $H > 1$. So for small enough $\lambda \geq 0$ this is bona fide.

Then the value $n(0, y)$ of the problem is given by S :

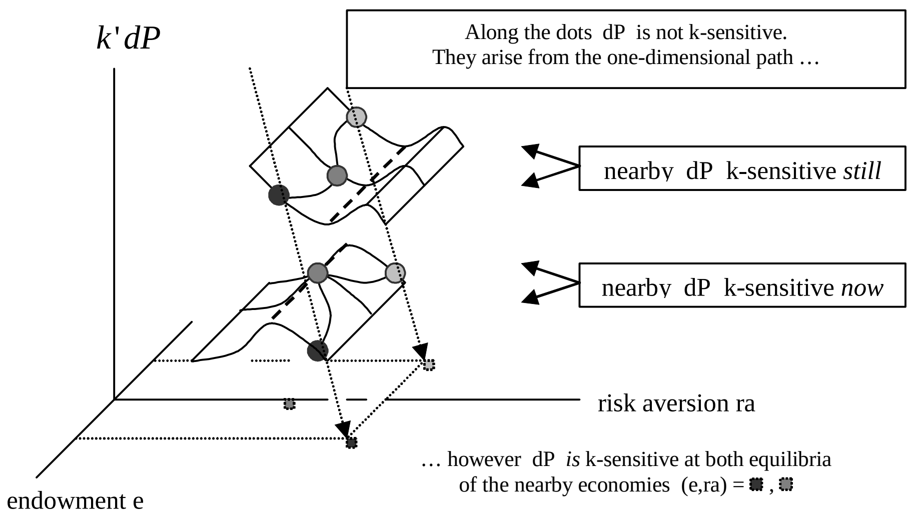
$$\begin{aligned} \|x-y\|_2^2 &= \left(1'x+s\right)^2 + \Sigma_S \left(x^i - y^i\right)^2 + \Sigma_{\backslash S} \left(x^i - y^i\right)^2 \\ &= \left(\frac{-\Sigma_S y^i}{H-|S|}\right)^2 + \Sigma_S \left(0 - y^i\right)^2 + \Sigma_{\backslash S} \left(\frac{\Sigma_S y^i}{H-|S|}\right)^2 \\ &= (H-|S|) \left(\frac{\Sigma_S y^i}{H-|S|}\right)^2 + \Sigma_S y^{i2} \\ &= \frac{\left(\Sigma_S y^i\right)^2}{H-|S|} + \Sigma_S y^{i2} \end{aligned}$$

Chapter 6

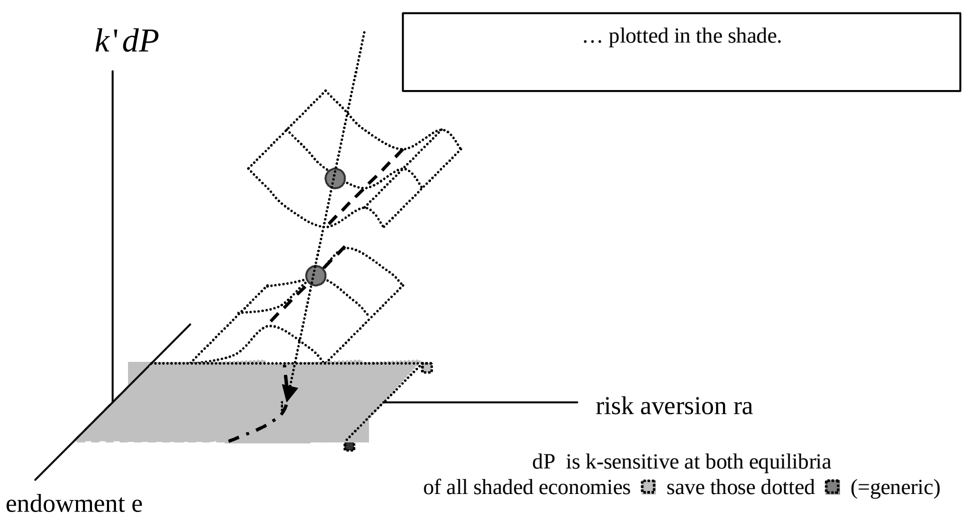
Picture of Generic Sensitivity of Price Adjustment to Risk Aversion



dP is not k-sensitive at both equilibria
of the economy (e,ra) = ▪ …



… however dP *is* k-sensitive at both equilibria
of the nearby economies (e,ra) = ▪ , ▪



dP is k-sensitive at both equilibria
of all shaded economies ▫ save those dotted ▪ (=generic)

Chapter 7

Mathematica Programs

7.1 Note

This documents two Mathematica programs for chapter 4's example, where utility is von Neumann-Morgenstern. In the simpler one the state index is a quadratic transformation of Cobb-Douglas utility; in the more elaborate one, the state index is a HARA transformation of CES utility. To find the Pareto improving tax rates from the envelope theorem, it is necessary to compute the price adjustment, and for this in turn to compute the derivative of demand. The former program has a closed formula for demand, and computes its derivative symbolically with Mathematica; the latter program has no closed formula for demand, and computes its derivative instead with chapter 1's Slutsky decompositions.

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