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The Re-sale Premium for Assets in
General Equilibrium

by

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Abstract: In general equilibrium with incomplete asset markets, many (but finite) time periods, re-trading of assets at each date and state of the world, and short sales constraints on asset holdings, the price of each asset is always at least as great as each agent's "fundamental" valuation - his marginal valuation of the future stream of payoffs from that asset. The price of asset j in state s is *strictly* greater than agent h 's fundamental valuation if and only if the short sales constraint for asset j binds in state s , or any state following s . The result generalizes to any constraint on asset holdings (possibly depending on prices and endowments) which has the property that agents can always hold *more* of any asset. This paper generalizes Harrison and Kreps' (1978) result (in a partial equilibrium, single good, risk neutral setting) that if agents have heterogeneous prior beliefs, and assets are re-traded, asset prices will always be at least as great as each agent's expected value of the returns of that asset, and typically strictly greater.

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1. Introduction

Harrison and Kreps [HK] (1978) showed a remarkable, intuitive and apparently since ignored results about asset pricing. If agents are risk neutral, have heterogeneous prior beliefs, and infinite endowment, and if an asset can be re-traded in each period, but cannot be held short, then the asset price in each state is always greater than or equal to each agent's expected value, and typically strictly greater. A simple example can explain the result. Suppose that the weather can either be wet or dry in the fall, and either wet or dry in the spring. If the weather is wet in both the fall and spring, a farmer's crop will be worth \$9 million. If the weather is dry in either season, it is worth nothing. The farmer sells his crop forward before the fall weather is realized. There are two risk neutral speculators who trade in his crop. One believes that there is a $2/3$ chance of a wet fall and a $1/3$ chance of a wet spring, while the other believes that there is a $1/3$ chance of a wet fall and a $2/3$ chance of a wet spring. The forward crop cannot be sold short, but it can be re-traded after the fall weather is realized, but before the spring weather is realized. Now unique market clearing prices can be identified by backward induction. If it was wet in the fall, then when re-trading occurs, the price must be \$6 million since the first trader believes there is a $2/3$ chance he will earn \$9 million. The second trader holds zero and would like to sell short, but is not allowed to. In the initial trading period, however, the second trader thinks there is $2/3$ chance that the price will be \$6 million after the fall, so the price must be \$4 million. The first trader holds nothing, and is short sales constrained. But notice that *each* agent's expected value of the crop in the initial trading round is \$2 million [$(1/3) \times (2/3) \times \9 million], half the competitive price.

In section 2, I give the general result for the risk neutral case. There is an arbitrary finite tree of states representing uncertainty, and a set of assets which pay out non-negative sums in some or all states. These long lived assets can be re-traded in every state. By finiteness, there are final states where asset prices must be zero. In the last period but one, all of each asset will be held by the agent(s) who has the highest expected value of that asset. In the last period but two, agents must consider both the dividend and the re-sale value in the next period. Again, the asset will be held by the agent(s) with the highest expected value, and his (their) expectation will be the price. Solving backwards, we see that the price must always be at least as great as each agent's expected value of the stream of dividends, with equality for some agent only if he values the asset the most at every possible future contingency. A formal argument is given why, in a large tree, it will typically be the case that the price is strictly higher than all agents' valuation. These results are essentially finite state/time analogues of Harrison and Kreps' result.

HK interpreted this result as a critique of "fundamentalism" and as a formalization of Keynes' notion of speculation in the *General Theory*: speculation occurs if an asset is bought for its short term

expected gain, at a price higher than the value of the discounted stream of expected returns. Another interpretation is that there is a "re-sale" premium factored into the price of an asset: an option value from the ability to re-sell over and above the "fundamental value". But how special is the result? The main purpose of this paper (section 3) is to derive a general equilibrium analogue of the HK result. Assets will now pay out in commodity bundles. Agents will have arbitrary concave utility functions. Now the price of each asset can be compared with each agent's marginal valuation (at his equilibrium consumption) of the stream of future commodity dividends from that asset. As before, the price of each asset is always greater than or equal to every agent's marginal valuation of that asset, with equality if and only if the agent is not short sales constrained in that asset in every possible future contingency.

I also show how the nature of restrictions on asset holdings can be generalized without changing the result. The result does not depend on a zero limit on short sales, or even a non-zero exogenous limit on short sales. The results holds whenever side constraints on asset holdings, in addition to the budget constraints, are imposed on agents' choices, as long as the set of allowed asset holdings is "unconstrained above", so that if some (possibly negative) asset vector is allowed, any larger vector must be allowed as well. It is shown in section 4 that this incorporates many kinds of exogenous and endogenous restrictions on asset holdings, including standard borrowing restrictions.

Because "assets" are real assets paying out in vectors of commodities, they can also be interpreted as durable goods. When buying a house, it is intuitive to assume that the option to re-sell (as opposed to merely renting it out) might be valuable. This too depends on endogenous and exogenous sales constraints.

The basic idea underlying the result is very simple and has appeared in many guises in the literature: if, in addition to budget constraints, agents face some other constraints from below on asset holdings, they will drive a wedge on the first order conditions (FOC) in a particular direction. Thus, in Zeldes (1989), liquidity constraints turn FOC equalities used in empirical work into inequalities. He and Pagès (1990) give an explicit dual interpretation of borrowing constraints.

In this paper, there is no asymmetric information, and so even though the price of some asset may be above every agent's *expected* marginal valuation of the payoffs of that asset in some state, it cannot be the case that the price is strictly higher than *every* agent's marginal valuation of the payoffs, conditional on the true final state, for *every* possible final state. A simple martingale argument shows that this cannot be the case. However, Allen, Morris and Postlewaite (1992) show that in the presence of asymmetric information, it is possible in a rational expectations equilibrium (with short sales constraints) as long as information is not fully revealed by prices in equilibrium. Allen et al. (1992) also contains a more detailed discussion of different notion of bubble.

This paper deals only with the finite horizon case. Either an infinite horizon, or asymmetric information as described above, breaks down the usual backwards induction type reasoning to solve for asset prices. "Speculative" phenomena occur in this paper because short sales constraints break down arbitrage arguments, *despite* the applicability of backwards induction type prices¹.

Finally, notice that important examples of assets whose price is strictly higher than the each agent's marginal valuation of their payoffs are nominal assets, paying out in fiat money (their payoffs have zero marginal value to all agents, their prices may be positive). Yet nominal assets can have positive value in general equilibrium with incomplete markets and finite time periods [Geanakoplos (1990)]. It is argued in section 3 that an extension of this paper to binding non-negative consumption constraints underlies this result.

2. The Risk Neutral Case

Harrison and Kreps (1978) analyzed this case with an infinite time horizon. Here, I give finite tree version of their results. This guarantees unique prices, and enables a natural extension beyond risk-neutrality in the next section.

Uncertainty is represented by a date-event tree of possible states. There are $T+1$ time periods, $t = 0, \dots, T$. There are a (finite) set of possible states of the world in each time period, S_0, \dots, S_T . We write $S_{t,t'} = S_t \cup S_{t+1} \cup \dots \cup S_{t'}$, where $t' \geq t$. A precedence mapping, $\alpha: S_{1,T} \rightarrow S_{0,T-1}$, is onto and satisfies $s \in S_t \Rightarrow \alpha(s) \in S_{t-1}$. Now if we assume that S_0 is a singleton, $S_0 = \{s_0\}$, α defines a date-event tree. We will define additional functions for future use. $\tau(s)$ is the time period in which state s occurs: $\tau(s) = t \Leftrightarrow s \in S_t$. $f(s)$ is the set of immediate followers of state s : $f(s) = \{s' \in S_{\tau(s)+1} \mid s = \alpha(s')\}$. Thus $s \in S_T \Rightarrow f(s) = \emptyset$. Assume, conversely, that $f(s) = \emptyset \Rightarrow s \in S_T$, so that every branch of the tree continues to period T . $F(s)$ is the set of all states which follow s : $F(s) = \{s' \in S_{\tau(s)+1,T} \mid s = \alpha^n(s') \text{ for some } n \geq 1\}$. $F_0(s)$ is the set of states which follow s , including s : $F_0(s) = F(s) \cup \{s\}$. Beliefs about S are given by a function $\pi: S \rightarrow \mathbb{R}_{++}$ satisfying $\sum_{s' \in f(s)} \pi(s') = \pi(s)$ and $\pi(s_0) = 1$. There is no discounting. Suppose there are H agents, $\{1, \dots, H\}$ and each agent h has some beliefs π^h . Agents consume $x^h(s)$ dollars in each state and maximize the expected value of consumption, $\sum_{s \in S} \pi^h(s) x^h(s)$. There are a collection of J assets, $\{1, \dots, J\}$. Asset j pays $a_j(s)$ dollars in state s . The assets are re-traded at every state. Agents cannot hold negative quantities of an asset. An equilibrium in this model

1. Kocherlakota (1992) has recently examined the role of short sales constraints in an infinite horizon model.

consists of a set of prices for the assets and asset holdings for the agents such that each agent maximizes expected return, subject to the short-sales constraint. Notice that an equivalent representation would have had agents consuming in the final period, and a riskless asset paying out one dollar in every state in the final period. Writing $q(s) = [q_1(s), \dots, q_J(s)]$ for the vector of asset prices, in dollar terms, in state s , and $\theta^h(s) = [\theta^h_1(s), \dots, \theta^h_J(s)]$ for agent h 's holding of the asset in state s , we have:

Theorem (equilibrium, risk neutral case) (q, x, θ) is an equilibrium if

$$q_j(s) = \max_{h=1, H} \sum_{s' \in f(s)} \frac{\pi^h(s')}{\pi^h(s)} [a_j(s') + q_j(s')]$$

$$\theta^h_j(s) > 0 \rightarrow (j, s) \notin C^h$$

$$\text{where } C^h = \left\{ (j, s) \mid h \notin \operatorname{argmax}_{h'=1, H} \left\{ \sum_{s' \in f(s)} \frac{\pi^{h'}(s')}{\pi^{h'}(s)} [a_j(s') + q_j(s')] \right\} \right\}$$

$$= \left\{ (j, s) \mid \sum_{s' \in f(s)} \frac{\pi^h(s')}{\pi^h(s)} [a_j(s') + q_j(s')] < q_j(s) \right\}$$

Proof This is special case of the theorem in the next section. It is a finite time/state version of the Harrison and Kreps (1978) result.

Notice that, by backward induction, prices are uniquely defined (final period asset prices are 0 since $f(s) = 0$ if $s \in S_T$). We are interested in comparing prices with "fundamental values". Define agent h 's valuation of assets in state s (in terms of the riskless asset) by

$$v^h(s) = \sum_{s' \in F(s)} \frac{\pi^h(s')}{\pi^h(s)} a(s')$$

Theorem (fundamental values, risk neutral case)

[1] $v_j^h(s) \leq q_j(s)$ for all $h \in H, s \in S$.

[2] $v_j^h(s) < q_j(s)$ if and only if $(j, s') \notin C^h$ for some $s' \in F_0(s)$

This result suggests that "typically", in a large tree, the price will be *strictly* higher than *every* agent's fundamental valuation of the asset, since all that is required for $q_j(s) > v_j^h(s)$ is that there be some possible contingency in the future following s in which agent h does not value asset j the most.

This intuition can be formalized in the following way. Suppose initially agents all had the same beliefs π . In this case, the price is always equal to each agent's expected value of the asset. Now suppose $a_j(s) \neq a_j(s')$ for some $s' \in f[\alpha(s)]$, for all $j=1, J, s \in S_T$ i.e. none of the J assets has become riskless in some state in period $T-1$. Also suppose that every non-terminal node has at least one follower [$f(s)$ is not a singleton for any s]. Now suppose each agent's beliefs is perturbed, uniformly and independently at each node, from π , i.e.

$$\delta^h(s) = \left\{ \frac{\pi^h(s')}{\pi^h(s)} - \frac{\pi(s')}{\pi(s)} \right\}_{s' \in f(s)} \in \mathbb{R}^{n(s)}$$

each $\delta^h(s)$ independently distributed uniformly on $[-\epsilon, \epsilon]^{n(s)} \cap \left\{ x \in \mathbb{R}^{n(s)} \mid \sum_{i=1}^{n(s)} x_i = 0 \right\}$,

$$\text{where } 0 < \epsilon < \min_{s \in S / \{s_0\}} \left\{ \min \left\{ \frac{\pi(s)}{\pi[\alpha(s)]}, 1 - \frac{\pi(s)}{\pi[\alpha(s)]} \right\} \right\}$$

and $n(s)$ is the number of states in $f(s)$

After the perturbation, agents will generically not agree on the expected value of the next period price plus payoff, so $(j, s) \notin C^h$ for exactly one h will be true with probability one, for each $s \in S/S_T$. For each h , the probability that $(j, s) \notin C^h$ is $1/H$. Now if $N(s)$ is the number of nodes in $F_0(s)/S_T$ (non-terminal nodes following s), the theorem tells us that the probability that $(j, s') \in C^h$ for some given h , for all $s' \in F_0(s)$, for some $s \in S_{0, T-1}$, is $(1/H)^{N(s)}$. So the probability that $(j, s') \in C^h$ for all $s' \in F(s)$, $s \in S_{0, T-1}$, for some $h=1, H$, is $(1/H)^{N(s)-1}$. So the probability that some asset j is priced higher than *every* agent's evaluation of it is $1 - (1/H)^{N(s)-1}$. (Notice that this formula correctly gives zero for $s \in S_{T-1}$).

This result is all the more surprising when we realize that a martingale asset pricing rule [Harrison and Kreps (1979)] in fact does hold in this example i.e. for any beliefs $\{\pi^h\}_{h=1, H}$, there exists beliefs π^* such that

$$q(s) = \sum_{s' \in F(s)} \frac{\pi^*(s')}{\pi^*(s)} a(s')$$

π^* can be constructed by letting beliefs over $f(s)$ at each node s be generated by the beliefs of any agent holding the asset, i.e.

$$\frac{\pi^*(s)}{\pi^*[\alpha(s)]} = \frac{\pi^h(s)}{\pi^h[\alpha(s)]}, \text{ for some } h \text{ s.t. } (j,s) \in C^h, \text{ for all } s \in S / \{s_0\}$$

In section 3, the intuition of this result is extended to risk-neutral agents trading long lived assets paying out in many goods, with no need to assume heterogeneous prior beliefs. It is useful to make clear, however, that the simple story described in this section is entirely consistent with *learning*. Suppose that the payoffs of each asset were determined by some Markov process. Asset j pays out one of (n_1, \dots, n_k) at each date. If asset j paid out n_k dollars in period t , there is probability ρ_{kk}^j that it will pay out n_k at date $t+1$. Suppose agents started off with different priors about the true Markov matrix R^j . Over time, if agents' priors put positive weight on the true value of R^j , agents' beliefs would converge to the true R^j , and prices would converge to fundamentals. But initially, prices might well exceed anyone's fundamentals. This is entirely consistent with one account of new issues of stocks:-

The prices of new issues, as of all securities, are set not by the appraisal of the typical investor, but by the small number who think highly enough of the investment merits of the new issue to include it in their portfolio. The divergence of opinion about a new issue are greatest when the stock is issued. Frequently the company has not started operations, or there is uncertainty about the success of new products or the profitability of a major business expansion. Over time, this uncertainty is reduced.... With the passage of time, and the reduction of uncertainty, the appraisal of the top x percent of the investors is likely to decline even if the average assessment is not changed. This would explain the poor performance of a group of new issues when compared to a group of stocks about which the uncertainty does not decrease over time.

Incidentally, if underwriters ignore the above arguments and price new issues on the basis of their own best estimates of the prices of comparable seasoned securities, they will typically underprice new issues. The mean of their appraisals will resemble the mean appraisal of the typical investor, and this will be below the appraisals of the most optimistic investors who actually constitute the market for the security. This may be a partial explanation for the underpricing of new issues by underwriters.²

2. Miller (1977)

3. The Re-sale Premium in General Equilibrium

This section extends the prices result of the previous section to an economy of risk-averse agents trading many goods. The same model of uncertainty is used. Now suppose there are L commodities, $\{1, \dots, L\}$ and J assets, $\{1, \dots, J\}$. Each agent will choose a consumption vector $\{x(s)\}_{s \in S} \in \mathbb{R}^{LS}$ and an asset vector $\{\theta(s)\}_{s \in S} \in \mathbb{R}^{JS}$. Each agent h has utility function, $u^h: \mathbb{R}_+^{LS} \rightarrow \mathbb{R}$, and endowment in each state, $e^h(s) \in \mathbb{R}_+^L$. Utilities are assumed to be strictly monotonic in each good, concave and differentiable. An asset pays off in a non-negative bundle of commodities in a given state; thus asset j 's payoff in state s can be written as a vector $A^j(s) \in \mathbb{R}_+^L$. The collection of J assets thus gives an $L \times J$ payoff matrix $A(s) \geq 0$. It is notationally convenient to add a state s_{-1} , preceding state s_0 [$\alpha(s_0) = \alpha(s_{-1})$], and allowing each agent to have an endowment (or state -1 holding) of assets, $\theta_{-1}^h \in \mathbb{R}_+^J$. There exist commodity prices $p = \{p(s)\}_{s \in S} \in \mathbb{R}_+^{LS}$ and asset prices $q = \{q(s)\}_{s \in S} \in \mathbb{R}_+^{JS}$. Since only relative prices in a given state matter, and commodity prices are strictly positive, let good L be the numeraire and $p_L(s) = 1$, for all $s \in S$.³ Now the agent's budget set (with S budget constraints) is:-

$$B(p, q, e^h, \theta^h(s_{-1})) = \{ (x, \theta) \in \mathbb{R}^{LS} \times \mathbb{R}^{JS} \mid p(s)[x(s) - e^h(s)] + q(s)[\theta(s) - \theta(\alpha(s))] \leq p(s)A(s)\theta[\alpha(s)] \text{ for all } s \in S \}$$

In addition to the budget constraints, there is some *asset constraint set*, $D \subset \mathbb{R}^{JS}$. Agents are restricted to choosing (x, θ) within the budget constraints, with θ in D . The only restriction that I will impose on D is that it must always allow agents to hold more of every asset: asset constraint set $D \subset \mathbb{R}^{JS}$ is *unconstrained above* if $\theta' \geq \theta$ and $\theta \in D$ implies $\theta' \in D$. Examples of a unconstrained above asset constraint set is the *no short sales* constraint set:

$$D = \{ \theta \in \mathbb{R}^{JS} \mid \theta \geq 0 \}$$

More generally, a *restricted short sales* constraint set imposes some exogenous restriction on how short you can go in any particular asset in any particular state:

$$D = \{ \theta \in \mathbb{R}^{JS} \mid \theta \geq \theta_* \}$$

3. This generalization of a canonical representation of general equilibrium with incomplete markets [Geanakoplos (1990)] to long-lived assets saves on notation, in doing so includes some redundancy. Assets pay off in state s_0 , even though those payoffs could equally well be included in agents' endowments. Similarly, assets are traded in final period states, S_T . There is no reason to hold them, however, and it will be a consequence of agents' maximization that $q(s) = 0$, for all $s \in S_T$.

The no short sales constraint required that you cannot go short in any asset. A weaker requirement is that you can never hold a portfolio that would ever have a negative payoff, i.e. a *positive portfolio value* constraint set:

$$D = \left\{ \theta \in \mathbb{R}^S \mid \begin{array}{l} [q(s) + p(s)A(s)]\theta[\alpha(s)] \geq 0, \\ \text{for all } s \in S / \{s_0\} \end{array} \right\}$$

It might be the case that holders of long positions required anyone going short against them hold collateral in their endowment to honor their commitment. This would give a *complete collateral* constraint set:

$$D = \left\{ \theta \in \mathbb{R}^S \mid \begin{array}{l} A(s)\theta[\alpha(s)] + e^h(s) \geq 0, \\ \text{for all } s \in S / \{s_0\} \end{array} \right\}$$

Clearly, we can imagine many variations on the above themes. In section 4, we will consider when *default* asset constraints - sets generated by a no default condition applied by backward induction - satisfy the unconstrained above property. Note that asset constraint sets can depend on data of the economy such as e^h and even on endogenous prices p and q . For the results that follow, it is only the unconstrained above property that matters.

In order to generalize the results of the previous section, we need a notion of the value of each asset to each agent in each state. Define the marginal valuation of asset j to agent h , at allocation x by:

$$v_j^h(s, x) = \frac{\sum_{s' \in F(s)} \frac{\partial u^h(x)}{\partial x(s')} A^j(s')}{\frac{\partial u^h(x)}{\partial x_L(s)}}$$

This represents agent h 's marginal valuation of the stream of payoffs from asset j , normalized by the marginal valuation of numeraire asset L in state s in order to make it in the same "units" as prices.

Let us give a characterization of which asset markets the agent is constrained in, at some maximizing choice. Consider a set of asset markets, $C \subset J \times S$. Suppose that (x^*, θ^*) maximizes $u_h(x)$ subject to $(x, \theta) \in B$, $x \in \mathbb{R}_+^{L^S}$ and $\theta_j(s) \geq \theta_j^*(s)$, for all (j, s) . Say that C is *sufficient* if (x^*, θ^*) also maximizes $u_h(x)$ subject to $(x, \theta) \in B$, $x \in \mathbb{R}_+^{L^S}$ and $\theta_j(s) \geq \theta_j^*(s)$, for all $(j, s) \in C$. Then $C^h(x^*, \theta^*) \subset J \times S$ is the set of binding constraints at (x^*, θ^*) if $C^h(x^*, \theta^*)$ is sufficient and is contained in all sufficient sets.

The proof of the theorem makes clear that such a set of binding constraints always exists (it is unique by construction).

Theorem If (x, θ) maximizes $u^h(x)$ subject to the budget constraints and $\theta \in D$ for some unconstrained above set D , and $x \gg 0$, then the marginal valuation of asset j in state s to h given allocation x is always less than or equal to the price of asset j in state s , and strictly less than if and only if h is short sales constrained in asset j in some state s' following s . Formally:-

If $(x, \theta) \in B(p, q, e^h, \theta^h(s_{-1}))$, $x \gg 0$ and $\theta \in D$, D unconstrained above, and there does not exist (x', θ') satisfying

$$[i] (x', \theta') \in B(p, q, e^h, \theta^h(s_{-1}))$$

$$[ii] x' \in \mathbb{R}_+^L$$

$$[iii] \theta \in D$$

$$[iv] u^h(x') > u^h(x)$$

Then $q_j(s) \geq v_j^h(s)$ for all j, s and $q_j(s) > v_j^h(s)$ if and only if $(j, s') \in C^h(x, \theta)$, for some $s' \in F_0(s)$

Proof

1. Let $D'' = \{ \theta' \in \mathbb{R}^S \mid \theta'_j(s) \geq \theta_j(s), \text{ for all } (j, s) \}$

2. D unconstrained above implies $D'' \subset D$.

3. So (x, θ) maximizes $u^h(x)$ subject $(x, \theta) \in B$, $x \in \mathbb{R}_+^L$ and $\theta \in D''$ and there exist multipliers $\lambda: S \rightarrow \mathbb{R}_+$ and $\mu: J \times S \rightarrow \mathbb{R}_+$ such that

$$\begin{aligned} \frac{\partial u^h}{\partial x(s)} &= \lambda(s)p(s), \text{ for all } s. \\ \lambda(s)q_j(s) - \sum_{s' \in F(s)} \lambda(s')[q_j(s') + p(s')A^j(s')] &= \mu_j(s), \text{ for all } j, s. \end{aligned}$$

4. $p_L(s) = 1 \Rightarrow \lambda(s) = \partial u^h(x) / \partial x_L(s)$, and u_h strictly increasing implies each $\lambda(s)$ strictly positive, so

$$\frac{\partial u^h(x)}{\partial x_L(s)} q_j(s) = \sum_{s' \in F(s)} \left\{ \frac{\partial u^h(x)}{\partial x(s')} A^j(s') + \lambda(s') q_j(s') \right\} + \mu_j(s), \text{ for all } j, s.$$

5. By progressive substitution, we now get:

$$\frac{\partial u^h(x)}{\partial x_L(s)} q_j(s) = \sum_{s' \in F(s)} \frac{\partial u^h(x)}{\partial x(s')} A^j(s') + \sum_{s' \in F_0(s)} \mu_j(s'), \text{ for all } j, s.$$

$$q_j(s) = v_j^h(s) + \left\{ \frac{1}{\frac{\partial u^h(x)}{\partial x_L(s)}} \right\} \sum_{s' \in F_0(s)} \mu_j(s'), \text{ for all } j, s.$$

6. Thus $q_j(s) \geq v_j^h(s)$, with strict inequality if and only if $\mu_j(s') > 0$, for some $s' \in F_0(s)$.
7. There exists a unique subset $C \subset J \times S$ such that there exist multipliers in (3) satisfying $C = \{ (j,s) \mid \mu_j(s) > 0 \}$ (if not, then 6 is violated). This ensures that $C^h(x, \theta)$ is unique and equal to $\{ (j,s) \mid \mu_j(s) > 0 \}$

Notice that there are no non-negativity constraints on consumption in the theorem. In fact, the result will easily extend, and re-sale premia for assets will exist [$q_j(s) > v_j^h(s)$] if and only if *either* agent h is short sales constrained in asset j today or in some future state $s' \in F_0(s)$ *or* if asset j pays out a positive amount of commodity l in some future state $s' \in F(s)$ [so $a_{jl}(s') > 0$], and agent h 's non-negative consumption constraint binds in good l in state s' . I have not included that extension in the main theorem because it is not clear what real interpretation can be given to a binding non-negative consumption constraint. However, in a technical sense, this result underlies one way of modelling money in general equilibrium and giving nominal assets, which pay out in money, positive value in a finite horizon model [Geanakoplos (1990)]. Suppose there is a commodity, fiat money, which does not effect any agent's utility. If this commodity is in zero net supply, it may have positive value in the final period T , but with every agent wanting to sell it and unable to, because they cannot consume negative money. If asset j is a nominal asset, paying out in money only, it must have zero marginal value always [$v_j^h(s) = 0$, for all h, s]. It can have positive price, however, because non-negative consumption constraints bind.

I have not defined an equilibrium concept. The theorem is true of any equilibrium concept where agents are maximizing utility subject to budget constraints and an unconstrained above restriction on asset holdings (it does not depend on market clearing, for example). Standard theorems give existence in general equilibrium with incomplete markets with many periods and re-trading of long lived assets. Radner (1972) proved existence for a restricted short sales asset constraint set. Duffie and Shafer (1986) proved generic existence when the asset constraint set is unrestrictive [$D = \mathbb{R}^{JS}$]. But note that asset constraint sets like the positive portfolio value constraint set discussed above, and others considered in section 4, depend on equilibrium prices. Standard arguments do not guarantee existence in such cases, but the theorem will be true if an equilibrium exists.

For the risk-neutral case, it was possible to derive three kinds of results. First, equilibrium prices were characterized as a function of the data of the economy. This cannot be done in the general case.

Second, the relation between agents' fundamental values of assets and prices was explicitly characterized in terms when short sales constraints bind. This result went through exactly as before in the general case. Third, conditions were given to ensure that short sales constraints do bind, and price bubbles do exist. With common priors, short sales constraints don't bind and bubbles don't exist. Perturb those common priors independently, and short sales constraints bind almost always and bubbles exist almost always. This also cannot be done in the general case, except for some very strong sufficient conditions. Jarrow (1980) showed that in a two period capital asset pricing model setting, where asset payoffs are normally distributed, if agents are expected utility maximizers and have a common prior, short sales constraints do not bind. If asset markets were complete in the sense that contingent commodity contracts (Arrow securities) were traded from state s_0 , and if the asset constraint set contained the complete collateral constraint set discussed above, so agents could sell their endowment in advance, then it can be shown that asset constraints will not bind.

4. "No Default" Asset Constraint Sets

Suppose that an agent who defaulted on his obligations faced some punishment. This punishment might be endogenous (exclusion from some or all asset markets, loss of some or all future endowment), or exogenous (a decrease in utility, for any given bundle of commodities). In a market with perfect information, agents would not be allowed to go short in an asset if they would ever have an incentive to default⁴. So for any given punishment rules, we can define the *no default* asset constraint set as the set of asset allocations where an agent will not have an incentive to default. This section explores when such *no default* asset constraint sets satisfy the *unconstrained above* property that was key in the previous section.

The *no short sales* and *positive portfolio value* constraints discussed in the previous section are examples of such *no default* asset constraint sets. If default in asset j in state s is punished by not allowing the agent to hold a negative position in that asset in the future, then a backward induction argument shows that agents can never be allowed to hold positive amount of that asset. If default in any asset in state s is punished by not allowing the agent to hold an asset position which will pay out a negative amount in any following state, then a backward induction argument implies the positive portfolio value constraint. These kind of restrictions on asset holdings, or - in the one good case - borrowing,

4. This is in contrast to general equilibrium models where default occurs in equilibrium [Dubey and Geanakoplos (1989)].

have been imposed in the international debt literature [Eaton and Gersovitz (1981)] and empirical work on consumption behavior [Zeldes(1989)] and elsewhere.

I will consider a class of cases where it is possible to provide a simple characterization of such no default asset constraint sets by backward induction. Suppose that agents' utility is additively separable across states, so $u^h(x) = \sum_s u_s^h(x(s))$. Then past consumption does not matter to agent h in any state s . His state s continuation indirect utility,

$$v^h(e_{F(s)}, p_{F_0(s)}, q_{F_0(s)}, E, I, s)$$

is a function of his endowment in future states (not including s), commodity and asset prices in future states (inclusive of s), the set of states where he defaults in states s , $E \subset \{1, \dots, J\}$, and his income I (after his decision of which markets to default in) in state s . His income will depend on which markets he defaults in:-

$$I(E, \theta[\alpha(s)]) = p(s)e^h(s) + \sum_{j \in E} [q_j(s) + p(s)A^j(s)]\theta_j[\alpha(s)]$$

Now for any punishment rule, continuation indirect utilities and the asset constraint set can be defined by backward induction on states. At each state, agents' continuation indirect utility takes into account not only punishments but also asset constraint sets at future states. The asset constraint set thus satisfies:-

$$D^h = \{\theta \in \mathbb{R}^{JS} | v^h(\dots, \theta, I(\theta, \theta[\alpha(s)]), s) \geq v^h(\dots, E, I(E, \theta[\alpha(s)]), s), \text{ for all } s \in S, E \subset \{1, \dots, J\}\}$$

Lemma Suppose [1] $E' \subset E \Rightarrow v^h(\dots, E', I, s) \geq v^h(\dots, E, I, s)$

[2] $I' \geq I \Rightarrow v^h(\dots, E, I', s) \geq v^h(\dots, E, I, s)$

[3] $v^h(\dots, \emptyset, I, s) \geq v^h(\dots, E, I', s)$ implies $v^h(\dots, \emptyset, I+I'', s) \geq v^h(\dots, E, I'+I'', s)$, for all $I'' \geq 0$

Then the asset constraint set is unconstrained above

Proof

Suppose $\theta' \in D^h$ and $\theta_j(s) = \theta'_j(s)$ for all $(j, s) \neq (j^*, s^*)$, $\theta_{j^*}(s^*) \geq \theta'_{j^*}(s^*)$

and $v^h(\dots, \theta, I(\theta, \theta[\alpha(s)]), s) \geq v^h(\dots, E, I(E, \theta[\alpha(s)]), s)$

Then either [i] $j \in E$ and $q_j(s) + p'(s)A(s) > 0$

or [ii] $j \in E$ and $q_j(s) + p'(s)A(s) \leq 0$

or [iii] $j \notin E$

Now $v^h(\dots, \theta, I(\theta, \theta[\alpha(s)]), s) \geq v^h(\dots, E, I(E, \theta[\alpha(s)]), s)$

by [1] if (i), by [2] if (ii), by [3] if (iii)

Conditions [1] and [2] in the lemma seem reasonable - the more you default, other things being equal, the more you will be punished [1] and an increase in income will always increase ex ante utility [2]. But [3] is rather strong, requiring that if you don't initially have an incentive to default, increasing your income will not give you an incentive to default. This would be violated, for example, if some exogenous punishment for default was more onerous when you were poor than when you were rich.

It is not, in general, possible to dispense with [3]. But it can be verified that it is sufficient if there is only one asset market ($J = 1$) or if, in each state, you must either default in all markets or in none.

5. Conclusion

There were three kinds of results about the risk neutral, heterogeneous prior case in section 2. First, prices can be explicitly derived by backward induction, with the price of each asset in each state equal to the highest (among the agents) expected value in the next period price. Second, the price of an asset in some state is strictly higher than each agent's fundamental value of the payoffs from the asset if and only if each agent is short-sales constrained in that asset in some possible future contingency. Third, for independent perturbations of beliefs, such bubbles will occur with probability close to one in large trees.

Only the second result - the price of an asset in some state is strictly higher than each agent's fundamental value of the payoffs from the asset if and only if each agent is short-sales constrained in that asset in some possible future contingency - generalizes to the general equilibrium case in section 3. But both kinds of results can be generalized beyond exogenous short sales limitations to arbitrary unconstrained above asset constraint sets.

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