Charles University
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CERGE-EI

Dissertation

Essays on Coordination Problems

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Prague June 2006

UMI Number: 3212504

UMI Microform 3212504

ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

To Markéta, Jolanka, and my parents.

Acknowledgements

I am grateful to my supervisors, Avner Shaked and Andreas Ortmann, for their inspiring guidance and constructive criticism, and to my family for their patience and for giving me creative distractions when I most needed them. I am indebted to many inspiring people for their time in discussing the work, especially Ken Binmore, Amil Dasgupta, Dirk Engelmann, Fuhito Kojima, Eugen Kováč and Hyun Shin. Laura Straková carefully edited all three chapters, greatly enhancing their readability. In the last stage, I benefited from insightful comments of Ken Binmore and Stephen Morris, who refereed the thesis. Last, I would like to thank two institutions, CERGE for enabling me to achieve an economics education in my home country, and LSE for its generous hospitality during my stays.

Introduction

The unifying topic of all three chapters of this dissertation is coordination. Equilibrium selection in coordination games is not yet fully understood, despite the fact that coordination failures have important policy consequences. Early opinions in economics were that rational players always choose a Pareto dominant equilibrium (Harsanyi and Selten, 1988), but experimental evidence as given for instance in Cooper, DeJong, Forsythe and Ross (1990), and newer theories of Carlsson and van Damme (1993), Kandori, Mailath and Rob (1993) have put that conjecture in question.

Carlsson and van Damme assume in their theory of global games that players are not perfectly informed on the coordination game they actually play; rather, they receive only imprecise signals about the payoff matrix. The resulting incomplete information game has a unique equilibrium even in the limit of extremely precise signals, and the theory predicts that people choose their action according to a risk dominance rather than a Pareto dominance criterion.

In the first two chapters I use global games as Lego blocks from which I build dynamic games that allow for a study of the interaction of seemingly distinct coordination problems. In the "Coordination Cycles" chapter I examine the case of investors facing a series of risky projects with positive externalities, i.e. a repeated coordination game. I assume that an investor, by choosing to invest today, risks instantaneous losses as well as her ability to participate in future stages: unsuccessful investment today can lead to bankruptcy. Fear of bankruptcy may motivate investors not to invest, especially in the days just before an expected boom. The amount of risk associated with investing at day t depends on future expected profits V_{t+1} from tomorrow on. And because equilibrium is selected according to the strategic risk associated with the actions, V_{t+1} acts like an endogenous sunspot. High probability of successful coordination tomorrow makes players more cautious today and hence decreases today's probability of successful investment. Unlike other models with self-fulfilling beliefs, this negative feedback between tomorrow and today leads to cycles which not only may happen, but must happen, because the presented model has a unique equilibrium.

In "Coordination in a Mobile World" I study the case of investors choosing between many simultaneously running projects, each of which is again a coordination problem. Each project is to a large extent independent, but its attractiveness depends on players' behavior in all other projects. In particular, the outside option in each project consists of a search for other of the projects and hence the option's value is determined by behavior in the whole set of projects. The global games framework again assures equilibrium uniqueness, which in turn allows for an analysis of comparative statics. Surprisingly, welfare is non-monotonic in the mobility of players. Lower mobility costs on the one hand allow players to find projects with better fundamentals, but because the outside option of each coordination problem increases, the willingness to risk investment decreases, which may override the direct positive effect. The whole dynamic game can be viewed as a rough model of globalization in which increasing mobility allows access to ever better projects but which also hinders local coordination.

In the final chapter I analyze public good games with punishment option. The model is inspired by Fehr and Gächter's (2000) experiments on public good games with a punishment stage in which players can pay to punish free-riders. As their experiments demonstrate, players are able to sustain long-run cooperation. I argue that if each of N players of such a game commits to spending one unit on the punishment of free-riders, the group can enforce contribution levels of N units, as only one individual deviator needs to be discouraged from free-riding in order to support the equilibrium with such high contributions. A small perturbation to the willingness to spend a unit on punishment transforms the prisoners' dilemma into a coordination game in which all contribute high levels and nobody dares to deviate, or in which everyone free-rides, which breaks up the total punishment into ineffectively small parts.

I analyze this coordination problem within the framework of Kandori, Mailath and Rob's (1993) theory of stochastic evolution. The authors consider an evolutionary process, such as best response dynamics, with several steady states. Additionally they assume that players occasionally, but rarely, "mutate" — deviate from the underlying evolutionary process and choose a random action. Such a model allows them to specify the steady state, where players stay most of the time in the long run.

I look for punishment rules that assure high contributions from near-selfish players in the long run evolutionary process. The key characteristic of such successful rules is that a single mutation suffices to induce a slight increase in the contribution norm but at least two mutations are needed for any decrease in the norm. Thus, if mutations are rare, an increase in the norm becomes arbitrarily more probable than a decrease and only high contributions prevail in the long run.

Contents

Chapter 1

Coordination Cycles

Abstract

I build a dynamic global game in which players repeatedly face a similar coordination problem. By choosing a risky action (invest) instead of an outside option (not invest), players risk instantaneous losses as well as payoffs from future stages in which they cannot participate if they go bankrupt. Thus, the total strategic risk associated with investment in a particular stage depends on the expected continuation payoff. High expected future payoffs make investment today riskier and therefore harder to coordinate, which decreases today's payoff. Expectation of successful coordination tomorrow undermines successful coordination today which leads to fluctuations of equilibrium behavior even if the underlying economic fundamentals happen to be stationary. The dynamic game inherits the equilibrium uniqueness of static global games.

Keywords: Coordination, Crises, Cycles and Fluctuations, Equilibrium Uniqueness, Global Games.
JEL classification: C72, C73 D8, E32.

Acknowledgements
The author thanks Amil Dasgupta, Bernardo Guimaraes, Fuhito Kojima, Eugen Kováč, Peter Kondor, Andreas Ortmann, Avner Shaked, Hyun Shin for their comments and Laura Straková for editing the manuscript. The title was suggested by Bernardo Guimaraes. The usual disclaimer applies.

1.1 Introduction

Consider investors facing a series of risky projects with positive externalities, each project being a coordination game with multiple equilibria. Assume that a player, by choosing to invest today, risks instantaneous losses as well as her ability to participate in future stages: unsuccessful investment today can lead to bankruptcy. Fear of bankruptcy may motivate a player not to invest, especially in the days just before an expected boom.

Formally, the total expected payoff of an investor i in period t is $u_t^i + V_{t+1} r_t^i$, where u_t^i is the instantaneous payoff, V_{t+1} is the common value of expected future profit, and r_t^i the continuation probability. The amount of strategic risk associated with an investment at time t depends on V_{t+1} and thus differs in each period. If players expect successful coordination (boom) in the near future they will hesitate to risk bankruptcy by investing today. If they expect coordination on not investing (slump) in the near future, they are more likely to invest today because bankruptcy is less worrisome. The negative feedback between tomorrow's and today's coordination leads, for some constellations of parameters, to cycles in the willingness to invest. The cycles are self-enforcing and arise without an external cause.

We assume instantaneous payoff u_t^i and continuation probability r_t^i such that each stage with the total payoff $u_t^i + V_{t+1} r_t^i$ is a global game. Global games, introduced by Carlsson and van Damme (1993) and elaborated by Morris and Shin (2003), link the outcome of a coordination game to the amount of strategic risk associated with the available actions. The global games approach enables us to solve the coordination problem of the last period and, by backward induction, of all periods.

Strategic complementarities resulting in multiple equilibria are common in many economic situations. Models with multiple equilibria and arbitrary self-fulfilling beliefs have been suggested to explain sudden shifts of the economy from one state to another. Complementarities have been used to model search (Diamond, 1982); bank runs (Diamond and Dybvig, 1983); currency attacks (Obstfeld, 1996); or business cycles (Farmer and Guo, 1994). Cooper (1999) provides a survey of coordination problems in macroeconomics.

The weakness of early coordination models was their weak predictive power. Without an additional selection principle, all equilibria were possible, and thus such models severed the natural link between fundamentals and economic outcomes. The global games literature filled the gap by showing that the multiplicity of equilibria in coordination games with complementarities is a peculiar consequence of the unrealistic assumption that the underlying economic fundamentals are common knowledge. If observation of fundamentals is noisy, the multiplicity of equilibria is eliminated and the fundamentals fully determine economic activity.

In global games models, economic outcomes change only if the fundamentals change (possibly by a small amount). Thus although the global games approach solves the indeterminacy of the self-fulfilling beliefs literature, it leaves no place for endogenous fluctuations unconnected to the evolution of the underlying fundamentals, and hence, it misses some of the attractive features of models based on multiple equilibria. In particular, models based on fundamentals have difficulty explaining the spread of crises among countries with uncorrelated fundamentals and no direct links (see e.g. Masson, 1998).

I present a model that exhibits the advantages of both approaches. Economic behavior changes only when the fundamentals pass a threshold; the changes are thus not arbitrary. However, although the thresholds are uniquely determined, they differ across periods. These fluctuations of thresholds can be interpreted as fluctuations of market sentiments; crises occur when these sentiments are too pessimistic compared to the realized fundamentals.

The existing dynamic global games do not alter the fundamentals-oriented explanation of fluctuations typical for their static predecessors. Burdzy, Frankel and Pauzner (2001) study a series of coordination problems in which fundamentals evolve according to a random walk and players experience frictions in changing their action.[1] Just as in static global games, the fluctuations of behavior in the unique equilibrium are driven by changes in the fundamentals. Chamley (1999) and Morris and Shin (1999) consider another dynamic link: past fundamentals serve as a public signal for the current period. These models also have unique equilibria in threshold strategies and fluctuations are again driven by fundamentals. They can exhibit path dependence; contingent on the history of the fundamentals, both investing and not investing can be the equilibrium action for the same current fundamentals. Nevertheless, switches between booms and slumps occur only if the fundamentals change and pass a threshold.

Angeletos, Hellwig and Pavan (2004) admit alternations "between crises and phases of tranquillity without changes in fundamentals" (p. 1). The fundamentals are assumed to be constant in the model and alternations are driven by the arrival of new information about the fundamentals.

Methodologically closest to our paper is a study of recursive global games by Giannitsarou and Toxvaerd (2003). The similarity lies in the recursive approach to the game and in the assumption that by their present actions players change their state, which influences their future payoffs. The major difference is that, although substantially more general in most details,

[1] Matsui (1999) and Oyama (2004) assume complementarities between actions of subsequent generations of an OLG model, which leads to similar results.

Giannitsarou and Toxvaerd allow for positive links between tomorrow's and today's investments only. Thus, their model generates endogenous growth or decline but not endogenous cycles. Alternations between booms and slumps have to be caused by a sudden change in the fundamentals. Toxvaerd (2004) presents another global game with a recursive structure that generates endogenous and monotone evolution of thresholds, but again it does not generate cyclical behavior.

The switches between booms and slumps in the model I present below are not only a consequence of the random evolution of fundamentals, but can be caused by the cyclical evolution of thresholds, or in other words, by the evolution of market sentiments that are the unique outcome of the model. The model combines the equilibrium uniqueness of the global games with the cyclicality of strategic delay models (e.g. Shleifer, 1986; Gale, 1995), where the delay models study situations in which players are motivated to delay investment to match the timing of others' investments.

Section 1.2 introduces cyclical evolution of the threshold in a basic model without a direct economic interpretation. I then amend the basic model to fit real economic problems in section 1.3. Section 1.4 concludes.

1.2 The Basic Game

Let us study a repeated coordination game in which risk-neutral players have one token each which they can invest in one of rounds $t \in \{1, \ldots, T\}$ (later I briefly discuss infinite horizon). There is a continuum of players with measure 1, indexed by $i \in [0, 1]$. In each period, players who still have the token decide whether to invest it or wait; $a_t^i \in \{I, NI\}$. (The constraint that players can invest only once simplifies the exposition. In section 1.3, successful investors are allowed to invest again and let the unsuccessful ones go bankrupt with some probability.)

I assume positive externalities of investment. Return at t increases with the measure of investments at t. The return at t also increases with (random) fundamentals θ_t, which are assumed to be i.i.d. with twice continuously differentiable c.d.f. $\Phi(.)$ on the real line; the associated p.d.f. is denoted by $\phi(.)$. The distribution of the fundamentals is simple. Yesterday's fundamentals do not offer any information about today's fundamentals; the model thus abstracts from social learning. Such an assumption effectively makes the stages more independent and thus the structure of the stage games more stationary: The game has nearly the same structure as repeated games; all the stage games are virtually identical, differing only in the continuation payoff V_{t+1}.

Following Morris and Shin (2003), the instantaneous payoff for not investing is 0 and for investing is

$$u(I, l_t, \theta_t) = \theta_t - 1 + l_t, \tag{1.1}$$

where l_t is a measure of the set of all players who have invested at t.[2] Players maximize the sum of discounted instantaneous payoffs $\sum_{t=1}^{T} \delta^t u_t$.

The measure of players is assumed to be constant in each round. For this reason, players who have already invested, and thus cannot invest in future rounds, are replaced by entrants in the benchmark model. This makes the game more stationary, entrants are not an additional source of fluctuations. Some justifications for the assumption can be found in the applications. In the applications 1.3.1 (currency attacks) and 1.3.2 (emerging market crises), players leave the game only when bankrupting and the constant measure of players can be interpreted as the amount of capital being fixed and only exchanging hands during the game. Payoff depends on the relative ratio of searching players in the application 1.3.3 (search), thus the effective measure of players is constant irrespectively of the history of play.

If players were to observe fundamentals perfectly, the game would exhibit a multiplicity of equilibria, since strategic complementarity makes investing, as well as non-investing, self-enforcing. For example, the last round stage has, except for the extreme values of θ_T, two pure Nash equilibria: in one everybody invests, and in the other nobody invests.

The set of fundamentals θ_t for which players coordinate on investment in stage t depends on the expected discounted continuation profit δV_{t+1}, which is the outside option of the coordination problem at t. The higher the outside option, the harder the coordination on the risky investment. An equilibrium selection tool is needed to map this influence one to one. I use the global games as formulated in Morris and Shin (2003); alternatively the concept of risk dominance proposed by Harsanyi and Selten (1988) could be used.

From now on I make the usual global games assumption that players do not observe fundamentals precisely. Each player receives in each stage t a private signal $x_t^i = \theta_t + \sigma\epsilon_t^i$ about the true fundamentals θ_t, where the idiosyncratic errors ϵ_t^i are independent across players and rounds and drawn from a continuous p.d.f. $f(.)$ with support on the real line, and with finite expectations $\int_{-\infty}^{\infty} z f(z) dz$. The c.d.f. is denoted by $F(.)$. Parameter σ denotes the size of the noise, and I am interested in which equilibrium will be selected as $\sigma \to 0$. All distributions and parameters of the game are common knowledge.

[2] The simple functional form of (1.1) is not substantial. Any static global game payoff can be used.

The information structure of such a multistage game is complicated.[3] Nevertheless, as the fundamentals across stages are independent, past outcomes do not provide any information on today's fundamentals. In fact, it will be proved that in equilibrium actions depend only on the current signal.

Let us solve this incomplete information game by backward induction and show that it has a unique equilibrium in the limit $\sigma \to 0$. The subgame of the last round T is a static coordination game described in the introduction of Morris and Shin (2003). For the sake of reference, I reproduce key proposition 2.2 of Morris and Shin (2003) in the appendix and refer to it below as the "Morris and Shin theorem". Morris and Shin show that, as $\sigma \to 0$, the unique strategy surviving the iterated elimination of dominated strategies in the last round subgame is a threshold strategy:

$$s_T^*(x) = \begin{cases} I & \text{if } x > \theta_T^*, \\ NI & \text{if } x < \theta_T^*, \end{cases} \tag{1.2}$$

with threshold θ_T^* such that $s_T^*(.)$ is the best reply to the belief according to which the measure L_T of investing players is distributed uniformly on $[0, 1]$. Morris and Shin (2003, p. 5) "dub such beliefs ... as being Laplacian, following Laplace's (1824) suggestion that one should apply a uniform prior to unknown events from the principle of insufficient reason". Such beliefs arise endogenously in global games for a player observing the threshold signal.

Given such "Laplacian" beliefs, threshold θ_T^* can be determined as an indifference point between investing, which pays $\int_0^1 (\theta - 1 + l)dl$, and not investing, which pays 0. The threshold at T is thus $\theta_T^* = \frac{1}{2}$.

Knowing the equilibrium of the last stage T, it is possible to compute the expected profit V_T. In the limit, as $\sigma \to 0$, all players invest if and only if the fundamentals $\theta_T > \theta_T^*$. In that case $l_T = 1$ and all receive $\theta - 1 + 1 = \theta$. If $\theta_T < \theta_T^*$ all players wait and receive 0. Thus:

$$V_T = E[\theta | \theta > \theta_T^*] + 0 \times Prob(\theta < \theta_T^*) = \int_{\theta_T^*}^{\infty} \theta \phi(\theta) d\theta + 0 \times \Phi(\theta_T^*). \tag{1.3}$$

Stage $T - 1$ is again a static global game in which the outside option payoff is δV_T rather than 0. The threshold at $T - 1$ is again an indifference point of a player with Laplacian beliefs: $\int_0^1 (\theta_{T-1}^* - 1 + l)dl = \delta V_T$ and hence $\theta_{T-1}^* = \frac{1}{2} + \delta V_T$. The backward induction can be applied further. Generally, (in the limit $\sigma \to 0$):

[3] The information set $I_t^i = \{x_1^i, \ldots, x_t^i, l_1, \ldots, l_{t-1}\}$ of player i in a round t is the history of her signals and the history of the aggregate investment l_t. Pure strategy $s = \{s_1, \ldots, s_T\}$ is a series of functions that assigns to a path of information sets $\{I_1^i, \ldots, I_T^i\}$ a path of actions $\{s_1(I_1^i), \ldots, s_T(I_T^i)\}$.

$$\theta_t^* = \vartheta(V_{t+1}) \equiv \frac{1}{2} + \delta V_{t+1}. \tag{1.4}$$

$$\begin{aligned} V_t &= G(V_{t+1}) \equiv E[\theta|\theta > \theta_t^*]Prob(\theta > \theta_t^*) + \delta V_{t+1} Prob(\theta < \theta_t^*) \\ &= \int_{\vartheta(V_{t+1})}^{\infty} \theta\phi(\theta)d\theta + \delta V_{t+1}\Phi(\vartheta(V_{t+1})), \end{aligned} \tag{1.5}$$

with the boundary condition $V_{T+1} \equiv 0$.

Formally, denote the whole game by Γ_σ:

Proposition 1.1. *For any $\epsilon > 0$ there exists $\overline{\sigma}$ such that for all $\sigma < \overline{\sigma}$ if strategy s survives iterated elimination of dominated strategies in the game Γ_σ then $s_t = NI$ for all $x_t \leq \vartheta(V_{t+1}) - \epsilon$ and $s_t = I$ for all $x_t \geq \vartheta(V_{t+1}) + \epsilon$ for all $t \in \{1, \ldots, T\}$ where the function $\vartheta(.)$ is defined in equation (1.4) and V_t are defined by mapping $G(.)$ in equation (1.5) and by a boundary condition $V_{T+1} = 0$.*

The proof, presented in the appendix, consists of checking all assumptions of the original Morris and Shin theorem 1.1 and of applying backward induction.

1.2.1 Evolution of Thresholds

How does the threshold θ_t^* evolve over time? Does it converge to a steady state if the length of the game goes to infinity? I show that (for some constellations of parameters) the thresholds necessarily fluctuate and the system never converges to a steady state.

The threshold θ_t^* is fully determined by the continuation values V_{t+1} which evolve according to the highly nonlinear mapping $G(.)$. An approximate picture of the mapping $G(.)$ is easy to plot if the noise of the prior distribution $\Phi(.)$ is small. Let us, without loss of generality, write $\theta = y + \tau\gamma_t$, where y is the expectation of θ_t and γ_t is a random component of θ_t. C.d.f. $\Phi(.)$ is nearly a step function if τ is small. Thus $G(V)$ is almost piecewise linear with quick transitions from one linear segment to another at such values of V that $\vartheta(V) = y$ (see figure 1.1).

What is the economic intuition behind $G(.)$ being approximately constant for small V, then quickly declining for medium values of V, and then moderately increasing for large V? An increase in V_{t+1} has two effects on V_t. The direct effect is positive: if the players coordinate on not investing, they receive a higher outside option δV_{t+1}. The strategic effect is an increase in the

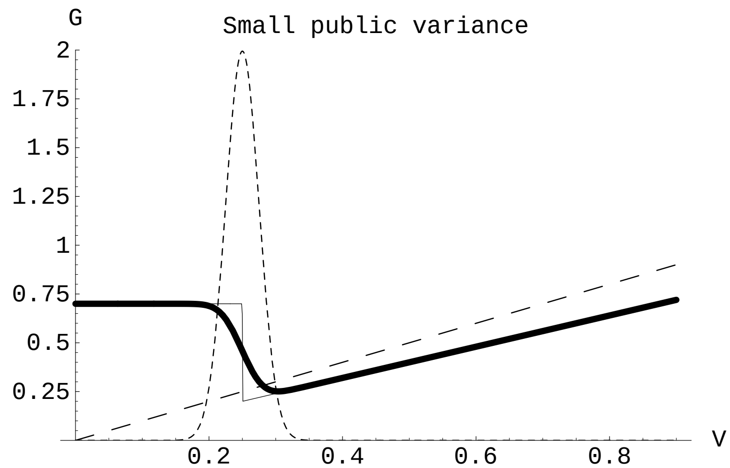


Figure 1.1: Thick line — mapping $G(.)$ for a nonzero variance of priors $\tau > 0$. Thin line — mapping $G(.)$ in a limit $\tau \to 0$. Dotted line — the probability density $\phi(\vartheta(V_{t+1}))$ at the threshold $\theta_T^* = \vartheta(V_{t+1})$. Dashed line — diagonal.

threshold $\theta_t^* = \frac{1}{2} + \delta V_{t+1}$. This decreases V_t because the players coordinate on investing at t for a smaller set of realizations of θ_t. The strategic effect is small if the threshold θ_t^* is far away from the average value y, because the probability density $\phi(\theta_t^*)$ is low and the impact of a small increase in the threshold on the total probability of coordination is small. If, however, V_{t+1} happens to be such that the threshold $\theta_t^* \approx y$ then the probability density is high and a small increase in the threshold disables the successful coordination in many states of the world, which substantially reduces expected profit. This explains the region of sharp decline of $G(.)$ in figure 1.1.

The steady states solve the equation $G(V) = V$. If $G(.)$ crosses the diagonal in the transition segment then $|G'(V^*)| \to \infty$ in the limit[4] as $\tau \to 0$ and the fixed point is unstable. The fixed point is unstable also for $\tau > 0$ as for τ sufficiently close to 0, $|G'(V^*)| > 1$. The algebraic condition for diagonal crossing $G(.)$ in the transition segment is $\frac{1}{2} < y < \frac{1}{2}/(1-\delta)$ and τ being sufficiently small. Thus the fixed point is unique and unstable and the threshold fluctuates for intermediate values of y. Simple numerical simulations revealed that regular periodic cycles as well as chaotic paths are possible for different model parameters. Note that the path does not explode; V_t is bounded from below by 0 and from above by $\int_{-\infty}^{\infty} \theta d\Phi(\theta)$.

If y is very low, the fundamentals are almost always bad, meaning the diagonal crosses $G(.)$ to the right from the transition segment. Thus the

[4]The ordered limit $\lim_{\tau\to 0, \sigma \to 0}$ has to be taken because the equilibrium uniqueness result holds only for σ being small compared to τ.

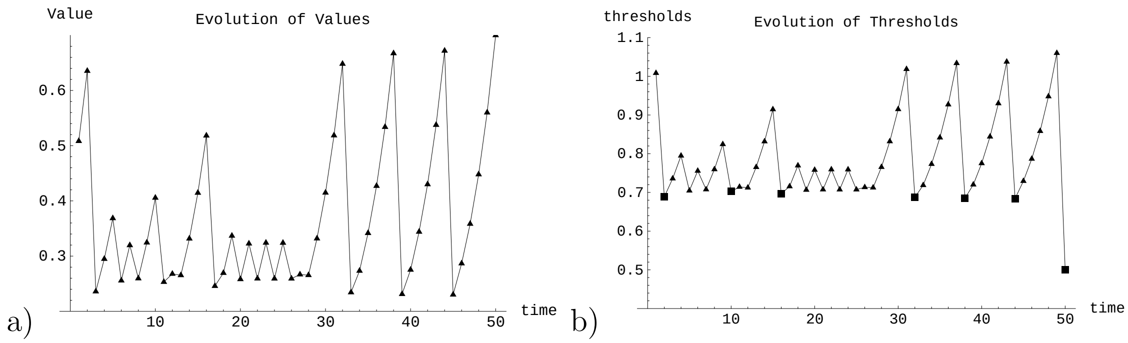


Figure 1.2: a) Evolution of the expected continuation values V_t generated by the mapping $G(.)$. b) Evolution of the thresholds $\theta_t^* = \vartheta(V_{t+1})$. The symbol ■ denotes periods in which players coordinated on investment for one particular realization of random fundamentals $\{\theta_1, \ldots, \theta_T\}$.

slope of $G(.)$ at the unique fixed point $V = 0$ is $\delta < 1$, and it is stable. The intuition is that for very bad priors the players almost never coordinate on the investment in the last period, hence $V_T \approx 0$ and the earlier periods are almost identical to the last period. Similarly, if y is so high that the fundamentals are almost always good, the diagonal crosses $G(.)$ to the left from the transition segment, where the slope of $G(.)$ is 0, and the unique fixed point $V \approx y$ is again stable. The intuition is that for very good priors the players nearly always coordinate on investing despite the high outside option δV.

Figure 1.2 depicts a numerical example of a fluctuating threshold path for particular parameters.[5] The coordination on investment is more probable in periods with low thresholds, but it depends also on the realizations of random fundamentals.

1.2.2 Time Horizon

A finite time horizon is unrealistic in any of the applications of section 1.3 below. Moreover, the values V_t are very sensitive to the specification of T. Two approaches can be taken. First, let us consider very long but finite games. The equilibrium uniqueness result holds for any T and if the mapping $G(.)$ has a unique and unstable fixed point, the fluctuations and nonstationary behavior are a necessary outcome for a game of any duration T.

Alternatively, let us consider an infinite game with t unbounded. In such a game, the equilibrium uniqueness result does not hold, as the boundary

[5] Prior beliefs distribution $N(0.6, 0.01^2)$ and $\delta = 0.8$.

condition $V_{T+1} = 0$ is lost. Nevertheless, the mapping $V_t = G(V_{t+1})$ still holds (in the limit $\sigma \to 0$). Thus, in the case of a unique unstable fixed point, although the sequence of V_t cannot be specified, the evolution will be nonstationary. Hence the testable prediction of the model is its nonstationarity rather than a particular prediction of equilibrium path.

The basic model serves as the simplest illustration of the dynamic global games I want to study. In the next section, I amend the instantaneous payoffs and the continuation structures of the basic model and study models with economic interpretations.

1.3 Applications

I present three illustrative models: a currency attacks model built on Morris and Shin (1998), a model of crises co-movement, and a model of search activity cycles. In all three applications players maximize a sum of discounted instantaneous payoffs. They face a series of coordination problems described by instantaneous payoffs and, moreover, their action at t influences their access to profits in future coordination problems. Players can be in one of two states {not bankrupt, bankrupt} in applications 1.3.1, 1.3.2, and {partnered, partnerless} in application 1.3.3. By a choice of action at t, players influence their state in future rounds as they did in the basic game where investing moved the player from the state with a token to the state without. I will admit more general situations in which the change of state is determined strategically; the probability $r(a_t^i, l_t, \theta_t)$ of the change of state will also depend on the aggregate of others' actions l_t.

I assume a continuum of players and that payoffs depend on the others' actions only through the aggregate action l_t. The absence of large players implies that the continuation value V_{t+1} is a common value for all players, which enables us to use a recursive formulation in all three applications. Thus, each stage payoff is a superposition of the instantaneous payoff and the continuation payoff. I study situations in which the total payoff $u_t^i + \delta V_{t+1} r_t^i$ satisfies the global game assumptions for any V_{t+1}, so each stage will have a unique equilibrium which allows to use backward induction.

All three applications share the information structure of the basic model.[6]

[6] To recall, economic fundamentals θ_t, $t \in \{1 \dots T\}$ are i.i.d., drawn from twice continuously differentiable c.d.f. $\Phi(.)$ on a real line. Players receive a private signal $x_t^i = \theta_t + \sigma\epsilon_t^i$ where the private errors ϵ_t^i are independent across players and rounds and drawn from continuous density $f(.)$ with support on the real line with finite expectations $\int_{-\infty}^{\infty} z f(z) dz$.

1.3.1 Currency Attacks

In the first application I inject economic content into the basic model and relax the assumption that players can invest only once. I extend the Morris and Shin (1998)[7] model of currency crises by adding a continuation structure — the unsuccessful speculators may go bankrupt and thus lose access to future profits.

Morris and Shin consider a currency pegged to an exchange rate e^* which, if the government does not protect the peg, will float to a rate $\zeta(\theta_t)$, where the function $\zeta(.)$ is continuous and increasing in θ_t. A continuum of speculators with measure 1 decide whether to sell the currency short or not. The transaction cost of short-selling is c. If the currency is devaluated, the short-selling pays a net profit $e^* - \zeta(\theta_t) - c$. The government defends the peg, but only if it is not too costly. The cost of defending increases with the measure of the short sales; the government will defend if the measure of attacking speculators is smaller than $a(\theta_t)$, which is continuous and increasing in θ_t. The instantaneous payoff for not attacking is 0. The instantaneous payoff for attacking is summarized by

$$u(I, l_t, \theta_t) = \begin{cases} e^* - \zeta(\theta_t) - c & \text{if } a(\theta_t) < l_t, \\ -c & \text{if } a(\theta_t) \geq l_t. \end{cases} \tag{1.6}$$

The authors assume dominance regions.[8] The informational structure is the same as in the basic game. The one-shot game can be solved by applying Morris and Shin theorem 1.1 because $u(I, l_t, \theta_t)$ is weakly monotone in θ_t and l_t.

I extend Morris and Shin (1998) by assuming that an unsuccessful speculation results in bankruptcy with probability b. Alternatively, it could be assumed that managers responsible for the attack decision get fired if the attack fails (Chevalier and Ellison, 1999) in which case they miss bonuses for future profits. In reality, the adverse consequences of losses may be more subtle than total bankruptcy or dismissal. Unsuccessful speculators may become constrained, which would limit their future short sales and hence future profits. The simplification of assuming only the possibility of a bankruptcy but not partial consequences is similar in nature to the simplification of assuming only the binary decision of attacking or not and abstracting from the amount of short sales.

[7] See also Heinemann (2000).

[8] The government devaluates for sufficiently bad fundamentals even without any speculators, and even a coordinated attack of all speculators will not lead to devaluation for sufficiently good fundamentals.

$r(a^i, l_t, \theta_t)$	$l_t > a(\theta_t)$	$l_t \leq a(\theta_t)$
I	0	$(1-b)$
NI	0	1

Table 1.1: Probability that the player will have an opportunity to attack in the next period.

The speculative capital of unsuccessful speculators is assumed to end up in the hands of other speculators after the bankruptcy, so the measure of the potential speculative capital is 1 in all rounds. Abandoning the peg makes further attacks impossible, so all players have zero future profits after the attack regardless of their action. The continuation probabilities $r(a^i, l_t, \theta_t)$ are summarized in table 1.1. Note that if the bankruptcy probability $b = 1$, the continuation structure becomes that of the basic game: a player can attack only once, as she either goes bankrupt or the peg is abandoned.

Proposition 1.2. *Proposition 1.1 applies with thresholds $\theta_t^* = \vartheta(V_{t+1})$ solving equation*

$$[1 - a(\theta)][e^* - \zeta(\theta)] - a(\theta)\delta b V_{t+1} = c. \tag{1.7}$$

The evolution of expected future payoffs is determined by mapping

$$G(V_{t+1}) = \int_{-\infty}^{\vartheta(V_{t+1})} (e^* - \zeta(\theta) - c)\,\phi(\theta)d\theta + \delta V_{t+1}[1 - \Phi(\vartheta(V_{t+1}))], \tag{1.8}$$

with a boundary condition $V_{T+1} = 0$.

Proof: In the recursive formulation, each round is a static global game with the payoff:

$$v_t(a_t, l_t, \theta_t) = u(a_t, l_t, \theta_t) + V_{t+1} r(a_t, l_t, \theta_t).$$

It is straightforward to verify that

$$\pi_t(l_t, \theta_t) \equiv v_t(I, l_t, \theta_t) - v_t(NI, l_t, \theta_t) = \begin{cases} e^* - \zeta(\theta_t) - c & \text{if } a(\theta_t) < l_t, \\ -c - \delta b V_{t+1} & \text{if } a(\theta_t) \geq l_t \end{cases}$$

satisfies[9] assumptions MS1–5 of the Morris and Shin theorem; MS6 — the finite expectation of errors was assumed directly (see the appendix for the assumptions MS1–6). Thus the Morris and Shin theorem 1.1 applies in all rounds. The threshold function $\vartheta(V_{t+1})$ defined in equation (1.7) that

[9]Note that $\pi(l, \theta)$ decreases with θ whereas, formally, the Morris and Shin theorem requires π increasing in θ. Such a situation can be accommodated by introducing $\tilde{\theta} = 1 - \theta$.

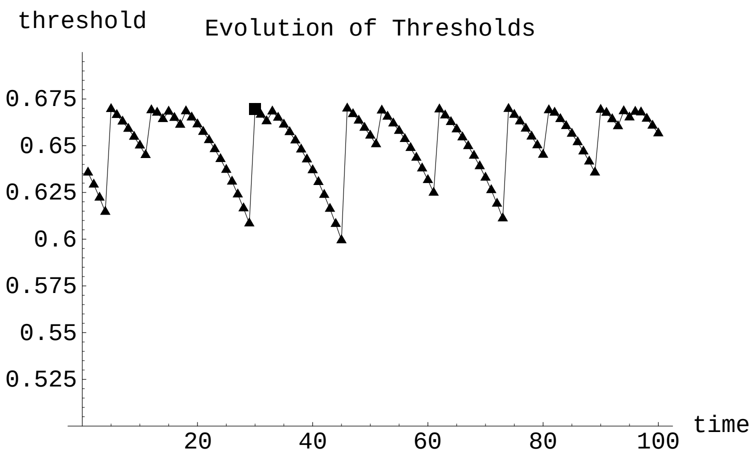


Figure 1.3: Evolution of the thresholds below which the speculators attack. The attack is more probable when the threshold is high. A (successful) attack has happened in the period denoted by the symbol ■, for a particular realization of the random fundamentals.

describes the thresholds is the unique solution to $\int_0^1 \pi_t(l,\theta)dl = 0$ (so the threshold strategy with the threshold $\vartheta(V_{t+1})$ is the best reply to the Laplacian belief).

Equation (1.8) describes that, in the limit of precise signals $\sigma \to 0$, all players coordinate on not attacking if $\theta_t > \theta_t^* = \vartheta(V_{t+1})$ and thus all receive the continuation payoff δV_{t+1} in such a case. All players attack at t if and only if the fundamentals are below θ_t^*, in which case they receive $e^* - \zeta(\theta) - c$.

Q.E.D.

To illustrate the result, let us study a numerical example with a specification of the exchange rate difference $e^* - \zeta(\theta)$ being constant and equal to 1. Function $a(\theta)$ describing the willingness of the government to protect the peg is set to $a(\theta) = \theta$. Equation (1.7) simplifies into $\vartheta(V_{t+1}) = \frac{1-c}{1+\delta b V_{t+1}}$ and equation (1.8) simplifies into $G(V_{t+1}) = \Phi(\vartheta(V_{t+1}))(1-c) + \delta V_{t+1}[1 - \Phi(\vartheta(V_{t+1}))]$. I plot the evolution of thresholds for particular parameters[10] in figure 1.3. Periods in which the threshold is high are windows of probable attacks because the speculators believe that others will attack even if the fundamentals of the economy are high.

[10]The prior distribution is $N(0.67, 0.001^2)$, $b = 0.5$, $c = 0.3$, $d = 0.9$.

1.3.2 Emerging Markets Crises — Co-movement

The models under study allow for a rigorous treatment of cycles of self-fulfilling beliefs and thus the framework is suitable for a study of beliefs-based contagion of crises. Two developing countries without any direct links but with a common set of investors are considered. I assume that investment in either of the two countries may cause bankruptcy. The consequences of the bankruptcy are the same, regardless of the country in which the unsuccessful investment has been realized — future profits in *both* countries are lost. Thus, the willingness to risk investment at date t is influenced in both countries by the *common* value V_{t+1}. High future profits, regardless of the country in which they would be realized, undermine coordination in both countries today, which causes co-movement of the willingness to invest in the otherwise independent countries.

Let us consider two emerging market countries A and B with economic fundamentals θ_A and θ_B respectively; each with a continuum of investment opportunities of measure 1. There is a continuum of investors of measure 1 of which each observes two investment opportunities — one in A and one in B. Investors can invest in a project they observe in country A or B or in both at each round t. The instantaneous payoff of an investor is the sum of returns from her current investments. The investments are, within each country, strategic complements: returns $R_c(l_{c,t}, \theta_{c,t})$, $c \in \{A, B\}$ increase with the level of investment $l_{c,t}$ and the fundamentals $\theta_{c,t}$ within each country. Return in A does not depend on investment $l_{B,t}$ or fundamentals $\theta_{B,t}$ and vice versa. The fundamentals $\theta_{A,t}$ and $\theta_{B,t}$ drawn from distribution $\phi_A(.)$ and $\phi_B(.)$, respectively, and are independent across countries and times, so the instantaneous payoffs cannot themselves alone provide any explanation for correlation in the economic outcomes.

To keep the problem within the simple global games framework I do not allow the players to choose the amount of investment; they choose in each country only whether to invest one unit or not. The existence of dominance regions is assumed[11] and the appropriate continuity[12] of $R(.,.)$. The payoff from not investing is 0.

Let us now introduce a continuation structure which will create a correlation between otherwise independent countries. I assume that investment in country c causes bankruptcy of the investor with probability b_c. Precisely, if a player does not invest in either of the countries, the probability of bankruptcy is 0; if she invests in one country c the probability is b_c; if in both countries

[11]For both countries $c \in \{A, B\}$ exist $\underline{\theta}_c$ and $\overline{\theta}_c$ and $\epsilon > 0$ such that: 1. $R_c(l, \theta) < -\epsilon$ for all $l \in [0, 1]$ and $\theta < \underline{\theta}_c$, and 2. $R_c(l, \theta) > \epsilon$ for all $l \in [0, 1]$ and $\theta > \overline{\theta}_c$.

[12]$\int_0^1 g(l)R(l, x)dl$ is continuous with respect to signal x and density $g(.)$.

the probability is $b_A + b_B$. The detailed mechanisms of the bankruptcy is not modelled. The bankruptcy is a black box for distress that a company (or manager) may meet in an emerging market and which may constrain a company's (manager's) future activities. Bankruptcy, although it was caused by a problem in *one* country, precludes the player from operating in *both* A and B in all future rounds.

The events in A and B that lead to bankruptcy are assumed to be independent. As this assumption makes countries more independent, it does not smuggle contagion into the model, and allows us to treat each country as a separate coordination problem, thus simplifying the analysis.

In the recursive formulation, the coordination problem of country $c \in \{A, B\}$ in period t is described by the payoff difference between investing and not investing:

$$\pi_{c,t}(l_{c,t}, \theta_{c,t}) = R_c(l_{c,t}, \theta_{c,t}) - b_c \delta V_{t+1}, \tag{1.9}$$

which constitutes two independent global games, one for each country. These can be solved in each round, so again backward induction can be used.

Proposition 1.3. *In the unique equilibrium (in the limit of precise signals $\sigma \to 0$) investors invest in country $c \in \{A, B\}$ at date t if and only if the fundamentals $\theta_{c,t}$ are above the threshold $\theta^*_{c,t}$. Both thresholds $\theta^*_{A,t} = \vartheta_A(V_{t+1})$ and $\theta^*_{B,t} = \vartheta_B(V_{t+1})$ are functions of a common continuation value V_{t+1}.*

The $\vartheta_c(V_{t+1})$ are θ^* solving

$$\int_0^1 R_c(\theta, l)dl = b_c \delta V_{t+1}. \tag{1.10}$$

The evolution of V_t is defined by the mapping

$$\begin{aligned} V_t = G(V_{t+1}) \equiv \; & \delta V_{t+1} + \int_{\vartheta_A(V_{t+1})}^{\infty} (R_A(\theta, 1) - b_A \delta V_{t+1}) \phi_A(\theta) d\theta + \\ & \int_{\vartheta_B(V_{t+1})}^{\infty} (R_B(\theta, 1) - b_B \delta V_{t+1}) \phi_B(\theta) d\theta, \end{aligned} \tag{1.11}$$

with a boundary condition $V_{T+1} = 0$.

Proof:

The incentive to invest in country c $\pi_{c,t}(l_{c,t}, \theta_{c,t})$ is described by equation (1.9) and satisfies assumptions MS1-5 of Morris and Shin theorem 1.1. The noise distribution $f(.)$ satisfies MS6. Therefore the coordination problems of both countries at each stage are global games, and the thresholds $\vartheta_c(V_{t+1})$ are the solutions of equation $\int_0^1 \pi_{c,t}(l, \theta)dl = 0$ which gives equation (1.10).

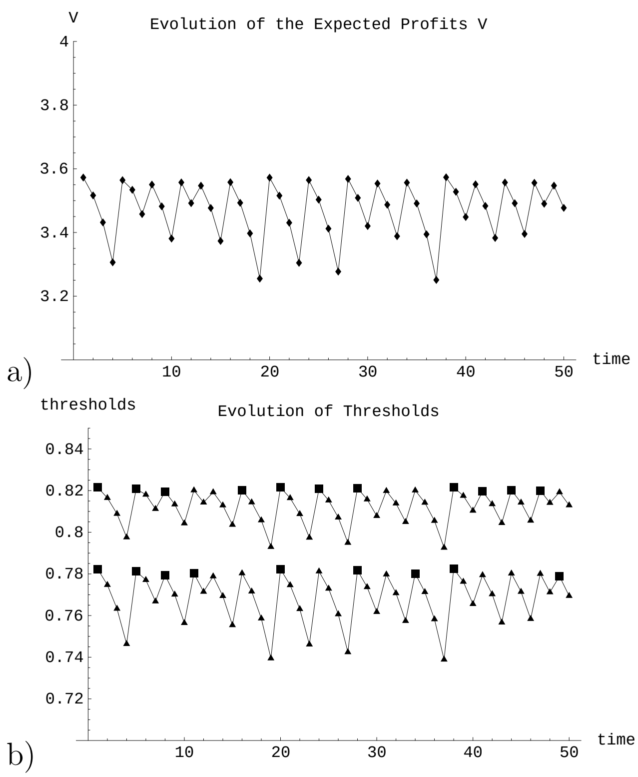


Figure 1.4: a) The evolution of the future expected profits V_{t+1} is common for both countries. b) As a consequence, the evolution of the thresholds θ^*_A and θ^*_B is correlated. The probability of crisis is high when the thresholds are high. The symbol ■ denotes crises for one particular realization of random fundamentals.

Equation (1.11) describes that, in the limit of precise signals, all players invest if and only if $\theta_{c,t} > \theta^*_{c,t} = \vartheta_c(V_{t+1})$ in which case they receive $R_c(1, \theta_{c,t})$ and go bankrupt with probability b_c.
Q.E.D.

I illustrate the result on an example with simple return functions

$$R_c(l, \theta) = \begin{cases} 1 - \gamma_c & \text{if } l > 1 - \theta, \\ -\gamma_c & \text{if } l \leq 1 - \theta. \end{cases}$$

The simple form of $R_c(.,.)$ enables us to compute the $\vartheta_c(.)$ defined in equation (1.10).

$$\vartheta_c(V_{t+1}) = \gamma_c + b_c \delta V_{t+1}.$$

The evolution is plotted in figure 1.4 for particular parameters.[13] The oscilla-

[13] $\gamma_A = 0.3$, $\gamma_B = 0.5$, prior beliefs distribution in country A is $N(0.78, 0.002^2)$; in

tion of thresholds around the averages of the domestic fundamentals in both countries is perfectly correlated. A high threshold above the country's average fundamentals means that crisis is probable, as investors will believe that others invest only if the realized fundamentals are high. The high thresholds can thus be interpreted as pessimistic market sentiments. I have generated the random fundamentals and marked the crises during which investors do not invest by ■. The occurrence of crises in both countries is correlated despite the lack of direct links among the countries.

The effect is similar to changes in the amount of strategic risk caused by a wealth increase and by the implied decrease of absolute risk aversion studied in Goldstein and Pauzner (2003). However, our fluctuations of strategic risk are caused by changes in the lottery rather than by changes in the risk attitude; our players are risk-neutral. Another difference is that our model has a reverse causality: profits tomorrow influence strategic risk today, whereas wealth accumulated yesterday influences risk aversion today in the Goldstein and Pauzner model. A crisis in A at t is not caused by a crisis in B at t or earlier in our model. Rather, the correlation of crises is caused by common expected future profits. Thus, the outcome of our model is a contagion in the broad sense of defining contagion as excess co-movement, but not in a narrow sense which requires a causality link from an earlier crisis to a later one.

1.3.3 Fluctuations of Search Activity

In the previous examples, the continuation structure was interpreted as a bankruptcy, or, more generally, as a reduced ability to act. In this section I study a model of one-sided search where the ability to search tomorrow is not decreased by any outcome of the search today, but if the search was successful today and a partner is found, a new search tomorrow is unnecessary (players need only one partner). Successful coordination today results in a reduced need for coordination tomorrow.

It is easier to find a business partner in a society where potential partners are actively searching for a partnership than in a society where nobody else searches. This strategic complementarity of search is stressed in the influential Diamond (1982) model. Chamley (1999) names search as his first example of a strategic complementarity. Ennis and Keister (2003) study the consequences of taxation on search activity and particularly the consequences on equilibrium selection. I offer a model in which the aggregate search activity fluctuates endogenously.

country B $N(0.82, 0.001^2)$, $\delta = 0.9$, $b_A = 0.15$, $b_B = 0.1$.

There is a continuum of identical players, each player needing a partner to produce. Players receive an instantaneous payoff 1 in each round in which they have a partner. After the payoff is received, the partnership survives into the next period with probability $0 < p < 1$ or dissolves with probability $1-p$. Players without a trading partner receive 0 instantaneous payoff, and they can search for a partner by incurring (stochastic) cost.[14]

I assume that the probability $m_t(l_t)$ of finding a partner at round t increases with a share l_t of searching players among the partnerless players. For simplicity, let $m_t(l_t) = l_t$. I also assume that m_t depends only on the relative share of searching players among the partnerless ones, not on their absolute number.[15] Thus, the measure of partnerless players is effectively renormalized to 1 in each round.

Proposition 1.4. *Proposition 1.1 applies with the threshold function*

$$\vartheta(\Delta_{t+1}) = \frac{\delta\Delta_{t+1}}{2}. \tag{1.12}$$

The evolution of the expected payoff advantage Δ_t of having a partner is determined by the mapping

$$G(\Delta_{t+1}) = 1 + \int_{-\infty}^{\vartheta(\Delta_{t+1})} \theta\phi(\theta)d\theta + [p - \Phi(\vartheta(\Delta_{t+1}))]\delta\Delta_{t+1}, \tag{1.13}$$

with a boundary condition

$$\Delta_{T+1} = 0.$$

Proof:

Lemma 1.1. $\Delta_t > 0$ *for all* t.

The proof of lemma 1.1 is found in the appendix.

In the recursive formulation, partnered players with a partner make no decisions and face the expected payoff

$$V_{e,t} = 1 + p\delta V_{e,t+1} + (1-p)\delta V_{u,t+1}. \tag{1.14}$$

[14]The costs are assumed to be sometimes prohibitively high, which implies the existence of the right dominance region, and that the costs are sometimes negative, which implies the left dominance region. This can be justified by government paying a subsidy for the search, which exceeds the true costs, or by an intrinsic motivation exceeding pecuniary costs.

[15]This can be justified in the following way: Partnerless players first simultaneously decide whether to prepare for future production by incurring cost θ_t. They are afterwards randomly matched to pairs and partnership is formed if both members of a pair are prepared.

Partnerless players face a coordination problem characterized by the payoff

$$u(a_t, l_t, \theta_t) = \begin{cases} l_t \delta V_{e,t+1} + (1 - l_t)\delta V_{u,t+1} - \theta_t & \text{if } a_t^i = I, \\ \delta V_{u,t+1} & \text{if } a_t^i = NI. \end{cases} \tag{1.15}$$

Partnerless players' incentive to search is

$$\pi_t(l_t, \theta_t) \equiv u(I, l_t, \theta_t) - u(NI, l_t, \theta_t) = \delta \Delta_{t+1} l_t - \theta_t,$$

where $\Delta_{t+1} \equiv V_{e,t+1} - V_{u,t+1}$. It is straightforward to verify[16] that $\pi_t(l_t, \theta_t)$ satisfies assumptions MS1–5 of Morris and Shin theorem 1.1 for any Δ_{t+1} and thus the theorem applies in all rounds. The threshold $\vartheta(V_{t+1})$ in equation (1.12) is the unique solution of $\int_0^1 \pi(l, \theta) dl = 0$.

Knowing the threshold, the expected profits of partnerless player $V_{u,t}$ can be recursively expressed:

$$V_{u,t} = \delta V_{e,t+1} \Phi(\vartheta(\Delta_{t+1})) - \int_{-\infty}^{\vartheta(\Delta_{t+1})} \theta \phi(\theta) d\theta + \delta V_{u,t}[1 - \Phi(\vartheta(\Delta_{t+1}))]. \tag{1.16}$$

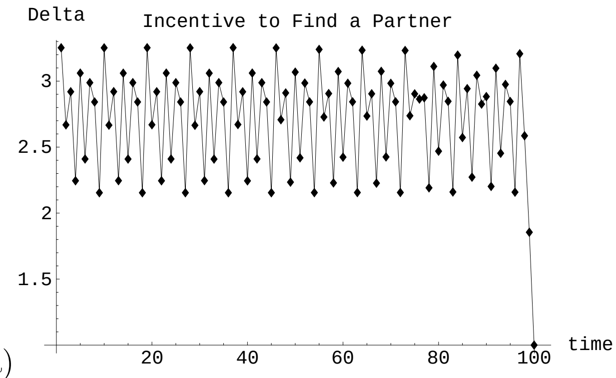
Function $G(\Delta_{t+1})$ in equation (1.13) can be found by subtracting equation (1.14) from equation (1.16).
Q.E.D.

I compute the evolution of advantage Δ_t of having a partner for particular[17] parameters. The value Δ_{t+1} oscillates. When the advantage Δ_{t+1} of being employed is high, players coordinate on searching even if the search costs are relatively high. Players coordinate on not searching even for relatively low search costs when Δ_{t+1} is low. Thus, the ratio of partnerless players will decrease with high probability when Δ_{t+1} is high and with low probability when Δ_{t+1} is low.

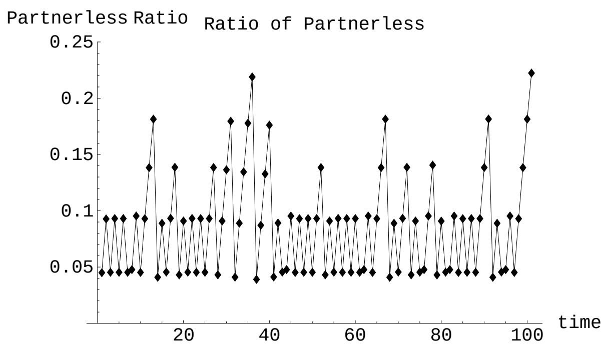
Like Diamond (1982), I have limited ourselves to a one-sided search model, in which the roles of employers and employees are not distinguished; rather, any pair of identical players could form a productive pair. The advantage is that the model stayed within the framework of simple global games in which all players are of the same type. Both our model and Diamond's model admit fluctuations in the measure of partnerless players, which Diamond interprets as unemployment fluctuations. However, whereas the fluctuations are a possible outcome of Diamond's model, they are a sure outcome in our model.

[16] Function π increases in l and decreases in θ whereas Morris and Shin theorem 1.1 requires π to be increasing in θ. Nevertheless, this can be accommodated by introducing $\tilde{\theta} = 1 - \theta$.

[17] The prior beliefs distribution is $N(1.3, 0.05^2)$, $p = 0.95$, $\delta = 0.9$; the ratio of partnerless players at $t = 0$ is 0.1.



a)



b)

Figure 1.5: a) Evolution of the advantage of being partnered $\Delta_t = V_{e,t} - V_{u,t}$. b) Evolution of the measure of partnerless players for a particular realization of the random fundamentals. A decrease is more probable when Δ_{t+1} is high.

1.4 Conclusion

Bankruptcy is worse prior to a boom than prior to a slump. Having a job today is more important if tomorrow's job prospects look grim, than if tomorrow looks bright. Fluctuations in the amount of strategic risk generated by the danger of termination are not only the consequences of cycles. Fluctuations can also cause cycles in an environment with strategic complementarities, since the probability of coordination on risky action decreases with the amount of strategic risk.

I have formalized this idea in a dynamic global game model which consists of a series of simple static global games. The non-trivial dynamic link between the rounds is that players influence not only their instantaneous payoff but also their ability to generate profits in the future. Successful coordination tomorrow increases the strategic risk associated with bankruptcy today and thus makes today's investment riskier. Coordination tomorrow thus undermines today's coordination, creating a negative feedback effect between tomorrow and today which leads to cycles. The dynamic model inherits attractive features of static global games: it is dominance solvable and thus the fluctuations unconnected to economic fundamentals not only may happen, but are a certain outcome of the model.

Interpretation of the unique equilibrium with a chaotic path is delicate. Such an equilibrium seems to be contradictory, because the slightest error in computation of thresholds would multiply greatly after a few iterations. Thus, it is extremely difficult to coordinate on such a chaotic equilibrium, and yet no stationary equilibrium exists. I do not expect to observe the behavior that exactly follows the predicted equilibrium path. Rather, the testable prediction is the non-existence of the stationary behavior. I believe that this prediction applies also for boundedly rational agents.

In future research, I wish to generalize the results for the case of autocorrelated distributions of fundamentals and for the case of nonstationary measure of players.

1.A Appendix

For convenience, I reproduce a condensed version of proposition 2.2 in Morris and Shin (2003):

Let there be a simultaneous move game with a continuum of players with measure 1, binary action space $a^i \in \{I, NI\}$ and payoff $u(a^i, l, \theta)$ where l is a measure of players playing I, and parameter θ is drawn from a continuously differentiable strictly positive density $\Phi(.)$ on the real line. Player i receives a signal $x^i = \theta + \sigma\epsilon^i$ where ϵ^i are independently drawn from a continuous density $f(.)$ with support on a real line. A (pure) strategy is a function $s : \mathbf{R} \to \{I, NI\}$, where $s(x)$ is the action chosen if a player observes signal x. Define $\pi(l, \theta) = u(I, l, \theta) - u(NI, l, \theta)$. The following assumptions are needed for the theorem:

MS1: **Action Monotonicity**: $\pi(l, \theta)$ is weakly increasing in l.

MS2: **State Monotonicity**: $\pi(l, \theta)$ is weakly increasing in θ.

MS3: **Strict Laplacian State Monotonicity**: There exists a unique $\theta^* \in \mathbf{R}$ such that $\int_0^1 \pi(l, \theta^*)dl = 0$.

MS4: **Uniform Limit Dominance**: There exist $\underline{\theta}$ and $\overline{\theta}$ and $\epsilon > 0$ such that 1. $\pi(l, \theta) < -\epsilon$ for all $l \in [0, 1]$ and $\theta < \underline{\theta}$ and 2. $\pi(l, \theta) > \epsilon$ for all $l \in [0, 1]$ and $\theta > \overline{\theta}$.

MS5: **Continuity**: $\int_0^1 g(l)\pi(l, x)dl$ is continuous with respect to signal x and density $g(.)$.

MS6: **Finite Expectations of Signals**: $E[z] = \int_{\infty}^{\infty} zf(z)dz$ is well defined.

Denote the game, satisfying MS1–MS6, by G_σ.

Theorem 1.1. (Morris and Shin): *Let θ^* be defined as in MS3. For any $\delta > 0$ there exists $\overline{\sigma} > 0$ such that for all $\sigma < \overline{\sigma}$, if strategy s survives iterated elimination of dominated strategies in the game G_σ then $s(x) = NI$ for all $x \leq \theta^* - \delta$ and $s(x) = I$ for all $x \geq \theta^* + \delta$.*

Proof in Morris and Shin (2003).

PROOF of proposition 1.1:

Assumption MS1-6 of the Morris and Shin theorem are satisfied in all stages:

$$\pi_t(l, \theta) \equiv u(I, l, \theta) - u(NI, l, \theta) = \theta - 1 + l - \delta V_{t+1}$$

is, for any V_{t+1}:
1. increasing in l;
2. increasing in θ;
3. there is a unique solution to $\int_0^1 \pi_t(l, \theta)dl = \theta - \frac{1}{2} + \delta V_{t+1} = 0$ which is $\vartheta(V_{t+1})$ of equation (1.4);
4. there exist uniformly dominant regions as π_t is linear in θ and bounded in l;
5. π_t is continuous in both l and θ;
6. The distribution of the error term is assumed to have a finite mean.

Therefore each stage can be solved as a global game and the unique threshold in each stage is $\theta_t^* = \vartheta(V_{t+1}) = \frac{1}{2} + \delta V_{t+1}$. Once knowing the continuation value $V_t + 1$ and the threshold $\theta_t^* = \vartheta(V_{t+1})$ the expected value V_t in the stage t can be expressed which is done in equation (1.5).
Q.E.D. (proposition 1.1)

PROOF of lemma 1.1:

Let us prove the statement by induction. Suppose $\Delta_{t+1} \equiv V_{e,t+1} - V_{u,t+1} \geq 0$. Then $\delta V_{u,t+1} \leq V_{u,t} \leq \delta V_{e,t+1}$. Using $V_{e,t} = 1 + \delta p V_{e,t+1} + \delta(1-p)V_{u,t+1}$ I get

$$\Delta_t \leq 1 + \delta p \Delta_{t+1},$$
$$\Delta_t \geq 1 - \delta(1-p)\Delta_{t+1}.$$

Let us denote the maximum and minimum of $\{\Delta_u\}_{u=t}^T$ by M and m. The equations imply

$$M \leq 1 + \delta p M,$$

$$m \geq 1 - \delta(1-p)M,$$

which gives

$$m \geq 1 - \frac{\delta(1-p)}{1-\delta p},$$

and the right hand side is positive for all $0 < p < 1$ and $0 < \delta < 1$.
Q.E.D. (lemma 1.1)

References

[1] Angeletos, G.M., Hellwig, A., Pavan, A., 2004. Information Dynamics and Multiplicity in Global Games of Regime Change, mimeo, http://www.econ.ucla.edu/people/papers/Hellwig/Hellwig279.pdf.

[2] Burdzy, K., Frankel, D.M., Pauzner, A., 2001. Fast Equilibrium Selection by Rational Players Living in a Changing World, Econometrica 69, 163–89.

[3] Carlsson, H., van Damme, E., 1993. Global Games and Equilibrium Selection, Econometrica 61, 989–1018.

[4] Chamley, C., 1999. Coordinating Regime Switches, Quart. J. Econ. 114, 869–905.

[5] Chevalier, J.A., Ellison, G., 1999. Career Concerns of Mutual Fund Managers, Quart. J. Econ. 114, 389–432.

[6] Cooper, R.W., 1999. Coordination Games: Complementarities and Macroeconomics, Cambridge University Press.

[7] Diamond, P., 1982. Aggregate Demand Management in Search Equilibrium, J. Polit. Economy 90, 881–894.

[8] Diamond, D.W, Dybvig, P.H., 1983. Bank Runs, Deposit Insurance, and Liquidity, J. Polit. Economy 91, 401-419.

[9] Farmer, R.E.A., Guo, J., 1994. Real Business Cycles and the Animal Spirits Hypothesis, J. Econ. Theory 63, 42–72.

[10] Ennis, H.M., Keister, T., 2003. Aggregate Demand Management with Multiple Equilibria, Federal Reserve Bank of Richmond Working Paper No. 03-04.

[11] Gale, D., 1995. Dynamic Coordination Games, Econ. Theory 5, 1–18.

[12] Giannitsarou, C., Toxvaerd, F., 2003. Recursive Global Games, Working Paper, Universidade Nova de Lisboa and Hebrew University of Jerusalem.

[13] Goldstein, I., Pauzner, A., 2003. Contagion of Self-Fulfilling Financial Crises Due to Diversification of Investment Portfolios, J. Econ. Theory 119, 151–183.

[14] Harsanyi, J., Selten, R., 1988. A General Theory of Equilibrium Selection in Games, MIT Press.

[15] Heinemann, F., 2000. Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks: Comment, Amer. Econ. Rev. 90, 316–318.

[16] Masson, P.R., 1998. Contagion-Monsoonal Effects, Spillovers, and Jumps Between Multiple Equilibria, IMF Working Paper no. 98/142.

[17] Matsui, A., 1999. Multiple Investors and Currency Crises, mimeo, University of Tokyo.

[18] Morris, S., Shin, H.S., 1998. Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks, Amer. Econ. Rev. 88, 587–597.

[19] Morris, S., Shin, H.S., 1999. A Theory of the Onset of Currency Attacks. In: Agenor, P.R., Vines, D., Weber, A. (Eds), Asian Financial Crisis: Causes, Contagion and Consequences, Cambridge University Press.

[20] Morris, S., Shin, H.S., 2003. Global Games: Theory and Applications. In: Dewatripont, M., Hansen, M., Turnovsky, S. (Eds), Advances in Economics and Econometrics (Proceedings of the Eighth World Congress of the Econometric Society), Cambridge University Press.

[21] Obstfeld, M., 1996. Models of Currency Crises with Self-fulfilling Features, Europ. Econ. Rev. 40, 1037–1047.

[22] Oyama, D., 2004. Booms and Slumps in a Game of Sequential Investment with the Changing Fundamentals, Japanese Econ. Rev. 55, 311–320.

[23] Shleifer, A., 1986. Implementation Cycles, J. Polit. Economy 94, 1163–1190.

[24] Toxvaerd, F., 2004. Strategic Merger Waves: A Theory of Musical Chairs, mimeo, http://ssrn.com/abstract=374281.

Chapter 2

Coordination in a Mobile World

Abstract

I study coordination failures in many simultaneously occurring coordination problems called projects. Players encounter one of these projects, but have an outside option to search for another of the projects. Drawing on the global games approach, I show that such a *mobile* game has a unique equilibrium which allows us to examine comparative statics. The endogeneity of the outside option value and of the search activity leads to non-monotonicity of welfare with respect to search costs; high mobility may hurt players. Moreover, outcomes of the mobile game are remarkably robust to changes in the exogenous parameters. In contrast to the "static" benchmark global game without a search option, successful coordination is frequent in the mobile game even for extremely poor distributions of economic fundamentals, and coordination failures are common even for extremely good distributions. The strategic consequences of the search option are robust to various modifications of the model.

Keywords: Coordination, Equilibrium Uniqueness, Global Games, Search, Mobility, Globalization.

Evolution.

JEL classification: C72, D82, D83.

Acknowledgements

The author thanks Jan Bena, Amil Dasgupta, Bernardo Guimaraes, Fuhito Kojima, Peter Kondor, Eugen Kováč, Andreas Ortmann, Avner Shaked, Hyun Shin and Flavio Toxvaerd for their comments. Laura Straková carefully edited the paper. The usual disclaimer applies.

2.1 Introduction

Coordination problems are common in economics (see e. g. Cooper, 1999), though typically they are modelled in isolation. For instance: players decide between a risky investment with returns increasing in the number of investors and a safe outside option which represents all other investment opportunities. In contrast, this paper studies coordination failures in many simultaneously occurring coordination problems and allows mobile players to move among them. More specifically, I consider projects, or coordination problems, and a set of players uniformly matched to projects at the beginning of the game. An outside option of a player who considers investing into project j consists of a search which allows her to join one of the other projects. Thus the outside option value in any coordination problem j is endogenously determined by players' behavior in all the other coordination problems. Similarly, the mass of observers of j considering investment depends on the coordination outcomes of all other coordination problems, because players rejecting any other project j' will search and may end up being matched to j.

Intuitively, the outside option value and number of observers influence the coordination outcome of each of the coordination problems. A valuable outside option lowers the attraction of a Pareto dominant but risky equilibrium and hence undermines successful coordination. On the opposite side a high number of observers enhances coordination, as it is easier to find other investors. These two externalities lead to non-trivial comparative statics and welfare effects. Searching players who have rejected project j impose a negative externality on other observers of j and a positive externality on observers of all other projects.

Both these causal links are difficult to analyze under the equilibrium multiplicity of coordination games. I therefore study the model using the global games approach, which uniquely predicts the coordination outcome for a given outside option and number of players. Comparative statics of the global games equilibrium is indeed in line with the causal links.[1] I depart from the standard global game, which is denoted as *static* and use as a benchmark, to build a *mobile* game, in which not only one but many projects are realized, each with economic fundamentals independently drawn from a prior distribution. Players receive an imprecise signal about the project's fundamentals they are matched to, and may move to another project if dissatisfied with the current signal.

[1] Global games were introduced by Carlsson and van Damme (1993) and further developed by Morris and Shin (2003). Heinemann, Nagel and Ockenfels (2004) test the theory experimentally, and although reject the global games threshold prediction, confirm the qualitative features of the predicted comparative statics.

Introduction of search allows us to analyze the welfare effects of increased mobility (decreased search costs). Counterintuitively, welfare is non-monotonic in mobility. The direct non-strategic effect is always positive as, ignoring strategic considerations, reaching a successful project is cheaper. However, the strategic effect is negative: smaller search costs increase the outside option value associated with the search, which undermines successful coordination. Thus some projects that would have succeeded had the search costs been high, fail if search costs are low. This negative strategic effect may outweigh the positive direct effect, and welfare may decrease with mobility.

The mobile game also has a natural *self-regulatory* property. Consider, for instance, a shift in the distribution of economic fundamentals towards poorer states of the world. This decreases the outside option value, as searching results in finding poorer projects. The lower value of the outside option enhances successful coordination, and this positive strategic effect partially counteracts the negative direct effect. Another channel through which the self-regulatory mechanism operates is the increasing mass of players observing each project: the more projects have poor fundamentals, the more players search. This makes coordination attempts more likely to succeed and thus helps to partially counteract the direct effect of the distribution's shift.

Below I show that this self-regulatory mechanism is powerful. Players frequently coordinate successfully on many projects even for distributions of fundamentals that almost preclude coordination in the static game. On the other hand, if the distribution of fundamentals is such that a project almost always succeeds in the static game, the value of the outside option in the mobile game is high and the mass of observers of each project is low as players need not search much. Thus, some coordination failures are to be expected in the mobile game. Because of this self-regulatory mechanism, an intermediate willingness to invest is typical for the mobile game equilibrium.

The above "general equilibrium" effects occur in a number of settings in which players actively choose the coordination problem they will participate in; thus our results can complement many of the existing global games applications. Let us apply the mobility extension to the model of currency attacks of Morris and Shin (1998). Allowing speculators to choose a currency they short-sell makes it possible to assess how the speculators' freedom to choose the currency influences their coordination on attacks. Another prominent example of coordination problems in economics is the game of foreign investors in emerging markets with increasing returns to scale. The model can be interpreted as a study of many such markets on which mobile investors operate. Our main result under this interpretation is that welfare is non-monotonic with respect to capital mobility.

The benchmark conclusion of two independent broad streams of literature, global games (Carlsson and van Damme, 1993) and stochastic stability concept (Kandori, Mailath and Rob, 1993), is that risk dominance rather than Pareto dominance selects the equilibrium in coordination games. Given this benchmark result, the influence of mobility on the coordination outcome has been examined in various papers belonging to the latter stream with the main conclusion that, if players are allowed to move and/or to choose with whom they interact, then a Pareto efficient equilibrium may prevail.[2] Goyal and Vega-Redondo (2005) vary the cost of link formation and find a similar welfare effect to the one I find: welfare is non-monotonic with respect to mobility — the efficient equilibrium prevails only at high cost — while if the cost of the link formation is low the risk dominant equilibrium prevails.

To our knowledge, mobility has not been studied within the global games literature. However, the outside option value is often varied exogenously in many global games applications, which leads to a similar tension between the positive direct effect and the negative strategic effect. Morris and Shin (2004 and section 2.3.1 in 2003) show that an increase in collateral may decrease debt value. Collateral is an outside option of creditors, so its increase undermines their ability to coordinate on (efficient) rolling over of the debt, which may outweigh the positive direct effect. Similarly, Goldstein and Pauzner (2005) study the influence of demand-deposit contracts on bank run probability. While the direct effect of higher short-term payment offered by banks is an increase in risk sharing, the strategic effect is negative as it increases the probability of panic-based bank runs.

Jeong (2003) and Burdett, Imai and Wright (2004) study "break-up" externalities which occur when matched players search for new partners while not taking into account the welfare loss of the abandoned partner. Jeong stresses the possible welfare improvements of mobility restrictions in environments with break-up externalities, which is in line with our main finding. Burdett et al. focus on the multiplicity of equilibria; if matched players search intensively, the partnerships become unstable and intensive search is the best response. While the basic model has a unique equilibrium, I encounter this self-fulfilling prophecy feature for the general payoff function in section 2.5.2. However, though the externality studied in our model is similar to break-up externalities, it is of a subtler form. While break-up externalities relate to players who cooperate on production, the searching player in our model leaves the project *before* production starts and the mere fact that she has stopped contemplating involvement in the project induces the negative externality on the rest of the project's observers.

Technically, the present paper combines the modelling frameworks of Das-

[2] E.g. Oechssler (1997); Mailath, Samuelson and Shaked (2000).

gupta (2005) and Steiner (2005). Dasgupta studies the effects of social learning on coordination failures by allowing players to delay investment and to learn from the behavior of early investors. Players delaying investment stay with the same project in Dasgupta's model, whereas they search for a new project in our model. Though the settings of both models are seemingly similar, the conclusions differ significantly. Social learning, central to Dasgupta's model, turns out to be irrelevant in our model (see section 2.4). Moreover, while the delay option unambiguously enhances welfare in Dasgupta's model, it may decrease welfare in ours.

Steiner (2005) considers a repeated coordination game in which players, by choosing to invest today, risk their instantaneous payoffs as well as their ability to participate in future projects — unsuccessful investment can cause bankruptcy. The fear of bankruptcy motivates players not to invest, especially just before an expected boom. This negative feedback between tomorrow's and today's coordination success leads to endogenous cycles in the willingness to invest.

Both Steiner (2005) and the model at hand are based on non-trivial effects of the endogenous outside option but they differ in timing[3] and interpretation. Steiner (2005) focuses on cycles endogenously arising in the equilibrium, whereas this paper emphasizes the self-regulatory properties of the mobile game and particularly the non-monotonicity of welfare with respect to mobility.

Section 2.2 describes the mobile game formally. I compute the equilibrium in the limit of precise signals in section 2.3, analyze its comparative statics, and contrast it to the static game equilibrium. The analysis of the general case away from the limit is relegated to appendix 2.A.1. In section 2.4 players are allowed to observe actions of early investors and find that social learning is irrelevant in the mobile game. I further demonstrate the robustness of the model in section 2.5 in which I vary the number of search rounds, consider general payoff functions, and allow players to direct their search toward better projects. Section 2.6 concludes.

2.2 The Model

Let us start by describing a simple coordination game in section 2.2.1 and then briefly introducing the benchmark global game, dubbed static game

[3]In the present model, there exist many projects simultaneously and returns are paid only after all search and investment takes place. In Steiner (2005), there is only one project in each round, and its returns are paid at the end of each round.

here, by adding noise to the observation of fundamentals. Then, having set the stage, I describe the mobile game in section 2.2.2.

2.2.1 The Static Game

There is a continuum of homogeneous risk-neutral players of measure 1 and one project; each player possesses one (indivisible) token and decides between investing or not investing into the project. The payoff of those who have invested is

$$R(\theta, l) = \begin{cases} 1 - c & \text{if } l \geq \theta, \\ -c & \text{if } l < \theta, \end{cases} \tag{2.1}$$

where l is the measure of investors and $0 < c < 1$ the sunk cost of investment. The project is said to succeed when $l \geq \theta$. The payoff for not investing is normalized to 0. The payoff function (2.1) is being used for its simplicity as the workhorse of the global games literature;[4] general payoff functions are studied in section 2.5.2. The payoff exhibits strategic complementarity; investment is more attractive if many players invest, which typically leads to equilibrium multiplicity. Clearly the game has, for non-extreme values of θ, two pure strategy equilibria in which nobody, respectively everybody, invests.

Building on Carlsson and van Damme (1993), Morris and Shin (2003) show that the equilibrium multiplicity disappears if a noise in observation of the project's parameters is assumed. I introduce this standard global games structure in the rest of this paragraph: θ is referred to as a state of economic fundamentals; it is a realization of a random variable Θ distributed according to $N(y, \tau^2)$; the c.d.f. of Θ is denoted by $\Phi(.)$. The players observe only an imprecise signal $x^i = \theta + \sigma\epsilon^i$ of the state θ which itself is unobserved. The parameter σ describes the size of the noise term. The error terms $\epsilon^i \sim N(0, 1)$ are independent across players. The c.d.f. of $N(0, 1)$ is denoted by $F(.)$. The random variable corresponding to realization x^i is X^i. Pure strategy is a function $s : \mathbb{R} \to \{0, 1\}$, which maps the signal x^i to an action where 0 corresponds to not investing and 1 to investing. This benchmark global game is labelled as a *static* game and denoted by $\Gamma_S(\sigma)$.

To avoid confusion, it is worth mentioning that the *higher* the value of θ the *worse* the fundamental, as more investors are needed for the success of the project. Some, but not all, global games papers use transformation $\tilde{\theta} = 1 - \theta$ which I do not use here.

[4]The payoff function (2.1) has been used in Morris and Shin (1999); Dasgupta (2005); Angeletos, Hellwig, and Pavan (2004), and in others.

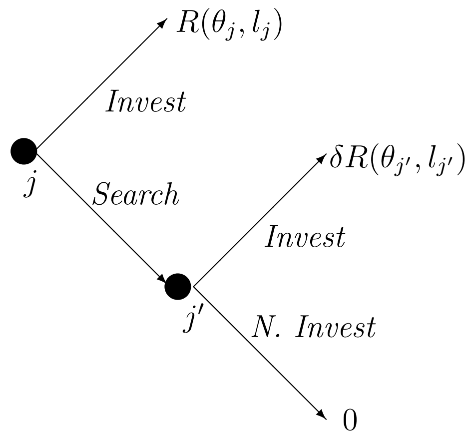


Figure 2.1: Structure of the mobile game — The player of a mobile game is matched to a project j and decides whether to invest or search. In the latter case, she is randomly matched to another project j' and decides whether to invest or not. (The diagram is not a formal game tree, as it does not depict moves of Nature and simultaneous moves of other players.)

2.2.2 The Mobile Game

The outside option payoff is treated exogenously and normalized to 0 in the static game; our next step is to endogenize it, which is achieved by considering many projects simultaneously and by allowing players to search for another project. More formally, there is a continuum of projects indexed by $j \in [0,1]$; each project has a state of fundamentals θ_j independently drawn from the distribution with c.d.f. $\Phi(.)$. Each project's state is fixed during the whole game. The game has two rounds. Players are randomly and uniformly matched to the projects at the beginning of round 1. The measure of players observing each project in round 1 after the matching is normalized to 1.[5] At round 1, after the players are matched to the projects, each player i observes a private signal $x_1^i = \theta_{j(i)} + \sigma_1 \epsilon_1^i$ about the fundamentals of project[6] $j(i)$ she is matched to. Each player chooses from:

To Invest into the project she observes in round 1. The player can take no further action afterwards.

To Search: The player continues to round 2, is randomly matched to an-

[5]There is a continuum of continua of players and thus the total measure of players is undefined. Formally, I should refer to a density, rather than to a measure of investors in a particular project. However, our formal impreciseness does not lead to confusion, because I never refer to a total measure of players in all projects. Occasionally, I will stress that I refer to measure *per* project.

[6]I will omit argument i and write simply j, but let us remember the matching process.

other of the existing projects j', and observes signal $x_2^i = \theta_{j'} + \sigma_2 \epsilon_2^i$ about the new project j'. Errors ϵ_1^i and $\epsilon_2^i \sim N(0,1)$ are independent across players and rounds.

Players who have searched decide in round 2 between investing and not investing into their new projects. The payoff of players who have invested into project j depends on its fundamentals θ_j, on the cumulative measure l_j of investment into j, and on the timing $t \in \{1,2\}$ of the investment:

$$u^i(t, l_j, \theta_j) = \delta^{t-1} R(\theta_j, l_j),$$

see figure 2.1. Players who have not invested in round 1 nor 2 receive payoff 0.

Note that the return depends on the *cumulative* measure of investments l_j into project j over both rounds. The return on the investment in round 2 is scaled down by the factor $0 < \delta < 1$ which should not be understood as a time discount factor because all payoffs are realized at the same moment, at the end of round 2. Rather, δ models implicit search costs: late investors (in round 2) may find the largest profitable investment opportunities being taken by early investors in round 1. Also, the late investors have less time to realize their investments, thus they will get less involved with the project. Alternatively, instead of discounting by δ, the search cost could be modelled by a fixed cost q that searching players incur. But, as expensive search cannot be mandatory, the players who do not want to invest or to search would have to be allowed a third, outside option. The simplicity of the two-action global games framework would be lost. Therefore, to enhance tractability, I model search costs by discounting.[7]

In the basic setting, players in round 2 observing j do not observe the measure of investors from round 1. The information sets of player i are histories of the signals: $I_1^i = \{x_1^i\}$, $I_2^i = \{x_1^i, x_2^i\}$. Later, in section 2.4, I consider social learning: players (imprecisely) observe the measure of previous investors, and I find our results to be robust to such a modification, which is in contrast with the study of social learning in the static game done by Dasgupta (2005).

Pure strategy of the mobile game is a pair of functions $a_1(x_1) : \mathbb{R} \to \{0,1\}$, $a_2(x_1, x_2) : \mathbb{R}^2 \to \{0,1\}$ that prescribe actions in round 1 and 2 contingent on the observed signals. A *threshold* strategy is a particularly simple pure strategy characterized by two thresholds x_1^*, x_2^* such that a player invests at

[7]Players in our setting can always choose not to invest in both rounds which assures 0, and thus I do not have to consider a particular outside option for players who wish not to engage in costly search.

round $t \in \{1, 2\}$ if and only if the observed signal x_t^i is below x_t^*. A *critical* state θ^* is such a state that only projects with $\theta_j < \theta^*$ succeed. Obviously, if players use sufficiently non-monotone strategies, the critical state may not exist. However, as shown below, players play threshold strategies and that the critical state exists in equilibrium. The whole game is called a *mobile* game and denoted by $\Gamma_M(\boldsymbol{\sigma})$, where $\boldsymbol{\sigma} \equiv (\sigma_1, \sigma_2)$ describes the size of noise terms in both rounds. I consider different noise sizes in both rounds primarily because it will be helpful in the analysis of social learning; otherwise it does not play any substantial role.

Players are not allowed to return to the project observed in round 1 after they have observed a project in round 2. Later, in section 2.5.1, I study a game with an infinite number of search rounds. In that game, returning to a previously observed project is always suboptimal, and thus it can be allowed without any consequences on the equilibrium.

Economic Example

I offer an economic example similar to Dasgupta's (2005) setting. There is a continuum of risky bonds indexed by $j \in [0, 1]$, whose returns increase with measure of investment; bond j returns $e^{r(l_j)(T-t)}$ at time T, where t is the time of investment, l_j is the cumulative investment into j over the whole time period and $r(l) = \underline{r} < 0$ if $l < \theta_j$ and $r(l) = \overline{r} > 0$ if $l \geq \theta_j$.

Investors possessing one dollar observe a random bond at $t = 0$ and the measure of investors per bond is 1. After observing a signal about the bond she is matched to, each investor decides whether to invest one dollar at $t = 0$ to the first bond she observes or whether to search for a new bond. However, the search lasts time t_s after which she observes a signal about the new bond and decides whether to invest at time t_s or not to invest at all. At time T, players who have not invested consume 1 while players who have invested at $t \in \{0, t_s\}$ consume $e^{r(l_j)(T-t)}$. This coincides with our model if $\underline{r}T = -c$, $\overline{r}T = 1 - c$, $\frac{T-t_s}{T} = \delta$ and the utility function is $u(.) = \ln(.)$.

2.3 Equilibrium

The key observation in the analysis of the mobile game is that each project can be treated as an independent coordination game with parameters induced by players' aggregate behavior in other projects. Let V_2 be the expected payoff in round 2, and n_2 be the measure of players per project continuing to round 2; the values V_2, n_2 are defined for any strategy profile. The interactions of project j's observers constitute a coordination game of two types of players: measure 1 has the outside option δV_2, measure n_2 has the outside option 0, and payoff for investment of all players is described by (2.1). Observers of each particular project interact in a global game and thus their equilibrium behavior is uniquely determined for any assumed values V_2, n_2: only projects with $\theta_j < \theta^*$ succeed, where the critical state θ^* is a function of V_2, n_2. Moreover, values V_2, n_2 are uniquely determined by θ^*, which leads to a system of equations with a unique solution.

Let us first analyze the mobile game informally in the limit $\boldsymbol{\sigma} \to \mathbf{0}$. The simplification of the limit case is that players receiving infinitely precise private signals x_t^i neglect their prior beliefs. The formal analysis is deferred to appendix 2.A.1, where I explicitly account for both prior distribution and the private signals, and only then take the limit $\boldsymbol{\sigma} \to \mathbf{0}$. I deal only with symmetric equilibrium in threshold strategies in this section and later prove that no other equilibria exist (also in appendix 2.A.1).

The following technical preliminary is needed. Denote $P^i_{\theta^*,t} \equiv Prob(\Theta_j < \theta^* | X_t^i)$, which is a posterior probability player i assigns to the success of the project she is matched to in round t; $P^i_{\theta^*,t}$ is a random variable that depends on the signal X_t^i player i receives.

Lemma 2.1. *Random variable $P^i_{\theta^*,t}$, conditional on $\Theta_j = \theta^*$ (which is unknown to the players), is distributed uniformly on $[0,1]$ in the limit $\boldsymbol{\sigma} \to 0$.*

Proof[8]: ignoring the prior distribution,

$$P^i_{\theta^*,t} = Prob(\Theta_j < \theta^* | X_t^i) = Prob(X_t^i - E_t^i < \theta^* | X_t^i) = Prob(E_t^i > X_t^i - \theta^*) = 1 - F\left(\frac{X_t^i - \theta^*}{\sigma_t}\right),$$

in round $t \in \{1, 2\}$. Then, for $p \in [0,1]$:

$$\begin{aligned} Prob(P^i_{\theta^*,t} < p | \Theta_j = \theta^*) &= Prob\left(1 - F\left(\frac{X_t^i - \theta^*}{\sigma_t}\right) < p | \Theta_j = \theta^*\right) \\ &= Prob\Big(\theta^* + \sigma_t F^{-1}(1-p) < X_t^i | \Theta_j = \theta^*\Big) \\ &= 1 - F\left(\frac{\big(\theta^* + \sigma_t F^{-1}(1-p)\big) - \theta^*}{\sigma_t}\right) = 1 - F\Big(F^{-1}(1-p)\Big) = p, \end{aligned}$$

which is the c.d.f. of the uniform distribution. □ (lemma 2.1)

Having established lemma 2.1, I now guess arbitrary equilibrium values $V_2 \geq 0$, $n_2 \geq 0$ and consider the interaction of players who have been matched

[8] Lemma 2.1 is a variation of Morris and Shin's (2003) argument of Laplacian beliefs.

to project j in round 1 or 2. The players observe x_t^i, form posterior probabilities p_t^i of the project's success that are realizations of $P_{\theta^*,t}^i$ and decide about investing into a lottery with expected payoff[9]

$$(1-c)p_t^i + (-c)(1-p_t^i) = p_t^i - c.$$

A player invests if she prefers such a lottery to δV_2 in round 1 or to 0 in round 2. Thus she invests if $p_1^i - c > \delta V_2$ in round 1 or, if $p_2^i - c > 0$ in round 2. Suppose the state of project j (unknown to the players) happens to be just equal to the critical state, $\theta_j = \theta^*$. Then, knowing the uniform distribution of p_t^i and the trigger probabilities $c + \delta V_2$ and c in round 1 and 2 one can compute the total measure of investors into j. The definition of the critical state implies that the measure of investment is just equal to θ^*:

$$(1-c-\delta V_2) + n_2(1-c) = \theta^*. \qquad (\textit{Crit.St.})$$

The expected payoff in round 2 is

$$V_2 = (1-c)\Phi(\theta^*), \qquad (\textit{Value})$$

because in the limit of precise signals all observers of projects with $\theta < \theta^*$ successfully invest and receive $1-c$, and other players do not invest and receive 0.

Using the law of large numbers, the measure of players per project not investing in round 1 and thus continuing into round 2 is

$$n_2 = 1 - \Phi(\theta^*). \qquad (\textit{Search})$$

Let us substitute equation (*Value*) and (*Search*) into (*Crit.St.*) and get

$$(1-c)[2-(1+\delta)\Phi(\theta^*)] - \theta^* = 0. \qquad (\textit{Modif.Crit.})$$

Equation (*Modif.Crit.*) has a unique solution because its left hand side is continuous, decreases in θ^* and is asymptotically linear in θ^*. Thus there is a unique equilibrium of the mobile game in the class of symmetric equilibria.

The shortcut of computing the equilibrium in the limit $\boldsymbol{\sigma} \to \mathbf{0}$ is justified by computing the symmetric equilibrium out of the limit, for $\boldsymbol{\sigma} > \mathbf{0}$. Moreover, I show that there is no other equilibrium than the symmetric one:

Proposition 2.1. *1. There exists $\overline{\sigma}$ such that if $\sigma_1 < \overline{\sigma}$ and $\sigma_2 < \overline{\sigma}$, then: the mobile game $\Gamma_M(\boldsymbol{\sigma})$ has a unique Bayesian Nash equilibrium, it is symmetric, and all players play threshold strategies.*

[9] Downsized by δ in round 2.

2. *$\theta^*(\boldsymbol{\sigma})$, $V_2(\boldsymbol{\sigma})$, and $n_2(\boldsymbol{\sigma})$ describing the unique equilibrium of $\Gamma_M(\boldsymbol{\sigma})$ converge in the limit $\boldsymbol{\sigma} \to \mathbf{0}$ to solution θ^*, V_2, and n_2 of the system of equations (Crit.St.), (Value) and (Search).*

The proof, found in appendix 2.A.1, has a structure typical for the global games literature. I first specify equations for a symmetric equilibrium in threshold strategies and show that these have a unique solution. Then I show, by an argument based on iterated dominance, that no other equilibrium exists: for any assumed equilibrium values V_2, n_2 a unique set of fundamentals with which projects succeed is found. Obviously, a project always succeeds if $\theta_j < 0$ and never does if $\theta_j > 2$ and I iteratively expand intervals of sure success/failure until they meet at the critical state θ^*, which is uniquely determined by the assumed values V_2, n_2 according to a critical state condition. This is a unique candidate for an equilibrium with the assumed values V_2, n_2 and if this truly is an equilibrium, the critical state θ^* has to generate the assumed values V_2, n_2 according to value and search conditions. Thus any equilibrium satisfies all conditions that specify the symmetric equilibrium and hence no other exists.

2.3.1 Comparative Statics

I examine comparative statics of the equilibrium in the limit $\boldsymbol{\sigma} \to 0$ described by equation (*Modif.Crit.*) with respect to the exogenous parameters c, y, δ:

Corollary 2.1. *The critical state θ^* decreases in c, δ and increases in y.*

Proof: the left hand side of (*Modif.Crit.*) decreases in c, δ, θ^* and because $\Phi(\theta^*) \equiv F(\frac{\theta^*-y}{\tau})$ it increases in y. The comparative statics of θ^* follows from the implicit function theorem. □ (corollary 2.1)

In particular, the set of successful projects shrinks with higher mobility, which decreases welfare. The welfare effects are examined in the next step:

Corollary 2.2. *Comparative statics of welfare with respect to c, δ, and y is as summarized in table 2.1.*

Parameter q	c	δ	y
Direct effect $\frac{\partial V}{\partial q}$	-	+	-
Strategic effect $\frac{\partial V}{\partial \theta^*}\frac{\partial \theta^*}{\partial q}$	-	-	+
Total effect $\frac{dV}{dq}$	-	-/+	-

Table 2.1: Overview of welfare effects.

The ex ante expected payoff V at the beginning of the game is

$$V = (1-c)F\left(\frac{\theta^* - y}{\tau}\right) + \left(1 - F\left(\frac{\theta^* - y}{\tau}\right)\right)\delta(1-c)F\left(\frac{\theta^* - y}{\tau}\right). \quad (2.2)$$

The total welfare effect with respect to parameters $q \in \{c, \delta, y\}$ consists of the direct non-strategic effect $\frac{\partial V}{\partial q}$ and the strategic effect $\frac{\partial V}{\partial \theta^*}\frac{\partial \theta^*}{\partial q}$ via the change of the critical state θ^*, so the proof of corollary 2.2 consists of computing the derivations, which I omit here. Note that $\frac{\partial V}{\partial \theta^*}$ is unambiguously positive, and hence the sign of the strategic effect is the same as $\frac{\partial \theta^*}{\partial q}$ specified in corollary 2.1. The total effect of the increase of y is unambiguously negative, despite the fact that the direct and strategic effects are of opposite signs, because derivative $\frac{\partial \theta^*}{\partial y}$ turns out to be smaller than 1 so $\theta^* - y$ decreases with y; hence the measure of successful projects decreases with y.

Let us summarize both corollaries verbally: increased mobility, measured by higher δ, makes players choosier (see figure 2.2a) because it increases the value of the outside option δV, and this negative strategic effect may outperform the positive direct effect (see figure 2.2b). Similarly, a decrease in the average project's quality, higher[10] y, makes players less choosy, as it decreases the outside option value and also increases search activity, which in turn increases the measure of observers of each project. Increase of c causes two strategic effects. A negative strategic effect, which already exists in the static game, makes players choosier because the profits from successful investment decrease, but this effect is partially counteracted by a positive strategic effect in the mobile game: larger c decreases the endogenous outside option value and increases search activity, both of which enhance successful coordination. The negative strategic effect always prevails and $\frac{\partial \theta^*}{\partial c}$ is unambiguously negative.

The comparative statics is simpler to analyze in a limit $\tau \to 0$ for which a closed form solution can be obtained.[11] The limit solution is reported in appendix 2.A.2, because the expressions, although in principle simple, are tiresome.

Corollary 2.3. *Welfare unambiguously decreases with increased mobility (higher δ) in the ordered limit $\tau \to 0, \boldsymbol{\sigma} \to \mathbf{0}$ (such that $\frac{\boldsymbol{\sigma}}{\tau} \to \mathbf{0}$).*

Proof can be found in appendix 2.A.2.

[10]Recall that θ_j is a measure necessary for the success of project j. Hence higher θ_j means worse quality of the project.

[11]I take the ordered limit $\lim_{\tau \to 0, \boldsymbol{\sigma} \to \mathbf{0}}$. This assures that $\frac{\boldsymbol{\sigma}}{\tau} \to \mathbf{0}$ and thus the equilibrium uniqueness holds.

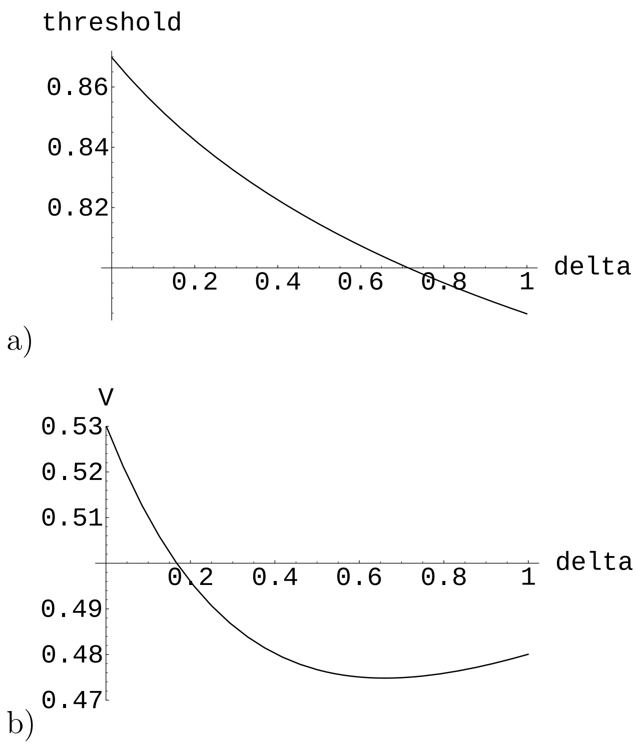


Figure 2.2: Welfare and comparative statics analysis of the mobile game, parameters: $c = 0.3$, $y = 0.8$, $\tau = 0.1$. **a)** $\theta^*(\delta)$, **b)** $V(\delta)$.

Though mobility is varied exogenously in our model, the finding of the welfare's non-monotonicity with respect to δ implies that, if welfare V decreases with δ, then were the players able to influence their mobility, they could find themselves in a prisoners' dilemma-like situation: each would benefit from a unilateral increase of mobility but a mutual increase would harm all.

2.3.2 Comparison of the Mobile and the Static Game — the Self-Regulatory Property

The mobile game is constructed in such a way that its outcomes are directly comparable with the static game outcomes because the measure of players per project is the same in both games, the fundamentals are drawn from the same distribution, and players can invest only once in both games. The solution to the static game is described by the following proposition:

Proposition 2.2. (Morris and Shin) *There exists $\overline{\sigma}$ such that the game $\Gamma_S(\sigma)$ is dominance solvable for all $\sigma < \overline{\sigma}$. The unique strategy surviving iterated elimination of dominated strategies is a threshold strategy*

$$s(x^i) = \begin{cases} 1 & \text{if } x < x^*_\sigma, \\ 0 & \text{if } x > x^*_\sigma, \end{cases}$$

where the threshold x^*_σ converges to $1 - c$ for $\sigma \to 0$.

Proof is in Morris and Shin (2003).[12]

Welfare in the static game is

$$V_{stat} = (1-c)F\left(\frac{\theta^*_{stat} - y}{\tau}\right),$$

and thus, if variance τ of the fundamentals' distribution is small, welfare V_{stat} declines sharply with y or c in the neighborhood of θ^*_{stat}. In contrast, the critical state θ^*_{mob} in the mobile game adjusts to an increase of y or c because players increase their thresholds — they become less choosy. As a result, V decreases with c and y markedly more slowly in the mobile game than in the static game; this self-regulatory property is depicted in figures 2.3a,b.

[12] The threshold in the limit $\sigma \to 0$ can be found by informal arguments similar to those behind the equation (*Crit.St.*): only players preferring lottery with expected payoff $(1-c)p^i + (-c)(1-p^i) = p^i - c$ to the safe outside option payoff 0 invest, and because the conditional probabilities p^i of the project's success are distributed uniformly on $[0, 1]$ if the state happens to be critical, the mass of players believing that $p^i > c$ is $1 - c$ which must coincide with the critical state.

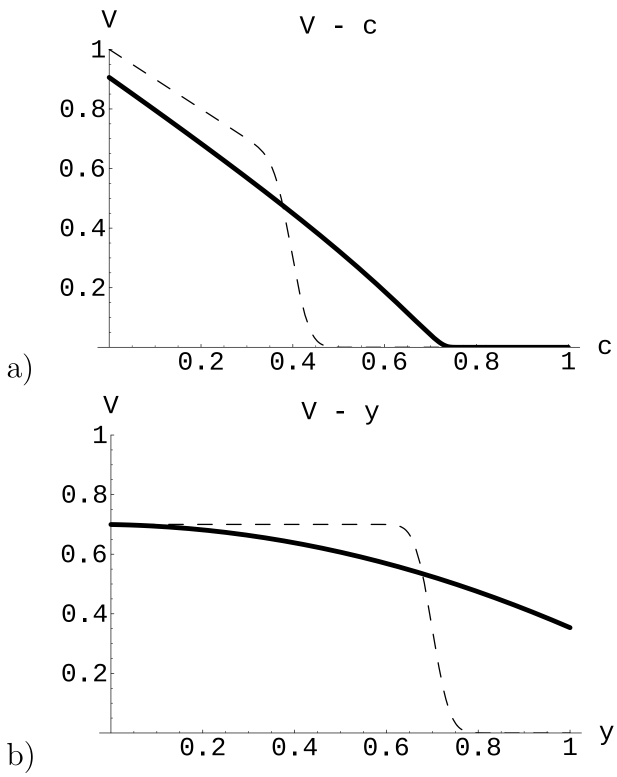


Figure 2.3: **a)** Comparison of $V(c)$ in the mobile game — thick line, and the static game — dashed line. (parameters: $\delta = 0.9$, $y = 0.6$, $\tau = 0.03$) **b)** Comparison of $V(y)$ in the mobile game — thick line, and the static game — dashed line. (parameters: $c = 0.3$, $\delta = 0.9$, $\tau = 0.03$)

Let us further examine the self-regulatory property analytically. Denote by P the equilibrium measure of successful projects $P = \Phi(\theta^*) \equiv F(\frac{\theta^* - y}{\tau})$; the welfare increases in P. Let us compare the dependence of P on y in the case of the static and the mobile game.[13] The derivative $\frac{\partial P_{stat}}{\partial y}$ can be computed straightforwardly in the case of the static game:

$$\frac{\partial P_{stat}}{\partial y} = -\frac{1}{\tau} f\left(\frac{\theta^*_{stat} - y}{\tau}\right),$$

hence for small τ, P_{stat} declines quickly when $y \approx \theta_{stat}$. In the case of the mobile game, equation (*Modif. Crit.*) gives:

$$P_{mob} = \Phi\Big((1-c)[2 - (1+\delta)P_{mob}]\Big). \tag{2.3}$$

The self-regulatory property is caused by the negative influence of P_{mob} on the right hand side of (2.3), which is absent in the static case. The derivative is

$$\frac{\partial P_{mob}}{\partial y} = \frac{-\frac{1}{\tau} f\left(\frac{\theta^*_{mob} - y}{\tau}\right)}{1 + \frac{1}{\tau} f\left(\frac{\theta^*_{mob} - y}{\tau}\right)(1-c)(1+\delta)}.$$

P.d.f. $\frac{1}{\tau} f\left(\frac{\theta^*_{mob} - y}{\tau}\right)$ is both in the numerator and in the denominator, and hence the derivative $\frac{\partial P_{mob}}{\partial y}$ does not diverge even for $\tau \to 0$ and $\theta^*_{mob} \approx y$. In fact, the derivative simplifies to $\frac{-1}{(1-c)(1+\delta)}$ in the limit $\tau \to 0$ and for non-extreme y (see appendix 2.A.2).

2.3.3 Limit of the Inefficient Search

Next, I examine the mobile game with very inefficient search, when $\delta \to 0$, and show that it does not approximate the static game. Let us consider the mobile game with parameter $\delta \equiv 0$, which is out of the assumed range of $\delta \in (0, 1)$ and thus proposition 3.1 does not hold. Obviously, players are indifferent between investing and not investing in round 2, which creates equilibrium multiplicity. The equilibrium in which nobody invests in round 2 can be associated with the equilibrium of the static game. However, this equilibrium is not approximated by the equilibrium of the mobile game as $\delta \to 0+$. The critical state $\theta^*_0 \equiv \lim_{\delta \to 0+} \theta^*(\delta)$ solves the limit of equation (*Modif. Crit.*):

$$(1-c)[2 - \Phi(\theta^*_0)] - \theta^*_0 = 0,$$

[13] The analysis of dependence of P on c is virtually identical, and hence omitted.

the solution of which differs from $\theta^*_{stat} = 1 - c$. This can be seen in figure 2.2a,b: $\lim_{\delta \to 0+} \theta(\delta) > \theta_{stat}$ and also $\lim_{\delta \to 0+} V(\delta) > 0$ whereas welfare of the static game would be virtually 0 for that setting of the parameters.[14] The intuition is that search increases the measure of observers of each project from 1 to $1 + n_2$, which enhances successful coordination (also) in round 1. Hence the critical state moves towards worse states and players are matched to a successful project in round 1 more often. Welfare in the mobile and the static game thus differs for purely strategic reasons for small δ and the difference does not disappear even if the gains from investment in round 2 are negligible but positive. Therefore, search options should not be ignored in the analysis of coordination problems even when search is very inefficient.

2.4 Social Learning

I have assumed until now that players matched to a project j in round 2 do not observe the measure of investment into j realized in round 1. This assumption is abandoned in this section and I find, somewhat surprisingly, that social learning is irrelevant in the mobile game. The game analyzed in this section remains as the mobile game described in section 2.2.2 except that, additionally, players matched to j in round 2 observe a signal z^i about the measure of investment $l_{j,1}$ into j in round 1. I assume Dasgupta's (2005) error structure which allows for analytical solution of the game:

$$z^i = F^{-1}(l_{j,1}) + \omega \xi^i, \tag{2.4}$$

where the error terms $\xi^i \sim N(0, 1)$ are independent across players and also independent of the error terms ϵ^i_t of the signals x^i_t. I argue below that ω depicts the informativeness of signal z^i compared to x^i_1; if $\omega = 1$ the two signals have the same informativeness.

A pure strategy is a pair of functions $a_1(x_1) : \mathbb{R} \to \{0, 1\}$, $a_2(x_1, x_2, z) : \mathbb{R}^3 \to \{0, 1\}$ which prescribe actions in rounds 1 and 2 conditional on the observed signals. This game is denoted as a *learning* game $\Gamma_L(\boldsymbol{\sigma})$. A *monotone* strategy is a pair of functions $a_1(x_1)$, $a_2(x_2, z)$ such that $a_1(.)$ is non-increasing, $a_2(x_2, z)$ is non-increasing in x_2, non-decreasing in z, and does not depend on x_1; hence x_1 is omitted from its arguments. Players are restricted to monotone strategies in this section. I find that the equilibrium in monotone strategies of the learning game coincides with the unique equilibrium of the mobile game in the limit $\boldsymbol{\sigma} \to \mathbf{0}$. Although an equilibrium in non-monotone strategies that differs from the equilibrium of the mobile game has not been ruled out, I have not found such.

Proposition 2.3.

[14]More precisely, it would be very small, $V \to 0$ in the limit $\tau \to 0$.

The learning game $\Gamma_L(\boldsymbol{\sigma})$ has a unique Bayesian Nash equilibrium in monotone strategies. This equilibrium is symmetric and converges to the equilibrium of the mobile game $\Gamma_M(\boldsymbol{\sigma})$ as $\boldsymbol{\sigma} \to \mathbf{0}$.

Proof: for any equilibrium in monotone strategies the critical value θ^* must exist because the measure of investors l_j monotonically decreases in θ_j. The existence of the critical state implies that the equilibrium is symmetric, because the maximization problem of each player is identical as it depends only on the common values of V_2, θ^* and on the exogenous parameters, and the best responses are strict.[15]

The measure of early investors is $l_{j,1} = F\left(\frac{x_1^* - \theta_j}{\sigma_1}\right)$ because only those who receive a signal below x_1^* invest. Define $\tilde{z}^i \equiv x_1^* - \sigma_1 z^i$ and because of the assumed error technology $\tilde{z}^i = \theta_j - \sigma_1 \omega \xi^i$. Thus, receiving signal z^i is equivalent to receiving signal $\tilde{z}^i$ about θ_j with error drawn from $N(0, (\sigma_1\omega)^2)$ and independent of error of signal x_2^i. Finally, players form sufficient statistics $\tilde{x}_2^i$ for θ_j using x_2^i and $\tilde{z}^i$:

$$\tilde{x}_2^i = \frac{\sigma_2^2 \tilde{z}^i + \sigma_1^2 \omega^2 x_2^i}{\sigma_2^2 + \sigma_1^2 \omega^2},$$

with an error term $(\tilde{x}_2^i - \theta_j) \sim N(0, \frac{\sigma_1^2 \sigma_2^2 \omega^2}{\sigma_2^2 + \sigma_1^2 \omega^2})$.

The equilibrium of the learning game therefore corresponds to the unique equilibrium of the mobile game with $\tilde{\boldsymbol{\sigma}} = (\sigma_1, \sqrt{\frac{\sigma_1^2 \sigma_2^2 \omega^2}{\sigma_2^2 + \sigma_1^2 \omega^2}})$. If $(\sigma_1, \sigma_2) \to \mathbf{0}$ then $\tilde{\boldsymbol{\sigma}} \to \mathbf{0}$ as well, so the equilibrium of the learning game converges to the limit equilibrium of the mobile game. $\square$ (proposition 2.3)

Social learning is irrelevant in the mobile game, whereas in Dasgupta (2005) social learning matters. This difference is due to the mobility present in our mobile game but not in Dasgupta's. Players delaying investment in Dasgupta's game remain in the same project as they were in round 1; they only gain additional information — the signal z^i. In contrast, the motivation to wait (search) in our model is to find a project with better fundamentals. Additional information z^i cannot be the decisive motivation for search, because if signal x_2^i is far away from θ^*, it is a sufficient guideline for the investment decision. The additional signal z^i is useful only if the distance of x_2^i from θ^* is of the order of σ_2, which has negligible probability for $\boldsymbol{\sigma} \to \mathbf{0}$. Hence giving players additional information z^i does not alter the mobile game equilibrium, because players are almost sure they will not need this information in round 2 in the limit $\boldsymbol{\sigma} \to \mathbf{0}$. In contrast, in Dasgupta's game, a player

[15] Precisely, players are indifferent between investing and not investing only when observing threshold signals, which happens with 0 probability.

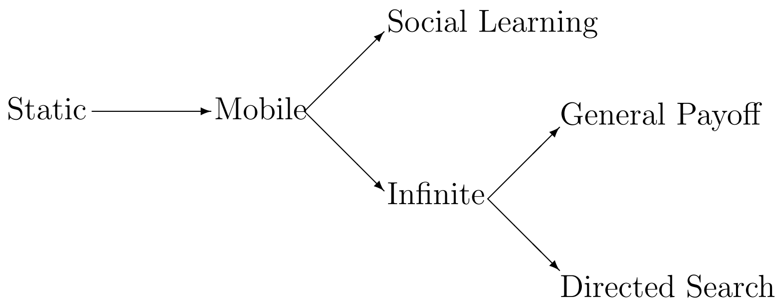


Figure 2.4: Modifications Structure — The benchmark static game is developed into a network of related models.

in need of additional information in round 1 knows that this information will be useful in round 2.

2.5 Robustness

I consider several other modifications of the mobile game and show that the qualitative features of the equilibrium are robust to them. See figure 2.4 for the relationships among individual modifications. In section 2.5.1, players search repeatedly. In section 2.5.2 the payoff function is generalized, and players are able to direct their search towards projects with better fundamentals in section 2.5.3.

2.5.1 Infinite Number of Search Rounds

The players of the mobile game have only one possibility of search. Are the results robust to a change in the number of search rounds? The following modification is examined: the game remains the same as the mobile game described in section 2.2.2 except it does not end after round 2. Instead, players decide in infinity of rounds, indexed by $t \in \mathbb{N}$, whether to invest into a currently observed project or to search and continue to round $t+1$. As in the mobile game, players can invest only once, hence they can search only until they invest and afterwards cannot take any further action. The return $R(\theta_j, l_j)$ of a project j depends on its fundamentals θ_j and on the cumulative investment l_j over all rounds. Payoffs of late investors are downsized to $\delta^{t-1}R(\theta_j, l_j)$, t being the time of investment. The payoff of players who never invest is normalized to 0. Player i who has continued to round t receives signal $x_t^i = \theta_j + \sigma\epsilon_t^i$, where j is the project she is matched to in round t and errors ϵ_t^i are independent across players and rounds. For the sake of simplicity, I assume the same value of σ in all rounds. This game is denoted as the *infinite* game.

I sketch the solution of the infinite game in the limit $\sigma \to 0$ in a similar manner as was done for the mobile game in section 2.3. Let V be the ex ante expected payoff at the beginning of the game and n be a measure of observers of any project cumulative over all rounds; n is a common value for all projects because the search is undirected. Consider the interaction of players matched to a project j in any of the rounds $t \in \mathbb{N}$. All observers of j are in the same situation, except the payoffs of those in round t are linearly re-scaled by factor δ^{t-1}, which does not alter their strategic position. Thus they are of the same type and simultaneously[16] decide between investing, which pays $R(\theta_j, l_j)$, and the outside option, which pays δV. Therefore, observers of j interact in a simple global game.

Denote player i's posterior probability of the project's success by $P^i_{\theta^*} \equiv Prob(\Theta_j < \theta^* | X^i)$ as in section 2.3 and let us reiterate that $P^i_{\theta^*}$ is distributed uniformly on $[0, 1]$ conditional on $\theta_j = \theta^*$. Player i invests if and only if she prefers the lottery of investment to the outside option:

$$P^i_{\theta^*}(1 - c) + (1 - P^i_{\theta^*})(-c) > \delta V,$$

which implies the critical mass condition:

$$(1 - c - \delta V)n = \theta^*. \qquad (\textit{Crit.st.'})$$

The value condition is

$$V = (1 - c)\Phi(\theta^*) + \delta V \left(1 - \Phi(\theta^*)\right). \qquad (\textit{Value'})$$

The measure of observers per project in round 1 is 1. Ratio $1 - \Phi(\theta^*)$ of the observers are matched to a project with $\theta > \theta^*$ so they continue into round 2. Out of these, the ratio $1 - \Phi(\theta^*)$ continue into round 3,... The cumulative measure of observers per project is

$$n = \sum_{t=1}^{\infty} \left(1 - \Phi(\theta^*)\right)^{t-1} = \frac{1}{\Phi(\theta^*)}, \qquad (\textit{Search'})$$

and because the search is undirected, each project is observed by the same measure of players.

Substitute (*Value'*) and (*Search'*) into (*Crit.st.'*) and get

$$(1 - c)\frac{1 - \delta}{[1 - \delta + \delta\Phi(\theta^*)]\Phi(\theta^*)} - \theta^* = 0. \qquad (\textit{Modif.Crit.'})$$

[16] The decision of players in all rounds can be can treated as simultaneous because players do not observe the measure of investments from previous rounds.

Equation (*Modif.Crit.'*) has a unique solution because its left hand side is continuous, decreases and is asymptotically linear in θ^*. Moreover, it decreases in c, δ, θ^* and because $\Phi(\theta^*) \equiv F(\frac{\theta^*-y}{\tau})$, it increases in y. The implicit function theorem implies that θ^* decreases in c, δ and increases in y, exactly as in the mobile game. The welfare effects are also the same as in the mobile game and table 2.2, which summarizes the signs of the welfare effects, remains valid.

The infinite game has no other equilibrium except the symmetric one: any assumed pair V and n imply a particular simple global game describing the interaction of players observing a project j. This global game has a unique equilibrium, with critical state θ^* depending on V and n according to (*Crit.st.'*). Thus any V and n imply a unique θ^*, and any θ^* implies a unique V and n according to equations (*Value'*) and (*Search'*). Hence any equilibrium has to satisfy the triplet of equations (*Crit.st.'*), (*Value'*) and (*Search'*), which has a unique solution.

As mentioned in section 2.2.2, a player of the infinite game never wishes to return to a project she has observed in an earlier round. If she has considered the expected payoff of some project inferior to search, then she will not change her opinion after any number of search rounds, as she does not learn anything new about the project nor about the underlying distribution of fundamentals. Thus, the possibility of returning to earlier projects can be introduced without any consequences on equilibrium behavior.

2.5.2 General Payoff Functions

Until now, I have been analyzing games with a particular return function (2.1). In this section I take first steps in examining the effects of mobility for a general return function. I do the analysis in the framework of the infinite game rather than the mobile game because the former is simpler to analyze, as all the players are of the same type, whereas in the mobile game the players of round 1 and 2 differ in their outside options.

The analyzed game is the same as the infinite game described in the previous section 2.5.1, except with a general return function $R(\theta_j, l_j)$. As in the previous section, I want each project to generate a simple global game with a unique equilibrium, conditional on V and n. To assure this, let us impose Morris and Shin's (2003) assumptions on $R(\theta_j, l_j)$, slightly modified to fit our setting: let $\mathcal{V}$ denote the positive part of the range of the return function $R(\theta_j, l_j)$.

MS1: *Action Monotonicity*: $R(\theta_j, l_j)$ is weakly increasing in l_j.

MS2[17]: *State Monotonicity*: $R(\theta_j, l_j)$ is weakly increasing in θ_j.
MS3: *Strict Laplacian State Monotonicity*: for any $V \in \mathcal{V}$, $n \in [1, +\infty)$, there exists a unique $\theta^* \in \mathbb{R}$ such that $\frac{1}{n}\int_0^n R(\theta^*, l)dl = \delta V$.
MS4: *Uniform Limit Dominance*: for any $V \in \mathcal{V}$, $n \in [1, +\infty)$, there exist $\underline{\theta}$ and $\overline{\theta}$ and $\epsilon > 0$ such that 1. $R(\theta, l) < -\epsilon + \delta V$ for all $l \in [0, n]$ and $\theta < \underline{\theta}$ and 2. $R(\theta, l) > \epsilon + \delta V$ for all $l \in [0, n]$ and $\theta > \overline{\theta}$.
MS5: *Continuity*: $\int_0^1 g(l)R(x, l)dl$ is continuous with respect to signal x and density $g(.)$.

Proposition 2.4. *Suppose MS1–MS5 are satisfied. Then, in the limit $\sigma \to 0$, all Bayesian Nash equilibria of the infinite game with the return function $R(\theta_j, l_j)$ are symmetric and in threshold strategies. Variables θ^*, V and n describing the equilibrium satisfy:*

$$\frac{1}{n}\int_0^n R(\theta^*, l)dl = \delta V, \qquad (Crit.st.g.)$$

$$V = \int_{\theta^*}^{+\infty} R(\theta, n)d\Phi(\theta) + \delta V\Phi(\theta), \qquad (Value.g.)$$

$$n = \frac{1}{1 - \Phi(\theta)}. \qquad (Search.g.)$$

Proof: values V and n are defined in any equilibrium. For any assumed pair V, n interaction of observers of any particular project is a simple global game with the payoff function $R(\theta_j, l_j)$, the outside option value δV, and the measure of players n. Because of the assumptions MS1 – MS5 and the normality of the errors' distribution, this simple global game satisfies proposition 2.2 in Morris and Shin (2003) and thus has a unique equilibrium with threshold θ^* satisfying equation (*Crit.st.g.*). Moreover, threshold θ^* determines expected value and search activity, which gives equations (*Value.g.*) and (*Search.g.*). □ (proposition 2.4)

However, proposition 2.4 does not guarantee equilibrium uniqueness. There is an example of the return function

$$R(\theta, l) = \theta - 1 + l^2, \qquad (2.5)$$

for which the system of equations (*Crit.st.g.*), (*Value.g.*) and (*Search.g.*) have multiple solutions (see figure 2.5). In such a case, each solution represents a symmetric equilibrium in threshold strategies differing in the endogenous

[17] Note that $R(\theta_j, l_j)$ as described in (2.1) is weakly decreasing in θ_j instead of increasing and thus MS2 is formally not satisfied. However, this can be accommodated by introducing $\tilde{\theta} = 1 - \theta$.

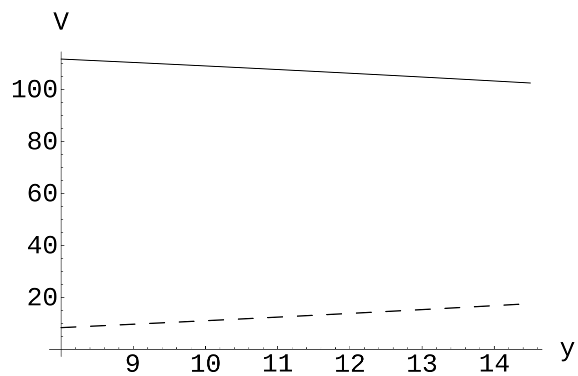


Figure 2.5: Payoff $R(\theta_j, l_j) = \theta_j - 1 + l^2$ generates equilibrium multiplicity. In one equilibrium, the full line, players invest only in projects of very high quality; hence search activity and measure of investment into successful projects are high, and returns of successful projects are very high. The incentive to search is thus high too. In the other equilibrium, the dashed line, players invest in projects of medium quality, search activity and measure of investment into successful projects is low, and hence successful projects have only medium returns and the motivation to search is low. Welfare in the first equilibrium decreases with improving distribution of fundamentals. Parameters: $\delta = .9$, $\tau = .01$.

values of the outside option and in the search activity and hence also in the threshold.

Though the uniqueness is not guaranteed generally, I pinpoint two simple classes of return functions, for which the equations (*Crit.st.g.*), (*Value.g.*) and (*Search.g.*) have a unique solution and hence the equilibrium uniqueness is guaranteed:

Corollary 2.4. *Let the return function, satisfying MS1-MS5, be of the form* $R(\theta, l) = p(\theta) + q(l)$ *where* $q(l)$ *is concave and the derivative* $q'(.)$ *exists. Then the game has a unique Bayesian Nash equilibrium.*

Proof is in appendix 2.A.3.

Another class of return functions guaranteeing equilibrium uniqueness is

$$R(\theta_j, l_j) = \begin{cases} \zeta(\theta_j) - c & \text{if } a(\theta_j) < l_j, \\ -c & \text{if } a(\theta_j) \geq l_j, \end{cases} \tag{2.6}$$

where $a(\theta_j)$ decreases and $\zeta(\theta_j)$ increases in θ_j. This return function generalizes the coordination problem induced by return function (2.1) and was studied in Morris and Shin's (1998) model of currency attacks.

Corollary 2.5. *Let the return function be of the form (2.6) and the derivatives* $a'(.)$, $\zeta'(.)$ *exist. Then the game has a unique Bayesian Nash equilibrium.*

Proof can be found in appendix 2.A.3.

The simple form of the particular return function (2.1) has allowed us to eliminate integrals in (*Crit.st.g.*) and (*Value.g.*), which is not possible for a general return function, and therefore I do not draw general conclusions about the comparative statics. However, examination of the return function (2.5) shows that the non-monotonicity of welfare with respect to δ is not a special feature of return function (2.1); it can be observed also in the case of (2.5).[18] Moreover, the return function (2.5) generates non-monotonicity with respect to y (see figure 2.5). The intuition is that worse distribution of fundamentals increases search activity and hence the measure of observers of each project. Thus the measure of investment into successful projects increases, and this positive strategic effect dominates the negative direct effect because the return steeply increases in the measure of investors.

[18]For instance, for parameters $y = 10$, $\delta = .9$, $\tau = .01$ and for the solution $\theta^* = .997$, $V = 11.0$, $n = 1.67$ welfare V locally decreases with δ.

2.5.3 Directed Search

To this point the search and matching to projects was assumed to be undirected, and hence each project has been observed by the same measure of players. This, although computationally convenient, is unrealistic. In this section, agents are let to direct their search toward better projects. As a consequence, the distribution of projects describing the matching outcome differs from the distribution of physically existing projects.

Let the fundamentals of physically existing projects be distributed according to p.d.f. $\phi(.)$, but let us assume in this section that players are able to influence the matching process such that they are matched to a project drawn from $\psi(.)$, with c.d.f. $\Psi(.)$. I assume return function 2.1 and use the framework of the infinite game; players can search in an infinite number of search rounds, and each search leads to a project drawn from $\psi(.)$. I also assume that $\phi(.)$ and $\psi(.)$ satisfy the monotone likelihood property, $\frac{\psi(.)}{\phi(.)}$ is decreasing. Accordingly, better projects, characterized by lower θ, are observed by more players than are worse projects. This game is dubbed a *directed search* game.

Proposition 2.5. *The directed search game has a unique BNE in the limit* $\sigma \to 0$.

Proof: The measure of observers $o(\theta_j)$ depends on project j's fundamentals θ_j. Let n be the measure of all searchers per project cumulatively over all rounds. The number n would also be the measure of observers of each project in the previous sections, but in this section the observers are distributed unevenly. Value n induces $o_n(\theta_j)$ observers of project j:

$$o_n(\theta_j) = n\frac{\psi(\theta_j)}{\phi(\theta_j)}. \tag{2.7}$$

Number n and the expected ex ante payoff V are defined for each strategy profile. Interaction among all observers of project j could be formalized as a simple global game in the previous sections. To proceed in the same way here, the measure of observers needs to be renormalized in order to avoid its dependence on θ_j. Interaction of $o_n(\theta_j)$ observers described by the return function

$$R(\theta_j, l) = \begin{cases} 1-c & \text{if } l \geq \theta_j, \\ -c & \text{if } l < \theta_j \end{cases}$$

can be equivalently described as the interaction of players with measure 1 and the return function

$$\tilde{R}_n(\theta_j, \tilde{l}) = \begin{cases} 1-c & \text{if } \tilde{l} \geq \frac{\theta_j}{o_n(\theta_j)}, \\ -c & \text{if } \tilde{l} < \frac{\theta_j}{o_n(\theta_j)}, \end{cases}$$

where $\tilde{R}_n(.,.)$ is defined on $\mathbb{R}\times[0,1]$. In other words, investment is measured in relative instead of absolute terms, and the return function are modified accordingly. Function $\tilde{R}_n(.,.)$ is non-decreasing in $\tilde{l}$ and non-increasing in θ on its definition range.[19] This modified description of the interaction associated with project j is a global game and satisfies conditions of theorem 2.2 in Morris and Shin (2003). Thus each assumed pair of values n, V generates a unique critical state θ^* according to

$$\int_0^1 \tilde{R}_n(\theta^*, l)dl = \delta V,$$

which can be simplified into

$$\frac{\theta^*}{o_n(\theta^*)} = 1 - c - \delta V. \qquad (\textit{Crit.st.DS.})$$

The critical state θ^* implies values V and n: The value condition is

$$V = (1-c)\Psi(\theta^*) + \big(1 - \Psi(\theta^*)\big)\delta V. \qquad (\textit{Value.DS.})$$

Measure n of searchers per project cumulatively over all rounds is

$$n = \sum_{t=1}^{\infty} \big(1 - \Psi(\theta^*)\big)^{t-1} = \frac{1}{\Psi(\theta^*)}. \qquad (\textit{Search.DS})$$

Note that the value and search condition depend on the distribution describing the matching process, not on the distribution describing the physical occurrence of states.

Substituting (*Value.DS.*), (*Search.DS*) and (2.7) into (*Crit.st.DS.*) gives:

$$(1-c)\,\frac{1-\delta}{\left[1-\delta+\delta\Psi(\theta^*)\right]\Psi(\theta^*)} - \frac{\theta^*}{\frac{\psi(\theta^*)}{\phi(\theta^*)}} = 0, \qquad (\textit{Modif.Crit.DS})$$

which has a unique solution as the left hand side of (*Modif.Crit.DS*) is continuous, positive for $\theta^* \leq 0$, decreasing for $\theta^* > 0$, and negative for sufficiently large θ. □ (proposition 2.5)

Comparative statics can be computed in the same way as in the case of the mobile or the infinite game: the left hand side of (*Modif.Crit.DS*) decreases in c, δ and θ^*, hence the solution θ^* decreases in c and δ. The comparative statics thus remains the same as in the case of the mobile and the infinite game. Furthermore, numerical solution of (*Modif.Crit.DS*) shows that welfare is non-monotonic in δ for some parameters.

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[19] Note the non-monotonicity of $\frac{\theta}{o_n(\theta)}$. However, $\frac{\theta}{o_n(\theta)}$ increases for $\theta \geq 0$ and though it can decrease for $\theta < 0$, it is then always negative and thus smaller than $\tilde{l} \in [0,1]$.

2.6 Conclusion

I have studied search among many simultaneous projects, each being a coordination problem. Players, dissatisfied with the signal about the project they currently observe, may search for another of the projects, but search is costly. This mobile game is an expansion of a simple benchmark global game, labelled a static game, from which it inherits equilibrium uniqueness allowing for examination of comparative statics. The mobile game has a "self-regulatory" property: any effects characteristic for the benchmark static game are partially counteracted by a strategic effect in the opposite direction through the endogenous changes of the outside option values and of the mass of observers of each project. Thus the occurrence of coordination failures is notably robust to the changes of exogenous parameters such as the distribution of fundamentals.

The self-regulatory mechanism implies that a project's coordination failure is determined not only by the absolute state of economic fundamentals but also by its relative ranking compared to other projects. This may explain the occurrence of investment crises despite substantial improvements in the distribution of countries' fundamentals over the past decades. In fact, I have found a payoff function for which an improvement in the distribution of fundamentals may increase the amount of coordination failures and decrease welfare. Improvement in the distribution decreases search activity which results in investment scattered among more projects, and this may outweigh the direct positive effects.

Similarly, welfare may decrease with mobility. The positive direct effect of lower search costs is counteracted by a negative strategic effect since lower search costs increase the outside option value, which hampers successful coordination. Again, the strategic effect may prevail and so the welfare is non-monotonic with respect to mobility. The result may be a prisoners' dilemma-like situation. While a fixed, exogenously given mobility was considered, real investors are able to unilaterally increase their mobility, which could be modelled as an increase of δ. Obviously, any investor would benefit from the unilateral increase, but the collective increase would harm all.

The qualitative features of the comparative statics of the mobile game seem to be robust to modifications; I have considered the possibility of social learning, infinite number of search opportunities, directed search, and general payoff functions satisfying strategic complementarity.

The mobile game is a realistic extension to many static global games applications. For instance, while Morris and Shin (1998) study a coordination game of speculators considering an attack on an isolated currency, the mobile game allows for the incorporation of parallel coordination problems of other currencies. The cost of search can be associated with the cost of acquiring the private signal x_j^i about currency j. It was shown that occurrence of successful coordination always decreases with lower search costs. Thus, the low cost of acquiring private signals about other currencies decreases the occurrence of currency attacks.

The possibility of analyzing the influence of mobility on coordination failures makes the model a useful framework for a study of globalization's consequences. Numerous projects succeed only if many agents coordinate their efforts. Globalization allows people skeptical about the risky project they were matched (born) with, to search for another risky opportunity. On the one hand, higher mobility allows agents to avoid risky projects with bad fundamentals; on the other hand, it lowers their ability to coordinate on risky investments. Either effect may prevail under certain circumstances. A firmer connection of the model to globalization processes is an opportunity for future research.

2.A Appendix

2.A.1 Proof of Proposition 3.1

1. I first formulate conditions for symmetric equilibrium in threshold strategies characterized by V_2, n_2, x_1^*, x_2^* and θ^* and later prove that no other equilibrium exists.

Critical state θ^* satisfies a *critical mass* condition: if the realized state of a project j is θ^*, the measure of players investing into j, because they have received a signal below x_t^*, must be precisely θ^*:

$$F\left(\frac{x_1^* - \theta^*}{\sigma_1}\right) + F\left(\frac{x_2^* - \theta^*}{\sigma_2}\right) n_2 = \theta^*. \qquad (crit.st.")$$

The players combine signals x_t^i and prior beliefs to form a posterior belief about the fundamental θ_j. Both the prior distribution and the distribution of errors are normal distributions and thus the posterior distribution in round $t \in \{1,2\}$ is also a normal distribution $N(e_t(x_t^i, y), u_t^2)$ where $e_t(x,y) \equiv \frac{\sigma_t^2 y + \tau^2 x}{\sigma_t^2 + \tau^2}$ and $u_t^2 \equiv \frac{\sigma_t^2 \tau^2}{\sigma_t^2 + \tau^2}$. Knowing the posterior distribution, the expected payoff for investing into j, conditional on signal x_t^i can be expressed:

$$(1-c)F\left(\frac{\theta^* - e_t(x_t^i, y)}{u_t}\right) - c\left[1 - F\left(\frac{\theta^* - e_t(x_t^i, y)}{u_t}\right)\right] = F\left(\frac{\theta^* - e_t(x_t^i, y)}{u_t}\right) - c.$$

A player observing x_1^* must be indifferent between investing and the outside option value δV_2. This gives an *indifference 1* condition:

$$F\left(\frac{\theta^* - e_1(x_1^*, y)}{u_1}\right) - c = \delta V_2. \qquad (indif.1)$$

A player observing x_2^* must be indifferent between investing and the outside option which is 0 in round 2. This gives an *indifference 2* condition:

$$F\left(\frac{\theta^* - e_2(x_2^*, y)}{u_2}\right) - c = 0. \qquad (indif.2)$$

The equilibrium value V_2 can be expressed in terms of θ^* as a solution of a nonstrategic maximization problem. Investing into j gives a lottery with expected payoff $Prob(\Theta_j < \theta^* | x_2) - c$ and players invest only if that is greater than 0. This gives a *value* condition:

$$V_2 = E[Max(Prob(\Theta_j < \theta^* | X_2^i) - c, 0)], \qquad (Value")$$

where the expectation is over unconditional distribution of X_2^i.

The unconditional distribution of signal x_1^i is $N(y, \tau^2 + \sigma_1^2)$. Players observing signal $x_1^i > x_1^*$ search, which gives a *search* condition:

$$n_2^* = 1 - F\left(\frac{x_1^* - y}{\sqrt{\tau^2 + \sigma_1^2}}\right). \qquad (Search")$$

I have specified a system of five equations (*crit.st."*), (*indif.1*), (*indif.2*), (*Value"*), and (*Search"*) for five unknowns θ^*, x_1^*, x_2^*, V_2 and n_2. Next, let us prove that this system has a unique solution if $\boldsymbol{\sigma}$ is sufficiently small. Express $x_1^* = \chi_1(\theta^*, V_2)$ as a function of θ^* and V_2 from (*indif.1*):

$$\chi_1(\theta^*, V_2) = (1 + \frac{\sigma_1^2}{\tau^2})\theta^* - \frac{\sigma_1}{\tau}\sqrt{\tau^2 + \sigma_1^2} F^{-1}(c + \delta V_2) - \frac{\sigma_1^2}{\tau^2} y, \qquad (2.8)$$

and $x_2^* = \chi_2(\theta^*)$ as a function of θ^* from (*indif.2*):

$$\chi_2(\theta^*) = (1 + \frac{\sigma_2^2}{\tau^2})\theta^* - \frac{\sigma_2}{\tau}\sqrt{\tau^2 + \sigma_2^2} F^{-1}(c) - \frac{\sigma_2^2}{\tau^2} y. \qquad (2.9)$$

Substitute (2.8) and (2.9) into (*crit.st."*):

$$F\left[-\frac{\sqrt{\tau^2 + \sigma_1^2}}{\tau} F^{-1}(c + \delta V_2) + \frac{\sigma_1}{\tau^2}(\theta^* - y)\right] +$$

$$F\left[-\frac{\sqrt{\tau^2 + \sigma_2^2}}{\tau} F^{-1}(c) + \frac{\sigma_2}{\tau^2}(\theta^* - y)\right] n_2 - \theta^* = 0, \qquad (2.10)$$

and denote the left hand side of (2.10) as $\Lambda(V_2, n_2, \theta^*)$. The function $\Lambda(V_2, n_2, \theta^*)$ increases in n_2 and decreases in V_2. It also decreases in θ^* for sufficiently small $\boldsymbol{\sigma}$ because the derivative $\frac{\partial \Lambda}{\partial \theta^*}$ is bounded above by $\frac{1}{\sqrt{2\pi}} \frac{\sigma_1 + n_2 \sigma_2}{\tau^2} - 1 \leq \frac{1}{\sqrt{2\pi}} \frac{max(\sigma_1, \sigma_2)}{\tau^2} 2 - 1$ which is negative for small $\boldsymbol{\sigma}$. The function $\Lambda(V_2, n_2, \theta^*)$ can be naturally interpreted as a measure of investment into j when the project's fundamentals θ_j (unknown to players) happens to be θ^* and V_2, n_2, θ^* are equilibrium values.

Our next aim is to eliminate unknowns V_2 and n_2 by expressing them as functions of θ^* in order to express Λ as a one-dimensional function of θ^*. The condition (*Value"*) has the form $V_2 = v(\theta^*)$ but variable $n_2 = \eta(x_1^*)$ is a function of x_1^* according to (*Search"*), so first I have to express x_1^* as a function of θ^*: $x_1^* = \tilde{\chi}_1(\theta^*) \equiv \chi_1(\theta^*, v(\theta^*))$. Function $v(\theta^*)$ increases in θ^* but monotonicity of $\tilde{\chi}_1(\theta^*)$ is not guaranteed:

$$\frac{d\tilde{\chi}_1}{d\theta^*} = \frac{\partial \chi_1}{\partial \theta^*} + \frac{\partial \chi_1}{\partial V_2} \frac{dv}{d\theta^*},$$

because the term $\frac{\partial \chi_1}{\partial \theta^*}$ is positive but the term $\frac{\partial \chi_1}{\partial V_2} \frac{dv}{d\theta^*}$ negative. However, the sign of the derivative $\frac{d\tilde{\chi}_1}{d\theta^*}$ is determined for sufficiently small σ_1 because the two terms are of different orders of magnitude. The term $\frac{\partial \chi_1}{\partial \theta^*} = 1 + \frac{\sigma_1^2}{\tau^2}$ is of order σ_1^0. The derivative $\frac{\partial \chi_1}{\partial V_2}$ is of order $\sigma_1 \tau$, and $\frac{dv}{d\theta^*}$ is of order $\frac{1}{\tau}$ because $v(\theta^*)$ increases from 0 to $1 - c$ within the increase of θ^* of order τ. So the term $\frac{\partial \chi_1}{\partial V_2} \frac{dv}{d\theta^*}$ is of order σ_1 and thus it is negligible compared to the first term $\frac{\partial \chi_1}{\partial \theta^*}$ for sufficiently small σ_1. This implies that $\tilde{\chi}_1(\theta^*)$ increases with θ^* for sufficiently small σ_1.

Condition (*Search"*) specifies that $n_2 = \eta(x_1^*)$ is a decreasing function of x_1^*, so $n_2 = \tilde{\eta}(\theta^*) \equiv \eta(\tilde{\chi}_1(\theta^*))$ increases in θ^*. Let us now substitute $V_2 = v(\theta^*)$, $n_2 = \tilde{\eta}(\theta^*)$ into (2.10) and get the equation with one unknown: $\lambda(\theta^*) \equiv \Lambda(v(\theta^*), \tilde{\eta}(\theta^*), \theta^*) = 0$. Given the monotonicity of $v(\theta^*)$ and $\tilde{\eta}(\theta^*)$ it is easy to check that $\lambda(\theta^*)$ decreases in θ^*. Moreover it is asymptotically linear in θ^* and continuous, therefore the equation $\lambda(\theta^*) = 0$ has a unique solution.

I have found a symmetric equilibrium in threshold strategies and have shown that there is only one of this kind for sufficiently small $\boldsymbol{\sigma}$. Next, let us show that, for sufficiently small $\boldsymbol{\sigma}$, no other equilibrium exists: each equilibrium generates values V_2, n_2 and a success set S of all values θ_j for which a project j succeeds. Note that V_2, n_2 and S are known by players in equilibrium.

Let us consider a project j and a random variable $P_{S,t}^i = Prob(\Theta_j \in S | X_t^i)$ that denotes the posterior probability of the project's success after player i

observes signal X_t^i. Let $k(V_2, n_2, S, \theta_j)$ be the measure of investors for given V_2, n_2, S and for the state of the project (unknown to players) being θ_j:

$$k(V_2, n_2, S, \theta_j) = Prob(P_{S,1}^i > c + \delta V_2 | \theta_j) + n_2 Prob(P_{S,2}^i > c | \theta_j).$$

Note that $k(.)$ increases in S; precisely $S \supseteq S' \Rightarrow k(V_2, n_2, S, \theta_j) \geq k(V_2, n_2, S', \theta_j)$.

Let $m(V_2, n_2, \theta', \theta_j) \equiv k(V_2, n_2, (-\infty, \theta'), \theta_j) - \theta_j$ be the measure of investors net of θ_j in a special case when the success set is an interval, $S = (-\infty, \theta')$. Note that $m(V_2, n_2, \theta^*, \theta^*) \equiv \Lambda(V_2, n_2, \theta^*)$ as $\Lambda(V_2, n_2, \theta^*)$ was formed from condition (*crit.st.”*) and hence it coincides with the definition of $m(V_2, n_2, \theta^*, \theta^*)$.

Next, assume that V_2, n_2 attain some particular values in equilibrium. I will show that there is a unique success set S compatible with this assumption: surely $S \supseteq (-\infty, 0)$ as the measure of investment $l_j \geq 0$. Moreover $\Lambda(V_2, n_2, 0) > 0$, hence $m(V_2, n_2, 0, 0) > 0$ and because the function m is continuous, there exists $\epsilon > 0$ such that $m(V_2, n_2, 0, \epsilon) > 0$. Value $m(V_2, n_2, 0, \epsilon)$ is a lower bound for the measure of equilibrium investment into a project with $\theta_j = \epsilon$ because the true success set contains $(-\infty, 0)$. Thus a project with $\theta_j = \epsilon$ surely succeeds. Thus a project surely succeeds for all $\theta \leq \epsilon$ because $m(V_2, n_2, \theta', \theta)$ decreases in θ. Hence $S \supseteq (-\infty, \epsilon)$. This argument can be iterated in the same manner and expand the interval of sure success further into the region of higher θ_j, up to the minimal θ' for which $m(V_2, n_2, \theta', \theta') = 0$, which is the minimal θ' solving $\Lambda(V_2, n_2, \theta') = 0$.

Symmetric arguments apply from above. The project never succeeds for $\theta > 2$ because 2 is the upper bound of observers of each project. Again, the interval of infeasible success can be expanded to (θ'', ∞), where θ'' is the maximal solution of $\Lambda(V_2, n_2, \theta'') = 0$.

$\Lambda(V_2, n_2, \theta)$ decreases in θ for any V_2, n_2 so equation $\Lambda(V_2, n_2, \theta) = 0$ has a unique solution, and therefore $\theta' = \theta''$. Hence any pair V_2, n_2 implies a unique critical state $\theta^*_{V_2,n_2}$ that satisfies equation (2.10). On the other hand, the critical state θ^* uniquely determines equilibrium values V_2, n_2 as functions $v(\theta^*)$ and $\tilde{\eta}(\theta^*)$. Therefore equilibrium values V_2, n_2 and θ^* must coincide with values of the unique symmetric equilibrium in threshold strategies and thus no other equilibrium exists.

2. Equations (2.10), (*Value”*) and (*Search”*) converge to equations (*Crit.St.*), (*Value*) and (*Search*) as $\boldsymbol{\sigma} \to \mathbf{0}$, hence their solution $\theta^*(\boldsymbol{\sigma})$, $V_2(\boldsymbol{\sigma})$, and $n_2(\boldsymbol{\sigma})$ converges to the solution of the latter equation system. □ (theorem 3.1)

2.A.2 Limit $\tau \to 0$

I find a closed form solution for the mobile game in the ordered limit $\tau \to 0$, $\boldsymbol{\sigma} \to \mathbf{0}$, where τ and $\boldsymbol{\sigma}$ approach 0 in such a way, that the private signals are

	$y < (1-c)(1-\delta)$	$(1-c)(1-\delta) < y < (1-c)2$	$(1-c)2 < y$
θ^*	$(1-c)(1-\delta)$	y	$(1-c)2$
$\Phi(\theta^*)$	1	$\frac{2-2c-y}{(1+\delta)(1-c)}$	0

Table 2.2: Closed form solution of the mobile game in the ordered limit $\tau \to 0$, $\boldsymbol{\sigma} \to \mathbf{0}$.

much more precise than the prior distribution, $\frac{\boldsymbol{\sigma}}{\tau} \to \mathbf{0}$.

The equilibrium is described by equation (*Modif.Crit.*) which I reproduce here for convenience:

$$(1-c)[2-(1+\delta)\Phi(\theta^*)] = \theta^*.$$

I solve (*Modif.Crit.*) by guessing and verifying:

- $\frac{\theta^*-y}{\tau} \ll 0 \Rightarrow \Phi(\theta^*) \to 0 \Rightarrow \theta^* \to (1-c)2 < y$,
- $\frac{\theta^*-y}{\tau} \gg 0 \Rightarrow \Phi(\theta^*) \to 1 \Rightarrow \theta^* \to (1-c)[1-\delta] > y$,
- $\theta^* \approx y \Rightarrow (1-c)[2-(1+\delta)\Phi(\theta^*)] = y \Rightarrow \Phi(\theta^*) = \frac{2-2c-y}{(1+\delta)(1-c)}$.

Table 2.2 summarizes the solution of equation (*Modif.Crit.*) in the limit $\tau \to 0$.

Next, I substitute $\Phi(\theta^*)$ into the welfare equation (2.2) and get a closed form expression for V. Welfare in the extreme regions is 0 respectively $1-c$. Welfare for the medium value of y is

$$V = \frac{2\,c^2\,(1+\delta^2) + c\,[-4+\delta^2\,(-4+y)+y-2\,\delta\,y] + (2-y)\,(1+\delta^2+\delta\,y)}{(1-c)\,(1+\delta)^2}.$$

It is possible to compute $\frac{dV}{d\delta}$ explicitly:

$$\frac{dV}{d\delta} = -\frac{(1-\delta)\,(2-2\,c-y)^2}{(1-c)\,(1+\delta)^3},$$

which is negative for all $\delta \in (0,1)$, so an increase in mobility unambiguously decreases welfare in the limit $\tau \to 0$. $\square$ (lemma 2.3)

2.A.3 Proof of Corollaries 2.4 and 2.5

Proof of corollary 2.4: unknown V can be eliminated by expressing it from (*Value.g.*) and substituting it into (*Crit.st.g.*). For the sake of simplifying tedious expressions I omit arguments of functions, but let us keep in mind that $n = \frac{1}{1-\Phi(\theta^*)}$ etc. I get

$$\frac{1}{n}\int_0^n Rdl = \frac{\delta}{1-\delta\Phi}\int_{\theta^*}^{+\infty} Rd\Phi(\theta). \tag{2.11}$$

Denote the left and right hand side of (2.11) by $LHS(\theta^*)$ and $RHS(\theta^*)$ and show that they satisfy the single-crossing property: A simple manipulation gives derivatives:

$$LHS'(\theta^*) = \left(-\int_0^n Rdl + nR\right)\phi + p', \tag{2.12}$$

$$RHS'(\theta^*) = \left(-R + nR_l + \underbrace{\frac{\delta}{1-\delta\Phi}\int_{\theta^*}^{+\infty} Rd\Phi(\theta)}_{(*)}\right)\frac{\delta\phi}{1-\delta\Phi} \tag{2.13}$$

Use the equality in (2.11) and replace the term (*) in (2.13) by $\frac{1}{n}\int_0^n Rdl$. Next, combining (2.12) and (2.13) find the difference of derivatives:

$$LHS'(\theta^*) - RHS'(\theta^*) = p' +$$
$$\left\{\underbrace{R\left(n(1-\delta\Phi)+\delta\right) - \int_0^n Rdl\left(1-\delta\Phi+\frac{\delta}{n}\right) - \delta nR_l}_{(**)}\right\}\frac{\phi}{1-\delta\Phi}.$$

Let us now show that $LHS'(\theta^*) - RHS'(\theta^*)$ is positive. Derivative p' is non-negative by assumption MS2, fraction $\frac{\phi}{1-\delta\Phi}$ is positive, and the term $(**)$ is positive because

$$(**) = \left(1-\delta\Phi+\frac{\delta}{n}\right)\underbrace{\left(Rn - \frac{R_l n^2}{2} - \int_0^n Rdl\right)}_{(I)} + \underbrace{(1-\delta)\frac{n^2}{2}}_{(II)},$$

where part (I) is positive as $R(\theta, l)$ is assumed to be concave with respect to l; part (II) is a residuum of the examined expression and it is positive. Therefore $LHS(\theta^*)$ crosses $RHS(\theta^*)$ always from below, and the single-crossing property implies uniqueness of the solution. □ (corollary 2.4)

Proof of corollary 2.5: equations (*Crit.st.g.*), (*Value.g.*) and (*Search.g.*) simplify into

$$\left(1 - \frac{a(\theta^*)}{n}\right)\zeta(\theta^*) - c = \delta V \tag{2.14}$$

$$V = \int_{\theta^*}^{+\infty} (\zeta(\theta^*) - c)d\Phi(\theta) + \Phi(\theta^*)\delta V \tag{2.15}$$

$$n = \frac{1}{1 - \Phi(\theta^*)} \tag{2.16}$$

After eliminating V and n one gets

$$[1 - a(1 - \Phi)]\zeta - c = \frac{\delta}{1 - \delta\Phi} \int_{\theta^*}^{+\infty} (\zeta - c)d\Phi(\theta). \tag{2.17}$$

Denote the left and right hand side of (2.17) by $LHS(\theta^*)$ and $RHS(\theta^*)$ and show that they satisfy the single-crossing property: The derivatives are

$$LHS'(\theta^*) = -a'(1 - \Phi)\zeta + a\phi\zeta + (1 - a(1 - \Phi))\zeta', \tag{2.18}$$

and $LHS'(\theta^*)$ is positive in an equilibrium, because $a' < 0$, $\zeta' > 0$ and in equilibrium $0 < \zeta(\theta^*)$, $0 < a(\theta^*) < n(= \frac{1}{1-\Phi})$.

$$RHS'(\theta^*) = \frac{-\delta}{1 - \delta\Phi}(\zeta - c)\phi + \underbrace{\frac{\delta^2\phi}{(1 - \delta\Phi)^2} \int_{\theta^*}^{+\infty} (\zeta - c)d\Phi(\theta)}_{(*)} \tag{2.19}$$

Use the equality in (2.17) and replace the term (*) in (2.19) by $\frac{\delta\phi}{1-\delta\Phi}[(1 - a(1 - \Phi))\zeta - c]$. A simple manipulation leads to

$$RHS'(\theta^*) = -\frac{\delta a\ (1 - \Phi)\zeta\phi}{1 - \delta\Phi}, \tag{2.20}$$

and hence $RHS'(\theta^*)$ is negative in equilibrium. Therefore the single-crossing property is satisfied and thus the solution is unique. □ (corollary 2.5)

2.A.4 Summary of the Main Notation

Exogenous parameters:	*Endogenous variables:*
c Sunk cost of investment.	V_2 Expected payoff in round 2.
δ Discount factor.	n_2 Measure of players observing each project in round 2.
σ_t^2 Variance of private signal at t.	x_t^* Threshold signal at round t.
τ^2 Variance of prior distribution.	θ^* Critical state.
y Average state of fundamentals.	l_j Cumulative investment into project j.
θ_j Fundamentals of project j.	
x^i Private signal of player i.	

Games analyzed:

Static game: A benchmark simple global game.

Mobile game: Same as the static game but players are allowed to search once for another project.

Learning game: Same as the mobile game but players in round 2 receive a signal about the amount of early investment from round 1.

Infinite game: Same as the mobile game but players are allowed to search infinitely many times.

General payoff: Same as the infinite game but a general payoff function satisfying strategic complementarity is assumed.

Directed Search game: Same as the infinite game but the search is directed, and hence better projects are observed more often.

References

[1] Angeletos G.M., A. Hellwig and A. Pavan, 2004, Information Dynamics and Multiplicity in Global Games of Regime Change, NBER Working Paper No. 11017.

[2] Burdett K., R. Imai and R. Wright, 2004, Unstable Relationships, Frontiers of Macroeconomics 1, Article 1.

[3] Carlsson H. and E. Van Damme, 1993, Global Games and Equilibrium Selection, Econometrica 61, 989–1018.

[4] Cooper R.W., 1999, Coordination Games: Complementarities and Macroeconomics, Cambridge University Press.

[5] Dasgupta A., 2005, Coordination, Learning, and Delay, London School of Economics, Financial Markets Group Discussion Paper No. 435.

[6] Heinemann F., R. Nagel and P. Ockenfels, 2004, The Theory of Global Games on Test: Experimental Analysis of Coordination Games with Public and Private Information, Econometrica 72, 1583–1599.

[7] Goldstein I. and A. Pauzner, 2005, Demand-Deposit Contracts and the Probability of Bank Runs, Journal of Finance 60, 1293–1328.

[8] Goyal S. and F. Vega-Redondo, 2005, Learning, Network Formation and Coordination, Games and Economic Behavior 50, 178–207.

[9] Jeong B., 2003, The Welfare Effects of Mobility Restrictions, Review of Economic Dynamics 6, 685–696.

[10] Kandori M., G.J. Mailath and R. Rob, 1993, Learning, Mutation, and Long Run Equilibria in Games, Econometrica 61, 29–56.

[11] Mailath G.J., L. Samuelson and A. Shaked, 2000, Endogenous Interactions, in: The Evolution of Economic Diversity, edited by U. Pagano and A. Nicita, Routledge.

[12] Morris S. and H.S. Shin, 1998, Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks, American Economic Review 88, 587–597.

[13] Morris S. and H.S. Shin, 1999, A Theory of the Onset of Currency Attacks, in: Asian Financial Crisis: Causes, Contagion and Consequences, edited by P.R. Agenor, D. Vines, and A. Weber, Cambridge University Press.

[14] Morris S. and H.S. Shin, 2003, Global Games: Theory and Applications, in: Advances in Economics and Econometrics (Proceedings of the Eighth World Congress of the Econometric Society), edited by M. Dewatripont, L. Hansen and S. Turnovsky, Cambridge University Press.

[15] Morris S. and H.S. Shin, 2004, Coordination Risk and the Price of Debt, European Economic Review 48, 133–153.

[16] Oechssler J., 1997, Decentralization and the Coordination Problem, Journal of Economic Behavior and Organization 32, 19–135.

[17] Steiner J., 2005, Coordination Cycles, CERGE-EI Working Paper No. 274.

Chapter 3

Strong Enforcement by a Weak Authority

Abstract

This paper studies the enforcement abilities of authorities with a limited commitment to punishing violators. Commitment of resources sufficient to punish only one agent is needed to enforce high compliance of an arbitrary number of agents. Though existence of other, non-compliance equilibria is generally inevitable, there exist punishment rules suitable for a limited authority to assure that compliance prevails in the long run under stochastic evolution.

Keywords: Commitment, Enforcement, Punishment, Stochastic Evolution.
JEL classification: C73, D64, H41.

Acknowledgements

I benefited from the comments of Kenneth Binmore, Fuhito Kojima, Simon Gächter, Werner Güth, Eugen Kováč, and Jaromír Kovařík. Dirk Engelmann, Andreas Ortmann, and Avner Shaked inspired me in numerous discussions. Laura Straková carefully edited the paper. The usual disclaimer applies.

Note

The paper builds on my earlier work "A Trace of Anger is Enough, on the Enforcement of Social Norms".

3.1 Introduction

Centralized authorities, such as governments, or decentralized ones, such as peers, use threats of punishment to enforce norms. However the authority, whether centralized or decentralized, achieves compliance only if it is able to commit to the punishment threat. Punishment is often costly, and hence an important determinant of the authority's success at enforcement is the amount of resources committed for punishment. In this paper I argue that both kinds of authorities are similar in that they can enforce high compliance of many agents with only few committed resources. The argument is as follows: suppose that the authority is *limited* in that it can commit only resources that suffice to punish just one agent by an amount higher than the agent's cost of compliance. Then, the authority's punishment threat induces among the agents a game with an equilibrium, in which all agents comply, as no agent wishes to deviate individually. For a centralized authority this implies that it is able to control an arbitrary number of subordinates as long as it is able to control one. Similarly, it is possible to apply this observation to decentralized peer enforcement in a public good game with punishment option. N players, each committing one unit for punishment, can enforce individual contributions of approximately N units, and can collect altogether approximately N^2 units.

However, even though a small punishment commitment may deter individual defectors from deviating, the existence of a non-compliance equilibrium would appear to be unavoidable. The committed resources are insufficient to punish all and therefore, if no agents comply, the punishment of each is small compared to the cost of compliance. Yet, as shown below, any limited authority may avoid the non-compliance equilibrium — at least in the long run — by choosing a proper punishment rule. The supporting argument is contingent on the authority's ability to punish colluding violators at least slightly. I divide authorities into two categories along this line. *Collusion-vulnerable* authorities cannot punish if all agents coordinate on the same level of non-compliance. Anger-based peer enforcement is a prime example, because punishing after a perfect collusion would require the punisher to be angry with peers who have perpetrated the same offense as herself.[1] *Collusion-resistant* authorities are able to punish by at least some amount even after a perfect collusion. The punishment of each colluding agent may be arbitrarily small so even an authority with limited committed resources can be collusion-resistant.

[1] Decentralized authority based on peer enforcement may more frequently belong to this category but even a centralized authority such as a government may be constrained, for instance politically, to punish agents unified in a common non-compliance action.

Let us first present a punishment rule which eliminates the non-compliance equilibria yet which is suitable for a limited collusion-resistant authority. The rule requires the authority to commit to punishing only the worst violator. In the case of a tie the authority divides the punishment equally among the worst violators which in turn induces a dominance solvable game among the agents. The lowest possible compliance level is dominated as it guarantees punishment, and an increase in compliance just above the second lowest level saves the violator from punishment. Elimination continues by induction until only high compliance levels remain in the strategy sets. The logic is similar to that in Abreu-Matsushima (1992) mechanism, in which a small punishment possibility is leveraged to a strong enforcement of truth-telling through an ingenious dominance solvable game.[2]

A collusion-vulnerable authority, in contrast, cannot use the above "punish-the-worst" rule as it requires slight punishment of all players even after a perfect collusion. An equilibrium in which no players comply inevitably exists under a collusion-vulnerable authority as no player can be punished in such an equilibrium. To assess which equilibrium prevails in the long run, I build a stochastic, evolutionary model along the lines of Young (1993) and Kandori, Mailath and Rob (1993). Agents occasionally but rarely deviate from their best responses and experiment with a random action. As demonstrated below, only a high level of compliance survives the evolution under a simple punishment rule.

This application of stochastic evolution is similar to that of Kandori (2003), who examines a public good game (without punishment option). Kandori, in line with psychological game theory, assumes intrinsic motivation to adhere to norms as long as others adhere to it and analyzes the resulting coordination problem. Occasional mutations — deviations from best responses — cause shifts of the norm. Downward shifts require fewer mutations than upward shifts in Kandori's model. As a result, high contribution levels eventually decay and only low contributions prevail in the long run, exactly as observed in experiments (see Ledyard, 1995). As shown below, adding a punishment option to the public good game reverses Kandori's result despite the fact that the commitment to punishment is limited. Small upward shifts of norms require fewer mutations than any downward shifts under a simple punishment rule. Therefore, for a low rate of mutations, shifts, conditional that they happen, are almost always upward and the stochastic evolution converges to high contribution levels. The evolution

[2] One of the differences is that while Abreu-Matsushima mechanism can implement any equilibrium of the underlying game, the punishment in our model forces players into a non-equilibrium behavior in the underlying game.

can be observed in the laboratory also in this case: the contribution level typically increases during public good experiments with punishment option (Fehr and Gächter, 2000).

The paper at hand does *not* examine where the authority's limited commitment ability comes from. For that reason, I choose a black box approach for the motivation of punishment. The authority is assumed to be able to commit to limited punishment. There is experimental evidence supporting this assumption for the case of peer enforcement (e.g. Fehr and Gächter, 2000, 2002; Yamagishi, 1986). Punishment is modelled in this paper as an automatic, limited reaction governed by a punishment rule which is a function of the individual compliance levels. The focus *is* on specifying rules assuring high compliance under the constraint of limited punishment.

The analysis starts by examining an optimal punishment rule suitable for a collusion-resistant authority in section 3.2. A collusion-vulnerable authority and its associated coordination problem is studied in section 3.3. Section 3.4 concludes.

3.2 Punishment Rule Suitable for a Limited Collusion-Resistant Authority

This section reproduces the model in Steiner (2005). It formalizes the introductory argument that a collusion-resistant authority can always avoid non-compliance equilibria. Though the authority of this section could be centralized or decentralized, the model is formulated in the former setting, as I discuss its connection to tax enforcement at the end of the section.

Each player $i \in \mathcal{I} = \{1, \dots, N\}$, $N \geq 1$, simultaneously chooses an action c_i from a common strategy set $S = \{0, \Delta, 2\Delta, \dots, L\Delta\}$, where Δ is sufficiently small, $\Delta < 1$, and $L\Delta \geq N$. The assumption of the dense grid is needed to enable a sufficiently small increase in compliance. The grid is used as a technically convenient approximation of the continuous strategy space, so the assumption is not substantial. The assumption $L\Delta \geq N$ assures that players are not physically precluded from high compliance. The action profile of all players is denoted by $\mathbf{c}$.

The authority has committed to a punishment rule $\mathbf{p}(.)$, $\mathbf{p} : S^N \to \mathbb{R}_+^N$ that allocates punishment $p_i(\mathbf{c}) \geq 0$ to each player i after the authority observes the realized strategy profile. The authority committed to the rule before the players choose actions and the commitment has been commonly observed by all players. The payoffs of the players are

$$u_i(\mathbf{c}) = -c_i - p_i(\mathbf{c}). \tag{3.1}$$

Thus c^i is interpreted as the cost of compliance net of individual benefits of the compliance, if these exist. $(\mathcal{I}, S^N, \{u^i\}_{i=1}^N)$ is the *punishment game.* Only the one-stage interaction of players is modelled here; the behavior of the (limited) authority is an assumption.

Enforcement of high compliance would be trivial if the authority could commit to any punishment rule. However, the authority is limited in the sense that it is at most able to commit to spending on punishment one unit per agent:

A1: $\frac{\sum_{i=1}^N p_i(\mathbf{c})}{N} \leq 1$ for any $\mathbf{c}$.

Despite assumption **A1**, there exists a punishment rule that induces a game with a unique equilibrium in which the actions of all players are approximately N. Denote the highest level below N by m_{cen}, the lowest action among players by l, and the second lowest by s with the convention that $l = s$ if there is more than one player with the lowest action. Let the punishment rule be

$$p_i(\mathbf{c}) = \begin{cases} \frac{N}{m_{cen}}\Big(\min(s, m_{cen}) - c_i\Big) & \text{if } c_i = l,\ l < s, \text{ and } c_i < m_{cen}, \\ 1 & \text{if } c_i = l,\ l = s, \text{ and } c_i < m_{cen}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

The marginal punishment, which is $\frac{N}{m_{cen}} > 1$ or $\frac{1}{\Delta} > 1$, suffices to motivate the player with the lowest action to increase her action, as long as the lowest action is below m_{cen}. Yet the total punishment expenditures are always at most N because the punishment is not too costly even in situations when many players coordinate on the same lowest level, as then $s = l$ and each colluder is punished only slightly. This exact punishment rule is not necessarily practiced in reality; Proposition 3.1 simply demonstrates that a rule inducing high compliance exists.

Proposition 3.1. *1. The punishment game with punishment rule (3.2) has a unique equilibrium with all N players playing m_{cen}.*

2. Punishment rule (3.2) satisfies assumption **A1***.*

Proof of Proposition 3.1. **1.** Actions larger than m_{cen} are dominated by m_{cen} because a player who has chosen at least m_{cen} is never punished. Moreover, the player with the lowest action below m_{cen} always wishes to increase her action by at least Δ because the increase of her compliance by Δ decreases her punishment by $\frac{N}{m_{cen}}\Delta > \Delta$ or by $1 > \Delta$. Hence, the lowest level, 0, is dominated by level Δ. After elimination of $\{0, \Delta, \ldots, k\Delta\}$, level $(k+1)\Delta$ is dominated by $(k+2)\Delta$ because $(k+1)\Delta$ would be the lowest action among the non-eliminated strategies, for $k = 0, \ldots, \frac{m_{cen}}{\Delta} - 2$. Thus, the game can be solved by iterated elimination of dominated strategies. Only m_{cen} survives this process.

2. There is either only one player with the lowest action, in which case she is the only one being punished. The punishment is largest in this case if $s = m_{cen}$ and $c_i = 0$. Then the punishment is $\frac{N}{m_{cen}} m_{cen} = N$. Or there may be many players with the lowest action, in which case $s = l$ and each punishment is 1. Thus the cost is at most 1 unit per player in both cases. □

A limited authority fulfilling **A1** cannot enforce higher actions than N, as this is the highest possible punishment it can inflict on a deviator. The "punish-the-worst" rule is thus the optimal rule.

Alm and McKee (2004) experimentally study several tax enforcement schemes and document that a rule similar to the "punish-the-worst" rule indeed elicits high compliance. The authors assume a coordination problem analogous to the one in the present model: audit probability increases with the difference between the average and agent's reported income. This models the use of the Discriminant Index Function (DIF) scores by the Internal Revenue Service in the United States. DIF is a statistical score indicating levels of suspiciousness of tax returns; those with above average DIF are more likely to be audited. Such an endogenous audit probability rule leads to a coordination game, in which full evasion by all agents constitutes an equilibrium. The experiment demonstrates that adding a small probability of a randomly allocated audit in the case of perfect collusion prevents coordination on full evasion. The intuition is the same as in the model of this section. Indeed, the experimental data show a gradual increase in compliance, as players try to escape the gradually increasing lowest position.

3.3 Punishment Rule Suitable for a Limited Collusion-Vulnerable Authority

This section examines long run sustainable compliance levels under a collusion-vulnerable authority. Unlike in the previous section, such an authority cannot assure high compliance in the short or medium run because zero compliance always constitutes an equilibrium. To compare the effectiveness of different punishment rules, I assume that players occasionally, but rarely, experiment with a randomly chosen action. I look for compliance levels that prevail in the long run.

For the sake of concreteness, the model is formulated in the setting of a public good game with punishment option which mimics in gross features the experiments in Fehr and Gächter (2000, 2002). The next subsection describes the evolution in a fixed group of players. A modification describing the evolution under a random matching protocol is given in subsection 3.3.2.

3.3.1 Partners Treatment

A fixed set of $N \geq 3$ risk-neutral players repeatedly plays the public good game with punishment option in rounds $t \in \mathbb{N}$, and each player i chooses a contribution level c_i^t from the common strategy set $S = \{0, \Delta, 2\Delta, \ldots, L\Delta\}$, $L\Delta \geq N$. S is of the same structure as in section 3.2 but a denser grid is required, $\Delta < \frac{1}{N-1}$. After the contributions $\mathbf{c}$ of all players are made and observed by everyone, players automatically assign punishment points to each other; p_j^i denotes the punishment i assigns to j.

The punishment $p_j^i(\mathbf{c}^{t-1}, \mathbf{c}^t)$ depends on the contribution levels of the previous and current rounds in this section; $p_j^i : S^N \times S^N \to \mathbb{R}_+$. By allowing mild history dependence, the model diverges from the experimental design of the partners treatment in Fehr and Gächter (2000), who excluded it in order to avoid reputation effects. The reputation effects are excluded here by assuming myopic behavior. I can therefore permit history-dependent punishment rules which are psychologically plausible and which allow higher contributions than do memoryless rules. Although longer memories could be considered, memory of length one turns out to be sufficient to support contribution levels of approximately N, which is the highest possible level. History dependence is not substantial for the qualitative results of the model. The enforceable contribution level increases linearly in the number of players even under a memoryless rule, but as $\sim \frac{N}{2}$ instead of $\sim N$. Only memoryless punishment rules are considered under the random matching setup in subsection 3.3.2.

Players play myopic best responses to the previous action profile in each round t.[3] That is, each player maximizes payoff under the punishment rules assuming that her opponents will carry over their contributions $\mathbf{c}^{t-1}$ from the last round:

$$c_i^t \in \arg\max_{c_i} \left\{ -c_i - \sum_{j \neq i} p_i^j \Big(\mathbf{c}^{t-1}, \big(c_i, c_{-i}^{t-1} \big) \Big) \right\}. \tag{3.3}$$

The public good does not enter the maximization problem; c_i is interpreted as the contribution costs net of the marginal increase of the public good. Also, the cost of the punishment does not enter the maximization problem although the agents bear the cost. The limited punishment is automatic

[3]The results would not be changed if players could adjust to their best responses only with a certain probability.

and thus is not part of the agents' decision problem. Alternatively, I could presuppose a behavioral utility function under which the limited punishment would be optimal, but the main claim is that a small willingness to punish leads to high contributions. The exact motivation to punish is outside the focus of this paper. The optimization problem (3.3) can be understood as a reduced form of a more complex optimization with the punishment stage already solved.

The strategy set S and the punishment rules $p^i_j(.,.)$ define a Markov process $(S^N, \mathbf{Q})$ where the transition matrix $\mathbf{Q}$ is determined by (3.3). Note that it is a memoryless process, despite the fact that the punishment rule is history dependent, because the optimization problem (3.3) depends only on the last round contribution profile $\mathbf{c}^{t-1}$. The pair $(S^N, \mathbf{Q})$ is the *unperturbed process*.

Assumption **A1** reformulated for the decentralized authority setting is:
A1': $\sum_{j \neq i} p^i_j \leq 1$ for all i and any $\mathbf{c}^{t-1}, \mathbf{c}^t$.
Assumption **A1'** is stricter than **A1** because it not only requires average expenses for the punishment to be below 1, but also individual expenses of each player to be below 1. The next assumption prohibits players from punishing peers that have contributed the same amount as themselves[4]:
A2: If $c^t_i = c^t_j$ then $p^i_j = 0$.

Assumption **A2** implies that $\mathbf{c} = \mathbf{0}$ is inevitably a steady state of the unperturbed process, so at worst a punishment rule does not induce any co-operation and at best there are multiple steady states. However, as demonstrated below, there exists a punishment rule under which increases of norms are much less demanding than decreases. Hence high contributions prevail in the long run.

In order to study the transitions between different steady states I introduce, following the framework of stochastic evolution of Kandori, Mailath and Rob (1993), occasional deviations from the unperturbed process: each player plays best response with probability $(1 - \epsilon)$ whereas with probability ϵ a "mutation" happens — the player chooses a random action from the uniform distribution on S. A *perturbed system* is a pair $(S^N, Q(\epsilon))$, where $Q(\epsilon)$ are the transition probabilities, with $\epsilon > 0$. The perturbed system has a unique invariant distribution μ^ϵ, which is close to $\mu^* \equiv \lim_{\epsilon \to 0} \mu^\epsilon$ for small ϵ. Ellison (2000) provides an intuitive "mutation counting" technique for the computation of μ^* based on the observation that step-by-step evolution passing through several intermediate states, with each step requiring few mutations, is quicker than a sudden evolutionary jump requiring the simultaneity of many mutations.

[4]Which implies that players never punish themselves.

I utilize Ellison's observation and design a punishment rule under which only one mutation is needed for an increase in contributions by one level, but a decrease by any number of levels requires more than one mutation. As a consequence, evolution reaches high contribution levels more quickly than it escapes it. This intuition is formally expressed in the following proposition. Let m_{par} be the highest contribution level below $N-2$ and denote by M_{par} the state in which all players contribute m_{par}.

Proposition 3.2. *There exists a punishment rule satisfying* **A1'**, **A2** *under which M_{par} is the unique stochastically stable state, and the expected waiting time to reach M_{par} is of order $O(\epsilon^{-1})$.*

Proof of Proposition 3.2. The proof is based on the following lemma and the theorem in Ellison (2000).

Lemma 3.1. *There exists a punishment rule satisfying* **A1'**, **A2** *for which:*

1. Any common contribution level $0 \leq \overline{c} \leq m_{par}$, $\overline{c} \in S$ constitutes a steady state of the unperturbed process.

1'. No other limit sets of the unperturbed process than those in 1. exist.

2. Deviation of only one player from a steady state with common contribution level $\overline{c}$ suffices to induce transition to the steady state with level $\overline{c}+\Delta$, for any $\overline{c} < m_{par}$, $\overline{c} \in S$.

3. Deviation of more than one player from a steady state with common contribution level $\overline{c}$ is needed to induce transition to a steady state with a lower level, for any $\overline{c} \leq m_{par}$, $\overline{c} \in S$.

Proof of Lemma 1 is given Appendix 3.A.

Having established Lemma 3.1, Proposition 3.2 is a consequence of Ellison's (2000) theorem that specifies the long run stochastically stable limit set in terms of radius and modified coradius. The radius $R(\Omega)$ is the number of mutations needed to escape Ω and hence property 3 in Lemma 3.1 and the fact that M_{par} is the highest steady state assures that $R(M_{par}) > 1$. The modified coradius $CR^{*}(\Omega)$ is the maximal modified number of mutations needed to reach Ω from other limit sets of the unperturbed process, where the modified number reflects that step-by-step evolution is more probable than sudden changes. In particular, a set Ω that is possible to reach through a series of one or zero mutation steps from anywhere has $CR^{*}(\Omega) = 1$; see Ellison (2000) for details. Property 2 in Lemma 3.1 guarantees that only one mutation is needed for transition from a steady state with level $\overline{c}$ to level $\overline{c} + \Delta$ and thus there is a path consisting of at most one mutation steps to M_{par} from any other state, and hence $CR^{*}(M_{par}) = 1$. According to theorem 2 in Ellison (2000), $R(M_{par}) > CR^{*}(M_{par})$ implies that M_{par} is the unique stochastically stable state. The same theorem specifies the waiting time as $O\left(\epsilon^{-CR^*(M_{par})}\right)$. □

Ellison provides an intuition for the speed of step-by-step evolution that translates naturally to the current setting: An increase in the norm by one contribution level is an ϵ probability event as it can be induced by one mutation. In contrast, a decrease in contribution level is an ϵ^2 or rarer event as at least two mutations are needed. Hence, conditional on a transition occurring, it is almost always an upward shift, for small ϵ.

It is worth noting that the waiting time $O(\epsilon^{-1})$ to reach M_{par} is of the least possible order. The contribution level enforceable by an authority limited by **A1'** and **A2** is bounded by $N-1$ because this is the maximal punishment a single deviator may suffer; thus the modified "punish-the-worst" rule induces a nearly optimal contribution level.

3.3.2 Strangers Treatment

The model of the partners treatment in the previous subsection describes evolution among a fixed set of players, evolving in isolation from the rest of the population. Alternatively, players may interact with different peers every round, in which case evolution occurs simultaneously in a large population, from which the groups are drawn anew each round. This subsection sketches evolution under the strangers treatment.

A population of KN risk-neutral players is randomly matched each round into $K \geq 2$ groups of $N \geq 2$ players to play the public good game with punishment option. The strategy set $S = \{0, \Delta, 2\Delta, \ldots, L\Delta\}$, $L\Delta \geq N$, is of the same structure as in sections 3.2 and 3.3.1 but the grid is denser, $\Delta < \frac{1}{KN-1}$. In each round, players can punish only the peers within the group they have been matched to and the punishment rules $p_j^i(\mathbf{c})$ are history *independent*, $p_j^i : S^N \to \mathbb{R}_+$. As in section 3.3.1, punishment rules are required to satisfy **A1'** and **A2**. The unperturbed process is again the best response dynamics and under the perturbed process, players choose the best response with probability $1-\epsilon$ and with probability ϵ choose a random action from the uniform distribution on S, as in section 3.3.1. Let m_{str} be the highest level below $(N-1)\frac{(K-1)N}{KN-1}$; it approaches $N-1$ for large K and N. Let M_{str} be the Markov state in which all players contribute m_{str}. The counterpart of Proposition 3.2 of subsection 3.3.1 is:

Proposition 3.3. *There exists a punishment rule satisfying* **A1'**, **A2**, *under which* M_{str} *is the unique stochastically stable state, and the expected waiting time to reach* M_{str} *is of order* $O(\epsilon^{-1})$.

Proof of Proposition 3.3. The punishment rule (3.4 in Appendix 3.A) without exception satisfies all four properties in Lemma 3.1.[5] Proof of property 1 and 1' remains unchanged. Property 2 is implied by the inequality $\Delta < \frac{1}{KN-1}$: suppose there is a single deviator j contributing more than the norm $\bar{c}$ prescribes. Then the probability that j will be matched with i is $\frac{N-1}{KN-1}$, and hence i's expected punishment is $\frac{1}{N-1}\frac{N-1}{KN-1}$, which equals the right hand side of the inequality. Hence the inequality assures that one deviator is sufficient to induce all other players to increase their contributions by Δ.

The inequality $m_{str} < (N-1)\frac{(K-1)N}{KN-1}$ implies property 3: suppose that $c_k = \bar{c} \leq m_{par}$ for all $k \notin \{i, j\}$ and $c_j < \bar{c}$. Then a conservative estimate of the slope of the expected punishment for player i is $\frac{N-1}{m_{str}}\frac{(K-1)N}{KN-1} > 1$ because $\frac{(K-1)N}{KN-1}$ is the probability that j will not be in i's group, thus i will be the only deviator in her group, and hence punished by $\frac{N-1}{m_{str}}(\bar{c} - c_i)$.

The properties of Lemma 3.1 imply $R(M_{str}) > 1$, $CR^*(M_{str}) = 1$ and Proposition 3.3 is a consequence of Ellison's (2000) theorem as it was in Proposition 3.2 of subsection 3.3.1. □

The models in this section are not literal models of Fehr and Gächter's (2000, 2002) experiments. Their grids of contribution levels in the strangers treatment experiments were not as dense as Proposition 3.3 requires, the information structure of the partners treatment in the (2000) experiment precluded history-dependent punishment, and, on the other hand, punishment was cheaper in the experiments than in the model. Also, while experimental subjects may have had a variety of motivations for contributing, the model focuses solely on the contributions enforced by the threats of punishment. A combination of Kandori's (2003) model of intrinsic motivation and the models at hand could provide even higher estimates of sustainable contribution equilibrium than do the present models alone.

The models suggest that the high contributions are due to the game's structure; that is, focusing the limited committed resources of all players on one potential deviator. Keeping the commitment ability fixed, the contributions increase linearly with the number of players. This insight is experimentally confirmed by Carpenter (2005), who documents positive group size effects in public good games with punishment option even after controlling for the marginal group return of contributions.

Of course, the game requires quite a bit of information: the actions of all players need to be monitored, which is feasible in small groups such as

[5] m_{par} needs to be replaced by m_{str} in the punishment rule and in Lemma 3.1.

work teams. Still, the effect can be noteworthy for a reasonable group size. Ten agents, each willing to spend only one unit for punishment, are able to collect at least $(10-2)\cdot 10 = 80$ units for a public good.

3.4 Conclusions

The models demonstrate that the commitment necessary for successful norms enforcement is small compared to the total cost of compliance of all agents. Agents in the compliance equilibrium consider deviating off the equilibrium *individually.* Hence, to support the compliance equilibrium, the authority needs only to be capable of substantially punishing one agent.

Nevertheless, other, non-compliance equilibria may exist. The main claim of the paper is that authorities can avoid these non-compliance equilibria by a proper punishment rule, even if their commitment capabilities are low. A punishment rule focusing on punishment of the worst offender creates competition among the agents and leads to a unique equilibrium with high compliance levels.

However, authorities using such a rule need to be able to punish perfectly colluding violators at least by a small amount, and many authorities fail to do so. Yet even such collusion-vulnerable authorities can avoid the non-compliance equilibria in the long run. They can introduce a punishment rule which deters revolts of a small fraction of players and enables a small fraction of players to initiate at least a tiny increase in compliance. Then, given a sufficiently small mutation rate, the increases are arbitrarily more times probable. High compliance prevails in the long run.

The prime application of the collusion-vulnerable authority model is the public good game with anger-driven punishment of free-riders. Even if the anger — a deviation from the *homo oeconomicus* framework — is limited, it can go a long way towards modifying equilibrium behavior. The public good game with punishment option is an instance of an institution that efficiently utilizes this behavioral deviation; a systematic search for other such institutions is needed.

3.A Proof of Lemma 1

Lemma 3.1. *There exists a punishment rule satisfying* **A1'**, **A2** *for which:*

1. Any common contribution level $0 \leq \bar{c} \leq m_{par}$, $\bar{c} \in S$ *constitutes a steady state of the unperturbed process.*

1'. No other limit sets of the unperturbed process than those in 1. exist.

2. Deviation of only one player from a steady state with common contribution level $\overline{c}$ suffices to induce transition to the steady state with level $\overline{c}+\Delta$, for any $\overline{c} < m_{par}$, $\overline{c} \in S$.

3. Deviation of more than one player from a steady state with common contribution level $\overline{c}$ is needed to induce transition to a steady state with a lower level, for any $\overline{c} \leq m_{par}$, $\overline{c} \in S$.

Proof of Lemma 3.1. let the definitions of l and s remain as in section 3.2. Consider a "modified punish-the-worst" rule:

$$p_j^i = \begin{cases} \frac{1}{m_{par}}\Big(\min(s, m_{par}) - c_j\Big) & \text{if } c_j = l,\ c_j < m_{par}, l < s, \text{ and } c_i > c_j, \\ \frac{1}{N-1} & \text{if } c_j = l,\ c_j < m_{par}, l = s, \text{ and } c_i > c_j, \\ 0 & \text{otherwise,} \end{cases} \tag{3.4}$$

except for situations when player k starts a rebellion against a norm of a common contribution level $\overline{c}$ in the previous round $t-1$, persists in that rebellion in round t, and player j joins the rebellion; then the remaining players concentrate on punishing the new free-rider j, not the old k. Formally, the exception states that (3.4) does not apply at t when at $t-1$ all players $i \neq k$ contributed some common level $c_i^{t-1} = \overline{c} \leq m_{par}$ and k contributed $c_k^{t-1} < \overline{c}$, and $c_k^t = c_k^{t-1}$, $c_j^t < \overline{c}$, and $c_i^t = \overline{c}$, for $i \notin \{j,k\}$ in round t. Then all $N-2$ players $i \notin \{j,k\}$ punish j in round t each by amount

$$p_j^i = \frac{1}{m_{par}}(\overline{c} - c_j^t).$$

The punishment rule suitable for the strangers treatment does not employ this exception.

This rule satisfies **A1'** because either player punishes only one of her peers in which case she spends at most $\frac{1}{m_{par}} m_{par} = 1$ or she punishes many players and then she spends at most $(N-1)\frac{1}{N-1} = 1$. The rule satisfies **A2** because it prescribes punishing only peers who have contributed less than the punisher. Let us verify that the modified "punish-the-worst" rule satisfies all four properties in Lemma 3.1:

1. The best response to $\mathbf{c}^{t-1} = \overline{\mathbf{c}}$ and $c_{-i}^t = \overline{c}$ is[6] $c_i^{t*} = \overline{c}$. Hence a state in which all players contribute $\overline{c}$ is a steady state of the unperturbed process.

2. Suppose $c_i^t > \overline{c}$, $c_j^t = \overline{c} < m_{par}$ for all $j \neq i$. The best response of $j \neq i$ is $\overline{c} + \Delta$, the best response of i is $\overline{c}$. Thus, at $t+1$, $c_i^{t+1} = \overline{c}$, $c_j^{t+1} = \overline{c} + \Delta$, and at $t+2$ all players contribute $\overline{c} + \Delta$ which becomes the new steady state of the unperturbed process.

[6] To avoid confusion, $c_i^{t-1} = \overline{c}$ for all i and $c_j^t = \overline{c}$ for all $j \neq i$.

3. Suppose that only one player has deviated from the common contribution level in round t; $c_j^t = \overline{c} \leq m_{par}$ for all $j \neq i$ and $c_i^t < \overline{c}$. Then the exemption applies in $t+1$ and the best response of all players in $t+1$ is to contribute $\overline{c}$.

1'. Consider a state $\mathbf{c}$ in which more than one contribution level is chosen. Let us distinguish two cases: in case **A**, $N-1$ players contribute some common $\overline{c}$ and the contribution of only one player differs from $\overline{c}$; case **B** includes all other situations. If **A** arises, players converge to a common contribution level $\overline{c}$ or $\overline{c}+\Delta$ within one or two rounds(see proofs of properties 2 and 3). In case **B**, the best response of each player i is to contribute $l_{-i} + \Delta > l$, where l_{-i} is the lowest contribution among i's opponents. Therefore the lowest contribution increases in those rounds when case **B** arises.[7][8] Thus in each round either l increases or **A** arises, and because the set of the contribution levels is finite, either l converges to m_{par} or **A** arises and under both eventualities players converge to a common contribution level. □

[7]This does not hold in situations described in the proof of property 2. Therefore the division of all situations into categories **A** and **B** is necessary.

[8]In the case of the adaptation of the proof for subsection 3.3.2 the best response is $l_{-i} + \Delta$ or higher.

References

[1] Abreu D. and H. Matsushima, Virtual Implementation in Iteratively Undominated Strategies, Econometrica 60, 993–1008.

[2] Alm J. and M. McKee, 2004, Tax Compliance as a Coordination Game, Journal of Economic Behavior & Organization 54, 297–312.

[3] Carpenter J., 2005, Punishing Free-Riders: How Group Size Affects Mutual Monitoring and the Provision of Public Goods, Games and Economic Behavior, forthcoming.

[4] Ellison G., 2000, Basins of Attraction, Long Run Stochastic Stability, and the Speed of Step-by-Step Evolution, Review of Economic Studies 67, 17–45.

[5] Fehr E. and S. Gächter, 2000, Cooperation and Punishment in Public Goods Experiments, American Economic Review 90, 980–994.

[6] Fehr E. and S. Gächter, 2002, Altruistic Punishment in Humans, Nature 415, 137–140.

[7] Kandori M., 2003, The Erosion and Sustainability of Norms and Morale, The Japanese Economic Review 54, 29–48.

[8] Kandori M., G. Mailath and R. Rob, 1993, Learning, Mutation, and Long Run Equilibria in Games, Econometrica 61, 29–56.

[9] Ledyard J., 1995, Public Goods: A Survey of Experimental Research, in: Kagel, J. and A. Roth, Eds, The Handbook of Experimental Economics, (Princeton University Press, Princeton) 111-194.

[10] Steiner J., 2005, A Trace of Anger is Enough, on the Enforcement of Social Norms, CERGE-EI Working Paper No. 246.

[11] Yamagishi T., 1986, The Provision of a Sanctioning System as a Public Good, Journal of Personality and Social Psychology 51, 110–116.

[12] Young P., 1993, The Evolution of Conventions, Econometrica 61, 57–84.