Common priors, Duality, and No-Trade

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Motivation and goal

- **Belief-based** representation of information is now pervasive in models of information transmission (Bayesian persuasion and information design)
- **Single-receiver**: Bayes plausibility (Kamenica and Gentzkow 2011, KG) characterizes the set of feasible distributions over receiver's posteriors
- **Multiple receivers**: a corresponding **general characterization** is missing:
  - Mathevet et al (2020) characterize (finite) distributions over hierarchies of beliefs rather implicitly
  - Arieli et al (2021) characterize distributions over first-order beliefs under binary states (no-trade)
  - Corrao (2021) characterizes distributions over first-order expectations with continuous states
  - *Bergemann and Morris (2013, 2016) and Bergemann et al (2018) characterize BCE distributions and their moments (linear best-response games)*
- **This paper** provides an explicit characterization of feasible distributions of higher-order beliefs (and their "coarsenings") in terms of moment inequalities with no-trade interpretations

This paper: distributions over higher-order beliefs

- **Observation**: Characterizing feasible distributions over hierarchies of beliefs amounts to study the implications of Common Prior (CP) assumptions
- Extensive literature provides characterization of CP (ex-ante and interim) implications in terms of no-trade conditions: Nau and McCardle (1990), Morris (1994), Samet (1998), Feinberg (2000)
  - **Existing results**: Abstract state-space (usually finite), characterize existence of CP rather than the set of feasible distributions
- We provide a characterization for **CP-feasible distributions** over payoff-relevant states and higher-order beliefs with a no-arbitrage interpretation:
  - A pair of priors over states and beliefs are CP-consistent iff they do not allow any arbitrage when seen as prices of **independent bets** that only depend respectively on states and beliefs

This paper: information design and robustness

- As an illustration, we revisit the **critical-path theorem** of Kajii and Morris (1997)
- Bounds on the probability of **common-p belief** in an event E in terms of the prior probability of E
  - No-trade interpretation of the critical bounds
  - Tighter lower bound than KM97
  - Extension to uncountable (but compact) spaces
- **Information robustness**: smallest probability that both players invest in an investment game attains the implied bounds
- **Information design**: If the designer's objective depends on players' hierarchies of beliefs and states then our characterization posits the problem as an (infinite-dimensional) linear program with moment constraints

This paper: coarsened types spaces

- However, equilibria in economic settings are often described by **coarser features** than the entire hierarchies of beliefs
- Motivated by this, we introduce **coarsened type spaces** where the types of the agents correspond to these coarsened features (e.g., first-order beliefs, expectations, or actions)
- The beliefs of each coarsened type are **not uniquely identified** and only need to satisfy given restrictions (e.g., obedience when types correspond to actions)
- We characterize the **distributions over coarsened types that are CP-consistent**:
  - First-order beliefs that can arise under any information structure for a given CP
  - Actions that can arise in any BCE
  - Partitions induced by belief operators (robust info design)
- Obtain moment restrictions on distributions over observable coarsenings that can falsify the CP assumption
- Simplify information design problems where designer's objective depends on these coarsenings

General incomplete information setting

- Finite set of agents $N = \{1, ..., n\}$
- Uncertain state of the world $\theta \in \Theta \subseteq \mathbb{R}^m$ with Θ compact (results extend to compact metric spaces)
- First-order beliefs of agent i: $p_i^1 \in P_i^1 := \Delta(\Theta)$
- Second-order beliefs of agent i: $p_i^2 \in P_i^2 := \Delta\left(\Theta \times \prod_{j \neq i} P_j^1\right)$, so on and so forth...
- Universal types $t_i \in T_i$ collect the entire (coherent) hierarchy of beliefs of agent i:

$$t_i = \left(p_i^1, p_i^2, ..., p_i^k, ...\right) \in \prod_{k \in \mathbb{N}} P_i^k$$

- Brandenburger and Dekel: T_i is a compact subset and there exists a (canonical) homeomorphism $g_i : T_i \longrightarrow \Delta(\Theta \times T_{-i})$ mapping universal types to beliefs and viceversa

Common prior and distribution of beliefs

- A common prior (over states) is a probability measure $\mu \in \Delta(\Theta)$ shared by all agents
- An information structure is a pair $\mathcal{I} = (S, \sigma)$ such that $S = \prod_{i \in N} S_i$ is the (product, measurable) signal space and

$$\sigma : \Theta \to \Delta(S)$$

  is a statistical experiment. Every agent i only observes the private realization $s_i \in S_i$
- A common prior $\mu \in \Delta(\Theta)$ and an information structure $\mathcal{I} = (S, \sigma)$ induce distributions $\pi_{\mu,\sigma} \in \Delta(\Theta \times T)$ and $\tau_{\mu,\sigma} \in \Delta(T)$ over universal types
- We aim to characterize

$$\Delta_{CP}(\mu) = \left\{\pi_{\mu,\sigma} \in \Delta(\Theta \times T) : \text{ for some } \mathcal{I} = (S, \sigma)\right\},$$

$$\mathcal{T}_{CP}(\mu) = \left\{\tau_{\mu,\sigma} \in \Delta(T) : \text{ for some } \mathcal{I} = (S, \sigma)\right\}$$

  as well as $\bigcup_{\mu \in \Delta(\Theta)} \mathcal{T}_{CP}(\mu)$

First-step: getting rid of information structures

Lemma

π is CP-consistent, that is $\pi \in \bigcup_{\mu \in \Delta(\Theta)} \Delta_{CP}(\mu)$, if and only if, for every $i \in N$, $g_i : T_i \to \Delta(\Theta \times T_{-i})$ is a version of the conditional probability of π given $t_i \in T_i$.

- **Immediate implication**: the following are equivalent

  (i) $\tau \in \Delta(T)$ is consistent with the common prior assumption (resp. with $\mu \in \Delta(\Theta)$)

  (ii) There exists $\pi \in \Delta(\Theta \times T)$ that admits $(g_i)_{i \in N}$ as versions of its conditional probabilities and $\mathrm{marg}_T \pi = \tau$ (resp. also $\mathrm{marg}_\Theta \pi = \mu$)

Trades

- State- and beliefs-contingent trades are profile of *continuous* functions $h = (h_i)_{i \in N} \in H := C(\Theta \times T)^N$
- Continuity needed for *countable additivity* of CPs (extension to *finitely additive* CPs with *bounded and measurable trades*)
- Consider a dummy agent i_0 with no information in the interim stage
- All the agents' preferences are *linear in money*

No-trade

- For trades $\left(h_{i_0}, (h_i)_{i \in N}\right)$ define:
  - **Feasibility**: $-h_{i_0}(\theta, t) \geq \sum_{i \in N} h_i(\theta, t)$ for every $(\theta, t) \in \Theta \times T$
  - **Acceptability**: $\int_{\Theta \times T_{-i}} h_i(\theta, t)\, dg(t_i)(\theta, t_{-i}) \geq 0$ for every $t_i \in T_i$ and $i \in N$

Definition

$\pi \in \Delta(\Theta \times T)$ satisfies **no-trade** if there does not exists a feasible and acceptable profile of trades $\left(h_{i_0}, (h_i)_{i \in N}\right)$ such that

$$
\begin{aligned}
\int_{\Theta \times T} h_j(\theta, t)\, d\pi(\theta, t) &\geq 0 \quad \text{for all } j \in N \cup \{i_0\}, \\
\int_{\Theta \times T} h_j(\theta, t)\, d\pi(\theta, t) &> 0 \quad \text{for some } j \in N \cup \{i_0\}.
\end{aligned}
$$

Zero-value trades

- **Alternative definition**: get rid of the dummy trader i_0 and replace it with an external trader who is still uniformed in the interim
- External trader offers $h = (h_i)_{i \in N} \in H$ to the agents who then choose whether to accept or not in the interim stage

Definition

A trade $h \in H$ is **zero-value** if, for every $t_i \in T_i$ and $i \in N$,

$$\int_{\Theta \times T_{-i}} h(\theta, t)\, dg(t_i)(\theta, t_{-i}) = 0.$$

Let $H_0 \subseteq H$ denote the set of zero-value trades.

- Every type of every agent is indifferent between accepting or rejecting a zero-value trade

Money pumps

Definition

$\pi \in \Delta(\Theta \times T)$ satisfies **no-money-pump** if, for every $h \in H_0$,

$$\int_{\Theta \times T} \sum_{i \in N} h_i(\theta, t)\, d\pi(\theta, t) \geq 0$$

- **Interpretation**: If $\int_{\Theta \times T} \sum_{i \in N} h_i(\theta, t)\, d\pi(\theta, t) < 0$, then an external trader with beliefs π can make a strictly positive expected profit by offering h to the agents

Common priors, no-money-pumps, and no-trade

Theorem

The following are equivalent:

(i) π *is CP-consistent, that is,* $\pi \in \bigcup_{\mu \in \Delta(\Theta)} \Delta_{CP}(\mu)$

(ii) π *satisfies no-money pump*

(iii) π *satisfies no-trade*

- Add the requirement that $\mathrm{marg}_{\Theta}\pi = \mu$ to (ii) and (iii) to obtain a characterization of $\Delta_{CP}(\mu)$

Superreplicating independent bets

- Next, focus only on marginals over higher-order beliefs (extension of Bayes-plausibility condition)
- Define the set

$$\mathcal{S} = \left\{ (\phi, \psi) \in C(\Theta) \times C(T) : \exists h \in H_0, \phi + \psi \geq \sum_{i \in N} h_i \right\}$$

- **Interpretation**: Suppose that the external trader has access to "independent" trades $(\phi, \psi) \in C(\Theta) \times C(T)$ that only depend either on the state θ or on the beliefs of the agents t

- The elements of $\mathcal{S}$ are those independent trades that *superreplicate* a portfolio of acceptable trades $h \in H_0$

Main characterization

Theorem

Fix $\mu \in \Delta(\Theta)$ *and* $\tau \in \Delta(T)$. *The following are equivalent:*

(i) *There exists an information structure* (S, σ) *such that* $\tau = \tau_{\mu,\sigma}$, *that is,* $\tau \in \mathcal{T}_{CP}(\mu)$

(ii) *For every* $(\phi, \psi) \in \mathcal{S}$,

$$\int_{\Theta} \phi(\theta)\, d\mu(\theta) + \int_{T} \psi(t)\, d\tau(t) \geq 0 \tag{1}$$

No-arbitrage interpretation

- We interpret condition (ii) as a no-arbitrage condition: suppose that μ and τ are the (linear) price functionals for independent trades $\phi \in C(\Theta)$ and $\psi \in C(T)$
- These prices correspond to the marginal distribution over states and beliefs due to fair pricing
- Suppose that there exist $(\phi, \psi) \in \mathcal{S}$ for some $h \in H_0$ such that (1) is not satisfied
- The external trader can then buy these two assets to obtain

$$-\left(\int_{\Theta} \phi(\theta)\, d\mu(\theta) + \int_{T} \psi(t)\, d\tau(t)\right) > 0$$

  and offer the profile of acceptable trades $(h_i)_{i \in N}$ to the agents
- Since (ϕ, ψ) superreplicate $(h_i)_{i \in N}$ pointwise, the external trader obtains a strictly positive profit

Single receiver: Bayes plausibility

- When $N = \{i\}$, we have $T = \Delta(\Theta)$
- Fix any $\phi \in C(\Theta)$ and define

$$\psi(t) = \mathbb{E}_t[\phi] \qquad \forall t \in \Delta(\Theta)$$

- We have

$$\mathbb{E}_t[\phi - \psi(t)] = 0 \qquad \forall t \in \Delta(\Theta)$$

so that $h(\theta, t) = \phi(\theta) - \psi(t)$ is a zero-value trade for the unique agent

- Our result gives

$$\int_\Theta \phi(\theta)\, d\mu(\theta) - \int_T \psi(t)\, d\tau(t) = 0 \iff \int \phi d\mu = \int \mathbb{E}_t[\phi]\, d\tau(t)$$

that is, Bayes plausibility

Formal Proof: Sion Maxmin

- Define the set

$$\Delta(\mu,\tau) = \left\{\pi \in \Delta(\Theta \times T) : \mathrm{marg}_{\Theta}\pi = \mu, \mathrm{marg}_{T}\pi = \tau\right\}$$

- Form previous Theorem, $\tau \in \mathcal{T}_{CP}(\mu)$ if and only if there exists $\pi \in \Delta(\mu,\tau)$ such that

$$\int_{\Theta\times T} \sum_{i\in N} h_i(\theta,t)\, d\pi(\theta,t) \geq 0 \qquad \forall h \in H_0,$$

  that is

$$\sup_{\pi\in\Delta(\mu,\tau)} \inf_{h\in H_0} \left\{\int_{\Theta\times T} \sum_{i\in N} h_i(\theta,t)\, d\pi(\theta,t)\right\} \geq 0$$

- $\Delta(\mu,\tau)$ weakly compact and convex, H_0 convex, objective function doubly linear $\Longrightarrow$ Apply Sion Maxmin Theorem

Formal Proof: Kantorovich Duality

- We then have $\tau \in \mathcal{T}_{CP}(\mu)$ if and only if

$$\inf_{h \in H_0} \sup_{\pi \in \Delta(\mu,\tau)} \left\{ \int_{\Theta \times T} \sum_{i \in N} h_i(\theta, t)\, d\pi(\theta, t) \right\} \geq 0$$

- Fix $h \in H_0$ and focus on inner maximization: optimal transport problem with marginals (μ, τ) and cost $c = -\sum h_i$
- Apply Kantorovich Duality to obtain

$$\begin{aligned} & \sup_{\pi \in \Delta(\mu,\tau)} \left\{ \int_{\Theta \times T} \sum_{i \in N} h_i(\theta, t)\, d\pi(\theta, t) \right\} \\ = & \inf_{(\phi,\psi) \in C(\Theta) \times C(T) : \phi + \psi \geq \sum_{h_i}} \left\{ \int_{\Theta} \phi d\mu + \int_T \psi d\tau \right\} \end{aligned}$$

- The result follows

Illustration: Critical-Path results

- Consider only two players $N = \{a, b\}$ and define simple Θ-events as

$$E \times T \subseteq \Theta \times T$$

for some event $E \subseteq \Theta$

- The belief operator of $i \in N$ is

$$B_i^p(E) = \{t_i \in T_i : g_i(E|t_i) \geq p\} \quad \forall p \in [0, 1]$$

- As usual we define

$$B_*^{p,1}(E) = B_*^p(E) = B_a^p(E) \times B_b^p(E)$$

and for all $n \in \mathbb{N}$

$$B_*^{p,n+1}(E) = B_*^p\left(B_*^{p,n}(E)\right)$$

- The common-p belief operator is $C^p(E) = \bigcap_{n \in \mathbb{N}} B_*^{p,n}(E) \subseteq T$

Illustration: Critical-Path results

Corollary

Fix $\mu \in \Delta(\Theta)$, a closed set $E \subseteq \Theta$, and $p \in (0, 1/2)$. For every $\tau \in \mathcal{T}_{CP}(\mu)$, we have

$$\underbrace{\frac{\mu(E)(1+p) - 3p}{1-2p}}_{KM97} \leq \underbrace{\frac{\mu(E) - 2p}{1-2p}}_{CS21} \leq \tau\left[C^p(E)\right] \leq \frac{1}{p}\mu(E),$$

where the lower bound is tight (if we consider finitely additive measures).

- Upper bound is simple. For lower bound we let $\psi = \mathbb{I}_{C^p(E)}$, $\phi = -\frac{\mathbb{I}_E - 2p}{1-2p}$, and find $h_0 \in H_0$ such that

$$\mathbb{I}_{C^p(E)}(t) - \frac{\mathbb{I}_E(\theta) - 2p}{1-2p} = \sum_{i \in N} h_i(t, \theta) \qquad \forall (\theta, t) \in \Theta \times T$$

- The lower bound then follows from the characterization theorem by approximating ψ and ϕ with continuous functions from above

Construction of the critical trade

- Define $\kappa_i : T_i \longrightarrow \mathbb{N} \cup \{\infty\}$ as

$$\kappa_i(t_i) = \begin{cases} 0, \text{ if } t_i \notin B_i^p(E) \\ k, \text{ if } t_i \in \left[B_i^p\right]^k(E) \text{ but } t_i \notin \left[B_i^p\right]^{k+1}(E) \\ \infty, \text{ if } t_i \in B_i^p(C^p(E)) \end{cases}$$

- For every $i \in N$, consider the trade

$$h_i(\theta, t) = \begin{cases} p & \textit{if} & \kappa_i(t_i) < \infty \text{ and } \theta \notin E \\ p & \textit{if} & \kappa_i(t_i) < \infty, \theta \in E \text{ and } \kappa_j(t_j) < \kappa_i(t_i) \\ p - \frac{1}{2} & \textit{if} & \kappa_i(t_i) < \infty, \theta \in E \text{ and } \kappa_j(t_j) = \kappa_i(t_i) \\ -(1-p) & \textit{if} & \text{if } \kappa_i(t_i) < \infty, \theta \in E \text{ and } \kappa_i(t_i) < \kappa_j(t_j) \\ 0 & \textit{if} & 0, \text{ if } \kappa_i(t_i) = \infty \end{cases}$$

Construction of the critical trade

- This trade is acceptable for both i and all t_i by construction
- Now let

$$\begin{aligned} F &= \{(t_a, t_b) \in T : \kappa_a(t_a) < \infty \text{ and } \kappa_b(t_b) < \infty\} \\ S &= \{(t_a, t_b) \in T : \kappa_i(t_i) = \infty \text{ and } \kappa_j(t_j) < \infty \text{ for some } i\} \end{aligned}$$

- Observe that

$$h_a(\theta, t) + h_b(\theta, t) = \begin{cases} 2 & \textit{if} \quad \theta \notin E \text{ and } t \in F \\ 1 & \textit{if} \quad \theta \notin E \text{ and } t \in S \\ 0 & \textit{if} \quad \theta \notin E \text{ and } t \in C^p(E) \\ 1-2p & \textit{if} \quad \theta \in E \text{ and } t \in F \\ p-1 & \textit{if} \quad \theta \in E \text{ and } t \in S \\ 0 & \textit{if} \quad \theta \in E \text{ and } t \in C^p(E) \end{cases}$$

and verify that $\mathbb{I}_{C^p(E)}(t) - \frac{\mathbb{I}_E(\theta) - 2p}{1-2p} = h_a(\theta, t) + h_b(\theta, t)$ for all (θ, t).

Information design: characterization

- General information design problem given common prior $\mu \in \Delta(\Theta)$

$$\mathcal{P} = \sup \left\{ \int_{\Theta \times T} V(\theta, t)\, d\pi(\theta, t) : \pi = \pi_{\mu,\sigma} \text{ for some } \mathcal{I} = (S, \sigma) \right\}$$

for some *continuous* objective function $V : \Theta \times T \to \mathbb{R}$

- **Example**: let $\mathcal{A} : T \rightrightarrows \Delta(A)$ be a *continuous* solution correspondence for a *compact-continuous* incomplete-info game $(A_i, U_i(a, \theta))_{i \in N}$, and define

$$V(\theta, t) = \min_{\alpha \in \mathcal{A}(t)} \hat{V}(\alpha(t), \theta)$$

for some continuous $\hat{V} : \Delta(A) \times \Theta \to \mathbb{R}$. This gives rise to robust design problems (e.g., Mathevet et al. 2020)

Information design: characterization

- Our characterization simplifies the problem to

$$\mathcal{P} = \sup \left\{ \int_{\Theta \times T} V(\theta, t)\, d\pi(\theta, t) : \forall h \in H_0, \mathbb{E}_\pi \left[\sum_i h_i \right] = 0 \right\}$$

- If the objective function is state independent, then

$$\mathcal{P} = \sup \left\{ \int_T V(t)\, d\tau(t) : \forall (\psi, \phi) \in \mathcal{S}, \mathbb{E}_\tau[\psi] \geq \mathbb{E}_\mu[\phi] \right\}$$

Information design: duality

- The dual information design problem is

$$\mathcal{D} = \inf_{\phi \in C(\Theta), h_0 \in H_0} \left\{ \int_\Theta \phi(\theta)\, d\mu(\theta) : \forall (\theta, t), \phi(\theta) \geq V(\theta, t) + \sum_{i \in N} h_i(\theta, t) \right\}$$

- We can always get rid of ϕ by defining

$$\phi(\theta) = \sup_{t \in T} \left\{ V(\theta, t) + \sum_{i \in N} h_i(\theta, t) \right\}$$

Theorem

We have:

1. *No Duality Gap:* $\mathcal{P} = \mathcal{D}$
2. *The pair* (π, h) *solve the primal and the dual problems if and only if*

$$t \in \arg\max_{\tilde{t} \in T} \left\{ V(\theta, \tilde{t}) + \sum_{i \in N} h_i(\theta, \tilde{t}) \right\} \quad \textit{for } \pi\textit{-almost all } (\theta, t)$$

Motivation for coarsened types

- Sometimes analyst interested in coarsened description of the hierarchies of beliefs (e.g., sufficient to describe equilibria)
- Coarsened types $x = (x_i)_{i \in N} \in X$ can be description of the agents' beliefs or behavior
- Distributions $\nu \in \Delta(X)$ over coarsened types $x \in X$ are potentially observable
- **Goal**: characterize $\pi \in \Delta(\Theta \times X)$ and $\nu \in \Delta(X)$ that are consistent with common prior assumption in terms of falsifiable implications

Generalization: coarsened type-spaces

- A coarsened type space is a structure $(X_i, \Delta_i)_{i \in N}$ where, for every $i \in N$,
  - X_i is a compact metric space of coarsened types
  - $\Delta_i : X_i \rightrightarrows \Delta(\Theta \times X_{-i})$ is a *closed and convex-valued correspondence* mapping types x_i to possible beliefs
- Examples:
  1. **Standard type space**: $X_i = T_i$ and $\Delta_i(t_i) = \{g_i(t_i)\}$ for all $t_i \in T_i$
  2. **First-order beliefs**: $X_i = P_i^1 = \Delta(\Theta)$ and

$$\Delta_i\left(p_i^1\right) = \left\{\gamma \in \Delta\left(\Theta \times P_{-i}^1\right) : \mathrm{marg}_\Theta \gamma = p_i^1\right\}$$

  3. **Kth-order beliefs**: $X_i = P_i^k$

$$\Delta_i\left(p_i^k\right) = \left\{\gamma \in \Delta\left(\Theta \times P_{-i}^k\right) : \forall l \leq k, \mathrm{marg}_{P_{-i}^l} \gamma = \mathrm{marg}_{P_{-i}^l} p_i^k\right\}$$

Coarsened type space: other examples

4. **Belief operators**: induce a partition of the universal type space

5. **Best-responses**: Consider a game with incomplete info $(A_i, U_i(a, \theta))_{i \in N}$ and let $X_i = A_i$ and

$$\Delta_i(a_i) = \left\{ \gamma \in \Delta(\Theta \times A_{-i}) : a_i \in \arg\max_{\tilde{a}_i \in A_i} \{\mathbb{E}_\gamma [U_i(\tilde{a}_i, \cdot)]\} \right\} \quad \forall a_i \in A_i$$

6. **Strategic type space**: coarsened types correspond to the sequences of action sets $x_i = \{A_i^n\}_{n \in \mathbb{N}}$ resulting from Interim Correlated Rationalizability

- We say that Δ_i is **linear** if the set $\Delta_i(x_i) \subseteq \Delta(\Theta \times X_{-i})$ is described by (potentially infinite) linear inequalities (half-spaces in the finite case)

Common priors over coarsened types

Definition

We say that $\pi \in \Delta(\Theta \times X)$ is CP-X-consistent if, for every $i \in N$, there exists a version of the conditional probability $(\pi_{x_i})_{x_i \in X_i}$ such that

$$\pi_{x_i} \in \Delta_i(x_i) \qquad \text{for } \pi\text{-almost all } x_i.$$

We say that $\nu \in \Delta(X)$ is CP-consistent if there exists a CP-consistent π such that $\mathrm{marg}_X \pi = \nu$.

- As before, we can also require consistency with a fixed prior over states $\mu \in \Delta(\Theta)$
- **First-order-belief coarsening**: the CP-consistent distributions $\nu \in \Delta(X)$ correspond to those that can be induced by an information structure (cf. Arieli et al 2021)
- **Best-response coarsening**: the CP-consistent distributions $\pi \in \Delta(\Theta \times A)$ correspond to Bayes correlated equilibria (BCE) of the underlying game (cf. Bergemann and Morris 2016, 2017)

Cautiously zero-value trades

- X-measurable trades $h = (h_i)_{i \in N} \in H_X = C(\Theta \times X)^N$ are trades that only depend on the state and the coarsened types
- Type $x_i \in X_i$ of agent i can evaluate h_i according to multiple beliefs $\gamma \in \Delta(x_i)$
- Consider the worst possible evaluation: for every $h_i \in C(\Theta \times X)$, define

$$\xi_i(h_i)(x_i) = \inf\{\mathbb{E}_\gamma[h_i(x_i, \cdot)] : \gamma \in \Delta_i(x_i)\} \quad \forall x_i \in X_i$$

Definition

A trade $h \in H_X$ is **cautiously zero-value** if, for every $x_i \in X_i$ and $i \in N$,

$$\xi_i(h_i)(x_i) = 0$$

- **Interpretation**: Every type x_i is indifferent between accepting or rejecting the trade under the worst possible belief, hence they will always weakly prefer to accept for every $\gamma \in \Delta_i(x_i)$
- Let $H_{X,0}$ denote the set of cautiously zero-value for coarsening X

Main characterization

Definition

$\pi \in \Delta(\Theta \times X)$ satisfies **no cautious money pump** if, for every $h \in H_{X,0}$,

$$\int_{\Theta \times X} \sum_{i \in N} h_i(\theta, x)\, d\pi(\theta, x) \geq 0$$

- **Interpretation**: If $\int_{\Theta \times X} \sum_{i \in N} h_i(\theta, x)\, d\pi(\theta, x) < 0$ for some $h \in H_{X,0}$, then an external trader with beliefs π can make a strictly positive expected profit by offering h to the agents

Theorem

Fix $\pi \in \Delta(\Theta \times X)$. The following are equivalent:

(i) *π is CP-X-consistent*

(ii) *π satisfies no cautious money pump*

- **Remark**: novel characterization for BCE in incomplete info games

Sketch of the proof: Strassen 65

- (i) $\Longrightarrow$ (ii) If π is CP-consistent π, then there exists a regular conditional probability $(\pi_{x_i})_{x_i \in X_i}$ of π such that $\pi_{x_i} \in \Delta(x_i)$ for π-almost all x_i

- For every $i \in N$, we then have

$$\int_{\Theta \times X} h_i(\theta, x)\, d\pi(\theta, x) \geq \inf\{\mathbb{E}_\gamma[h_i(x_i, \cdot)] : \gamma \in \Delta_i(x_i)\}$$

- Next, fix $h \in H_0$ and observe that

$$\sum_{i \in N} \int_{\Theta \times X} h_i(\theta, x)\, d\pi(\theta, x) \geq \sum_{i \in N} \int_{X_i} \xi_i(h_i)(x_i)\, d\mathrm{marg}_{X_i}\pi(x_i) = 0$$

proving the implication

Sketch of the proof: Strassen 65

- (ii) $\implies$ (i) Fix $h_i \in C(\Theta \times X)$ and define

$$\hat{h}_i(\theta, x) = h_i(\theta, x) - \xi_i(h_i)(x_i) \qquad \forall i \in N$$

- Next argue that $\hat{h} \in H_0$ and apply no cautious money pump to conclude that

$$\int_{\Theta \times X} h_i(\theta, x)\, d\pi(\theta, x) \geq \int_{X_i} \xi_i(h_i)(x_i)\, d\mathrm{marg}_{X_i}\pi(x_i) \qquad \forall i \in N$$

- Finally, Theorem 3 in Strassen 65 implies that, for every $i \in N$, there exists a regular conditional probability $\pi_{x_i} \in \Delta(x_i)$ for π-almost all x_i

Simpler characterization for linear coarsenings

- $E \subseteq \Theta \times X$ is an i-event if $E = E_i \times E_{-i}$ for some $E_i \subseteq X_i$ and $E_{-i} \subseteq \Theta \times E_{-i}$

Theorem

Let $(\Delta_i)_{i \in I}$ be linear and fix $\pi \in \Delta(\Theta \times X)$. The following are equivalent:

(i) *π is CP-consistent*

(ii) *For every $i \in N$ and every i-event $E = E_i \times E_{-i}$, we have*

$$\pi(E_i \times E_{-i}) \geq \int_{E_i} \min\{\gamma(E_{-i}) : \gamma \in \Delta_i(x_i)\}\, d\pi_i(x_i)$$

- **Sketch**: The proof is similar to the previous one by replacing Theorem 4 of Strassen to his Theorem 3
  - The non-trivial part is to show that, for every $x_i \in X_i$, the set-function

$$E_{-i} \mapsto \min\{\gamma(E_{-i}) : \gamma \in \Delta_i(x_i)\}$$

is supermodular in the inclusion order, which is necessary to invoke Strassen's result

CP-consistent supports

- What coarsened types (e.g. actions) are consistent with the common prior assumption?

Corollary

Fix a compact $S \subseteq \Theta \times X$. The following are equivalent:

(i) *There exists a CP-consistent $\pi \in \Delta(\Theta \times X)$ such that $\pi(S) = 1$*

(ii) *For every $h \in H_{X,0}$, we have*

$$\sup_{(\theta,x)\in S} \sum_{i\in N} h_i(\theta, x) \geq 0$$

- **Sketch**: Point (i) can be expressed as a maxmin problem, then use Sion (compactness of S) to obtain result

CP-consistent marginals

- Define

$$\mathcal{S}_X = \left\{ (\phi, \psi) \in C(\Theta) \times C(X) : \exists h \in H_{X,0}, \phi + \psi \geq \sum_{i \in N} h_i \right\}$$

Corollary

Fix $\mu \in \Delta(\Theta)$ and $\nu \in \Delta(X)$. The following are equivalent:

(i) *There exists a CP-consistent $\pi \in \Delta(\Theta \times X)$ such that $marg_\Theta \pi = \mu$ and $marg_X \pi = \nu$*

(ii) *For every $(\phi, \psi) \in \mathcal{S}_X$, we have*

$$\int_\Theta \phi(\theta)\, d\mu(\theta) + \int_T \psi(x)\, d\nu(x) \geq 0$$

- No-arbitrage interpretation as before

Common-prior-free characterization

- In particular, $\nu \in \Delta(X)$ is CP-consistent for some $\mu \in \Delta(\Theta)$ if and only if

$$\int_X \max_{\theta \in \Theta} \left[\sum_{i \in N} h_i(\theta, x) \right] d\nu(x) \geq 0$$

- Generalizes the main result in Arieli et al. (2021) to continuous states and arbitrary (coarsened) type spaces

Illustration: first-order expectations

- Let $\Theta = X_i = [0, 1]$ and consider the first-order expectation coarsening with

$$\Delta_i(x_i) = \left\{\gamma \in \Delta(\Theta \times X_{-i}) : \mathbb{E}_\gamma\left[\tilde{\theta}\right] = x_i\right\}$$

- A sufficient class of cautiously zero-value trades is given by

$$h_i(\theta, x) = q_i(x_i)(\theta - x_i) \qquad q_i \in C(A_i)$$

- Obtain result in Arieli et al (2021): $\nu \in \Delta(X)$ is CP-consistent for some $\mu \in \Delta(\Theta)$ if and only if

$$\int_X \left\{\sum_{i \in N} q_i(x_i)\, x_i - \left[\sum_{i \in N} q_i(x_i)\right]^+\right\} d\nu(x) \leq 0$$

for all $q_i \in C([0, 1])$ and $i \in N$

Illustration: smooth incomplete information games

- Consider an incomplete information game with $\Theta = [0,1]$, $A_i = [0,1]$ and payoff functions U_i are smooth and strictly concave in a_i
- The belief-correspondence is

$$\Delta_i(a_i) = \left\{ \gamma \in \Delta(\Theta \times A_{-i}) : \mathbb{E}_\gamma \left[\frac{\partial}{\partial a_i} U_i(a_i, \cdot) \right] = 0 \right\}$$

- A sufficient class of cautiously zero-value trades is given by

$$h_i(\theta, a) = q_i(a_i) \frac{\partial}{\partial a_i} U_i(\theta, a) \qquad q_i \in C(A_i)$$

- Interpretation: trades are proportional to the marginal utility of the players (cf. Nau and McCardle 90 characterization of correlated equilibrium)

Illustration: smooth incomplete information games

Corollary

The distribution over actions $\nu \in \Delta(A)$ *is a BCE for some common prior* $\mu \in \Delta(\Theta)$ *if and only if*

$$\int_A \max_{\theta \in \Theta} \left\{ \sum_{i \in N} q_i(a_i) \frac{\partial}{\partial a_i} U_i(\theta, a) \right\} d\nu(a) \geq 0$$

for every $(q_i)_{i \in N} \in \prod_{i \in N} C(A_i)$.

- If the marginal utility of every i is affine $\frac{\partial}{\partial a_i} U_i(\theta, a) = \theta - \beta_i(a)$, then the previous condition becomes

$$\int_A \left\{ \sum_{i \in N} q_i(a_i) \beta_i(a) - \left[\sum_{i \in N} q_i(a_i) \right]^+ \right\} d\nu(a) \geq 0$$

generalizing Arieli et al (2021) to incomplete information games

Implications for the single-receiver case

- Assume that $N = \{i\}$. For every $f \in C(\Theta)$, define the U-concavification of f as

$$f^U(a) = \max_{\lambda,\bar{\theta},\underline{\theta}:\lambda \frac{\partial}{\partial a} U(\bar{\theta},a)+(1-\lambda)\frac{\partial}{\partial a} U(\underline{\theta},a)} \left\{\lambda f(\bar{\theta}) + (1-\lambda) f(\underline{\theta})\right\} \qquad \forall a \in A$$

Corollary

Fix $\mu \in \Delta(\Theta)$. The distribution over actions $\nu \in \Delta(A)$ is implementable by an information structure if and only if

$$\int_A f^U(a)\, d\nu(a) \geq \int_\Theta f(\theta)\, d\mu(\theta) \qquad \forall f \in C(\Theta)$$

- For $\frac{\partial}{\partial a} U(\theta, a) = (\theta - a)$ this reduces to standard convex ordering $\mu \succsim_{cvx} \nu$
- Corrao, Wolitzky, and Kolotilin (2021): use duality approach to solve the single-receiver persuasion problem

Conclusion and future research

- Provided a no-trade characterization of feasible distributions over higher-order beliefs under CP
- Introduced language of coarsened type space and characterized CP-implications
- This allowed to unify and revisit several scattered results in information design and information economics
- Propose a dual approach to implementation and optimal design of information
- **Future research:** The (simple) math trick was to express conditional moments conditions in terms of unconditional ones (Econometricians know better)
- Same trick can be used to characterize other conditional moment conditions:
  - Truthful reporting in communication equilibria and mechanism design
  - Inscrutability principle in mechanism design with informed principal
  - REE and Self-confirming equilibrium

A differential characterization

- Define the cost function $R : \Delta(\Theta) \times \Delta(X) \to \overline{\mathbb{R}}_+$

$$R(\mu, \nu) = - \inf_{h_0 \in H_{X,0}} \sup_{\pi \in \Delta(\mu,\nu)} \left\{ \int_{\Theta \times X} \sum_{i \in N} h_i(\theta, x) \, d\pi(\theta, x) \right\}$$

- **Interpretation**: Capture a measure of "distance" between μ and ν with respect to the cautiously zero-value trades $h \in H_{X,0}$
- Define the operators

$$I_\mu(\psi) = \min_{\nu \in \Delta(X)} \left\{ \int_X \psi(x) \, d\nu(x) + R(\mu, \nu) \right\} \qquad \forall \psi \in C(X)$$

and

$$I_\nu(\phi) = \min_{\mu \in \Delta(\Theta)} \left\{ \int_\Theta \phi(\theta) \, d\mu(\theta) + R(\mu, \nu) \right\} \qquad \forall \phi \in C(\Theta)$$

A differential characterization

- These operators evaluate independent bets ψ and ϕ under the worst possible distributions with higher penalization for those that are "distant" from μ and ν
- Decision theory under uncertainty: I_μ and I_ν represent variational preferences
- These operators are concave and 1-Lipschitz continuous

Corollary

We have:

$$\{\nu \in \Delta(X) : \textit{outcomes } \nu \textit{ consistent with common prior } \mu\} = \partial I_\mu(0)$$

and

$$\{\mu \in \Delta(\Theta) : \textit{common priors } \mu \textit{ consistent with outcome } \nu\} = \partial I_\nu(0)$$

Extreme points

- Let $\Delta^X_{CP}(\mu)$ denote the set of CP-consistent $\pi \in \Delta(\Theta \times X)$ such that $\text{marg}_\Theta \pi = \mu$

Theorem

Fix $\mu \in \Delta(\Theta)$ and define

$$\hat{H}_{X,0} = \left\{ \phi + \sum_{i \in N} h_i \in C(\Theta \times X) : \phi \in C(\Theta), h \in H_{X,0} \right\}.$$

The following are equivalent:

(i) *π is an extreme point of $\Delta^X_{CP}(\mu)$*

(ii) *$\text{marg}_\Theta \pi = \mu$ and $\hat{H}_{X,0}$ is dense in $\mathcal{L}_1(\pi)$*

Version of conditional probability

- We say that $g_i : T_i \to \Delta(\Theta \times T_{-i})$ is a version of the conditional probability of $\pi \in \Delta(\Theta \times T)$ given T_i if

$$\int_{\Theta \times T} h(\theta, t)\, d\pi(\theta, t) = \int_{T_i} \left[\int_{\Theta \times T_{-i}} h(\theta, t)\, dg(t_i)(\theta, t_{-i}) \right] d\mathrm{marg}_{T_i}(\pi)(t_i) \tag{2}$$

for all $i \in N$ and all $h \in C(\Theta \times T)$.

Moment conditions and information design

- Consider a (possibly multidimensional) bounded objective function $V : \Theta \times T \to \mathbb{R}$
- Recall that $\Delta_{CP}(\mu) = \{\pi \in \Delta(\Theta \times T) :$ for some $\mathcal{I} = (S, \sigma)\}$
- We aim to characterize the set of feasible moments:

$$\mathcal{V}_{CP}(\mu) = \left\{\mathbb{E}_{\mu}[V] \in \mathbb{R} : \pi \in \Delta_{CP}(\mu)\right\} \subseteq \mathbb{R}$$

Theorem

Fix $v \in \mathbb{R}^m$. The following are both equivalent to $v \in \mathcal{V}_{CP}(\mu)$:

(i) *For every $h \in H_0$, there exist $\lambda \in [0, 1]$ and $(\theta_0, t_0), (\theta_1, t_1) \in \Theta \times T$ such that the collections of vectors $V(\theta_0, t_0) \neq V(\theta_1, t_1)$ and*

$$\begin{aligned} \lambda V(\theta_0, t_0) + (1 - \lambda) V(\theta_1, t_1) &= v, \\ \lambda h(\theta_0, t_0) + (1 - \lambda) h(\theta_1, t_1) &\geq 0. \end{aligned}$$