Nonlinear Fixed Points and Stationarity: Economic Applications*

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Abstract

We consider the fixed points of nonlinear operators that naturally arise in games and general equilibrium models with endogenous networks, dynamic stochastic games, in models of opinion dynamics with stubborn agents, and financial networks. We study limit cases that correspond to high coordination motives, infinite patience, vanishing stubbornness and small exposure to the real sector in the applications above. Under monotonicity and continuity assumptions, we provide explicit expressions for the limit fixed points. We show that, under differentiability, the limit fixed point is linear in the initial conditions and characterized by the Jacobian of the operator at any constant vector with an explicit and linear rate of convergence. Without differentiability, but under additional concavity properties, the multiplicity of Jacobians is resolved by a representation of the limit fixed point as a maxmin functional evaluated at the initial conditions. In our applications, we use these results to characterize the limit equilibrium actions, prices, and endogenous networks, show the existence and give the formula of the asymptotic value in a class of zero-sum stochastic games with a continuum of actions, compute a nonlinear version of the eigenvector centrality of agents in networks, and the characterize the equilibrium loss evaluations in financial networks.

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1 Introduction

Nonlinear fixed-point equations are ubiquitous in economic models including the ones that characterize general equilibrium prices, Nash equilibria, continuation values in dynamic games (Shapley equation), steady states under social learning, recursive preferences, and equilibrium loss evaluations in financial networks. Often, these fixed points are indexed by a key economic parameter $\beta \in (0, 1)$ capturing, for example, strength of coordination motives, patience, and stubbornness, with the comparative statics for β close to 1 playing a prominent role. The problem of solving for these nonlinear fixed points has been tackled with different tools across these applications without a unifying approach.

In this paper, we first highlight a few key mathematical properties shared by all these classes of nonlinear fixed-point equations: monotonicity, translation invariance, and normalization. These properties generalize the ones of linear averaging operators for which the structure of corresponding fixed-point equations is well known. In fact, for the linear case it is in general possible to derive a closed-form expression for the fixed point at each β and in particular for the limit as β goes to 1, yielding a rate of convergence as well. These expressions are often interpreted as (Bonacich or eigenvector) centrality measures of agents within the context of, for example, models of production networks (e.g., Long and Plosser [33]) or coordination games (e.g., Ballester et al. [6]) and social learning on networks (Golub and Jackson [25]).

However, in all the aforementioned applications nonlinearities naturally arise due to economic forces. For example, in production network models both relaxing the assumption of Cobb-Douglas production functions (e.g., Baqaee and Farhi [8]) and/or allowing for endogenous networks (e.g., Acemoglu and Azar [1] and Kopytov et al. [32]) generate nonlinearities in the equation describing equilibrium prices. Similarly, in coordination games on networks when we relax the assumption of quadratic payoffs and/or allow for endogenous link formation (e.g., Sadler and Golub [38]) the resulting Nash equilibria are characterized by nonlinear fixed points.

In models of non Bayesian social learning, as soon as we move from the simple DeGroot heuristic to the class of richer models proposed by Cerreia-Vioglio et al. [12], nonlinearities in aggregation arises. Similarly, regulation requires banks not to evaluate loss at (the linear) expected value, but using robust scenario-conditional loss assessments, as the one considered in Adrian and Brunnermeier [3].

Moreover, in some other applications such as stochastic games and recursive preferences the maximization defining the value functions already induces nonlinearities (e.g., Sorin [42]). Yet, in all these cases our three properties are still satisfied. Thus, we exploit this common structure to derive properties of the nonlinear fixed point the most important of which is a closed-form expression for the limit as β approaches 1.

These expressions admit a natural interpretation as nonlinear versions of the (linear) centrality measures above. Along the way, we derive additional results extending the conclusions obtained for the linear case.

Formally, in this paper we consider an operator $T : \mathbb{R}^k \to \mathbb{R}^k$ where $\mathbb{R}^k$ is endowed with the supnorm $\| \|_\infty$. Let e be the vector whose components are all 1. We assume that:

1. T is *normalized*, that is, $T(he) = he$ for all $h \in \mathbb{R}$;
2. T is *monotone*, that is, $x \geq y$ implies $T(x) \geq T(y)$ for all $x, y \in \mathbb{R}^k$;
3. T is *translation invariant*, that is, $T(x + he) = T(x) + he$ for all $x \in \mathbb{R}^k$ and for all $h \in \mathbb{R}$.

As we already pointed out, these three properties are often satisfied in applications in Economics and Computer Science where T is seen as either a best-response map, or a value function, or an opinion aggregator, or a robust risk measure. Clearly, for these maps the set of fixed points/equilibria of T, denoted by $E(T)$, contains all the constant vectors, denoted by D, that is, $D \subseteq E(T)$.

In these applications, the interest is in the following fixed points equations (with variable y). Given $x \in \mathbb{R}^k$ and $\beta \in (0,1)$,

$$T((1-\beta)x + \beta y) = y \tag{1}$$

and

$$(1-\beta)x + \beta T(y) = y \tag{2}$$

it is routine to show that the two equations have each a unique solution (cf. Lemma 1). We denote such solutions by x_β and $\tilde{x}_\beta$, respectively, to highlight their dependence on x and β. The goal of this paper is to provide conditions that guarantee that $\lim_{\beta \to 1} x_\beta$ and $\lim_{\beta \to 1} \tilde{x}_\beta$ exist, characterize their value, and also comment on the rate of convergence. We prove and state our results for x_β, the solution of equation (1), but we also show that the results immediately extend to $\tilde{x}_\beta$ (cf. Remark 1).

We next introduce the linear case which is well known. This will provide a useful benchmark to which we can compare our contributions.

Example 1 We begin by observing that further assuming T linear is equivalent to impose that $T(x) = Wx$ for all $x \in \mathbb{R}^k$ where W is a (row)-stochastic matrix. Given $x \in \mathbb{R}^k$ and $\beta \in (0,1)$, let $x_{\beta,W}$ be the (unique) vector satisfying:

$$T((1-\beta)x + \beta x_{\beta,W}) = x_{\beta,W}. \tag{3}$$

By induction and passing to the limit, it is routine to show that

$$x_{\beta,W} = (1-\beta)\sum_{t=0}^{\infty}\beta^t W^{t+1}x \qquad \forall x\in\mathbb{R}^k, \forall\beta\in(0,1)\,. \tag{4}$$

By the Hardy-Littlewood Theorem paired with the Mean Ergodic Theorem, this implies $\lim_{\beta\to1} x_{\beta,W}$ exists and belongs to $E(T)$.

Assume now that the unique fixed points of T are the constant vectors, that is, $D = E(T)$. This is equivalent to assume that the matrix W has a unique left Perron-Frobenius eigenvector γ_W, that is, $\gamma_W^{\mathrm{T}}W = \gamma_W^{\mathrm{T}}$ and γ_W is a probability vector. In this case, we can conclude that $\lim_{\beta\to1} x_{\beta,W}$ is a constant vector whose value can be computed by observing that

$$\langle\gamma_W, x_{\beta,W}\rangle = (1-\beta)\sum_{t=0}^{\infty}\beta^t\left\langle\gamma_W, W^{t+1}x\right\rangle = \langle\gamma_W, x\rangle \qquad \forall x\in\mathbb{R}^k, \forall\beta\in(0,1)\,. \tag{5}$$

We conclude by commenting on the rate of convergence of $\{x_{\beta,W}\}_{\beta\in(0,1)}$. Fix again $x\in\mathbb{R}^k$ and $\beta\in(0,1)$. Since γ_W is a probability vector, we have that $\min x_{\beta,W} \leq \langle\gamma_W, x_{\beta,W}\rangle \leq \max x_{\beta,W}$. Since $\langle\gamma_W, x_{\beta,W}\rangle = \langle\gamma_W, x\rangle$, this implies that $\min x_{\beta,W} \leq \langle\gamma_W, x\rangle \leq \max x_{\beta,W}$ and, in particular, $\|x_{\beta,W} - \langle\gamma_W, x\rangle e\|_\infty \leq \max x_{\beta,W} - \min x_{\beta,W}$. In other words, bounding the rate of convergence of $x_{\beta,W}$ can be achieved by bounding the range of $x_{\beta,W}$. ▲

Thus, the main takeaways of the linear case are three:

1. $\lim_{\beta\to1} x_\beta$ exists;
2. If $E(T) = D$, then we have that

   $$\lim_{\beta\to1} x_\beta = \langle\gamma_W, x\rangle e \qquad \forall x\in\mathbb{R}^k$$

   where γ_W is the unique left Perron-Frobenius eigenvector of the representing matrix W;
3. In this case, the rate of convergence of x_β is controlled by the rate to which the range of x_β, $\mathrm{Rg}(x_\beta)$, goes to 0.

Our contributions are to generalize these findings well beyond the linear case. We here discuss an important example. To fix ideas, assume that T is *concave*, rather than linear. If $E(T) = D$, we again have that $\lim_{\beta\to1} x_\beta$ exists (cf. Theorem 2). If $E(T) = D$ and T is also differentiable around 0 with partial derivatives that are "nicely" bounded away from 0 when nonnull, then

$$\lim_{\beta\to1} x_\beta = \langle\gamma, x\rangle e \qquad \forall x\in\mathbb{R}^k$$

where γ is the unique left Perron-Frobenius eigenvector of the Jacobian of T at 0 (cf. Corollary 1). Finally, if T has a Jacobian which is Lipschitz continuous, then the rate of convergence of x_β is controlled by the rate to which $\mathrm{Rg}\,(x_\beta)$ goes to 0 and $\mathrm{Rg}\,(x_\beta)$ goes to 0 at least linearly fast (cf. Theorems 4 and 5). In the paper, we go well beyond the concave case, which we actually use to study more general functionals (cf. Propositions 1 and 1).

In the second part of the paper we provide economic applications for these results. First, we consider two models of endogenous network formation applied to general equilibrium in a production economy and a coordination game. In both cases, the parameter β captures the intrinsic coordination motives of the agents and, under the assumptions of Cobb-Douglas production functions and quadratic costs of effort, the fixed-point equations characterizing the equilibria are linear. However, when the agents are allowed to choose their neighbors structure, either in a costly or constrained way, the equilibria fixed-point equations become nonlinear (and in general nondifferentiable) yet still satisfying all of our assumptions. With this, we completely characterize the limit equilibrium as $\beta \to 1$ with respect to (generally nonlinear) measures of centrality of the agents. This allows us to extend some of the comparative statics on the equilibrium from the linear case to the differentiable case and to obtain new ones for the nondifferentiable case. Second, we study the classic issue of existence and characterization of the asymptotic value for zero-sum stochastic games (cf. Sorin [42]). We observe that the Shapley equation characterizing the value of the game for every level of the discount factor is a particular case of our fixed point condition, thereby enabling us to use our abstract results to provide a novel characterization of the asymptotic value in terms of the value of a static zero-sum game. We then apply our results to an extension of the dynamic opinion aggregation model in networks of Cerreia-Vioglio et al. [12] that allow for vanishing stubbornness. Finally, we consider an equilibrium model of interconnected financial institutions that evaluate their losses with respect to coherent risk measures. In this case, the limit $\beta \to 1$ captures the idea of increasing financial interconnectedness and our results imply that the robustness concerns of the banks vanish in this limit, exposing all of them to possible model misspecification and large unforeseen losses.

2 Convergence and limit characterization

In this section, we provide our main results on the existence and characterization of the limit fixed point. Consider $x \in \mathbb{R}^k$ and $\beta \in (0,1)$, the next lemma–a routine application of the Banach contraction principle–shows that (1) admits a unique solution, denoted by x_β.

Lemma 1 *Let T be normalized, monotone, and translation invariant, $\beta \in (0,1)$, and $x \in \mathbb{R}^k$. There exists unique $x_\beta \in \mathbb{R}^k$ such that*

$$T\left((1-\beta)x + \beta x_\beta\right) = x_\beta.$$

Differentiability Our first result covers the case of continuously differentiable operators. It is easy to see that, for every normalized, monotone, and translation invariant operator T that is differentiable at $z \in \mathbb{R}^k$, the Jacobian of T at z is a stochastic matrix, hence nonnegative. We say that a nonnegative matrix $M \in \mathbb{R}_+^{k\times k}$ is regular if and only if it is nontrivial and its essential indices form a single essential class.[1] The positive entries of M represent a directed graph over k nodes and, if this graph is strongly connected, M is regular.

Theorem 1 *Let T be normalized, monotone, and translation invariant. If T is continuously differentiable in a neighborhood of 0 and the Jacobian of T at 0 is regular then*

$$\lim_{\beta\to 1} x_\beta = \langle \gamma, x\rangle e \qquad \forall x \in \mathbb{R}^k$$

where γ is the unique left Perron-Frobenius eigenvector of the Jacobian of T at 0.

The left Perron-Frobenius eigenvector of a linear operator is often interpreted as a centrality measures (cf. eigenvector centrality, Jackson [29]) for the nodes $\{1, ..., k\}$, an interpretation justified by the considerations in Example 1. Theorem 1 extends this interpretation to the (differentiable) nonlinear case and identifies the relevant matrix for which the eigenvector centrality must be computed: the Jacobian of T at 0.

Concavity We can dispense with the assumption of differentiability, if we impose concavity. To this extent, we introduce some notation and terminology. We denote by $\partial T : \mathbb{R}^k \rightrightarrows \mathbb{R}^k$ the superdifferential correspondence.[2] Similarly to the differentiable case, for every normalized, monotone, translation invariant, and concave operator T and $z \in \mathbb{R}^k$, $\partial T(z)$ is a set of stochastic matrices.

Let $\mathcal{W}$ be the set of $k \times k$ stochastic matrices. Given $W \in \mathcal{W}$, we denote by

$$\Gamma(W) = \left\{\gamma \in \Delta : \gamma^{\mathrm{T}} W = \gamma^{\mathrm{T}}\right\}$$

[1] Nontriviality means that M does not have a zero row. We report the standard definitions of essential index and essential class in Appendix A.1.

[2] Given a function $\varphi : \mathbb{R}^k \to \mathbb{R}$ and $z \in \mathbb{R}^k$, recall that a vector γ is an element of $\partial\varphi(z)$, that is, a superdifferential of φ at z if and only if $\langle \gamma, y - z\rangle \geq \varphi(y) - \varphi(z)$ for all $y \in \mathbb{R}^k$. We denote by $\partial T(z)$ the collection of all $k \times k$ matrices whose i-th row belongs to $\partial T_i(z)$.

the collection of all left W-invariant probability vectors. It is routine to show that $\Gamma(W)$ is nonempty, convex, and compact. If W has a unique left Perron-Frobenius eigenvector γ_W, that is, W is regular, then $\Gamma(W) = \{\gamma_W\}$. Given a subset $\mathcal{M} \subseteq \mathcal{W}$, we denote by $\Gamma(\mathcal{M})$ the set $\cup_{W \in \mathcal{M}} \Gamma(W)$. In particular, if $\mathcal{M}$ is closed, then $\Gamma(\mathcal{M})$ is compact.

Theorem 2 *Let T be normalized, monotone, and translation invariant. If T is concave and $E(T) = D$, then*

$$\lim_{\beta \to 1} x_\beta = \min_{\gamma \in \Gamma(\partial T(0))} \langle \gamma, x \rangle e \qquad \forall x \in \mathbb{R}^k.$$

Moreover, if all the elements of $\partial T(0)$ are regular, then $\Gamma(\partial T(0))$ is the collection of left Perron-Frobenius eigenvectors of the superdifferential of T at 0.

Differently from the differentiable case, here the limit fixed point is not necessarily a linear function of x. Still, the key objects to characterize its value remain the (collection of) left Perron-Frobenius eigenvectors of the superdifferential of T at 0. Thus, the set $\Gamma(\partial T(0))$ can be interpreted as a collection of eigenvector centralities associated with the nonlinear operator T. Moroever, concavity of T implies that this multiplicity is resolved by the minimum evaluation of x across all the eigenvector centralities.

The condition $E(T) = D$ may seem hard to check, but the following sufficient condition turns out to be useful in the applications.

Lemma 2 *Let T be normalized, monotone, and translation invariant. If T is concave and there exists $W \in \partial T(0)$ such that W is strongly connected, then $E(T) = D$.*

Concavity allows also to improve Theorem 1. In fact, we only need differentiability at 0 without explicitly asking the derivative to be continuous near 0.

Corollary 1 *Let T be normalized, monotone, and translation invariant. If T is concave, differentiable at 0 and the Jacobian of T at 0 is regular then*

$$\lim_{\beta \to 1} x_\beta = \langle \gamma, x \rangle e \qquad \forall x \in \mathbb{R}^k$$

where γ is the unique left Perron-Frobenius eigenvector of the Jacobian of T at 0.

Nice star-shaped operators In certain applications, the operator T is neither continuously differentiable nor concave: for example, the Shapley operator in Zero-sum stochastic games (cf. Section 4), which characterizes the value of the game, has a maxmin structure. Yet, in these applications, the operator T is defined as the maximum of a collection of concave operators, which is the case we analyze next.

Formally, consider a family of operators $\{S_\alpha\}_{\alpha\in\mathcal{A}}$ each of which satisfies the assumptions of Theorem 2. Define $T : \mathbb{R}^k \to \mathbb{R}^k$ by[3]

$$T(x) = \sup_{\alpha\in\mathcal{A}} S_\alpha(x).$$

It is immediate to show that T is normalized, monotone, and translation invariant. We say that $\{S_\alpha\}_{\alpha\in\mathcal{A}}$ is *nice* if and only if the previous sup is achieved for all $x \in \mathbb{R}^k$, that is, for each $x \in \mathbb{R}^k$ there exists $\alpha_x \in \mathcal{A}$ such that $T(x) = S_{\alpha_x}(x)$.

Given $x \in \mathbb{R}^k$, $\beta \in (0,1)$, and $\alpha \in \mathcal{A}$, denote by $x_{\beta,\alpha}$ the unique point satisfying $S_\alpha((1-\beta)x + \beta x_{\beta,\alpha}) = x_{\beta,\alpha}$. For each $\alpha \in \mathcal{A}$, define $\varphi_\alpha : \mathbb{R}^k \to \mathbb{R}$ by

$$\varphi_\alpha(x) = \min_{\gamma\in\Gamma(\partial S_\alpha(0))} \langle\gamma, x\rangle \qquad \forall x \in \mathbb{R}^k.$$

As shown by Theorem 2, $\varphi_\alpha(x)e = \lim_{\beta\to1} x_{\beta,\alpha}$ for all $x \in \mathbb{R}^k$ and for all $\alpha \in \mathcal{A}$. The next result shows that when $\{S_\alpha\}_{\alpha\in\mathcal{A}}$ is nice, $\lim_{\beta\to1} x_\beta$, defined for the operator T, exists and is given by the sup of the evaluations $\{\varphi_\alpha(x)\}_{\alpha\in\mathcal{A}}$.

Proposition 1 *If* $\{S_\alpha\}_{\alpha\in\mathcal{A}}$ *is nice, then* $\lim_{\beta\to1} x_\beta = \sup_{\alpha\in\mathcal{A}} \varphi_\alpha(x)e$.

The importance of the above result is twofold. First, in some applications, a collection $\{S_\alpha\}_{\alpha\in A}$ forms the primitives of the problem and the operator T is derived from it (see, e.g., the Shapley operator in Section 4). Second, Proposition 1 will be useful in proving a result about the rate of convergence (Proposition 6).

In the next sections, we consider several applications of the results above. We start with an application to endogenous network formation in general equilibrium and coordination games models. Here, the limit $\beta \to 1$ captures the idea that the importance of own idiosyncratic factors becomes negligible compared to that of the external inputs or coordination with coplayers, respectively. We next move to zero-sum stochastic games, where $\beta \to 1$ is interpreted as the limit for infinite patience of the players. Finally, we consider an application to an equilibrium model of interconnected financial institutions that use coherent risk measure to evaluate the riskiness of their positions. Here, the limit $\beta \to 1$ captures the idea that the institutions are highly interconnected and that the financial sector dominates the underlying real one. Some of these applications will feature the alternative fixed point equation (2). The next remark explains why our results also apply to that case.[4]

[3] For a given collection of vectors $\{z_\alpha\}_{\alpha\in\mathcal{A}} \subseteq \mathbb{R}^k$, we let $\sup_{\alpha\in\mathcal{A}} z_\alpha \in \mathbb{R}^k$ denote the supremum with respect to the coordinatewise order. Therefore,

$$T_i(x) = \sup_{\alpha\in\mathcal{A}} S_{\alpha,i}(x) \qquad \forall x \in \mathbb{R}^k, \forall i \in \{1, ..., k\}.$$

[4] In addition, some of the applications will feature convex rather concave operators. It is easy to see that Theorem 2 holds as written, replacing concave with convex and min with max.

Remark 1 *Let T be normalized, monotone, and translation invariant and consider $x \in \mathbb{R}^k$ and $\beta \in (0,1)$. As we mentioned, many applications in Network Theory (e.g., Sections 3 and 5) feature the alternative fixed point $\tilde{x}_\beta$ of equation (2). This alternative fixed point can be directly computed once we have x_β. In particular, we have that $\tilde{x}_\beta = (1-\beta)x + \beta x_\beta$. Moreover, if $\lim_{\beta \to 1} x_\beta$ exists, then $\lim_{\beta \to 1} \tilde{x}_\beta$ exists and $\lim_{\beta \to 1} x_\beta = \lim_{\beta \to 1} \tilde{x}_\beta$.*

3 Application I: Endogenous networks

In this section, we consider general equilibria and coordination games on networks with endogenous links. In both cases, when we fix a network structure, the induced equilibrium map is linear, a feature highly exploited in the literature of production networks (e.g., Long and Plosser [33] and Acemoglu et al. [2]) and coordination games (e.g., Ballester et al. [6] and Golub and Morris [26]). However, the endogeneity of the network structure introduces nonlinearities in the equilibrium map, thereby complicating the equilibrium analysis. The nonlinear fixed point equation that we have studied in Section 2 implicitly defines the equilibrium maps of both applications, allowing us to characterize it for high-coordination motives among firms and players respectively.

3.1 Production networks

Following Acemoglu and Azar [1] and Kopytov et al. [32], we consider a static and frictionless model of *production network* among cost-minimizing firms with Cobb-Douglas production functions and endogenous networks. We completely characterize the equilibrium prices and outputs as the relative importance of the firms' idiosyncratic shocks vanish.

Consider a finite set of firms $\{1, ..., k\}$ each of which produces a potentially different output. Firm i can choose a set of weights $w_i \in \Delta$ specifying both the set of inputs from the other firms that are used in production and how these inputs are to be combined. Moreover, each firm uses an external input that is irreproducible by any other firm and whose productivity and importance in the production function are fixed. This can be either labor or another factor that is produced outside the economy we analyze.

Following [32], we fix a productivity shifter $S_i : \Delta \to [0,1]$ that depends on the technology w_i selected. Given the level of inputs from the external factor and from the firms in the economy $Q_i = \left(Q_{i0,} (Q_{ij})_{j=1}^k\right) \in \mathbb{R}_+^{k+1}$ and technology $w_i \in \Delta$, the production function of firm i is

$$F_i(Q_i, w_i) = S_i(w_i)\,\xi(\beta, w_i)\,(Z_i Q_{i0})^{(1-\beta)} \prod_{j=1}^{k} Q_{ij}^{\beta w_{ij}}$$

where $Z_i > 0$ is the productivity relative to the external factor for firm i, $\beta \in (0,1)$ is the common intensity of the external factor, and $\xi(\beta, w_i)$ is a normalization constant that only depends on the overall technology (β, w_i) of firm i.[5] Each firm selects both a technology $w_i \in \Delta$ and levels of all inputs Q_i needed given the technology selected. For example, if $w_{ij} = 0$ then the input from j is not relevant for i's production. Let $\mathcal{W}^*$ denote the set of strongly connected stochastic matrices. We maintain the following assumptions on the productivity shifters.

Assumption: The profile of productivity shifters $S = (S_i)_{i=1}^k$ is such that each S_i is upper semicontinuous, log-concave, and there exists $W \in \mathcal{W}^*$ such that

$$S_i(w_i) = 1 \quad \forall i \in \{1, .., k\}. \tag{6}$$

Upper semicontinuity and log-concavity are technical conditions that guarantees existence of equilibrium and are always imposed in this literature, often times in the stronger form of continuity and strict log-concavity. The last part of the assumption involves the set of most efficient technologies. For a given S, define the set

$$\operatorname{argmax}(S) = \{W \in \mathcal{W} : S_i(w_i) = 1 \quad \forall i \in \{1, .., k\}\}.$$

The production networks $W \in \operatorname{argmax}(S)$ are the most efficient ones since the production of each firm is not being shifted down by a discount factor. Equation (6) says that there exist efficient technologies and that at least one efficient technology induces a strongly connected network. Instead, all those production networks $W \in \mathcal{W}$ such that $S_i(w_i) = 0$ for some $i \in \{1, ..., k\}$ are either extremely inefficient or unfeasible. The next examples illustrate natural settings where our assumption is satisfied.

Example 2 When all the feasible technologies are efficient, we have that the productivity shifter of each i is $S_i = 1_{C_i}$ an indicator function over a nonempty, convex, and compact set $C_i \subseteq \Delta$ of technologies. In this case, $\operatorname{argmax}(S)$ is the set of all stochastic matrices whose i-th row belongs to C_i. The condition in equation (6) implies that at least one feasible configuration is strongly connected. When each C_i is a singleton, $\{w_i^0\}$, for some $W^0 \in \mathcal{W}^*$, we have that $\operatorname{argmax}(S) = \{W^0\}$, that is, the production network is exogenously fixed and we get back the standard Cobb-Douglas model of Long and Plosser [33] and Acemoglu et al. [2]. Differently, Kopytov et al. [32] consider

[5] The normalization constant

$$\xi_i(\beta, w_i) = (1-\beta)^{-(1-\beta)} \prod_{j=1}^{k} (\beta w_{ij})^{-\beta w_{ij}}$$

is the same as the one [32] (see their Footnote 8). Differently form [32], our productivity shock Z_i is relative to the external factor Q_{i0} as opposed to be Hicks-neutral.

a continuously differentiable and strictly log-concave function productivity shifter. In [32] the leading example is

$$S_i(w_i) = \exp\left(-\sum_{j=1}^{k} \kappa_{ij} \left(w_{ij} - w_{ij}^0\right)^2\right) \qquad \forall w_i \in \Delta \tag{7}$$

where $W^0 \in \mathcal{W}^*$ is the efficient production network for the economy and κ is a positive matrix of weights capturing the cost, in terms of productivity, of moving the j-th input share away from its ideal value. Following a parallel logic, we can replace the quadratic distance in equation (7) with another "distance" such as the relative entropy to obtain

$$S_i(w_i) = \exp\left(-\lambda_i R\left(w_i || w_i^0\right)\right). \tag{8}$$

In this case, $R(\cdot||\cdot)$ is the relative entropy while $W^0 \in \mathcal{W}^*$ and $\lambda_i > 0$. In both these smooth cases, we have $\operatorname{argmax}(S) = \{W^0\}$. In general, this is the case every time that each S_i is strictly log-concave (as in [32]). ▲

Next, we proceed with the description of the general equilibrium. Firms are price takers and act in a perfect-competition economy. We normalize the price of the external factor to 1 and, given a vector $P \in \mathbb{R}_+^k$ of inputs' prices and a feasible technology $w_i \in \Delta$, the cost-minimization problem for firm i, producing at least 1 unit of output, is defined by

$$K_i(P, w_i) = \min_{Q_i \in \mathbb{R}_+^{k+1}} \left\{ Q_{i0} + \sum_{j=1}^{k} Q_{ij} P_j : F_i(Q_i, w_i) \geq 1 \right\} \qquad \forall i \in \{1, ..., k\}. \tag{9}$$

Because each firm can choose its technology w_i so to minimize their unitary cost, the equilibrium zero-profit condition is

$$P_i = \min_{w_i \in \Delta} K_i(P, w_i) = \min_{(w_i, Q_i) \in \Delta \times \mathbb{R}_+^{k+1}} \left\{ Q_{i0} + \sum_{j=1}^{k} Q_{ij} P_j : F_i(Q_i, w_i) \geq 1 \right\} \quad \forall i \in \{1, ..., k\}. \tag{10}$$

Note that the above equilibria are β dependent. In particular, for each $\beta \in (0,1)$, an equilibrium is given by a vector of prices $P \in \mathbb{R}_+^k$, a matrix of inputs $Q \in \mathbb{R}_+^{k \times (k+1)}$, and a network structure $W \in \mathcal{W}$. In the triple (P, W, Q), the vector P solves the fixed point equation (10) and the pair (w_i, Q_i) solves the cost-minimization problem in the right hand-side of equation (10).

Following the same steps in [32], the fixed point condition for equilibrium log-prices can be written as

$$p_i = (1-\beta) x_i + \beta \min_{w_i \in \Delta} \left\{ \sum_{j=1}^{k} w_{ij} p_j + \frac{1}{\beta} c_i(w_i) \right\} \qquad \forall i \in \{1, ..., k\} \tag{11}$$

where $p_i = \ln(P_i)$, $x_i = \ln(1/Z_i)$, and $c_i(w_i) = \ln(1/S_i(w_i))$.[6] It is standard to show that, for each $\beta \in (0,1)$, there exists a unique vector of log-prices p_β that solves the fixed point equation (11) and therefore a unique vector of equilibrium prices P_β. Given these prices, the equilibrium network and quantities are not unique in general due to the fact that each firm might have multiple optimal technologies, that is,

$$\operatorname*{argmin}_{w_i \in \Delta} \left\{ \sum_{j=1}^{k} w_{ij} p_{\beta,j} + \frac{1}{\beta} c_i(w_i) \right\} \tag{12}$$

might not be single-valued. When S_i is strictly log-concave, as in [32], it follows that c_i is strictly convex and there exists a unique minimizer $w_{\beta,i}$ in equation (12). This in turn uniquely pins down the equilibrium inputs Q_β. We next characterize the vector of equilibrium prices in the limit for a vanishing intensity of the external factor.

Proposition 2 *The limit equilibrium vector of prices is constant across firms and*

$$\lim_{\beta \to 1} p_{\beta,i} = \min_{\gamma \in \Gamma(\operatorname{argmax}(S))} \langle \gamma, x \rangle \qquad \forall i \in \{1, ..., k\}.$$

Moreover, if S_i is continuously differentiable and strictly log-concave for all $i \in \{1, ..., k\}$ with $\operatorname{argmax}(S) = \{W^0\}$, *then*

$$\lim_{\beta \to 1} p_{\beta,i} = \langle \gamma_{W^0}, x \rangle, \ \lim_{\beta \to 1} w_{\beta,i} = w_i^0, \ \lim_{\beta \to 1} Q_{\beta,i0} = 0, \textit{ and } \lim_{\beta \to 1} Q_{\beta,ij} = w_{ij}^0.$$

This result establishes that, when the share of the external factor vanishes, only the eigenvector centralities of the efficient technologies determine the relative importance of the firms' productivity shocks on the equilibrium prices. Notably, due to competition, the firms' centralities are determined by the efficient networks that optimally minimize the equilibrium prices.

So far, we considered endogenous production networks under a Cobb-Douglas production function. Another potential source of nonlinearity recently studied in this literature comes from generalizing the production function to the class of nested CES (see for example Baqaee and Farhi [8] and Carvalho and Tahbaz-Salehi [10]). It turns out that our method can also be applied in this case. For simplicity, fix a production network $W \in \mathcal{W}^*$, and assume that the production function of firm i is

$$F_i(Q_i) = \hat{\xi}_i(\beta, w_i)(Z_i Q_{i0})^{(1-\beta)} \left(\sum_{j=1}^{k} w_{ij}^{1/\sigma_i} Q_{ij}^{(\sigma_i - 1)/\sigma_i} \right)^{\beta\sigma_i/(\sigma_i - 1)}$$

where σ_i is the elasticity of substitution among the inputs of the firm and $\hat{\xi}_i(\beta, w_i)$ is a normalization constant that only depends on β and the fixed technology w_i. It

[6] See Online Appendix F.1 for the details.

is standard to show that in this case the log prices are uniquely characterized by the fixed-point condition[7]

$$p_{\beta,i}^{NCS} = (1-\beta)\,x_i + \beta\frac{1}{1-\sigma_i}\ln\left(\sum_{j=1}^{k} w_{ij}\exp\left((1-\sigma_i)\,p_{\beta,j}^{NCS}\right)\right) \qquad \forall i \in \{1,...,k\}\,.$$

The limit fixed point of this equation is covered by our Theorem 1, and we then have the following result.

Proposition 3 *The limit equilibrium vector of prices is constant across firms and*

$$\lim_{\beta\to 1} p_{\beta,i}^{NCS} = \langle \gamma_{W^0}, x\rangle \qquad \forall i \in \{1,...,k\}\,.$$

Therefore, the nonlinear effects emphasized in Baqaee and Farhi [8] matter only for sizeable dependence on the external factor that is not tradeable within the production network.

3.2 Coordination games

We consider a finite set of agents $N = \{1,...,n\}$ playing a complementary-effort game on an endogenous network. Each agent $i \in N$ chooses how much effort to exercise in the partnership with other agents: $a_i \in \mathbb{R}_+$. The benefit of effort is directly proportional to a linear combination of her ability $x_i \in \mathbb{R}_+$ with a weighted average of the efforts exercised by her neighbors. The cost of effort is instead quadratic, a feature that will guarantee linearity of the best response for a given network structure.

Formally, given a fixed weighted and directed network $W \in \mathcal{W}$, the payoff of agent i for every profile of actions $a = (a_i)_{i=1}^{n}$ is

$$u_i\,(a, w_i) = a_i\left((1-\beta)\,x_i + \beta\sum_{j=1}^{n} w_{ij}a_j\right) - \frac{a_i^2}{2}$$

where $\beta \in (0,1)$ captures the relative importance of complementary efforts over the personal skills of every agent. In what follows, we consider two different cases of endogenous networks. In both cases, we assume that each feasible network structure W has two features: (i) there is no self-link, that is, $w_{ii} = 0$ and (ii) W is strongly connected. The first assumption is standard in coordination games on networks (cf. [6] and [26]). We discuss the relevance of the second assumption below. Let us denote the set of stochastic matrices satisfying both (i) and (ii) with $\mathcal{W}_0^*$.

[7]See Online Appendix A of [10].

Costly link formation Here we assume that, before choosing her effort, each agent i chooses her weighted links $w_i \in \Delta_n$. This is costly and the cost function of agent i is denoted by $c_i : \Delta_n \to [0, \infty]$. In particular, those weighted networks $W \in \mathcal{W}$ such that $c_i(w_i) = 0$ for all $i \in N$ are the free one. Instead, all those weighted networks $W \in \mathcal{W}$ such that $c_i(w_i) = \infty$ for some $i \in N$ are unfeasible. We maintain the following assumption on the cost functions.

Assumption: The profile of cost functions $c = (c_i)_{i=1}^n$ is such that (i) each feasible network structure W has no self-link, (ii) each c_i is lower semicontinuous and there is a strongly connected W with[8]

$$w_i \in co\left(c_i^{-1}(0)\right) \qquad \forall i \in \{1, ..., k\}. \tag{13}$$

The first assumption is standard in coordination games on networks (cf. Ballester, Calvó-Armengol, and Zenou [6] and Golub and Morris [26]). Lower semicontinuity of the cost functions is a technical condition that guarantees existence of a well-defined best response map for the coordination game. In turn, equation (13) says that each agent has at least a free vector of weights and that there is a strongly connected matrix that can be obtained by mixing the free networks. The next example illustrates a setting where our assumption on the cost functions is satisfied.

Example 3 (Free Network) Assume that the agents are connected on a baseline unweighted and strongly connected network represented by a graph $G \in \{0, 1\}^{n \times n}$ with $g_{ii} = 0$. Maintaining the links specified in G is free for all the agents. However, they can costly form new links in addition to the ones in G. In particular, there is a fixed cost $k > 0$ for each addition link that player i forms on top of the baseline ones. We next show how this particular case of costly link formation can be represented by a profile of cost functions $(c_i)_{i=1}^n$ satisfying our assumption. Let $N_i(G) \subseteq \{1, ..., n\}$ denote the set of neighbors of i in the graph G. For every $i \in N$, define the set of uniform weights

$$D_i(G) = \left\{ \frac{1}{|N_i|} \sum_{j \in N_i} \delta_j \in \Delta_n : N_i(G) \subseteq N_i \subseteq N \setminus \{i\} \right\}$$

and the cost function $c_i : \Delta_n \to [0, \infty]$ as

$$c_i(w_i) = k \left| \{j \in N : w_{ij} > 0\} \setminus N_i(G) \right| + \mathbf{I}_{D_i(G)}(w_i)$$

where $k \in \mathbb{R}_+$ and $\mathbf{I}_{D_i(G)}$ is equal to 0 if $w_i \in D_i(G)$ and ∞ otherwise. On the one hand, it is easy to see that c_i is lower semicontinuous. On the other hand, the uniform network $W(G)$ defined by $w_i(G) = \frac{1}{|N_i(G)|} \sum_{j \in N_i(G)} \delta_j$ for all $i \in N$ is free for every player. Therefore $(c_i)_{i=1}^n$ satisfy our assumption. ▲

[8]For every set $K \subseteq \Delta_n$, for example $K = c_i^{-1}(0)$, we let $co(K)$ denote the convex hull of K.

For a fixed profile of cost functions c, we assume that the total payoff of each player $i \in N$ given a profile of efforts $a \in \mathbb{R}_+^n$ and weighted links $w_i \in \Delta_n$ is $u_i(a, w_i) - a_i c_i(w_i)$. In words, the total cost of forming and maintaining the link is increasing and linear in the effort chosen. The assumption that the effort and the weighted links are complementary in increasing the total cost of the player has been already considered by Sadler and Golub [38] in the same context of endogenous link formation, we refer to them for a detailed motivation.

We next analyze the best response map of the total game of choosing both the weighted links and the effort. In particular, observe that neither the payoff function u_i nor the cost function c_i of i depend on the links chosen by the other agents. Therefore, given a conjecture $a_{-i} \in \mathbb{R}_+^{n-1}$ about the effort of the other agents, player i solves

$$\max_{a_i \in \mathbb{R}_+} \max_{w_i \in \Delta_n} \left\{ u_i(a, w_i) - a_i c_i(w_i) \right\}.$$

Observe that the previous maximization problem can be rewritten as

$$\max_{a_i \in \mathbb{R}_+} \left\{ a_i \max_{\tilde{w}_i \in \Delta_n} \left\{ \beta \sum_{j=1}^{n} \tilde{w}_{ij} a_j - c_i(\tilde{w}_i) \right\} - \frac{a_i^2}{2} \right\}.$$

Therefore the induced objective function is still quadratic with respect to the choice variable a_i, hence the unique best response can be still characterized by the first-order conditions. In general, this implies that a profile of efforts $a \in \mathbb{R}_+^n$ and a weighted network $W \in \mathcal{W}$ form a Nash equilibrium of the total game if and only if, for every $i \in N$,

$$a_i = (1-\beta) x_i + \beta \max_{\tilde{w}_i \in \Delta_n} \left\{ \sum_{j=1}^{n} \tilde{w}_{ij} a_j - \frac{c_i(\tilde{w}_i)}{\beta} \right\} \tag{14}$$

and

$$w_i \in \operatorname*{argmax}_{\tilde{w}_i \in \Delta_n} \left\{ \sum_{j=1}^{n} \tilde{w}_{ij} a_j - \frac{c_i(\tilde{w}_i)}{\beta} \right\}.$$

The first condition is a standard fixed-point equation on the profile actions a. The main difference with respect to the game with a fixed weighted network is the nonlinearity of the fixed point equation. However, we show below that it can be still analyzed through the results of the previous sections. The second condition instead requires that the equilibrium network is a best response for each player given the efforts chosen by the others.

It is not hard to see that, for every $\beta \in (0,1)$, there exists a unique equilibrium profile of efforts $a_\beta \in \mathbb{R}_+^n$ that solves the fixed-point equation (14). We aim to characterize the limit for high coordination motives $\lim_{\beta \to 1} a_\beta$. First, define

$$\Gamma(c) = \left\{ \gamma \in \Delta_n : \exists W \in \mathcal{W} : \forall i \in N, w_i \in co\left(c_i^{-1}(0)\right), \gamma^{\mathrm{T}} = \gamma^{\mathrm{T}} W \right\},$$

the set of all the eigenvector centralities of networks that are free. With this we have the following result.

Proposition 4 *The limit equilibrium profile of efforts is well defined, constant across players, and equal to* $\lim_{\beta\to 1} a_{\beta,i} = \max_{\gamma\in\Gamma(c)} \langle\gamma, x\rangle$ *for every* $i \in \{1, ..., k\}$.

The main implication is that the most central agent in any of the limit equilibrium networks are those that are at the same time most efficient (higher x_i) and cheaper to link with.

Example 4 (Free Networks Continued) Observe that $W(G)$ is the unique equilibrium network consistent with the limit for $\beta \to 1$. Moreover, it is well known that the eigenvector centrality of $W(G)$ is given by

$$\gamma_i(G) = \frac{|N_i(G)|}{\sum_{j\in N} |N_j(G)|} \qquad \forall i \in N.$$

With this, we have $\Gamma(c) = \{\gamma(G)\}$, hence that

$$\lim_{\beta\to 1} a_{\beta,i} = \frac{\sum_{j\in N} |N_j(G)|\, x_j}{\sum_{j\in N} |N_j(G)|}.$$

Therefore, the common equilibrium effort is relatively higher if the agents who are relatively more efficient (i.e., high x_i) are also those that are more central in the baseline network. ▲

4 Application II: Zero-sum stochastic games

In this section, we consider zero-sum stochastic games with finitely many states and a continuum of actions for both players. We closely follow the textbook formalization of Sorin [42, Chapter 5].

There are two players repeatedly interacting in a zero-sum game under uncertainty. We identify the two player as the *maximizer* and the *minimizer*. Time is discrete $t \in \mathbb{N}$ and at each period the game is at a state drawn from a finite set Ω. At the end of each period, an outcome r from a finite set $R \subseteq \mathbb{R}$ realizes and the maximizer gets payoff r and the minimizer gets $-r$. The set of feasible actions for the maximizer and the minimizer are respectively denoted by S and Q, two compact metric spaces. Both the outcome at period t and the state at period $t+1$ depend on players' actions and the state at period t. Formally, this is described by a *continuous* transition map $\rho : S \times Q \times \Omega \to \Delta(R \times \Omega)$. With a small abuse of notation, we also use ρ to denote its linear extension $\rho : \Delta(S) \times \Delta(Q) \times \Omega \to \Delta(R \times \Omega)$ to mixed actions as well as the corresponding marginal distributions over R and Ω.[9] With this, define the state-dependent one-period expected reward $g : \Delta(S) \times \Delta(Q) \times \Omega \to \mathbb{R}$ as

$$g(\hat{s}, \hat{q}, \omega) = \sum_{r' \in R} r' \rho(\hat{s}, \hat{q}, \omega)(r') \qquad \forall \hat{s} \in \Delta(S), \forall \hat{q} \in \Delta(Q), \forall \omega \in \Omega.$$

This setting is equivalent to the more standard one where there are no outcomes and the primitive objects are a transition function $\rho : S \times Q \times \Omega \to \Delta(\Omega)$ and a one-period expected reward function $g : S \times Q \times \Omega \to \mathbb{R}$ (e.g., Sorin [42, Chapter 5]). We explicitly keep track of the outcomes so to obtain a cleaner limit characterization using our methods.

Following the standard analysis of zero-sum stochastic games, we consider two different cases: (i) the one-period game is infinitely repeated and the agents maximize their discounted expected payoffs with common discount factor $\beta \in (0,1)$; (ii) the one-period game is repeated only t times and the agents maximize the time average of their expected payoffs.

In case (i), it is well known that, for each discount factor $\beta \in (0,1)$, the value of the game $v^\beta \in \mathbb{R}^\Omega$ exists and is the unique solution of the Shapley equation (see, e.g., Neyman and Sorin [36, Theorem 2 of Chapter 8]):

$$v^\beta_\omega = \max_{\hat{s} \in \Delta(S)} \min_{\hat{q} \in \Delta(Q)} \left\{ (1-\beta) g(\hat{s}, \hat{q}, \omega) + \beta \sum_{\omega' \in \Omega} v^\beta_{\omega'} \rho(\hat{s}, \hat{q}, \omega)(\omega') \right\} \qquad \forall \omega \in \Omega. \tag{15}$$

Similarly, in case (ii), for every length $t \in \mathbb{N}$, the value of the game $v^t \in \mathbb{R}^\Omega$ exists and satisfies the following recursive equation:

$$v^t_\omega = \max_{\hat{s} \in \Delta(S)} \min_{\hat{q} \in \Delta(Q)} \left\{ \frac{1}{t} g(\hat{s}, \hat{q}, \omega) + \frac{t-1}{t} \sum_{\omega' \in \Omega} v^{t-1}_{\omega'} \rho(\hat{s}, \hat{q}, \omega)(\omega') \right\} \qquad \forall \omega \in \Omega. \tag{16}$$

We say that the game has an *asymptotic value* (cf. Sorin [42]) if and only if both $\lim_{\beta \to 1} v^\beta$ and $\lim_t v^t$ exist and coincide.[10] Our abstract analysis of nonlinear fixed points yields the existence of the asymptotic value and its explicit form under a minimal connectedness assumption.

We first need some preliminary definitions. Let $\Sigma_S = \Delta(S)^\Omega$ and $\Sigma_Q = \Delta(Q)^\Omega$ denote the set of stationary mixed strategies of the agents and, for all $\sigma_S \in \Sigma_S$ and

[9] The linear extension of ρ is defined as usual:

$$\rho(\hat{s}, \hat{q}, \omega)(r, \omega') = \int_S \int_Q \rho(s, q, \omega)(r, \omega') \, d\hat{s}(s) \, d\hat{q}(q),$$

for all $\hat{s} \in \Delta(S)$ and $\hat{q} \in \Delta(Q)$.

[10] The derivations of equations (15) and (16) can be found in Sorin [42, Propositions 5.2 and 5.3].

$\sigma_Q \in \Sigma_Q$, let $W(\sigma_S, \sigma_Q)$ denote the transition matrix between state-outcome pairs with entries given by

$$w_{(r,\omega),(r',\omega')}(\sigma_S, \sigma_Q) = \rho(\sigma_S(\omega), \sigma_Q(\omega), \omega)(r', \omega') \qquad \forall (r, \omega), (r', \omega') \in R \times \Omega.$$

Let $n = |R \times \Omega|$. Next we state the crucial assumption that allows us to apply our result to the current stochastic-game setting.

Assumption A: For every $\sigma_S \in \Sigma_S$ there exists a strategy $\sigma_Q \in \Sigma_Q$ such that the matrix $W(\sigma_S, \sigma_Q)$ is strongly connected.

In words, we assume that for every strategy σ_S of the maximizing player there is a strategy of the minimizing player that makes the network of outcome-state pairs strongly connected. Moreover, these baseline links are such that there exists a unique essential class of essential pairs (r, ω). This implies that each $W(\sigma_S, \sigma_Q)$ admits a unique Perron-Frobenius eigenvector, denoted by $\gamma(\sigma_S, \sigma_Q) \in \Delta(R \times \Omega)$. With the same abuse of notation as before we use the same symbol $\gamma(\sigma_S, \sigma_Q)$ for its marginal over outcomes.

Proposition 5 *Under Assumption A, the game has an asymptotic value that is independent of the state and such that*

$$\lim_{\beta \to 1} v^\beta = \lim_t v^t = \left(\sup_{\sigma_S \in \Sigma_S} \min_{\sigma_Q \in \Sigma_Q} \sum_{r \in R} r\gamma(\sigma_S, \sigma_Q)(r) \right) e.$$

This result extends the standard result on the existence and characterization of the asymptotic value of zero-sum stochastic games from the finite case to the class of games considered in the current section (see for example Sorin [42, Propositions 5.12-5.14]). As in the finite case, the asymptotic value coincides with the value of static zero-sum game with expected payoffs given by the stationary distributions generated by the players' strategies. Importantly, compared to other similar conditions employed in the literature, we allow for the possibility that for each strategy of the maximizing player a single (and possibly different) strategy of the minimizing player makes the outcome-state pair strongly connected .

Whenever Q is a singleton, we obtain a Markov decision process (MDP) where the minimizer is optimally controlling her cost. With this, Proposition 5 collapses to average-cost optimality for MDPs with a continuum of actions and finitely many states, that is,

$$\lim_{\beta \to 1} v^\beta = \lim_t v^t = \left(\sup_{\sigma_S \in \Sigma_S} \sum_{r \in R} r\gamma(\sigma_S)(r) \right) e.$$

Finally, observe that, in this setting, Theorem 4 can be applied and would deliver an estimate on how the value of the game depends on the current state for every $\beta \in (0, 1)$.

5 Application III: Opinion aggregation with stubbornness

Cerreia-Vioglio et al. [12] consider a finite set of agents $i \in \{1, ..., k\}$ and let $x \in \mathbb{R}^k$ denote an arbitrary profile of opinions for the agents. An opinion is just a real number that can be interpreted as the estimate of agent i about some fundamental parameter of interest or the intensity with which an individual agrees with a certain policy. Under this interpretation, they assume that the opinions of the agent evolve according to the operator T, that is, if the current profile of opinions is x, then the profile of opinions in the next period is $T(x)$. With this, the sequence of iterates $\{T^t(x)\}_{t=1}^{\infty}$ corresponds to the sequence of profile of opinions in the population over time. For example, when $T = W$ is linear, we obtain the celebrated DeGroot's model [18] of opinion aggregation of experts.[11] In particular, Golub and Jackson [25] interpreted W as a directed and weighted network where the entry w_{ij} represents the weighted link from j to i. In general, motivated by the fact agents may use opinion aggregators reflecting their attraction or aversion for extreme opinions, [12] introduce several classes of nonlinear opinion aggregators T. For example, when

$$T_i(x) = \frac{1}{\lambda_i} \ln \left(\sum_{j=1}^{k} w_{ij} \exp(\lambda_i x_j) \right) \tag{17}$$

for some fixed set of weighted links $w_i \in \Delta$ and a parameter $\lambda_i \in \mathbb{R}$, it is possible to model agents with heterogeneous attractions for high ($\lambda_i > 0$) or low ($\lambda_i < 0$) opinions while maintaining the underlying linear network structure. Alternatively, we can altogether relax the existence of a single network structure and consider opinion aggregators such as

$$T_i(x) = \alpha_i \min_{w_i \in C_i} \langle w_i, x \rangle + (1 - \alpha_i) \max_{w_i \in C_i} \langle w_i, x \rangle \tag{18}$$

where $C_i \subseteq \Delta$ is a compact and convex set of possible weighted links to i and $\alpha_i \in [0, 1]$ is a parameter capturing the relative attraction of i for high or low opinions. It is routine to show that if each element T_i of T is defined as in equations (17) or (18), then T is monotone, normalized, and translation invariant.

Friedkin and Johnsen [21] proposed a variation of the DeGroot's model where the agents have a degree of stubbornness with respect to their initial opinions. Here we extend Friedkin and Johnsen's model of stubbornness by considering nonlinear opinion aggregators T with the functional properties introduced above. Formally, we assume that, for every period $t \in \mathbb{N}$, the profile of opinions in the population is

$$\tilde{x}_i^t = (1 - \beta) x_i + \beta T_i\left(\tilde{x}^{t-1}\right) \qquad \forall i \in \{1, ..., k\} \tag{19}$$

[11] See for example Golub and Jackson [25] for a detailed analysis of this model.

where $\beta \in (0,1)$ is a fixed parameter capturing the degree of stubbornness in the population and $x = \tilde{x}^0$ is the profile of initial opinions. In words, each agent i aggregates the last-period opinions $\tilde{x}^{t-1}$ with her opinion aggregator T_i and then mixes the resulting aggregate with her original opinion, using the common weight β. When $T = W$ is linear, we exactly obtain Friedkin and Johnsen's model. In general, it is easy to see that the sequence of opinions $\{\tilde{x}^t\}_{t=1}^{\infty}$ converges to the unique fixed point $\tilde{x}_\beta$ defined in equation (2), which then corresponds to the long-run profile of opinions of the agent under stubbornness β. Provided that it exists, the $\lim_{\beta \to 1} \tilde{x}_\beta$ corresponds to the profile of long-run opinions of the agents as the stubbornness friction is vanishing.[12]

The results of Section 2 can be applied to this setting. For example, consider the standard Friedkin and Johnsen model with linear $T = W$ having a single LPF eigenvector γ_W and compare it to an alternative opinion aggregator $\tilde{T}$ where the agents have the same network structure W but aggregate opinions according to equation (17) for some profile of parameters $(\lambda_1, ..., \lambda_k) \in \mathbb{R}^k$. Because $\tilde{T}$ is continuously differentiable with $J_{\tilde{T}}(0) = W$, Theorem 1 states that, regardless of the value of $(\lambda_1, ..., \lambda_k)$, the opinions of the agents will converge to the same consensus $\langle \gamma_W, x \rangle$ for both T and $\tilde{T}$.

Alternatively, consider the opinion aggregator T defined as in equation (18) such that, for every $i \in \{1, ..., k\}$, we have $\alpha_i = \alpha$ and

$$C_i = \left\{ (1-\varepsilon)\, w_i^0 + \varepsilon w_i : w_i \in \Delta \right\}$$

for some $\alpha \in [0,1]$, $\varepsilon \in [0,1)$, and stochastic matrix $W \in W$ with $A(W)$ regular. Also fix a vector of initial opinions $x \in \mathbb{R}^k$ and define

$$\hat{x}^\varepsilon = \varepsilon \left[I - (1-\varepsilon)\, W \right]^{-1} x.$$

Observe that T can be written as $T = (1-\varepsilon)\, W + \varepsilon S$ where

$$S_i(z) = \alpha \min_{j \in \{1,...,k\}} z_j + (1-\alpha) \max_{j \in \{1,...,k\}} z_j \qquad \forall z \in \mathbb{R}^k, \forall i \in \{1, ..., k\}.$$

Therefore, Proposition 1 implies that the long-run opinions as the stubbornness vanishes converge to the consensus

$$\alpha \min_{j \in \{1,...,k\}} \hat{x}_j^\varepsilon + (1-\alpha) \max_{j \in \{1,...,k\}} \hat{x}_j^\varepsilon.$$

In words, we first need to compute the vector of opinions of the agents obtained by applying the matrix of ε-weighted Bonacich centralities of W to x and then linearly combine the maximum and the minimum of the opinions so obtained.

[12] Banerjee and Compte [7] provided a game-theoretic foundation for this limit by considering a noisy version of the Friedkin and Johnsen's model where the agents choose once and for all the stubbornness weight to assign to their initial opinion so to maximize the accuracy of their long-run opinion. They show that as the noise vanishes, the symmetric equilibrium weight converges to zero, that is $\beta \to 1$, providing an alternative foundation for the limit we study.

6 Application IV: Recursive preferences

Following the seminal work of Marinacci and Montrucchio [34], in this section, we consider *recursive preferences* over uncertain streams of consumption characterized by fixed-point equation. Similarly to Al-Najjar and Shmaya [5], we focus on the stationary limit $\beta \to 1$ (i.e., infinitely patient decision maker) that we characterize exploiting our results.

Consider a finite state space Ω and assume that time is discrete $t \in \mathbb{N}$. The agent needs to evaluate an infinite stream of state-dependent consumption that is i.i.d. across dates. This is represented by a vector $z \in \mathbb{R}^\Omega$ giving for each realized state ω the amount of consumption $z_\omega \in \mathbb{R}$ for the agent. We assume that the agent knows that the data-generating process is i.i.d. but she is potentially uncertain on the exact form of the one-period marginal $p^* \in \Delta(\Omega)$. The agent has recursive preferences given by a state-dependent utility $v \in \mathbb{R}^\Omega$ defined by

$$v_\omega = W\left(z_\omega, S_\omega(v)\right)$$

where $W : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an aggregator and $S : \mathbb{R}^\Omega \to \mathbb{R}^\Omega$ is a state-dependent certainty equivalent. An important case is the one of the Epstein-Zin aggregator $W_\beta(z_0, z_1) = \phi^{-1}\left((1-\beta)\phi(z_0) + \beta\phi(z_1)\right)$ for some $\beta \in [0,1]$ and some continuous and strictly increasing $\phi : \mathbb{R} \to \mathbb{R}$. With this, the fixed-point equation becomes

$$v_\omega^\beta = \phi^{-1}\left((1-\beta)\phi(z_\omega) + \beta\phi\left(S_\omega\left(v^\beta\right)\right)\right),$$

which we can rewrite as

$$x^\beta = (1-\beta)x_\omega + \beta T_\omega\left(x^\beta\right)$$

by letting $x_\omega^\beta = \phi\left(v_\omega^\beta\right)$, $x_\omega = \phi(z_\omega)$, and $T_\omega = \phi\left(S_\omega\left(\phi^{-1}\right)\right)$ for all $\omega \in \Omega$. With this, our results can be directly applied to characterize a the infinitely patient evaluation of the consumption stream under a large class of preferences under uncertainty.

7 Application V: Financial networks

In this section, we consider an equilibrium model of systemic risk where a group of financial institutions are exposed to idiosyncratic losses and hold cross-capital interdependencies. Following the approach of Adrian and Brunnermeier [3] in modeling systemic risk, we assume that the banks' evaluation of uncertain return losses conditional on each macroeconomic scenario are performed using conditional risk measures (see also Detlefsen and Scandolo [19]).

Consider a finite number of banks $K = \{1, ..., |K|\}$ and states of the world $\Omega = \{1, ..., |\Omega|\}$. We interpret each state $\omega \in \Omega$ as a possible macroeconomic scenario that determines the nature of the idiosyncratic losses of the banks. Formally, fix $x \in \mathbb{R}^{K\times\Omega}$ and interpret $x_{k,\omega}$ as the realization of the idiosyncratic return loss in real-economy assets for bank k in state ω. Each bank k is endowed with a partition $\Pi_k \subseteq 2^{\Omega}$ of the states representing the coarsened scenarios considered by k. In particular, we assume that Π_k is finer than the partition induced by the random variable $x_k \in \mathbb{R}^{\Omega}$ representing the idiosyncratic return loss of k, that is, each bank is able to discern the scenarios determining their own return losses in real-economy assets. Moreover, we assume that $\{\Pi_k\}_{k\in K}$ are common knowledge. Finally, the banks are connected in a financial network represented by a strongly connected stochastic matrix $M \in [0,1]^{K\times K}$ whose entries correspond to the financial interdependencies among the banks: $m_{k,k'} \in [0,1]$ is the fraction of the exposure of bank k to the financial sector that accrues to bank k' and $m_{k,k} = 0$ for all $k \in K$.

Each bank k has to evaluate their total return loss in each of the considered scenarios, that is, for each cell of Π_k. The total return loss of each bank is a combination of the realized idiosyncratic return loss and the estimated return loss induced by the exposures to the other banks. For every $k, k' \in K$, let $y_{k,k'} \in \mathbb{R}^{\Omega}$ denote the state-contingent return loss of bank k' conjectured by bank k. In particular, $y_{k,k'}$ is a random variable that is measurable with respect to $\Pi_{k'}$. With this, the total conjectured return loss of bank k for state ω is

$$(1-\beta)\, x_{k,\omega} + \beta \sum_{k'\in K} m_{k,k'} y_{k,k',\omega}, \tag{20}$$

where $\beta \in (0,1)$ captures the intensity of cross exposure of the banks.

The total conjectured return loss in equation (20) is still a random variable from the point of view of bank k since each $y_{k,k'}$ is only measurable with respect to $\Pi_{k'}$. We endow each bank k with a *conditional risk measure* $V_k : \Omega \times \mathbb{R}^{\Omega} \to \mathbb{R}$ that quantifies each possible uncertain prospect in terms of return loss, conditional to each scenario considered by bank k. Following Detlefsen and Scandolo [19], we assume that each V_k is measurable with respect to Π_k and such that, for every $\omega \in \Omega$, the functional $V_k(\omega, \cdot)$ is normalized, monotone decreasing, convex, cash invariant, that is

$$V_k(\omega, \ell + ke) = V(\omega, \ell) - k \qquad \forall \omega \in \Omega, \forall \ell \in \mathbb{R}^{\Omega}, \forall k \in \mathbb{R},$$

and information regular (or consequentialist), that is,

$$V_k\left(\omega, \ell 1_{\Pi(\omega)} + h 1_{\Pi(\omega)^c}\right) = V(\omega, \ell) \quad \forall \omega \in \Omega, \forall \ell, h \in \mathbb{R}^{\Omega}.$$

By [19, Theorem 1], this conditional risk measure admits the following representation

$$V_k(\omega, \ell) = \max_{p \in \Delta(\Omega)} \left\{ -\sum_{\tilde{\omega}\in\Omega} \ell_{\tilde{\omega}} p_{\tilde{\omega}} - c_{k,\omega}(p) \right\} \qquad \forall \omega \in \Omega, \forall \ell \in \mathbb{R}^{\Omega},$$

where, for every $\omega \in \Omega$, the function $c_{k,\omega} : \Delta(\Omega) \to [0, \infty]$ is grounded, convex, lower semicontinuous, and such that $c_{k,\omega}(p) < \infty$ implies that $p \in \Delta(\Pi_k(\omega))$.

Given conjectures $\{y_{k,k'}\}_{k' \in K \setminus \{k\}}$, the risk of bank k in state ω is given by

$$V_k\left(\omega, (1-\beta)\, x_k + \beta \sum_{k' \in K \setminus \{k\}} m_{k,k'} y_{k,k'}\right)$$

where the equality follows from the fact that x_k is Π_k-measurable and information regularity.

The model is closed by the equilibrium assumption that the conjecture $y_{k,k'}$ of bank k about the return losses of the bank k' in state ω is equal to bank k own assessment of the return losses in the scenario $\Pi_{k'}(\omega)$. Formally, for every level of connectedness β, the vector of return losses $x^\beta \in \mathbb{R}^{K \times \Omega}$ is an equilibrium if and only if

$$x^\beta_{k,\omega} = -V_k\left(\omega, (1-\beta)\, x_k + \beta \sum_{k' \in K} m_{k,k'} x^\beta_{k'}\right) \qquad \forall (k, \omega) \in K \times \Omega. \tag{21}$$

Observe that this is a particularly prudential benchmark: bank k uses a prudential risk measure to evaluate return losses in which the contribution $x^\beta_{k'}$ is itself the result of a prudential (i.e., risk-measure) evaluation.

Fixed-point conditions such as the one in equation (21), in linear or nonlinear form, are pervasive in equilibrium analysis of financial networks (see for example Elliott et al. [20] and the survey Jackson and Pernoud [30]). In particular, as β goes to 1, the return losses from financial interdependencies dominate the idiosyncratic return losses from own real assets.

Next, define the concave operator $T : \mathbb{R}^{K \times \Omega} \to \mathbb{R}^{K \times \Omega}$ as

$$T_{(k,\omega)}(z) = \min_{p \in \Delta(\Omega)} \left\{ \sum_{(k',\omega') \in K \times \Omega} m_{k,k'} p_{\omega'} z_{k',\omega'} + c_{k,\omega}(p) \right\} \qquad \forall z \in \mathbb{R}^{K \times \Omega}.$$

Under the mild connectedness assumption that $E(T) = D \subseteq \mathbb{R}^{K \times \Omega}$, Theorem 2 implies that the limit risk $\lim_{\beta \to 1} x^\beta$ exists and is independent of the realized fundamental state as well as of the bank's identity.

This result has particularly strong implications for the case of *smooth divergence risk measures* with respect to a common ex-ante probabilistic model. Formally, suppose that the banks share the same *full support* probabilistic model $p^0 \in \Delta(\Omega)$ in the ex-ante stage and then update conditional on their private information. For example, bank k in state ω has interim belief $p^0(\cdot|\Pi_k(\omega))$. Therefore, in the interim stage, the conditional risk measure of bank k in state ω is

$$V_k(\omega, \ell) = \max_{p \in \Delta(\Omega)} \left\{ -\sum_{\tilde{\omega} \in \Omega} \ell_{\tilde{\omega}} p_{\tilde{\omega}} - D_k\left(p || p^0(\cdot|\Pi_k(\omega))\right) \right\} \qquad \forall \ell \in \mathbb{R}^\Omega$$

where $D_k(\cdot||\cdot) : \Delta(\Omega) \times \Delta(\Omega) \to [0, \infty]$ is a divergence that is essentially strictly convex (cf. Maccheroni et al. [35]). The standard example of such divergences is the relative entropy. Let $\mu \in \Delta(K)$ denote the unique left Perron-Frobenius eigenvector of M.

Corollary 2 *We have that*

$$\lim_{\beta \to 1} x^{\beta} = \left(\sum_{k \in K} \mu_k \left(\sum_{\omega \in \Omega} p^0_{\omega} x_{k,\omega} \right) \right) e.$$

This result follows by Corollary 1 and Golub and Morris [26, Proposition 3]. It shows that the limit equilibrium exists, is independent on the state-bank index, and coincides with the convex linear combination of the ex-ante linear expectation of the banks' return losses with weights given by the eigenvector centrality of the network. Therefore, as $\beta \to 1$, the return losses declared by all the banks tend to ignore completely their concern for robustness converging to an aggregated probabilistic evaluation of the return losses. This result is even more surprising in light of two facts: (i) as stressed above equation (21) is a very conservative benchmark, as uses a prudential risk measure to evaluate return losses that are themselves result of a prudential assessment by the other institutions, (ii) concern for robustness, indexed by the divergences $D_k(\cdot||\cdot)$, can be heterogeneous across the banks.

The result has also important implications whenever the common ex-ante probabilistic model p^0 of the banks is highly misspecified. Indeed, suppose that the banks are aware of the possibility of misspecification and evaluate their return losses with robust risk measures such as the divergence ones. Even in this case, high connectedness and equilibrium reasoning can completely offset the caution used in the evaluations and lead the banks to declare return losses that become closer and closer to their original misspecified expectations.

8 Additional results

In this section, we provide additional results complementary to our main convergence result and illustrate them by revisiting some of the economic applications proposed.

8.1 Convergence for general star-shaped operators

The result about nice star-shaped operators generalize in the following way. Consider a normalized, monotone, and translation invariant operator T. It is immediate to see that it is Lipschitz continuous of order 1. By Rademacher's Theorem, T is (Frechet) differentiable on a subset $\mathcal{D}$ of $\mathbb{R}^k$ whose complement has (Lebesgue) measure 0. Denote by $T_i : \mathbb{R}^k \to \mathbb{R}$ the i-th component of T. Since T is monotone, we have that $\frac{\partial T_i}{\partial x_j}(x) \geq 0$

for all $i,j \in \{1,...,k\}$ and for all $x \in \mathcal{D}$. Since T is also translation invariant, we have that $\sum_{j=1}^{k} \frac{\partial T_i}{\partial x_j}(x) = 1$ for all $i,j \in \{1,...,k\}$ and for all $x \in \mathcal{D}$. With this, we define an adjacency matrix $\underline{A}(T)$ for the operator T, that is, $\underline{a}_{ij} \in \{0,1\}$. Given $i,j \in \{1,...,k\}$, we set

$$\underline{a}_{ij} = 1 \iff \exists \varepsilon_{ij} \in (0,1) \text{ s.t. } \frac{\partial T_i}{\partial x_j}(x) \geq \varepsilon_{ij} \quad \forall x \in \mathcal{D}. \tag{22}$$

In words, $\underline{a}_{ij}$ is defined to be 1 if and only if the partial derivative $\frac{\partial T_i}{\partial x_j}$ is bounded away from 0, whenever it exists. If we think of $\underline{A}(T)$ as representing a directed graph over k nodes, $\underline{A}(T)$ is regular whenever the graph is strongly connected.

Theorem 3 *Let T be normalized, monotone, and translation invariant. If T is star-shaped and $\underline{A}(T)$ is regular, then $\lim_{\beta\to 1} x_\beta$ exists for all $x \in \mathbb{R}^k$.*

8.2 Fit of the approximation

The goal of this section is to provide estimates on the rate of convergence of the nets $\{x_\beta\}_{\beta\in(0,1)}$ and $\{\tilde{x}_\beta\}_{\beta\in(0,1)}$. In order to achieve this, we observe that all our previous results are for operators whose fixed points are the constant vectors. Conceptually, this makes the quantities

$$\max \tilde{x}_\beta - \min \tilde{x}_\beta \text{ and } \max x_\beta - \min x_\beta$$

interesting. In fact, both converge to zero as β goes to 1. We first bound these two quantities and then use them to provide an estimate for the rate of convergence. Perhaps interestingly, computing these bounds does not require to know that $\{x_\beta\}_{\beta\in(0,1)}$ and $\{\tilde{x}_\beta\}_{\beta\in(0,1)}$ converge.

8.2.1 Range

Given a vector $y \in \mathbb{R}^k$, we denote by $\operatorname{Rg} y$ the quantity $\max_{i\in\{1,...,k\}} y_i - \min_{i\in\{1,...,k\}} y_i$. We define

$$\delta = \min_{i,j:\underline{a}_{ij}=1} \inf_{x\in\mathcal{D}} \frac{\partial T_i}{\partial x_j}(x).$$

Next, consider the adjacency matrix $\underline{A}(T) \vee I$ which coincides with $\underline{A}(T)$ with the possible exception of the diagonal where the diagonal entries of $\underline{A}(T) \vee I$ are all 1. We can define the quantity

$$t_T = \min\left\{t \in \mathbb{N} : \left(\underline{A}(T) \vee I\right)^t \text{ has a strictly positive column}\right\}.$$

It is well known that if $\underline{A}(T)$ is regular, then t_T is well defined. Moreover, if $\underline{A}(T)$ is strongly connected, one can show that $t_T \leq k-1$ where k is the dimension of the space (see, e.g., [28, Theorem 8.5.9]). In proving Theorem 4, we provide a sharper, yet more convoluted, bound compared to the one reported below. Nevertheless, in both cases, the rate to which the ranges of x_β and $\tilde{x}_\beta$ shrink to 0 are linear.

Theorem 4 *Let T be normalized, monotone, and translation invariant. If $\underline{A}(T)$ is regular, then*

$$\operatorname{Rg}(x_\beta) \leq \operatorname{Rg}(\tilde{x}_\beta) \leq (1-\beta)(1+\kappa_T)\operatorname{Rg}(x) \quad \forall x \in \mathbb{R}^k, \forall \beta \in (0,1)$$

where

$$\kappa_T = \frac{1+\delta}{\min\left\{\frac{1}{t_T}, \left(\frac{\delta}{1+\delta}\right)^{2t_T}\right\}}.$$

8.2.2 Rate of convergence

In this section, we prove that x_β converges at least linearly fast to its limit, provided some extra conditions, differentiability or positive homogeneity, hold. The constants that appear in the statements below are the same defined in the section above. We first consider maps which are differentiable and their Jacobian is Lipschitz continuous with constant L. More formally, it is natural to view the gradient of each component $T_i : \mathbb{R}^n \to \mathbb{R}$ as an element of the dual of $\mathbb{R}^n$. Therefore, we use $\| \, \|_1$ to compute the norm of the gradient of T_i.[13] We say that the Jacobian of T is Lipschitz continuous (with constant L) if and only if

$$\|\nabla T_i(x) - \nabla T_i(y)\|_1 \leq L\|x-y\|_\infty \quad \forall x, y \in \mathbb{R}^k, \forall i \in \{1, ..., k\}.$$

Theorem 5 *Let T be normalized, monotone, and translation invariant. If T has a Lipschitz continuous Jacobian and $\underline{A}(T)$ is regular, then*

$$\|x_\beta - \langle\gamma, x\rangle e\|_\infty \leq (1-\beta)(1+\kappa_T)\left(1 + \frac{(1+\delta)^{t_T-1}}{\delta^{t_T}} t_T L \|x\|_\infty\right)\operatorname{Rg}(x)$$

for all $x \in \mathbb{R}^k$ and for all $\beta \in (0,1)$ where γ is the unique left Perron-Frobenius eigenvector of the Jacobian of T at 0.

In particular, the result above allows us to conclude that convergence happens at a linear rate. We have a similar result also in the positive homogeneous case.

Proposition 6 *Let T be normalized, monotone, and translation invariant. If T is positively homogeneous and $\underline{A}(T)$ is regular, then for each $x \in \mathbb{R}^k$*

$$\|\bar{x} - x_\beta\|_\infty \leq (1-\beta)(1+\kappa_T)\operatorname{Rg}(x) \quad \forall \beta \in (0,1)$$

where $\bar{x} = \lim_{\beta \to 1} x_\beta$.

[13] Recall that $\|x\|_1 = \sum_{i=1}^k |x_i|$ for all $x \in \mathbb{R}^k$.

8.2.3 Application: ε-equilibria and approximation in endogenous networks

Beyond bounding the distance between the exact equilibrium with a fixed β and our limit formula for the equilibrium, our results on the rate of convergence can also be used to establish that coordinating on the constant equilibrium profile equal to the limit value is an ε-equilibrium for low ε. Importantly, such ε can be derived from the primitives of the problem. This implies that as long as there are minimal incentives towards not breaking the homogeneous status quo, the limit behavior is stable even with high β. We develop this idea in the setting of Proposition 4, but of course a similar point could be made in any application where $\underline{A}(T)$ is regular and either T is continuously differentiable in a neighborhood of 0 or T is positive homogeneous. Formally, define

$$T_i^c(y) = (1-\beta)\, x_i + \beta \max_{w_i \in \Delta_n} \left\{ \sum_{j=1}^{n} w_{ij} a_j - c_i(w_i) \right\} \qquad \forall a \in \mathbb{R}_+^n, \forall i \in N,$$

and

$$\hat{\xi}(\beta, c, x) = (1-\beta)(2+\kappa_{T^c}) \left(1 + \frac{(1+\delta)^{t_{T^c}}}{\delta^{t_{T^c}}} \left(1 - \frac{\delta}{1+\delta}\right) t_{T^c} L \|x\|_\infty \right) \mathrm{Rg}(x)$$

Proposition 7 *In the setting of Proposition 4, suppose that T is continuously differentiable in a neighborhood of 0, $\underline{A}(T)$ is regular, and fix $\beta \in (0,1)$. The constant profile of efforts $a^* \in \mathbb{R}_+^n$ given by*

$$a_i^* = \max_{\gamma \in \Gamma(c)} \langle \gamma, x \rangle \qquad \forall i \in N$$

is an ε-equilibrium of the game for all $\varepsilon > \hat{\xi}(\beta, c, x)$.

8.3 Computing the fixed point

For some applications, most notably the production networks one considered in Section 3.1, it is of independent interest to derive a formula for the fixed points even for values of β relatively far from 1. The next result provides such a formula in the concave case.

Proposition 8 *Let T be normalized, monotone, and translation invariant. If there exists a collection $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ such that S_α is monotone and $S_\alpha \geq T$ for all $\alpha \in \mathcal{A}$ and for each $x \in \mathbb{R}^k$ there exists $\alpha_x \in \mathcal{A}$ such that $S_{\alpha_x}(x) = T(x)$, then*

$$x_{\beta,T} = \min_{\alpha \in \mathcal{A}} x_{\beta, S_\alpha} \qquad \forall \beta \in (0,1), \forall x \in \mathbb{R}^k.$$

As in the linear case, the fixed point is not constant across entries and is less clean than the one from the limit case, but nevertheless turns out to be handy in relevant examples.

Example 5 Suppose each S_α is such that $S_\alpha(x) = W_\alpha x + h_\alpha$ for all $x \in \mathbb{R}^k$ where $W_\alpha \in W$ and $h_\alpha \in \mathbb{R}^k$. Recall that $x_{\beta,W_\alpha} = (1-\beta)\sum_{t=0}^{\infty} \beta^t W_\alpha^{t+1} x$ and $W_\alpha((1-\beta)x + \beta x_{\beta,W_\alpha}) = x_{\beta,W_\alpha}$. Define $\hat{x}_{\beta,W_\alpha} = x_{\beta,W_\alpha} + \sum_{t=0}^{\infty} \beta^t W_\alpha^t h_\alpha$. We next show that $\hat{x}_{\beta,W_\alpha} = x_{\beta,S_\alpha}$. Note that

$$\begin{aligned} S_\alpha\left((1-\beta)x + \beta\hat{x}_{\beta,W_\alpha}\right) &= W_\alpha\left((1-\beta)x + \beta\hat{x}_{\beta,W_\alpha}\right) + h_\alpha \\ &= W_\alpha\left((1-\beta)x + \beta x_{\beta,W_\alpha} + \sum_{t=0}^{\infty}\beta^{t+1}W_\alpha^t h_\alpha\right) + h_\alpha \\ &= W_\alpha\left((1-\beta)x + \beta x_{\beta,W_\alpha}\right) + \sum_{t=0}^{\infty}\beta^{t+1}W_\alpha^{t+1}h_\alpha + h_\alpha \\ &= x_{\beta,W_\alpha} + \sum_{t=0}^{\infty}\beta^t W_\alpha^t h_\alpha = \hat{x}_{\beta,W_\alpha}, \end{aligned}$$

proving that $\hat{x}_{\beta,W_\alpha} = x_{\beta,S_\alpha}$. ▲

8.4 Beyond normalization

So far we have always assumed that the operator T is monotone, translation invariant, and normalized. In some applications, normalization is not a natural property, hence it is natural to ask whether it can be relaxed. In this section, we show that our main results extend to operators that do not satisfy normalization, provided that some additional regularity conditions are satisfied.

Fix an operator $T : \mathbb{R}^k \to \mathbb{R}^k$ that is monotone, translation invariant, and such that

$$\emptyset \neq E(T) \subseteq \left\{x^* + ce \in \mathbb{R}^k : c \in \mathbb{R}\right\},$$

for some $x^* \in \mathbb{R}^k$. First, observe that we can still define the operator graph $\underline{A}(T)$ for T even if it is not normalized. Next, define $T^* : \mathbb{R}^k \to \mathbb{R}^k$ as

$$T^*(x) = T(x^* + x) - x^*.$$

Also, for every $\beta \in (0,1)$ define x_β^* and $\tilde{x}_\beta^*$ respectively as the unique solutions of the two equations (1) and (2) with respect to T^*.

Lemma 3 *We have $x_\beta = (x - x^*)_\beta^* + x^*$ for all $\beta \in (0,1)$ and all $x \in \mathbb{R}^k$. In particular, we have*

$$\lim_{\beta\to 1}\tilde{x}_\beta = \lim_{\beta\to 1}x_\beta = \lim_{\beta\to 1}(x - x^*)_\beta^* + x^*.$$

Lemma 4 *The following facts hold:*

1. *T^* is normalized and such that $E(T^*) = D$, moreover $\underline{A}(T^*) = \underline{A}(T)$.*

2. *If T is continuously differentiable in a neighborhood of x^*, then T^* is continuously differentiable in a neighborhood of* 0.

3. *If T is concave, then T^* is concave.*

4. *If T is such that,*

$$T\left(\left(1-\lambda\right)x^{*}+\lambda x\right)\geq\left(1-\lambda\right)T\left(x^{*}\right)+\lambda T\left(x\right)$$

for all $x \in \mathbb{R}^k$ and $\lambda \in (0,1)$, then T^* is star-shaped.

8.5 Vectorial $\boldsymbol{\beta}$

Given $k \in \mathbb{N}$, we endow $\mathbb{R}^k$ with the supnorm. We consider a nonexpansive operator $T : \mathbb{R}^k \to \mathbb{R}^k$. Given $\boldsymbol{\beta} \in (0,1)^k$ and $x \in \mathbb{R}^k$, consider the equation

$$y = T\left(\left(1-\boldsymbol{\beta}\right)x + \boldsymbol{\beta}y\right).$$

It is easy to verify that indeed there is only one point y, which we call $x_{\boldsymbol{\beta}}$, that satisfies the above equation.[14] If T is normalized, monotone, and translation invariant, then $\min_{i\in\{1,\dots,k\}} x_i \leq x_{\boldsymbol{\beta}} \leq \max_{i\in\{1,\dots,k\}} x_i$, that is, $\left\|x_{\boldsymbol{\beta}}\right\|_\infty \leq \left\|x\right\|_\infty$ for all $\boldsymbol{\beta} \in (0,1)^k$.

Lemma 5 *Let $T : \mathbb{R}^k \to \mathbb{R}^k$ be normalized, monotone, and translation invariant. If $\boldsymbol{\beta}, \boldsymbol{\gamma} \in (0,1)^k$, then*

$$\left\|x_{\boldsymbol{\gamma}} - x_{\boldsymbol{\beta}}\right\|_\infty = \left\|x_{\boldsymbol{\beta}} - x_{\boldsymbol{\gamma}}\right\|_\infty \leq 2\frac{\left\|\boldsymbol{\beta}-\boldsymbol{\gamma}\right\|_\infty}{1-\left\|\boldsymbol{\gamma}\right\|_\infty}\left\|x\right\|_\infty.$$

When the vector $\boldsymbol{\beta}$ is constant, we might just denote it by β: the constant value it takes. In general, we set $\overline{\boldsymbol{\beta}} = \max_{i\in\{1,\dots,k\}} \beta_i$ and $\underline{\boldsymbol{\beta}} = \min_{i\in\{1,\dots,k\}} \beta_i$.

Lemma 6 *Let $T : \mathbb{R}^k \to \mathbb{R}^k$ be normalized, monotone, and translation invariant and a net $\{\boldsymbol{\beta}_\alpha\}_{\alpha\in A} \subseteq (0,1)^k$ such that $\boldsymbol{\beta}_\alpha \to e$. If $\lim_{\beta\to1} x_\beta = \bar{x}$ and*

$$\overline{\boldsymbol{\beta}_\alpha} - \underline{\boldsymbol{\beta}_\alpha} = o\left(1-\overline{\boldsymbol{\beta}_\alpha}\right),$$

then $\lim_\alpha x_{\boldsymbol{\beta}_\alpha} = \bar{x}$.

[14] Consider the map $T_{\boldsymbol{\beta},x} : \mathbb{R}^k \to \mathbb{R}^k$ defined by $T_{\boldsymbol{\beta},x}(y) = T\left(\left(1-\boldsymbol{\beta}\right)x + \boldsymbol{\beta}\,y\right)$ for all $y \in \mathbb{R}^k$. It follows that for each $y, z \in \mathbb{R}^k$

$$\begin{aligned}\left\|T_{\boldsymbol{\beta},x}(y) - T_{\boldsymbol{\beta},x}(z)\right\|_\infty &\leq \left\|\left(1-\boldsymbol{\beta}\right)x + \boldsymbol{\beta}\,y - \left(1-\boldsymbol{\beta}\right)x - \boldsymbol{\beta}z\right\|_\infty \\ &= \left\|\boldsymbol{\beta}\left(y-z\right)\right\|_\infty \leq \left(\max_{i\in\{1,\dots,k\}} \beta_i\right)\left\|y-z\right\|_\infty,\end{aligned}$$

proving that $T_{\boldsymbol{\beta},x}$ is a $\max_{i\in\{1,\dots,k\}} \beta_i$-contraction.

A Appendix: Preliminaries

A.1 Nonnegative matrices

Given a nonnegative matrix $M \in \mathbb{R}_{+}^{k\times k}$, let $\underline{A} = \underline{A}(M) \in \{0,1\}^{k\times k}$ be the adjacency matrix of M. For every $t \in \mathbb{N}$, denote by $\underline{a}_{ij}^{(t)}$ the ij-th entry of $\underline{A}^t$. We write $i \overset{A}{\to} j$ if and only if $\underline{a}_{ij}^{(t)} > 0$ for some $t \in \mathbb{N}$. We also write $i \overset{A}{\longleftrightarrow} j$ if and only if $i \overset{A}{\to} j$ and $j \overset{A}{\to} i$. An index i is *essential* if and only if for each j

$$i \overset{A}{\to} j \implies i \overset{A}{\longleftrightarrow} j,$$

otherwise is *inessential*. For each essential i, define $[i] = \left\{ j : i \overset{A}{\longleftrightarrow} j \right\}$. Note that given two essential indexes i and j we have that either $[i] = [j]$ or $[i] \cap [j] = \emptyset$. Moreover, given i, j such that $i \overset{A}{\longleftrightarrow} j$, i is essential if and only if j is. Thus, we can partition $\{1, ..., k\}$ in classes of essential indexes and a, possibly empty, subset of inessential indexes.

A.2 Clarke differential

Given $z \in \mathbb{R}^k$, we denote by $\partial_C T_i(z)$ the Clarke differential of the i-th component of T at z. If T_i is concave, it is well known that $\partial_C T_i(z)$ coincides with $\partial T_i(z)$ where the latter is the usual superdifferential of convex analysis (see, e.g., [17, Proposition 2.2.7]). Therefore

$$W \in \partial_C T(z) \implies W(y - z) \geq T(y) - T(z) \qquad \forall y \in \mathbb{R}^k. \tag{23}$$

Since T is normalized, monotone, and translation invariant, we have that $\partial_C T_i(z + he) = \partial_C T_i(z)$ for all $i \in \{1, ..., k\}$, for all $z \in \mathbb{R}^k$, and for all $h \in \mathbb{R}$. In particular, we also have that $\partial_C T(he) = \partial_C T(0)$ for all $h \in \mathbb{R}$. Observe that $\partial_C T_i(z)$ is a collection of probability vectors. We denote by $\partial_C T(z)$ the collection of all $k \times k$ (stochastic) matrices whose i-th row belongs to $\partial_C T_i(z)$.[15] Recall that by [17, Propositions 2.1.2 and 2.1.5], the correspondence $\partial_C T_i : \mathbb{R}^k \rightrightarrows \mathbb{R}^k$ is nonempty-, convex-, compact-valued, and upper hemicontinuous.

For every normalized, monotone, and translation invariant T, also denote as $\hat{\partial}_C T$ the generalized Jacobian of T as defined in [17, Proposition 2.6.2]:

$$\hat{\partial}_C T(x) = \operatorname{co}\left\{ \gamma \in \mathbb{R}^k : \gamma = \lim_k J_T\left(z^k\right) \text{ s.t. } z^k \to z \text{ and } z^k \in \mathcal{D} \right\}$$

where for every $z \in \mathcal{D}$, $J_T(z)$ denotes the (usual) Jacobian of T at z.

[15] The notion of generalized gradient we use for real-valued functions coincides with the one of Clarke, but our derived notion of generalized Jacobian for operators is larger than the one of Clarke (see, e.g., [17, Proposition 2.6.2]). With an abuse of notation, we still denote it by $\partial_C T(z)$.

B Appendix: A representation result

In this appendix, we consider a functional $g : \mathbb{R}^k \to \mathbb{R}$ which is normalized, monotone, translation invariant, and star-shaped.[16] The objective is to prove that such a functional can be rewritten as the max of a collection $\{g_\alpha\}_{\alpha \in \mathcal{A}}$ of normalized, monotone, translation invariant, and concave functionals. Results of this form have appeared in Decision Theory (see, e.g., Chandrasekher, Frick, Iijima, and Le Yaouanq [15]), Mathematical Finance (see, e.g., Castagnoli, Cattelan, Maccheroni, Tebaldi, and Wang [11, Theorem 2]), and Mathematics (see, e.g., Rubinov and Dzalilov [37]). The version we need for this paper is slightly different from what is available in the literature and it is a refinement of [11], whose techniques we also exploit. Compared to their Theorem 5, we obtain a version in which $\{g_\alpha\}_{\alpha \in \mathcal{A}}$ "inherits the derivatives" of g: a property which we badly need for our convergence results.

Proposition 9 *Let $g : \mathbb{R}^k \to \mathbb{R}$. The following statements are equivalent:*

(i) The functional g is normalized, monotone, translation invariant, and star-shaped;

(ii) There exists a family $\{g_\alpha\}_{\alpha \in \mathcal{A}}$ of normalized, monotone, translation invariant, and concave functionals such that

$$g(x) = \max_{\alpha \in \mathcal{A}} g_\alpha(x) \qquad \forall x \in \mathbb{R}^k. \tag{24}$$

Moreover, $\{g_\alpha\}_{\alpha \in \mathcal{A}}$ can be chosen to be such that $\bar{\text{co}}\left(\partial_C g_\alpha\left(\mathbb{R}^k\right)\right) \subseteq \bar{\text{co}}\left(\partial_C g\left(\mathbb{R}^k\right)\right)$ for all $\alpha \in \mathcal{A}$.

Before proving the statement, we need to introduce an ancillary object. Given $g : \mathbb{R}^k \to \mathbb{R}$, define the binary relation $\succsim_g^*$ by

$$x \succsim_g^* y \overset{def}{\iff} g(\lambda x + (1-\lambda) z) \geq g(\lambda y + (1-\lambda) z) \qquad \forall \lambda \in (0,1], \forall z \in \mathbb{R}^k.$$

It is immediate to see that $x \succsim_g^* y$ implies that $g(x) \geq g(y)$. By Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi [13] and if g is normalized, monotone, and continuous, we have that there exists a closed convex set $C_g \subseteq \Delta$ such that

$$x \succsim_g^* y \iff \langle \gamma, x \rangle \geq \langle \gamma, y \rangle \qquad \forall \gamma \in C_g. \tag{25}$$

Moreover, if $\succsim^\circ$ is another conic binary relation such that

$$x \succsim^\circ y \implies g(x) \geq g(y),$$

[16] With a small abuse of terminology, we use the same name for similar properties that pertain to functionals and operators.

then $\succsim^\circ$ is a subrelation of $\succsim_g^*$, that is, $x \succsim^\circ y$ implies $x \succsim_g^* y$.[17] Recall that if g is normalized, monotone, and translation invariant $\partial_C g(x) \subseteq \Delta$ for all $x \in \mathbb{R}^k$. By Ghirardato and Siniscalchi [22, Theorem 2], in this case, we have that $C_g = \bar{\text{co}}\left(\partial_C g\left(\mathbb{R}^k\right)\right)$ where $\partial_C g\left(\mathbb{R}^k\right) = \cup_{x\in\mathbb{R}^k}\partial_C g(x)$.

Proof. (i) implies (ii). Define $P = \left\{x \in \mathbb{R}^k : x \succsim_g^* 0\right\}$. It is immediate to see that P is a nonempty, closed, and convex cone. Define $\mathcal{A} = \left\{z \in \mathbb{R}^k \backslash \{0\} : g(z) = 0\right\}$. For each $z \in \mathcal{A}$ define $U_z = \text{co}(\{0, z\}) + P$.[18] We say that a functional $\tilde{g} : \mathbb{R}^k \to \mathbb{R}$ is $\succsim_g^*$-monotone if and only if $x \succsim_g^* y$ implies $\tilde{g}(x) \geq \tilde{g}(y)$. Since $x \geq y$ implies $x \succsim_g^* y$, we have that $\succsim_g^*$-monotonicity yields standard monotonicity.

Step 1. For each $z \in \mathcal{A}$ the set U_z is a nonempty, convex, and closed set such that

1. $0, z \in U_z$;
2. *if $x \in U_z$, then $g(x) \geq 0$;*
3. *if $y \succsim_g^* x \in U_z$, then $y \in U_z$;*
4. *if $h > 0$, then $-he \notin U_z$.*

Proof of the Step. Since $0, z \in \text{co}(\{0, z\})$ and $0 \in P$ and $\text{co}(\{0, z\})$ is convex and compact and P is convex and closed, we have that $0, z \in U_z = \text{co}(\{0, z\}) + P$ is nonempty, convex, and closed, and, in particular, point 1 holds. If $x \in U_z$, then there exist $\lambda \in [0, 1]$ and $y \in P$ such that $x = \lambda z + (1-\lambda)0 + y$. Since g is star-shaped and $g(z) = 0$, we have that $g(\lambda z + (1-\lambda)0) = g(\lambda z) \geq \lambda g(z) = 0$. Since $y \in P$, we have that $x = \lambda z + (1-\lambda)0 + y \succsim_g^* \lambda z + (1-\lambda)0$, yielding that $g(x) \geq g(\lambda z + (1-\lambda)0) \geq 0$, proving point 2. Next, consider $x, y \in \mathbb{R}^k$ such that $y \succsim_g^* x \in U_z$, that is, $y - x \succsim_g^* 0$ and $x \in U_z$. Since $x \in U_z$, then there exist $\lambda \in [0, 1]$ and $\hat{y} \in P$ such that $x = \lambda z + \hat{y}$. Since P is a convex cone, it follows that $y = x + (y - x) = \lambda z + (\hat{y} + y - x) \in \text{co}(\{0, z\}) + P = U_z$. Finally, by contradiction, assume that $h < 0$ and $-he \in U_z$. By point 2 and since g is normalized, $0 > -h = g(-he) \geq 0$, a contradiction. □

Step 2. For each $z \in \mathcal{A}$ the functional $g_z : \mathbb{R}^k \to \mathbb{R}$, defined by

$$g_z(x) = \max\{h \in \mathbb{R} : x - he \in U_z\} \qquad \forall x \in \mathbb{R}^k,$$

is well defined, normalized, $\succsim_g^*$-monotone, translation invariant, concave, and such that $g_z(z) = 0$ as well as $g_z(z') \leq 0$ for all $z' \in \mathcal{A}$.

Proof of the Step. Fix $z \in \mathcal{A}$. Consider $x \in \mathbb{R}^k$. Define $I_x = \{h \in \mathbb{R} : x - he \in U_z\}$. Since U_z is convex and closed, I_x is a closed interval. Next we show that I_x is bounded

[17] The binary relation $\succsim^\circ$ is conic if and only if there exists a subset $\tilde{C} \subseteq \Delta$ such that $x \succsim^\circ y$ if and only if $\langle \gamma, x\rangle \geq \langle \gamma, y\rangle$ for all $\gamma \in \tilde{C}$.

[18] The construction of [11] differs from ours in that the cone added to $\text{co}(\{0, z\})$ is $\mathbb{R}_+^k$.

from above. Let $h \geq \|x\|_\infty + \|z\|_\infty$. Since $z \neq 0$ and g is normalized and monotone, note that $g(x - \|x\|_\infty e - \|z\|_\infty e) \leq g(-\|z\|_\infty e) = -\|z\|_\infty < 0$. By point 2 above, we have that $x - \|x\|_\infty e - \|z\|_\infty e \notin U_z$. By point 3 and since $x - (\|x\|_\infty + \|z\|_\infty) e \geq x - he$, we can conclude that $x - he \notin U_z$, proving that I_x is bounded from above. Since $C_g \subseteq \Delta$ is compact, consider $h \in \mathbb{R}$ such that $-h \geq \max_{\gamma \in C} \{\langle \gamma, z \rangle - \langle \gamma, x \rangle\} \in \mathbb{R}$. By points 1 and 3 and the characterization of $\succsim_g^*$, it follows that $x - he \succsim_g^* z \in U_z$, proving that $x - he \in U_z$ and I_x is nonempty. Since I_x is a nonempty, closed, and bounded from above interval, we have that $\sup I_x$ is well defined and attained, proving that g_z is well defined. In particular, $x - g_z(x) e \in U_z$ for all $x \in \mathbb{R}^k$.

Consider $z' \in \mathcal{A}$. By point 2 and since $z' \in \mathcal{A}$ and $z' - g_z(z') e \in U_z$, we have that $g(z') - g_z(z') = g(z' - g_z(z') e) \geq 0$, that is, $0 = g(z') \geq g_z(z')$. By point 1, if $z' = z$, then $0 \in I_z$ and $g_z(z) \geq 0$. Since z' was arbitrarily chosen, we can conclude that $g_z(z') \leq 0$ for all $z' \in \mathcal{A}$ and $g_z(z) = 0$. Consider $x, y \in \mathbb{R}^k$ such that $x \succsim_g^* y$. By point 3, (25), and the definition of g_z, we have that $x - g_z(y) e \succsim_g^* y - g_z(y) e \in U_z$, yielding that $g_z(y) \in I_x$ and $g_z(x) \geq g_z(y)$, that is, g_z is $\succsim_g^*$-monotone. Consider $x \in \mathbb{R}^k$ and $h \in \mathbb{R}$. By definition of g_z, we can conclude that

$$(x + he) - (g_z(x) + h) e = x - g_z(x) e \in U_z.$$

This implies that $g_z(x) + h \in I_{x+he}$ and, in particular, $g_z(x + he) \geq g_z(x) + h$. Since x and h were arbitrarily chosen, we have that

$$g_z(x + he) \geq g_z(x) + h \qquad \forall x \in \mathbb{R}^k, \forall h \in \mathbb{R}.$$

This yields that $g_z(x + he) = g_z(x) + h$ for all $x \in \mathbb{R}^k$ and for all $h \in \mathbb{R}$. Finally, consider $x, y \in \mathbb{R}^k$ and $\lambda \in (0, 1)$. By definition of g_z, we have that $x - g_z(x) e, y - g_z(y) e \in U_z$. Since U_z is convex, this implies that $\lambda x + (1 - \lambda) y - (\lambda g_z(x) + (1 - \lambda) g_z(y)) e \in U_z$, yielding that $g_z(\lambda x + (1 - \lambda) y) \geq \lambda g_z(x) + (1 - \lambda) g_z(y)$ and proving that g_z is concave.

To sum up, g_z is well defined, $\succsim_g^*$-monotone, translation invariant, and concave. Consider $x = 0$. By point 1, we have that $0 \in U_z$, yielding that $0 \in I_0$ and, in particular, $g_z(0) \geq 0$. By point 4, we have that $I_0 \subseteq (-\infty, 0]$, proving that $g_z(0) \leq 0$, that is, $g_z(0) = 0$. Since g_z is translation invariant and $g_z(0) = 0$, it follows that $g_z(he) = g_z(0 + he) = g_z(0) + h = h$ for all $h \in \mathbb{R}$, that is, g_z is normalized, proving the step. $\square$

We can prove the implication. Consider the family of functionals $\{g_z\}_{z \in \mathcal{A}}$ of Step 2. Each g_z is normalized, $\succsim_g^*$-monotone (in particular, monotone), translation invariant, and concave. Consider $x \in \mathbb{R}^k$. Since g and g_z are normalized for all $z \in \mathcal{A}$, if $x = he$ for some $h \in \mathbb{R}$, we have that $g(x) = h = g_z(x)$ for all $z \in \mathcal{A}$, that is, $g(x) = \max_{z \in \mathcal{A}} g_z(x)$. If x is not a constant vector, define $\bar{z} = x - g(x) e$. Note that $\bar{z} \neq 0$. Since g is translation invariant, we have that $\bar{z} \in \mathcal{A}$. By Step 2, we have that $g_z(\bar{z}) \leq 0 = g_{\bar{z}}(\bar{z}) = 0 = g(\bar{z})$

for all $z \in \mathcal{A}$. Since each g_z is translation invariant, we have that

$$g(x) - g(x) = g(x - g(x)e) = g(\bar{z}) = \max_{z\in\mathcal{A}} g_z(\bar{z}) = \max_{z\in\mathcal{A}} g_z(x - g(x)e)$$
$$= \max_{z\in\mathcal{A}} \{g_z(x) - g(x)\} = \max_{z\in\mathcal{A}} g_z(x) - g(x),$$

proving (24).

(ii) implies (i). It is trivial.

Consider $\{g_z\}_{z\in\mathcal{A}}$ as in the proof of (i) implies (ii). Fix $z \in \mathcal{A}$. By Step 2, we have that g_z is $\succsim_g^*$-monotone. This implies that $x \succsim_g^* y$ implies $x \succsim_{g_z}^* y$. By the Hahn-Banach Theorem, this yields that $\bar{\text{co}}\left(\partial_C g\left(\mathbb{R}^k\right)\right) = C_g \supseteq C_{g_z} = \bar{\text{co}}\left(\partial_C g_z\left(\mathbb{R}^k\right)\right)$, proving the last part of the statement. ■

Given a family of normalized, monotone, translation invariant, and concave operators $\{S_\alpha\}_{\alpha\in\mathcal{A}}$ such that $E(S_\alpha) = D$ for all $\alpha \in \mathcal{A}$, we say that it represents $T : \mathbb{R}^k \to \mathbb{R}^k$ (and is nice) if and only if $T(x) = \sup_{\alpha\in\mathcal{A}} S_\alpha(x)$ for all $x \in \mathbb{R}^k$ (and the previous sup is achieved for all $x \in \mathbb{R}^k$, that is, for each $x \in \mathbb{R}^k$ there exists $\alpha_x \in \mathcal{A}$ such that $T(x) = S_{\alpha_x}(x)$).

Corollary 3 *Let T be normalized, monotone, and translation invariant. If T is star-shaped and $\underline{A}(T)$ is regular, then there exists a collection $\{S_\alpha\}_{\alpha\in\mathcal{A}}$ which represents T, is nice, and such that $\underline{A}(S_\alpha)$ is regular for all $\alpha \in \mathcal{A}$. Moreover, if T is positively homogeneous, then S_α can be chosen to be positively homogeneous for all $\alpha \in \mathcal{A}$.*

Proof. Note that T_i is normalized, monotone, translation invariant, and star-shaped (resp. positively homogeneous) for all $i \in \{1, ..., k\}$. By Proposition 9, we have that for each $i \in \{1, ..., k\}$ there exists a family $\{S_{\alpha_i}\}_{\alpha_i\in\mathcal{A}_i}$ of normalized, monotone, translation invariant, and concave functionals (resp. and positively homogeneous) such that

$$T_i(x) = \max_{\alpha_i\in\mathcal{A}_i} S_{\alpha_i}(x) \qquad \forall x \in \mathbb{R}^k \tag{26}$$

and $\bar{\text{co}}\left(\partial_C S_{\alpha_i}\left(\mathbb{R}^k\right)\right) \subseteq \bar{\text{co}}\left(\partial_C T_i\left(\mathbb{R}^k\right)\right)$ for all $\alpha_i \in \mathcal{A}_i$. Define $\mathcal{A} = \Pi_{i=1}^k \mathcal{A}_i$ and for each $\alpha \in \mathcal{A}$ define $S_\alpha : \mathbb{R}^k \to \mathbb{R}^k$ to be such that its i-th component coincides with S_{α_i} for all $i \in \{1, ..., k\}$. It is immediate to see that S_α is normalized, monotone, translation invariant, and concave (resp. and positively homogeneous) for all $\alpha \in \mathcal{A}$. Since $\bar{\text{co}}\left(\partial_C S_{\alpha_i}\left(\mathbb{R}^k\right)\right) \subseteq \bar{\text{co}}\left(\partial_C T_i\left(\mathbb{R}^k\right)\right)$ for all $\alpha_i \in \mathcal{A}_i$ and for all $i \in \{1, ..., k\}$, it follows that $\underline{A}(S_\alpha) \geq \underline{A}(T)$ for all $\alpha \in \mathcal{A}$. By Lemmas 10 and 11 and since $\underline{A}(T)$ is regular, this implies that $\underline{A}(S_\alpha)$ is regular and $E(S_\alpha) = D$ for all $\alpha \in \mathcal{A}$. By (26) and since $\mathcal{A}$ has a product structure, we have that

$$T(x) = \sup_{\alpha\in\mathcal{A}} S_\alpha(x) \qquad \forall x \in \mathbb{R}^k$$

and for each $x \in \mathbb{R}^k$ there exists $\alpha_x \in \mathcal{A}$ such that $T(x) = S_{\alpha_x}(x)$. We can conclude that $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ represents T, is nice, and such that $\underline{A}(S_\alpha)$ is regular for all $\alpha \in \mathcal{A}$. Moreover, if T is positively homogeneous, S_α can be chosen to be positively homogeneous for all $\alpha \in \mathcal{A}$, proving also the second part of the statement. ■

C Appendix: Proofs of Section 2

The next result shows that, given $z \in \mathbb{R}^k$, the value of $T(z)$ can be calculated alternatively by computing Wz where W is a "replicating" stochastic matrix that belongs to the Clarke differential of T.

Proposition 10 *Let T be normalized, monotone, and translation invariant. For each $z \in \mathbb{R}^k$ and for each $\hat{h} \in \mathbb{R}$, there exists a stochastic matrix $W_{z,\hat{h}}$ such that $T(z) = W_{z,\hat{h}}z$ and $W_{z,\hat{h}} \in co\left(\cup_{\lambda\in[0,1]}\hat{\partial}_C T\left(\lambda z + (1-\lambda)\hat{h}e\right)\right)$.*

Proof. Consider $z \in \mathbb{R}^k$ and $\hat{h} \in \mathbb{R}$. By Clarke [17, Theorem 2.6.5] and since T is normalized and Lipschitz continuous, there exists $W \in co\left(\cup_{\lambda\in[0,1]}\hat{\partial}_C T\left(\lambda z + (1-\lambda)\hat{h}e\right)\right)$ such that

$$T(z) - \hat{h}e = T(z) - T\left(\hat{h}e\right) = W\left(z - \hat{h}e\right) = Wz - \hat{h}e$$

and the statement follows. ■

A similar result applies under our more permissive definition of the Clarke differential of an operator.

Proposition 11 *Let T be normalized, monotone, and translation invariant. For each $z \in \mathbb{R}^k$ and for each $\hat{h} \in \mathbb{R}$, there exists a stochastic matrix $W_{z,\hat{h}}$ such that $T(z) = W_{z,\hat{h}}z$ and $w^i_{z,\hat{h}} \in \partial_C T_i\left(\lambda_{i,z,\hat{h}}z + \left(1-\lambda_{i,z,\hat{h}}\right)\hat{h}e\right)$ where $\lambda_{i,z,\hat{h}} \in [0,1]$ for all $i \in \{1,...,k\}$.*

Proof. Consider $z \in \mathbb{R}^k$ and $\hat{h} \in \mathbb{R}$. By Lebourg's Mean Value Theorem (see, e.g., Clarke [17, Theorem 2.3.7]) and since T is normalized and Lipschitz continuous, for each $i \in \{1,...,k\}$ there exists $w^i \in \partial_C T_i\left(\lambda_{i,z,\hat{h}}z + \left(1-\lambda_{i,z,\hat{h}}\right)\hat{h}e\right)$ where $\lambda_{i,z,\hat{h}} \in [0,1]$ such that

$$T_i(z) - \hat{h} = T_i(z) - T_i\left(\hat{h}e\right) = \left\langle w^i, z - \hat{h}e\right\rangle = \left\langle w^i, z\right\rangle - \hat{h}.$$

If we define $W_{z,\hat{h}}$ to be such that its i-th row is w^i, then the statement follows. ■

The next preliminary result guarantees that the adjacency matrices of the replicating matrices and of the generalized Jacobians of T inherit the property of regularity of $\underline{A}(T)$. Moreover, it provides a quantitative lower bound for the entries of the replicating matrices. The first property will be exploited in proving that $\{x_\beta\}_{\beta\in(0,1)}$ and $\{\tilde{x}_\beta\}_{\beta\in(0,1)}$

converge while the latter will be useful in the proofs that elaborate on the rate of such convergence. Given a stochastic matrix W, we define the adjacency matrix $A(W)$ to be such that $a_{ij} = 1$ if and only if $w_{ij} > 0$ and $a_{ij} = 0$ otherwise. Recall that (see, e.g., [17, Theorem 2.5.1])

$$\partial_C T_i(z) = \operatorname{co}\left\{\gamma \in \mathbb{R}^k : \gamma = \lim_k \nabla T_i\left(z^k\right) \text{ s.t. } z^k \to z \text{ and } z^k \in \mathcal{D}\right\}. \tag{27}$$

Proposition 12 *Let T be normalized, monotone, and translation invariant. The following statements are true:*

1. *If $\underline{A}(T)$ is regular, then $A\left(W_{z,\hat{h}}\right)$ is regular for all $z \in \mathbb{R}^k$ and for all $\hat{h} \in \mathbb{R}$. Moreover, we have that*

$$\min_{i,j:\underline{a}_{ij}=1} w_{ij} \geq \min_{i,j:\underline{a}_{ij}=1} \inf_{x\in\mathcal{D}} \frac{\partial T_i}{\partial x_j}(x) \tag{28}$$

where w_{ij} is the ij-th entry of $W_{z,\hat{h}}$.

2. *If $\underline{A}(T)$ is regular, then $A(W)$ is regular for all $W \in \partial_C T(0)$. Moreover, we have that*

$$\min_{i,j:\underline{a}_{ij}=1} w_{ij} \geq \min_{i,j:\underline{a}_{ij}=1} \inf_{x\in\mathcal{D}} \frac{\partial T_i}{\partial x_j}(x).$$

As a consequence, if $\underline{A}(T)$ is regular, we have that $W_{z,\hat{h}}$ has a unique left Perron-Frobenius eigenvector for all $z \in \mathbb{R}^k$ and for all $\hat{h} \in \mathbb{R}$. We denote it by $\gamma_{z,\hat{h}}$.

Proof. 1. By (22) and (27), we have that $\gamma_j \geq \varepsilon_{ij} > 0$ for all $\gamma \in \partial_C T_i(z)$, for all $i,j \in \{1,...,k\}$ such that $\underline{a}_{ij} = 1$, and for all $z \in \mathbb{R}^k$. By definition of $W_{z,\hat{h}}$ and w_{ij}, and Clarke [17, Proposition 2.6.2], this implies that $w_{ij} \geq \varepsilon_{ij} > 0$ for all $i,j \in \{1,...,k\}$ such that $\underline{a}_{ij} = 1$ and $A\left(W_{z,\hat{h}}\right) \geq \underline{A}(T)$ for all $z \in \mathbb{R}^k$ and for all $\hat{h} \in \mathbb{R}$. Since $\underline{A}(T)$ is regular, we can conclude that $A\left(W_{z,\hat{h}}\right)$ is regular for all $z \in \mathbb{R}^k$ and for all $\hat{h} \in \mathbb{R}$. By (22) and since $w_{ij} \geq \varepsilon_{ij}$ for all $i,j \in \{1,...,k\}$ such that $\underline{a}_{ij} = 1$, we have that (28) follows.

2. By (22) and (27), we have that $\gamma_j \geq \varepsilon_{ij} > 0$ for all $\gamma \in \partial_C T_i(0)$, for all $i,j \in \{1,...,k\}$ such that $\underline{a}_{ij} = 1$, and for all $z \in \mathbb{R}^k$. By definition of $\partial_C T(0)$, this implies that $A(W) \geq \underline{A}(T)$ for all $W \in \partial_C T(0)$. Since $\underline{A}(T)$ is regular, we can conclude that $A(W)$ is regular. Similarly to before, we can conclude that $\min_{i,j:\underline{a}_{ij}=1} w_{ij} \geq \min_{i,j:\underline{a}_{ij}=1} \inf_{x\in\mathcal{D}} \frac{\partial T_i}{\partial x_j}(x)$. ■

C.1 Convergence

Given $T : \mathbb{R}^k \to \mathbb{R}^k$, define $T_{\beta,x} : \mathbb{R}^k \to \mathbb{R}^k$ by $T_{\beta,x}(y) = T((1-\beta)x + \beta y)$. Clearly, the fixed points of $T_{\beta,x}$ are the solutions of (1).

Lemma 7 *Let T be normalized, monotone, and translation invariant. If $\beta \in (0,1)$ and $x \in \mathbb{R}^k$, then $T_{\beta,x}$ is a β-contraction. In particular, for each $\beta \in (0,1)$ and for each $x \in \mathbb{R}^k$, there exists unique $x_\beta \in \mathbb{R}^k$ such that*

$$T^t_{\beta,x}(y) \to x_\beta \quad \forall y \in \mathbb{R}^k,\ T_{\beta,x}(x_\beta) = x_\beta,\ \text{and}\ \|x_\beta\|_\infty \leq \|x\|_\infty . \tag{29}$$

Proof. Fix $\beta \in (0,1)$ and $x \in \mathbb{R}^k$. We prove that $T_{\beta,x}$ is a β-contraction. Since T is Lipschitz continuous of order 1, we have that for each $y, z \in \mathbb{R}^k$

$$\|T_{\beta,x}(y) - T_{\beta,x}(z)\|_\infty = \|T((1-\beta)x + \beta y) - T((1-\beta)x + \beta z)\|_\infty \leq \beta \|y - z\|_\infty$$

proving that $T_{\beta,x}$ is a β-contraction. By the Banach contraction principle, for each $y \in \mathbb{R}^k$ we have that $T^t_{\beta,x}(y) \to x_\beta$ as well as $T_{\beta,x}(x_\beta) = x_\beta$ where x_β is the unique fixed point of $T_{\beta,x}$. Finally, since T is normalized and Lipschitz continuous of order 1, observe that for all $y \in \mathbb{R}^k$

$$\|T_{\beta,x}(y)\|_\infty = \|T((1-\beta)x + \beta y) - T(0)\|_\infty \leq \|(1-\beta)x + \beta y\|_\infty \leq (1-\beta)\|x\|_\infty + \beta\|y\|_\infty .$$

By induction, this implies that $\left\|T^t_{\beta,x}(x)\right\|_\infty \leq \|x\|_\infty$ for all $t \in \mathbb{N}$. By passing to the limit, (29) follows. ■

Proof of Lemma 1. It follows from Lemma 7. ■

Consider the set L of limit points of $\{x_\beta\}_{\beta\in(0,1)}$.[19] By construction and since $\{x_\beta\}_{\beta\in(0,1)}$ is bounded, the set L is closed and bounded. We define $\liminf_{\beta\to1} x_\beta = \inf L$ and $\limsup_{\beta\to1} x_\beta = \sup L$ where inf and sup are computed coordinatewise.

The next simple lemma yields that the limit points of $\{x_\beta\}_{\beta\in(0,1)}$ are fixed points of T and so are $\liminf_{\beta\to1} x_\beta$ and $\limsup_{\beta\to1} x_\beta$, provided $E(T) = D$.

Lemma 8 *If T is normalized, monotone, and translation invariant, then $L \subseteq E(T)$. Moreover, if $E(T) = D$, then $\liminf_{\beta\to1} x_\beta, \limsup_{\beta\to1} x_\beta \in E(T)$.*

We now provide various lower and upper bounds for $\liminf_{\beta\to1} x_\beta$ and $\limsup_{\beta\to1} x_\beta$.

Lemma 9 *Let T be normalized, monotone, and translation invariant. If $E(T) = D$, then*

$$\max_{\gamma\in\Gamma\left(\hat{\partial}_C T(0)\right)} \langle\gamma, x\rangle e \geq \limsup_{\beta\to1} x_\beta \geq \liminf_{\beta\to1} x_\beta \geq \min_{\gamma\in\Gamma\left(\hat{\partial}_C T(0)\right)} \langle\gamma, x\rangle e \qquad \forall x \in \mathbb{R}^k.$$

[19] That is, $\bar{x} \in L$ if and only if there exists $\{x_{\beta_n}\}_{n\in\mathbb{N}} \subseteq \{x_\beta\}_{\beta\in(0,1)}$ such that $\beta_n \to 1$ and $x_{\beta_n} \to \bar{x}$.

Proof. Consider a sequence $\{x_{\beta_n}\}_{n\in\mathbb{N}} \subseteq \{x_\beta\}_{\beta\in(0,1)}$ such that $\beta_n \to 1$ and $x_{\beta_n} \to \bar{x}$, that is in symbols, $\bar{x}$ is a limit point of $\{x_\beta\}_{\beta\in(0,1)}$ and $\bar{x} \in L$. By Lemma 8 and since $E(T) = D$, we have that $\bar{x} \in L \subseteq E(T)$ and $\bar{x} = \bar{h}e$ for some $\bar{h} \in \mathbb{R}$. Consider $\tilde{x}_{\beta_n} = (1-\beta_n)x + \beta_n x_{\beta_n}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ consider also $W_{\tilde{x}_{\beta_n},\bar{h}}$ as in Proposition 10. We have that for each $n \in \mathbb{N}$

$$W_{\tilde{x}_{\beta_n},\bar{h}}((1-\beta_n)x + \beta_n x_{\beta_n}) = W_{\tilde{x}_{\beta_n},\bar{h}}\tilde{x}_{\beta_n} = T(\tilde{x}_{\beta_n}) = T((1-\beta_n)x + \beta_n x_{\beta_n}) = x_{\beta_n}.$$

By (3) and Example 1, we have that $x_{\beta_n} = x_{\beta_n, W_{\tilde{x}_{\beta_n},\bar{h}}}$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ consider $\gamma_n \in \Gamma\left(W_{\tilde{x}_{\beta_n},\bar{h}}\right)$. By definition of $\Gamma\left(W_{\tilde{x}_{\beta_n},\bar{h}}\right)$ and (4), we have that

$$\langle \gamma_n, x_{\beta_n}\rangle = \left\langle \gamma_n, x_{\beta_n, W_{\tilde{x}_{\beta_n},\bar{h}}}\right\rangle = \langle \gamma_n, x\rangle \qquad \forall n \in \mathbb{N}. \tag{30}$$

Since $\left\{W_{\tilde{x}_{\beta_n},\bar{h}}\right\}_{n\in\mathbb{N}}$ is a sequence of stochastic matrices, it admits a subsequence $\left\{W_{\tilde{x}_{\beta_{n_l}},\bar{h}}\right\}_{l\in\mathbb{N}}$ such that $W_{\tilde{x}_{\beta_{n_l}},\bar{h}} \to W$. Similarly, since $\{\gamma_{n_l}\}_{l\in\mathbb{N}}$ is a sequence of probability vectors, it admits a subsequence $\left\{\gamma_{n_{l(r)}}\right\}_{r\in\mathbb{N}}$ such that $\gamma_{n_{l(r)}} \to \bar{\gamma}$. Since $\gamma_{n_{l(r)}} \in \Gamma\left(W_{\tilde{x}_{\beta_{n_{l(r)}}},\bar{h}}\right)$ for all $r \in \mathbb{N}$, we can conclude that

$$\bar{\gamma}^{\mathrm{T}} W = \lim_r \gamma^{\mathrm{T}}_{n_{l(r)}} W_{\tilde{x}_{\beta_{n_{l(r)}}},\bar{h}} = \lim_r \gamma^{\mathrm{T}}_{n_{l(r)}} = \bar{\gamma}^{\mathrm{T}},$$

that is, $\bar{\gamma} \in \Gamma(W)$.

We next prove that the correspondence $z \mapsto \cup_{\lambda\in[0,1]}\hat{\partial}_C T\left(\lambda z + (1-\lambda)\bar{h}e\right)$ is closed. Let $\{z_n, \rho_n\}_{n\in\mathbb{N}}$ with $\rho_n \in \cup_{\lambda\in[0,1]}\hat{\partial}_C T\left(\lambda z_n + (1-\lambda)\bar{h}e\right)$ for all $n \in \mathbb{N}$ and $\{z_n, \rho_n\}_{n\in\mathbb{N}} \to \{\hat{z}, \hat{\rho}\}_{n\in\mathbb{N}}$. Since $[0,1]$ is compact, there is a subsequence $\{z_{n_l}, \rho_{n_l}\}_{l\in\mathbb{N}}$ and a $\{\lambda_{n_l}\}_{l\in\mathbb{N}} \subseteq [0,1]$ with $\rho_{n_l} \in \hat{\partial}_C T\left(\lambda_{n_l} z_{n_l} + (1-\lambda_{n_l})\bar{h}e\right)$ for all $l \in \mathbb{N}$ and $\lambda_{n_l} \to \hat{\lambda}$. Since, by Clarke [17, Proposition 2.6.2], the correspondence $z \mapsto \hat{\partial}_C T(z)$ is closed,

$$\hat{\rho} \in \hat{\partial}_C T\left(\hat{\lambda}\hat{z} + \left(1-\hat{\lambda}\right)\bar{h}e\right) \subseteq \cup_{\lambda\in[0,1]}\hat{\partial}_C T\left(\lambda\hat{z} + (1-\lambda)\bar{h}e\right)$$

proving that $z \mapsto \cup_{\lambda\in[0,1]}\hat{\partial}_C T\left(\lambda z + (1-\lambda)\bar{h}e\right)$ is closed.

Therefore, by Aliprantis and Border [4, Theorems 17.11 and 17.35],

$$z \mapsto co\left(\cup_{\lambda\in[0,1]}\hat{\partial}_C T\left(\lambda z + (1-\lambda)\bar{h}e\right)\right)$$

is upper hemicontinuous. Since $\tilde{x}_{\beta_{n_{l(r)}}} \to \bar{h}e$, we can conclude that

$$W \in co\left(\cup_{\lambda\in[0,1]}\hat{\partial}_C T\left(\lambda\bar{h}e + (1-\lambda)\hat{h}e\right)\right) = \hat{\partial}_C T\left(\bar{h}e\right) = \hat{\partial}_C T_i(0).$$

Since $\bar{\gamma} \in \Gamma(W)$, this implies that $\bar{\gamma} \in \Gamma(W) \subseteq \Gamma\left(\hat{\partial}_C T(0)\right)$. By (30) and since $x_{\beta_{n_{l(r)}}} \to \bar{x} = \bar{h}e$, it follows that

$$\bar{h} = \langle\bar{\gamma}, \bar{x}\rangle = \lim_r \left\langle \gamma_{n_{l(r)}}, x_{\beta_{n_{l(r)}}}\right\rangle = \lim_r \left\langle \gamma_{n_{l(r)}}, x\right\rangle = \langle\bar{\gamma}, x\rangle,$$

that is,

$$\sup_{\gamma\in\Gamma\left(\hat{\partial}_C T(0)\right)} \langle\gamma, x\rangle e \geq \langle\bar{\gamma}, x\rangle e = \bar{h}e = \bar{x} = \bar{h}e = \langle\bar{\gamma}, x\rangle e \geq \inf_{\gamma\in\Gamma\left(\hat{\partial}_C T(0)\right)} \langle\gamma, x\rangle e$$

Since $\bar{x}$ was arbitrarily chosen, we can conclude that

$$\sup_{\gamma\in\Gamma\left(\hat{\partial}_C T(0)\right)} \langle\gamma, x\rangle e \geq \bar{x} \geq \inf_{\gamma\in\Gamma\left(\hat{\partial}_C T(0)\right)} \langle\gamma, x\rangle e \qquad \forall \bar{x} \in L. \tag{31}$$

Since $E(T) = D$, we have that $\liminf_{\beta\to 1} x_\beta, \limsup_{\beta\to 1} x_\beta \in L$. By (31) applied to $\limsup_{\beta\to 1} x_\beta$ and $\liminf_{\beta\to 1} x_\beta$ and since $\limsup_{\beta\to 1} x_\beta \geq \liminf_{\beta\to 1} x_\beta$, we obtain that $\sup_{\gamma\in\Gamma\left(\hat{\partial}_C T(0)\right)} \langle\gamma, x\rangle e \geq \limsup_{\beta\to 1} x_\beta \geq \liminf_{\beta\to 1} x_\beta \geq \inf_{\gamma\in\Gamma\left(\hat{\partial}_C T(0)\right)} \langle\gamma, x\rangle e$. Since $\hat{\partial}_C T(0)$ is closed, we have that $\Gamma\left(\hat{\partial}_C T(0)\right)$ is compact, yielding that the above sup and inf are achieved and thus proving the statement. ■

Proposition 13 *Let T be normalized, monotone, and translation invariant. If $E(T) = D$, then*

$$\max_{\gamma\in\Gamma(\partial_C T(0))} \langle\gamma, x\rangle e \geq \limsup_{\beta\to 1} x_\beta \geq \liminf_{\beta\to 1} x_\beta \geq \min_{\gamma\in\Gamma(\partial_C T(0))} \langle\gamma, x\rangle e \qquad \forall x \in \mathbb{R}^k.$$

Proof. It follows immediately from Lemma 9 and Clarke [17, Proposition 2.6.2]. ■

Given a normalized, monotone, and translation invariant operator $T : \mathbb{R}^k \to \mathbb{R}^k$, we define the adjacency matrix $\underline{A}(T, 0)$ by

$$\underline{a}_{ij} = 1 \iff \min_{W\in\partial_C T(0)} w_{ij} > 0.$$

In what follows, we will show that the regularity of $\underline{A}(T, 0)$ implies $E(T) = D$. Before moving there, we make few simple observations.

Lemma 10 *Let T be normalized, monotone, and translation invariant. The following statements are true:*

1. *If $\underline{A}(T)$ is regular, so is $\underline{A}(T, 0)$.*
2. *If T is continuously differentiable at 0, $\underline{A}(T, 0)$ is regular if and only if $A(J_T(0))$ is regular.*

Proof. 1. For ease of notation, set $A = \underline{A}(T)$ and $B = \underline{A}(T, 0)$. Recall that $a_{ij} = 1$ if and only if $\inf_{x\in\mathbb{R}^k} \min_{W\in\partial_C T(x)} w_{ij} > 0$. This implies that $b_{ij} \geq a_{ij}$ for all $i, j \in \{1, ..., k\}$, that is, $\underline{A}(T, 0) = B \geq A = \underline{A}(T)$, yielding the statement.

2. Since T is continuously differentiable at 0, we have that $\partial_C T(0) = \{J_T(0)\}$, yielding that $\underline{A}(T, 0) = A(J_T(0))$ and proving the statement. ■

Lemma 11 *Let T be normalized, monotone, and translation invariant. If $\underline{A}(T,0)$ is regular, then $E(T) = D$.*

Proof. By construction, note that $A(W) \geq \underline{A}(T,0)$ for all $W \in \partial_C T(0)$. We begin by proving the statement when $\underline{A}(T,0)$ is also assumed to be aperiodic. Assume that there exists $x \in E(T) \backslash D$ such that $T(x) = x$. It follows that $T^t(x) = x$ for all $t \in \mathbb{N}$. By induction and [17, point e of Proposition 2.6.2 and Theorem 2.6.6], we also have that

$$\partial_C T^t(0) \subseteq \left\{W \in \mathcal{W} : W = \Pi_{s=1}^t W_s \text{ s.t. } \{W_s\}_{s=1}^t \subseteq \partial_C T(0)\right\} \qquad \forall t \in \mathbb{N}. \qquad (32)$$

Since $A = \underline{A}(T,0)$ is regular and aperiodic, it follows that there exist $\bar{t} \in \mathbb{N}$ and $h \in \{1, ..., k\}$ such that $a_{ih}^{(\bar{t})} > 0$ for all $i \in \{1, ..., k\}$. By (32) and since $A(W) \geq \underline{A}(T,0)$ for all $W \in \partial_C T(0)$, we have that $w_{ih} > 0$ for all $i \in \{1, ..., k\}$ and for all $W \in \partial_C T^{\bar{t}}(0)$. Consider now $\bar{\imath}, \bar{\jmath} \in \{1, ..., k\}$ such that $x_{\bar{\imath}} = \min_{l \in N} x_l$ and $x_{\bar{\jmath}} = \max_{l \in N} x_l$. It follows that $x_{\bar{\jmath}} > x_{\bar{\imath}}$ and $\bar{\imath} \neq \bar{\jmath}$. We have two cases:

1. $x_h < x_{\bar{\jmath}}$. Define $z = x - x_{\bar{\jmath}} e \leq 0$ and $y = x_{\bar{\jmath}} e$. Define $y_n = y + \frac{1}{n} z$ for all $n \in \mathbb{N}$. By Lebourg's Mean Value Theorem (see, e.g., [17, Theorem 2.3.7]), for each $n \in \mathbb{N}$ there exists $\lambda_n \in (0,1)$ and $w_n \in \partial_C T_{\bar{\jmath}}^{\bar{t}}(\lambda_n y_n + (1-\lambda_n) y)$ such that

   $$T_{\bar{\jmath}}^{\bar{t}}(y_n) - T_{\bar{\jmath}}^{\bar{t}}(y) = \langle w_n, y_n - y \rangle = \frac{1}{n} \langle w_n, z \rangle .$$

   Since $y_n \to y$, we have that $\lambda y_n + (1-\lambda_n) y \to y$. By [17, point b of Proposition 2.1.5] and since $\{w_n\}_{n \in \mathbb{N}} \subseteq \Delta$, there exists a subsequence $\{w_{n_m}\}_{m \in \mathbb{N}} \subseteq \{w_n\}_{n \in \mathbb{N}}$ such that $w_{n_m} \to w \in \partial_C T_{\bar{\jmath}}^{\bar{t}}(0)$. Since $w_{ih} > 0$ for all $i \in \{1, ..., k\}$ and for all $W \in \partial_C T^{\bar{t}}(0)$ and since $w_{n_m} \to w$, $w_h > 0$ and, in particular, there exists $\bar{m} \in \mathbb{N}$ such that $w_{n_m,h} > 0$ for all $m \geq \bar{m}$. Since T is normalized and monotone and $T^{\bar{t}}(x) = x$, so is $T^{\bar{t}}$ and it follows that

   $$\begin{aligned} 0 = x_{\bar{\jmath}} - x_{\bar{\jmath}} = x_{\bar{\jmath}} - T_{\bar{\jmath}}^{\bar{t}}(x) = x_{\bar{\jmath}} - T_{\bar{\jmath}}^{\bar{t}}(y+z) \geq T_{\bar{\jmath}}^{\bar{t}}(y) - T_{\bar{\jmath}}^{\bar{t}}\left(y + \frac{1}{n_{\bar{m}}} z\right) \\ = \frac{1}{n_{\bar{m}}} \langle w_{n_{\bar{m}}}, -z \rangle \geq -\frac{1}{n_{\bar{m}}} w_{n_{\bar{m}},h} z_h = \frac{1}{n_{\bar{m}}} w_{n_{\bar{m}},h} \left(x_{\bar{\jmath}} - x_h\right) > 0, \end{aligned}$$

   a contradiction.

2. $x_h > x_{\bar{\imath}}$. Define $z = x - x_{\bar{\imath}} e \geq 0$ and $y = x_{\bar{\imath}} e$. Define $y_n = y + \frac{1}{n} z$ for all $n \in \mathbb{N}$. By Lebourg's Mean Value Theorem (see, e.g., [17, Theorem 2.3.7]), for each $n \in \mathbb{N}$ there exists $\lambda_n \in (0,1)$ and $w_n \in \partial_C T_{\bar{\imath}}^{\bar{t}}(\lambda_n y_n + (1-\lambda_n) y)$ such that

   $$T_{\bar{\imath}}^{\bar{t}}(y_n) - T_{\bar{\imath}}^{\bar{t}}(y) = \langle w_n, y_n - y \rangle = \frac{1}{n} \langle w_n, z \rangle .$$

Since $y_n \to y$, we have that $\lambda y_n + (1-\lambda_n) y \to y$. By [17, point b of Proposition 2.1.5] and since $\{w_n\}_{n\in\mathbb{N}} \subseteq \Delta$, there exists a subsequence $\{w_{n_m}\}_{m\in\mathbb{N}} \subseteq \{w_n\}_{n\in\mathbb{N}}$ such that $w_{n_m} \to w \in \partial_C T_{\bar{\imath}}^{\bar{t}}(0)$. Since $w_{ih} > 0$ for all $i \in \{1,...,k\}$ and for all $W \in \partial_C T^{\bar{t}}(0)$ and since $w_{n_m} \to w$, $w_h > 0$ and, in particular, there exists $\bar{m} \in \mathbb{N}$ such that $w_{n_m,h} > 0$ for all $m \geq \bar{m}$. Since T is normalized and monotone and $T^{\bar{t}}(x) = x$, so is $T^{\bar{t}}$ and it follows that

$$
\begin{aligned}
0 = x_{\bar{\imath}} - x_{\bar{\imath}} = T_{\bar{\imath}}^{\bar{t}}(x) - x_{\bar{\imath}} = T_{\bar{\imath}}^{\bar{t}}(y+z) - x_{\bar{\imath}} &\geq T_{\bar{\imath}}^{\bar{t}}\left(y + \frac{1}{n_{\bar{m}}} z\right) - T_{\bar{\imath}}^{\bar{t}}(y) \\
= \frac{1}{n_{\bar{m}}} \langle w_{n_{\bar{m}}}, z \rangle \geq \frac{1}{n_{\bar{m}}} w_{n_{\bar{m}},h} z_h = \frac{1}{n_{\bar{m}}} w_{n_{\bar{m}},h} (x_h - x_{\bar{\imath}}) > 0,
\end{aligned}
$$

a contradiction.

Cases 1 and 2 prove that $E\left(T^{\bar{t}}\right) = D$. Since $D \subseteq E(T) \subseteq E\left(T^{\bar{t}}\right) = D$, we have that $E(T) = D$, provided $\underline{A}(T,0)$ is also assumed to be aperiodic. For the general case, set $T_\lambda = \lambda I + (1-\lambda) T$ where $\lambda \in (0,1)$. It is immediate to see that T_λ is normalized, monotone, translation invariant and such that $\underline{A}(T_\lambda, 0) \geq \underline{A}(T,0) \vee I$. Since $\underline{A}(T,0)$ is regular, this implies that $\underline{A}(T_\lambda, 0)$ is regular and aperiodic. Since $E(T) = E(T_\lambda)$, the previous part of the proof yields that $E(T) = E(T_\lambda) = D$, proving the statement. ■

Proof of Theorem 1. Since T is continuously differentiable in a neighborhood of 0, so is each T_i. By [17, p. 32 and Proposition 2.2.4], it follows that $\partial_C T_i(0)$ is a singleton for all $i \in \{1,...,k\}$. This implies that $\partial_C T(0)$ is a singleton and coincides with the Jacobian of T at 0. In particular, $\Gamma(\partial_C T(0))$ is the singleton given by γ: the unique left Perron-Frobenius eigenvector of the Jacobian of T at 0. By Lemmas 10 and 11, $E(T) = D$. Therefore, by Proposition 13 we can conclude that

$$
\langle \gamma, x \rangle e \geq \limsup_{\beta \to 1} x_\beta \geq \liminf_{\beta \to 1} x_\beta \geq \langle \gamma, x \rangle e,
$$

proving that $\lim_{\beta \to 1} x_\beta = \langle \gamma, x \rangle e$. ■

Proposition 14 *Let T be normalized, monotone, and translation invariant. If T is concave and $E(T) = D$, then*

$$
\min_{\gamma \in \Gamma(\partial_C T(0))} \langle \gamma, x \rangle \geq \min_{\gamma \in \Gamma(\partial_C T(0))} \langle \gamma, x_\beta \rangle \quad \forall x \in \mathbb{R}^k, \forall \beta \in (0,1) \tag{33}
$$

and

$$
\min_{\gamma \in \Gamma(\partial_C T(0))} \langle \gamma, x \rangle e \geq \limsup_{\beta \to 1} x_\beta \qquad \forall x \in \mathbb{R}^k.
$$

Proof. Let $x \in \mathbb{R}^k$. Consider $W \in \partial_C T(0)$ and $\bar{\gamma} \in \Gamma(W)$. Define $S : \mathbb{R}^k \to \mathbb{R}^k$ by $S(y) = Wy$ for all $y \in \mathbb{R}^k$. By (23), it follows that $S(y) \geq T(y)$ for all $y \in \mathbb{R}^k$. By induction, we have that $S^t_{\beta,x} \geq T^t_{\beta,x}$ for all $t \in \mathbb{N}$ and for all $\beta \in (0,1)$. By Lemma 7, if we define by $x_{\beta,W}$ the unique fixed point of $S_{\beta,x}$, this implies that $x_{\beta,W} \geq x_\beta$ for all $\beta \in (0,1)$. By definition of $\Gamma(W)$ and $\Gamma(\partial_C T(0))$ and (4) and since $\bar{\gamma} \in \Gamma(W)$ and $W \in \partial_C T(0)$, we also have that $\bar{\gamma} \in \Gamma(W) \subseteq \Gamma(\partial_C T(0))$ and

$$\langle \bar{\gamma}, x \rangle = \langle \bar{\gamma}, x_{\beta,W} \rangle \geq \langle \bar{\gamma}, x_\beta \rangle \geq \min_{\gamma \in \Gamma(W)} \langle \gamma, x_\beta \rangle \geq \min_{\gamma \in \Gamma(\partial_C T(0))} \langle \gamma, x_\beta \rangle \quad \forall \beta \in (0,1).$$

Since W, $\bar{\gamma}$, and x were arbitrarily chosen, we have that

$$\min_{\gamma \in \Gamma(\partial_C T(0))} \langle \gamma, x \rangle \geq \min_{\gamma \in \Gamma(\partial_C T(0))} \langle \gamma, x_\beta \rangle \quad \forall x \in \mathbb{R}^k, \forall \beta \in (0,1),$$

proving (33). Fix $x \in \mathbb{R}^k$ again. Observe that the function $\varphi : \mathbb{R}^k \to \mathbb{R}$, defined by $\varphi(y) = \min_{\gamma \in \Gamma(\partial_C T(0))} \langle \gamma, y \rangle$ for all $y \in \mathbb{R}^k$, is normalized, monotone, and translation invariant. In particular, φ is Lipschitz continuous. By Lemma 8 and since $E(T) = D$, we have that $\limsup_{\beta \to 1} x_\beta$ is a limit point of $\{x_\beta\}_{\beta \in (0,1)}$, that is, there exists a sequence $\{x_{\beta_n}\}_{n \in \mathbb{N}} \subseteq \{x_\beta\}_{\beta \in (0,1)}$ such that $\beta_n \to 1$ and $\lim_n x_{\beta_n} = \limsup_{\beta \to 1} x_\beta$ and $\limsup_{\beta \to 1} x_\beta = \bar{h}e$ for some $\bar{h} \in \mathbb{R}$. By (33) and since φ is continuous, we can conclude that

$$\min_{\gamma \in \Gamma(\partial_C T(0))} \langle \gamma, x \rangle \geq \lim_n \min_{\gamma \in \Gamma(\partial_C T(0))} \langle \gamma, x_{\beta_n} \rangle = \lim_n \varphi(x_{\beta_n}) = \varphi(\bar{h}e) = \bar{h},$$

proving the statement. ■

Proof of Theorem 2. By Lemma 7 and Propositions 13 and 14, we have that $\liminf_{\beta \to 1} x_\beta \geq \min_{\gamma \in \Gamma(\partial_C T(0))} \langle \gamma, x \rangle e \geq \limsup_{\beta \to 1} x_\beta$ and $\lim_{\beta \to 1} x_\beta = \min_{\gamma \in \Gamma(\partial_C T(0))} \langle \gamma, x \rangle$, proving the first part of the statement. The second one immediately follows. ■

Proof of Lemma 2. Suppose by contradiction that there exists $y \in E(T) \setminus D$. Then, by equation (23), we have $T(y) \leq Wy$. Let $I^* = \operatorname{argmax}_{i \in \{1,...,k\}} y_i$. Since $y \notin D$, $\{1,...,k\} \setminus I^* \neq \emptyset$. Since W is strongly connected, there is $i^* \in I^*$ and $j^* \in \{1,...,k\} \setminus I^*$ such that $w_{i^*j^*} > 0$. But then $T_{i^*}(y) \leq \sum_{j=1}^k w_{ij} y_j \leq w_{i^*j^*} y_{j^*} + (1 - w_{i^*j^*}) \max_{i \in \{1,...,k\}} y_i = w_{i^*j^*} y_{j^*} + (1 - w_{i^*j^*}) y_{i^*} < y_{i^*}$, a contradiction with $T_{i^*}(y) = y_{i^*}$. ■

Proof of Corollary 1. If T is differentiable at 0, so is each T_i. Since T is concave, so is each T_i. It follows that $\partial_C T_i(0) = \partial T_i(0)$ is a singleton for all $i \in \{1,...,k\}$. This implies that $\partial_C T(0)$ is a singleton and coincides with the Jacobian of T at 0. Since the Jacobian of T at 0 is regular, $\Gamma(\partial_C T(0))$ is a singleton given by the unique left Perron-Frobenius eigenvector of the Jacobian of T at 0. By Theorem 2, we can conclude that $\lim_{\beta \to 1} x_\beta = \langle \gamma, x \rangle e$. ■

Observe that if T is nice we have that $E(T) = D$. To see this, consider $x \in E(T)$. By construction of T, there exists $\bar{\alpha} \in A$ such that $x = T(x) = S_{\bar{\alpha}}(x)$, yielding that $x \in E(S_{\bar{\alpha}}) = D$. This shows that $E(T) \subseteq D$. The opposite inclusion follows from normalization. In order to prove Proposition 1, we first provide two ancillary lemmas which give bounds on $\liminf_{\beta\to 1} x_\beta$ and $\limsup_{\beta\to 1} x_\beta$. These bounds are in terms of the limits of the operators S_α whose sup gives T.

Lemma 12 *If $\{S_\alpha\}_{\alpha\in\mathcal{A}}$ is nice, then $\liminf_{\beta\to 1} x_\beta \geq \sup_{\alpha\in\mathcal{A}} \varphi_\alpha(x) e$ for all $x \in \mathbb{R}^k$.*

Proof. By construction, we have that $T(y) \geq S_\alpha(y)$ for all $y \in \mathbb{R}^k$ and for all $\alpha \in \mathcal{A}$. By induction and since S_α and T are monotone, this implies that $T^t_{\beta,x}(y) \geq S^t_{\alpha,\beta,x}(y)$ for all $t \in \mathbb{N}$, for all $\beta \in (0,1)$, for all $x, y \in \mathbb{R}^k$, and for all $\alpha \in \mathcal{A}$. By passing to the limit and Lemma 7, this implies that $x_\beta \geq x_{\beta,\alpha}$ for all $\beta \in (0,1)$, for all $x \in \mathbb{R}^k$, and for all $\alpha \in \mathcal{A}$. By Theorem 2 and since $E(T) = D$, it follows that $\liminf_{\beta\to 1} x_\beta \geq \liminf_{\beta\to 1} x_{\beta,\alpha} = \lim_{\beta\to 1} x_{\beta,\alpha} = \varphi_\alpha(x) e$ for all $x \in \mathbb{R}^k$ and for all $\alpha \in \mathcal{A}$, proving the statement. ■

Lemma 13 *If $\{S_\alpha\}_{\alpha\in\mathcal{A}}$ is nice, then $\sup_{\alpha\in\mathcal{A}} \varphi_\alpha(x) e \geq \limsup_{\beta\to 1} x_\beta$ for all $x \in \mathbb{R}^k$.*

Proof. Fix $x \in \mathbb{R}^k$ and $\beta \in (0,1)$. By construction of T and definition of x_β and since $\{S_\alpha\}_{\alpha\in\mathcal{A}}$ is nice, we have that there exists $\alpha_\beta \in \mathcal{A}$

$$S_{\alpha_\beta}((1-\beta)x + \beta x_\beta) = T((1-\beta)x + \beta x_\beta) = x_\beta.$$

By Lemma 7, it follows that $x_\beta = x_{\beta,\alpha_\beta}$. By Proposition 14 and since β was arbitrarily chosen, we have that for each $\beta \in (0,1)$

$$\sup_{\alpha\in\mathcal{A}} \varphi_\alpha(x) \geq \varphi_{\alpha_\beta}(x) = \min_{\gamma\in\Gamma\left(\partial_C S_{\alpha_\beta}(0)\right)} \langle \gamma, x \rangle \geq \min_{\gamma\in\Gamma\left(\partial_C S_{\alpha_\beta}(0)\right)} \left\langle \gamma, x_{\beta,\alpha_\beta} \right\rangle = \min_{\gamma\in\Gamma\left(\partial_C S_{\alpha_\beta}(0)\right)} \langle \gamma, x_\beta \rangle. \tag{34}$$

By Lemma 8 and since $E(T) = D$, we have that there exists a sequence $\{x_{\beta_n}\}_{n\in\mathbb{N}} \subseteq \{x_\beta\}_{\beta\in(0,1)}$ such that $\beta_n \to 1$ and $\limsup_{\beta\to 1} x_\beta = \lim_n x_{\beta_n} = \bar{h}e$ for some $\bar{h} \in \mathbb{R}$. For each $n \in \mathbb{N}$ consider $\gamma_n \in \Gamma\left(\partial_C S_{\alpha_{\beta_n}}(0)\right)$ such that $\langle \gamma_n, x_{\beta_n} \rangle = \min_{\gamma\in\Gamma\left(\partial_C S_{\alpha_{\beta_n}}(0)\right)} \langle \gamma, x_{\beta_n} \rangle$. We have that $\sup_{\alpha\in\mathcal{A}} \varphi_\alpha(x) \geq \langle \gamma_n, x_{\beta_n} \rangle$ for all $n \in \mathbb{N}$. Since $\{\gamma_n\}_{n\in\mathbb{N}} \subseteq \Delta$, there exists a subsequence $\{\gamma_{n_l}\}_{l\in\mathbb{N}} \subseteq \{\gamma_n\}_{n\in\mathbb{N}}$ such that $\gamma_{n_l} \to \bar{\gamma} \in \Delta$. We can conclude that

$$\sup_{\alpha\in\mathcal{A}} \varphi_\alpha(x) \geq \lim_l \left\langle \gamma_{n_l}, x_{\beta_{n_l}} \right\rangle = \left\langle \bar{\gamma}, \bar{h}e \right\rangle = \bar{h},$$

proving that $\sup_{\alpha\in\mathcal{A}} \varphi_\alpha(x) e \geq \bar{h}e = \limsup_{\beta\to 1} x_\beta$. Since $x \in \mathbb{R}^k$ was arbitrarily chosen, the statement follows. ■

Proof of Proposition 1. By Lemmas 12 and 13, we have that $\lim_{\beta\to 1} x_\beta = \sup_{\alpha\in\mathcal{A}} \varphi_\alpha(x) e$ for all $x \in \mathbb{R}^k$. ■

Proof of Remark 1. Define $\tilde{T}_{\beta,x}(y) = (1-\beta)x + \beta T(y)$ for all $y \in \mathbb{R}^k$. It is easy to show that $\tilde{T}_{\beta,x}$ is a β-contraction (see, e.g., [24, Theorem 11.3]). By the Banach contraction principle and since $\tilde{T}_{\beta,x}$ is also a β-contraction, for each $y \in \mathbb{R}^k$ we have that $\tilde{T}^t_{\beta,x}(y) \to \tilde{x}_\beta$ as well as $\tilde{T}_{\beta,x}(\tilde{x}_\beta) = \tilde{x}_\beta$ where $\tilde{x}_\beta$ is the unique fixed point of $\tilde{T}_{\beta,x}$. Fix $x \in \mathbb{R}^k$ and $\beta \in (0,1)$. Set $\hat{x}_\beta = (1-\beta)x + \beta x_\beta$. By definition of $\tilde{T}_{\beta,x}$ and $\hat{x}_\beta$ as well as x_β, we have that

$$\tilde{T}_{\beta,x}(\hat{x}_\beta) = (1-\beta)x + \beta T(\hat{x}_\beta) = (1-\beta)x + \beta T((1-\beta)x + \beta x_\beta) = (1-\beta)x + \beta x_\beta = \hat{x}_\beta.$$

Since $\tilde{x}_\beta$ is the unique fixed point of $\tilde{T}_{\beta,x}$ and x and β were arbitrarily chosen, we can conclude that $\tilde{x}_\beta = \hat{x}_\beta = (1-\beta)x + \beta x_\beta$ for all $\beta \in (0,1)$ and for all $x \in \mathbb{R}^k$. The second part of the statement follows by taking the limit for β when it exists. ■

D Appendix: Proofs of Sections 3, 4, and 6

D.1 Endogenous networks

Proof of Proposition 2. Observe that $c_i = \ln(1/S_i) = -\ln(S_i)$ and that $\ln(S_i)$ is the composition of two upper semicontinuous functions where ln is monotone increasing, so it is upper semicontinuous, and thus c_i is lower semicontinuous. It is also convex as S_i is log-concave. Since there is w_i with $S_i(w_i) = 1$, $c_i^{-1}(0) \neq \emptyset$. Therefore, it is easy to see that $p \mapsto \min_{w_i : S_i(w_i)=1} \sum_{j=1}^k w_{ij} p_j$ and $p \mapsto \min_{w_i \in \Delta} \left\{ \sum_{j=1}^k w_{ij} p_j + c_i(w_i) \right\}$ are well defined, normalized, monotone, translation invariant, and concave.

To prove the result, we are going to use two ancillary fixed point equations:

$$\hat{p}_{\beta,i} = (1-\beta)x_i + \beta \min_{w_i \in \Delta} \left\{ \sum_{j=1}^k w_{ij} \hat{p}_{\beta,j} + c_i(w_i) \right\} \qquad \forall i \in \{1, ..., k\}$$

and

$$p'_{\beta,i} = (1-\beta)x_i + \beta \min_{w_i : S_i(w_i)=1} \sum_{j=1}^k w_{ij} p'_{\beta,j} \qquad \forall i \in \{1, ..., k\}.$$

It is easy to see that for all $\beta \in (0,1)$ and $i \in \{1, ..., k\}$, $p'_{\beta,i} \geq p_{\beta,i} \geq \hat{p}_{\beta,i}$. Since by Theorem 2, equation (6), and Lemmas 1 and 2 for all $i \in \{1, ..., k\}$, $\lim_{\beta \to 1} \hat{p}_{\beta,i} = \lim_{\beta \to 1} p'_{\beta,i} = \min_{\gamma \in \Gamma(\operatorname{argmax}(S))} \langle \gamma, x \rangle$, the first part of the result follows.

For the second part of the result, $\lim_{\beta \to 1} p_{\beta,i} = \langle \gamma_{W^0}, x \rangle$ immediately follows from the first part. Suppose by contradiction that for some $i \in \{1, ..., k\}$ we do not have $\lim_{\beta \to 1} w_{\beta,i} = w_i^0$. Since Δ is compact, the sequence $\lim_{\beta \to 1} w_{\beta,i}$ admits a converging subsequence $\{w_{\beta_n,i}\}_{n \in \mathbb{N}}$ with limit $\hat{w}_i \neq w_i^0$. But this, by the lower semicontinuity of c_i, means that

$$\langle \gamma_{W^0}, x \rangle + c_i(\hat{w}_i) \leq \lim_{n \to \infty} \sum_{j=1}^k w_{\beta_n,ij} \hat{p}_{\beta,j} + c_i(w_{\beta_n,i}) \leq \lim_{n \to \infty} \sum_{j=1}^k w_{ij}^0 \hat{p}_{\beta,j} + c_i\left(w_i^0\right) = \langle \gamma_{W^0}, x \rangle$$

a contradiction with $c_i(\hat{w}_i) > 0$. With this, that $\lim_{\beta\to 1} Q_{\beta,i0} = 0$, and $\lim_{\beta\to 1} Q_{\beta,ij} = w_{ij}^0$ follows from equations (47). ■

Proof of Proposition 3. It follows immediately by Lemma 2 and Corollary 1. ■

Proof of Proposition 4. For every $i \in \{1,...,k\}$, let $\bar{c}_i$ be the convexification of c_i. Observe that for all $i \in \{1,...,k\}$ and $a \in \mathbb{R}^k$

$$(1-\beta)x_i + \beta \max_{\tilde{w}_i \in \Delta_n} \left\{ \sum_{j=1}^n \tilde{w}_{ij} a_j - \frac{c_i(\tilde{w}_i)}{\beta} \right\} = (1-\beta)x_i + \beta \max_{\tilde{w}_i \in \Delta_n} \left\{ \sum_{j=1}^n \tilde{w}_{ij} a_j - \frac{\bar{c}_i(\tilde{w}_i)}{\beta} \right\}$$

by Theorem 3 of [14]. To prove the result, we are going to use two ancillary fixed point equations:

$$\hat{a}_{\beta,i} = (1-\beta)x_i + \beta \max_{\tilde{w}_i \in \Delta_n} \left\{ \sum_{j=1}^n \tilde{w}_{ij} \hat{a}_{\beta,j} - \bar{c}_i(\tilde{w}_i) \right\} \qquad \forall i \in \{1,...,k\}$$

and

$$a'_{\beta,i} = (1-\beta)x_i + \beta \max_{\tilde{w}_i \in \Delta_n} \sum_{j=1}^n \tilde{w}_{ij} a'_{\beta,j} \qquad \forall i \in \{1,...,k\}.$$

It is easy to see that for all $\beta \in (0,1)$ and $i \in \{1,...,k\}$, $a'_{\beta,i} \leq a_{\beta,i} \leq \hat{a}_{\beta,i}$. Since by Theorem 2, equation (6), and Lemmas 1 and 2 for all $i \in \{1,...,k\}$, $\lim_{\beta\to 1} \hat{a}_{\beta,i} = \lim_{\beta\to 1} a'_{\beta,i} = \max_{\gamma\in\Gamma(c)} \langle \gamma, x \rangle$. ■

D.2 Zero-sum stochastic games

Proof of Proposition 5. For every $\hat{s} \in \Delta(S)$, define the operator $H(\cdot,\hat{s}) : \mathbb{R}^{R\times\Omega} \to \mathbb{R}^{R\times\Omega}$ as

$$H_{r,\omega}(z,\hat{s}) = \min_{\hat{q}\in\Delta(Q)} \left\{ \sum_{(r',\omega')\in R\times\Omega} z_{r',\omega'} \rho(\hat{s},\hat{q},\omega)(r',\omega') \right\} \qquad \forall (r,\omega) \in R\times\Omega, \forall z \in \mathbb{R}^{R\times\Omega}.$$

Moreover, define the operator $T : \mathbb{R}^{R\times\Omega} \to \mathbb{R}^{R\times\Omega}$ as

$$T_{r,\omega}(z) = \max_{\sigma\in\Sigma_S} H_{r,\omega}(z,\sigma(\omega)) \qquad \forall (r,\omega) \in R\times\Omega, \forall z \in \mathbb{R}^{R\times\Omega}.$$

Observe that, for every $\sigma \in \Sigma_S$, the operator $H(\cdot,\sigma)$ is monotone, normalized, translation invariant, positive homogeneous, and concave. Moreover, since $\Delta(Q)$ is compact and ρ is continuous, it follows by the Maximum theorem that, for every $z \in \mathbb{R}^{R\times\Omega}$, the map $\sigma \mapsto H(z,\sigma)$ is continuous. Given that $\Delta(S)$ is compact, Assumption A and Lemma 2, it follows that $\{H(\cdot,\sigma)\}_{\sigma\in\Sigma_S}$ is nice. Moreover, observe that by construction we have

$$H_{r,\omega} = H_{r',\omega} \qquad \forall r,r' \in R, \forall \omega \in \Omega.$$

Therefore, for all $\sigma \in \Sigma_S$ we have

$$\begin{aligned}
\partial H(0,\sigma) &= \left\{ \begin{array}{c} W \in \mathcal{W}_{R\times\Omega} : \exists \hat{\sigma} \in \Delta(Q)^{R\times\Omega}, \forall (r,\omega), (r',\omega') \in R \times \Omega, \\ w_{(r,\omega),(r',\omega')} = \int_{\Delta(Q)} \rho(\sigma(\omega), \hat{q}, \omega)(r',\omega')\, d\hat{\sigma}_{r,\omega}(\hat{q}) \end{array} \right\} \\
&= \left\{ \begin{array}{c} W \in \mathcal{W}_{R\times\Omega} : \exists \tilde{\sigma} \in \Sigma_Q, \forall (r,\omega), (r',\omega') \in R \times \Omega, \\ w_{(r,\omega),(r',\omega')} = \int_{\Delta(Q)} \rho(\sigma(\omega), \hat{q}, \omega)(r',\omega')\, d\tilde{\sigma}_{\omega}(\hat{q}) \end{array} \right\} \\
&= \left\{ W(\sigma,\tilde{\sigma}) \in \mathcal{W}_{R\times\Omega} : \tilde{\sigma} \in \Sigma_Q \right\}.
\end{aligned}$$

Next, define $x \in \mathbb{R}^{R\times\Omega}$ as $x_{r,\omega} = r$. Also, for every $\beta \in (0,1)$, define $x^\beta \in \mathbb{R}^{R\times\Omega}$ as $x^\beta_{r,\omega} = v^\beta_\omega$, and observe that it is the unique solution of the fixed-point equation

$$\begin{aligned}
x^\beta_{r,\omega} = v^\beta_\omega &= \max_{\hat{s}\in\Delta(S)} \min_{\hat{q}\in\Delta(Q)} \left\{ (1-\beta)\, g(\hat{s},\hat{q},\omega) + \beta \sum_{\omega'\in\Omega} v^\beta_{\omega'} \rho(\hat{s},\hat{q},\omega)(\omega') \right\} \\
&= \max_{\hat{s}\in\Delta(S)} \min_{\hat{q}\in\Delta(Q)} \left\{ (1-\beta) \sum_{r'\in R} r' \rho(\hat{s},\hat{q},\omega)(r') + \beta \sum_{\omega'\in\Omega} v^\beta_{\omega'} \rho(\hat{s},\hat{q},\omega)(\omega') \right\} \\
&= \max_{\hat{s}\in\Delta(S)} \min_{\hat{q}\in\Delta(Q)} \left\{ (1-\beta) \sum_{(r',\omega')\in R\times\Omega} \rho(\hat{s},\hat{q},\omega)(r',\omega')\, x_{r',\omega'} + \beta \sum_{(r',\omega')\in R\times\Omega} \rho(\hat{s},\hat{q},\omega)(r',\omega')\, x^\beta_{r',\omega'} \right\} \\
&= \max_{\hat{s}\in\Delta(S)} \min_{\hat{q}\in\Delta(Q)} \left\{ \sum_{(r',\omega')\in R\times\Omega} \left((1-\beta)x + \beta x^\beta\right)_{r',\omega'} \rho(\hat{s},\hat{q},\omega)(r',\omega') \right\} \\
&= \max_{\hat{s}\in\Delta(S)} H_{r,\omega}\left((1-\beta)x + \beta x^\beta, \hat{s}\right) \\
&= \max_{\sigma\in\Sigma_S} H_{r,\omega}\left((1-\beta)x + \beta x^\beta, \sigma(\omega)\right) = T_{r,\omega}\left((1-\beta)x + \beta x^\beta\right) \qquad \forall (r,\omega) \in R\times\Omega.
\end{aligned}$$

Also, for every $t \in \mathbb{N}$, define $x^t \in \mathbb{R}^{R\times\Omega}$ as $x^t_{r,\omega} = v^t_\omega$, and observe that, for all $t \in \mathbb{N}$ and for all $(r,\omega) \in R\times\Omega$,

$$x^t_{r,\omega} = v^t_\omega = \max_{\sigma\in\Sigma_S} H_{r,\omega}\left(\frac{1}{t}x + \frac{t-1}{t}x^\beta, \sigma(\omega)\right) = T_{r,\omega}\left(\frac{1}{t}x + \frac{t-1}{t}x^\beta\right).$$

Therefore, by Propositions 1 and **??**, it follows that

$$\lim_{\beta\to 1} x^\beta = \sup_{\sigma\in\Sigma_S}\left(\min_{\gamma\in\Gamma(\partial H(0,\sigma))} \sum_{(r',\omega')\in R\times\Omega} x_{r',\omega'}\gamma_{r',\omega'}\right) e = \left(\sup_{\sigma\in\Sigma_S} \min_{\tilde{\sigma}\in\Sigma_Q} \sum_{r\in R} r\gamma(\sigma,\tilde{\sigma})(r)\right) e.$$

The equality between $\lim_{\beta\to1} x^\beta$ and $\lim_t x^t$ follows from Theorem 1 in Ziliotto [45].[20] ■

[20] Ziliotto [45] proves the equality between the two limits without any additional connectedness structure for the operator, i.e., without $E(T) = D$. For a simpler proof in our case, observe that since $\{x^t\}_{t\in\mathbb{N}_0}$ is bounded, that is $\sup_{t\in\mathbb{N}_0} \|x^t\|_\infty < \infty$, $\{x^t\}_{t\in\mathbb{N}_0}$ has at least a limit point. Using the same techniques of [41, Lemma 1], one can show that $\lim_t \left\|x^{t+1} - x^t\right\|_\infty = 0$. Note that $\left\|T(x^t) - x^{t+1}\right\|_\infty \leq 2(1-\beta_t)\sup_{\tau\in\mathbb{N}_0} \|x^\tau\|_\infty$ for all $t \in \mathbb{N}_0$, yielding that $\lim_t \|T(x^t) - x^t\|_\infty = 0$. Since $E(T) = D$, this implies that all the limit points of $\{x^t\}_{t\in\mathbb{N}_0}$ belong to D and, in particular, $\lim_s \|x^{t_s}\|_\infty = h$ if and only if $\lim_s x^{t_s} = he$. By [31, Theorem 1.1] (see also [43, Theorem 1]) and since $\lim_{\beta\to1} x_\beta \in D$, we have that $\lim_{\beta\to1} \|x_\beta\|_\infty = \lim_t \|x^t\|_\infty = \inf_{y\in\mathbb{R}^k} \|T(x+y) - y\|_\infty$, proving that $\lim_{\beta\to1} x_\beta = \lim_t x^t$.

E Appendix: Proofs of Section 8

E.1 Star-Shaped Operators

Proof of Theorem 3. Note that T_i is normalized, monotone, translation invariant, and star-shaped for all $i \in \{1, ..., k\}$. By Proposition 9, we have that for each $i \in \{1, ..., k\}$ there exists a family $\{S_{\alpha_i}\}_{\alpha_i \in \mathcal{A}_i}$ of normalized, monotone, translation invariant, and concave functionals such that

$$T_i(x) = \max_{\alpha_i \in \mathcal{A}_i} S_{\alpha_i}(x) \qquad \forall x \in \mathbb{R}^k \tag{35}$$

and $\bar{\text{co}}\left(\partial_C S_{\alpha_i}\left(\mathbb{R}^k\right)\right) \subseteq \bar{\text{co}}\left(\partial_C T_i\left(\mathbb{R}^k\right)\right)$ for all $\alpha_i \in \mathcal{A}_i$. Define $\mathcal{A} = \Pi_{i=1}^k \mathcal{A}_i$ and for each $\alpha \in \mathcal{A}$ define $S_\alpha : \mathbb{R}^k \to \mathbb{R}^k$ to be such that its i-th component coincides with S_{α_i} for all $i \in \{1, ..., k\}$. It is immediate to see that S_α is normalized, monotone, translation invariant, and concave for all $\alpha \in \mathcal{A}$. Since $\bar{\text{co}}\left(\partial_C S_{\alpha_i}\left(\mathbb{R}^k\right)\right) \subseteq \bar{\text{co}}\left(\partial_C T_i\left(\mathbb{R}^k\right)\right)$ for all $\alpha_i \in \mathcal{A}_i$ and for all $i \in \{1, ..., k\}$, it follows that $\underline{A}(S_\alpha) \geq \underline{A}(T)$ for all $\alpha \in \mathcal{A}$. By Lemmas 10 and 11 and since $\underline{A}(T)$ is regular, this implies that $\underline{A}(S_\alpha)$ is regular and $E(S_\alpha) = D$ for all $\alpha \in \mathcal{A}$. By (35) and since $\mathcal{A}$ has a product structure, we have that

$$T(x) = \sup_{\alpha \in \mathcal{A}} S_\alpha(x) \qquad \forall x \in \mathbb{R}^k$$

and for each $x \in \mathbb{R}^k$ there exists $\alpha_x \in \mathcal{A}$ such that $T(x) = S_{\alpha_x}(x)$. We can conclude that $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ is nice. By Proposition 1, the statement follows. ■

E.2 Range

Given the additivity and homogeneity properties of max and min, it is routine to check that

$$\text{Rg}(\lambda y + \mu z) \leq \lambda \,\text{Rg}(y) + \mu \,\text{Rg}(z) \quad \forall \lambda, \mu \in \mathbb{R}_+, \forall y, z \in \mathbb{R}^k. \tag{36}$$

In particular, since $\tilde{x}_\beta = (1 - \beta) x + \beta x_\beta$ for all $x \in \mathbb{R}^k$ and for all $\beta \in (0, 1)$, this implies that

$$\text{Rg}(\tilde{x}_\beta) \leq (1 - \beta) \,\text{Rg}(x) + \beta \,\text{Rg}(x_\beta) \quad \forall x \in \mathbb{R}^k, \forall \beta \in (0, 1). \tag{37}$$

If T is normalized and monotone, we have that $\left(\min_{i \in \{1,...,k\}} y_i\right) e \leq T(y) \leq \left(\max_{i \in \{1,...,k\}} y_i\right) e$ for all $y \in \mathbb{R}^k$, thus

$$\text{Rg}(T(y)) \leq \text{Rg}(y) \qquad \forall y \in \mathbb{R}^k. \tag{38}$$

By definition of $\tilde{x}_\beta$ and x_β, we have that $T(\tilde{x}_\beta) = x_\beta$ for all $x \in \mathbb{R}^k$ and for all $\beta \in (0, 1)$ and

$$\text{Rg}(x_\beta) \leq \text{Rg}(\tilde{x}_\beta) \quad \forall x \in \mathbb{R}^k, \forall \beta \in (0, 1). \tag{39}$$

Thus, for our purposes, (37) and (39) show that we can alternatively either study $\mathrm{Rg}(x_\beta)$ or $\mathrm{Rg}(\tilde{x}_\beta)$. Since some results are easier to be derived just focusing on one of the two, we will extensively use these inequalities to go back and forth $\mathrm{Rg}(x_\beta)$ and $\mathrm{Rg}(\tilde{x}_\beta)$. We begin with two ancillary lemmas.

Lemma 14 *Let $T : \mathbb{R}^k \to \mathbb{R}^k$ be such that there exist a stochastic matrix W and $\varepsilon \in (0,1]$ such that*

$$T(y) = \varepsilon W y + (1-\varepsilon) S(y) \qquad \forall y \in \mathbb{R}^k \tag{40}$$

where $S : \mathbb{R}^k \to \mathbb{R}^k$ is normalized, monotone, and translation invariant. If there exist $h \in \{1, ..., k\}$ and $\hat{t} \in \mathbb{N}$ such that $w_{ih}^{(\hat{t})} > 0$ for all $i \in \{1, ..., k\}$, then

$$\mathrm{Rg}(x_\beta) \leq \frac{1}{1 + \frac{\delta^{\hat{t}} (\beta\varepsilon)^{\hat{t}} (1-\beta\varepsilon)}{(1-\beta)\left(1-(\beta\varepsilon)^{\hat{t}}\right)}} \mathrm{Rg}(x) \quad \forall \beta \in (0,1), \forall x \in \mathbb{R}^k$$

where $\delta = \min_{i,j:w_{ij}>0} w_{ij}$.

Proof. Recall that $\tilde{x}_\beta = (1-\beta)x + \beta x_\beta$ for all $x \in \mathbb{R}^k$ and for all $\beta \in (0,1)$. Given $x \in \mathbb{R}^k$ and $\beta \in (0,1)$, recall also that $T_{\beta,x} : \mathbb{R}^k \to \mathbb{R}^k$ is defined by $T_{\beta,x}(y) = T((1-\beta)x + \beta y)$ for all $y \in \mathbb{R}^k$. By Lemma 7, x_β is a fixed point of $T_{\beta,x}$ and so of $T_{\beta,x}^t$ for all $t \in \mathbb{N}$.

Step 1. For each $x \in \mathbb{R}^k$, for each $\beta \in (0,1)$, and for each $t \in \mathbb{N}$

$$T_{\beta,x}^t(x_\beta) = (1-\beta)\varepsilon \sum_{\tau=0}^{t-1} (\beta\varepsilon)^\tau W^{\tau+1} x + (\beta\varepsilon)^t W^t x_\beta + (1-\varepsilon) \sum_{\tau=0}^{t-1} (\beta\varepsilon)^\tau W^\tau S(\tilde{x}_\beta).$$

Proof of the Step. We proceed by induction.

Initial Step. If $t = 1$, then

$$\begin{aligned}
T_{\beta,x}(x_\beta) &= T((1-\beta)x + \beta x_\beta) \\
&= (1-\beta)\varepsilon W x + \beta\varepsilon W x_\beta + (1-\varepsilon) S((1-\beta)x + \beta x_\beta) \\
&= (1-\beta)\varepsilon \sum_{\tau=0}^{t-1} (\beta\varepsilon)^\tau W^{\tau+1} x + (\beta\varepsilon)^t W^t x_\beta + (1-\varepsilon) \sum_{\tau=0}^{t-1} (\beta\varepsilon)^\tau W^\tau S(\tilde{x}_\beta),
\end{aligned}$$

proving the initial step.

Inductive Step. Assume the statement holds for t. We show it holds for $t+1$. By (40) and inductive hypothesis $T^t_{\beta,x}(x_\beta) = x_\beta$, we have that

$$
\begin{aligned}
T^{t+1}_{\beta,x}(x_\beta) &= T\left((1-\beta)x + \beta T^t_{\beta,x}(x_\beta)\right) \\
&= \varepsilon W\left((1-\beta)x + \beta T^t_{\beta,x}(x_\beta)\right) + (1-\varepsilon) S\left((1-\beta)x + \beta T^t_{\beta,x}(x_\beta)\right) \\
&= (1-\beta)\varepsilon W x + \beta\varepsilon W T^t_{\beta,x}(x_\beta) + (1-\varepsilon) S\left((1-\beta)x + \beta x_\beta\right) \\
&= (1-\beta)\varepsilon W x \\
&+ \beta\varepsilon W\left((1-\beta)\varepsilon \sum_{\tau=0}^{t-1} (\beta\varepsilon)^\tau W^{\tau+1} x + (\beta\varepsilon)^t W^t x_\beta + (1-\varepsilon)\sum_{\tau=0}^{t-1}(\beta\varepsilon)^\tau W^\tau S(\tilde{x}_\beta)\right) \\
&+ (1-\varepsilon) S(\tilde{x}_\beta) \\
&= (1-\beta)\varepsilon W x + (1-\beta)\varepsilon \sum_{\tau=0}^{t-1}(\beta\varepsilon)^{\tau+1} W^{\tau+2} x + (\beta\varepsilon)^{t+1} W^{t+1} x_\beta \\
&+ (1-\varepsilon)\sum_{\tau=0}^{t-1}(\beta\varepsilon)^{\tau+1} W^{\tau+1} S(\tilde{x}_\beta) + (1-\varepsilon) S(\tilde{x}_\beta) \\
&= (1-\beta)\varepsilon \sum_{\tau=0}^{t}(\beta\varepsilon)^\tau W^{\tau+1} x + (\beta\varepsilon)^{t+1} W^{t+1} x_\beta + (1-\varepsilon)\sum_{\tau=0}^{t}(\beta\varepsilon)^\tau W^\tau S(\tilde{x}_\beta),
\end{aligned}
$$

proving the inductive step.

Step 1 follows by induction. □

Step 2. For each $z \in \mathbb{R}^k$

$$\mathrm{Rg}\left(W^{\hat{t}} z\right) \le \left(1-\delta^{\hat{t}}\right)\mathrm{Rg}(z)$$

where $\delta = \min_{i,j:w_{ij}>0} w_{ij}$.

Proof of the Step. Let $z \in \mathbb{R}^k$ and define $y = W^{\hat{t}} z$. Consider $i_1, i_2 \in \{1,...,k\}$ such that $y_{i_1} = \max_{i\in\{1,...,k\}} y_i$ and $y_{i_2} = \min_{i\in\{1,...,k\}} y_i$. Define also $z^\star = \max_{i\in\{1,...,k\}} z_i$ and $z_\star = \min_{i\in\{1,...,k\}} z_i$. Define $\tilde{\delta} = \min_{i\in\{1,...,h\}} w^{(\hat{t})}_{ih} \in (0,1)$ where $w^{(\hat{t})}_{ih}$ is the ih-th entry of $W^{\hat{t}}$. Note that

$$
\begin{aligned}
\mathrm{Rg}\left(W^{\hat{t}} z\right) = \mathrm{Rg}(y) = y_{i_1} - y_{i_2} &= \sum_{j=1}^k w^{(\hat{t})}_{i_1 j} z_j - \sum_{j=1}^k w^{(\hat{t})}_{i_2 j} z_j \\
&\le \left(1 - w^{(\hat{t})}_{i_1 h}\right) z^\star - \left(1 - w^{(\hat{t})}_{i_2 h}\right) z_\star + \left(w^{(\hat{t})}_{i_1 h} - \tilde{\delta}\right) z_h + \tilde{\delta} z_h - \left(w^{(\hat{t})}_{i_2 h} - \tilde{\delta}\right) z_h - \tilde{\delta} z_h \\
&\le \left(1 - w^{(\hat{t})}_{i_1 h}\right) z^\star - \left(1 - w^{(\hat{t})}_{i_2 h}\right) z_\star + \left(w^{(\hat{t})}_{i_1 h} - \tilde{\delta}\right) z^\star - \left(w^{(\hat{t})}_{i_2 h} - \tilde{\delta}\right) z_\star \\
&\le \left(1-\tilde{\delta}\right)(z^\star - z_\star) = \left(1-\tilde{\delta}\right)\mathrm{Rg}(z).
\end{aligned}
$$

Next, by induction, it is immediate to see that $\min_{i,j:w^{(t)}_{ij}>0} w^{(t)}_{ij} \ge \left(\min_{i,j:w_{ij}>0} w_{ij}\right)^t = \delta^t$ for all $t \in \mathbb{N}$. Since $w^{(\hat{t})}_{ih} > 0$ for all $i \in \{1,...,k\}$, we can conclude that $\tilde{\delta} \ge \delta^{\hat{t}}$, proving the statement. □

By Steps 1 and 2 as well as (36), (37), and (38) and since $T_{\beta,x}^{\hat{t}}(x_\beta) = x_\beta$ and the composition of normalized and monotone operators is normalized and monotone, we have that for each $x \in \mathbb{R}^k$ and for each $\beta \in (0,1)$

$$\begin{aligned}
\operatorname{Rg}(x_\beta) &= \operatorname{Rg}\left(T_{\beta,x}^{\hat{t}}(x_\beta)\right) \\
&= \operatorname{Rg}\left[(1-\beta)\,\varepsilon\sum_{\tau=0}^{\hat{t}-1}(\beta\varepsilon)^\tau W^{\tau+1}x + (\beta\varepsilon)^{\hat{t}} W^{\hat{t}} x_\beta + (1-\varepsilon)\sum_{\tau=0}^{\hat{t}-1}(\beta\varepsilon)^\tau W^\tau S(\tilde{x}_\beta)\right] \\
&\leq (1-\beta)\,\varepsilon\sum_{\tau=0}^{\hat{t}-1}(\beta\varepsilon)^\tau \operatorname{Rg}\left(W^{\tau+1}x\right) + (\beta\varepsilon)^{\hat{t}}\operatorname{Rg}\left(W^{\hat{t}}x_\beta\right) + (1-\varepsilon)\sum_{\tau=0}^{\hat{t}-1}(\beta\varepsilon)^\tau \operatorname{Rg}\left(W^\tau S(\tilde{x}_\beta)\right) \\
&\leq (1-\beta)\,\varepsilon\sum_{\tau=0}^{\hat{t}-1}(\beta\varepsilon)^\tau \operatorname{Rg}(x) + (\beta\varepsilon)^{\hat{t}}\operatorname{Rg}\left(W^{\hat{t}}x_\beta\right) + (1-\varepsilon)\sum_{\tau=0}^{\hat{t}-1}(\beta\varepsilon)^\tau \operatorname{Rg}(\tilde{x}_\beta) \\
&\leq (1-\beta)\,\varepsilon\sum_{\tau=0}^{\hat{t}-1}(\beta\varepsilon)^\tau \operatorname{Rg}(x) + (\beta\varepsilon)^{\hat{t}}\operatorname{Rg}\left(W^{\hat{t}}x_\beta\right) + (1-\varepsilon)(1-\beta)\sum_{\tau=0}^{\hat{t}-1}(\beta\varepsilon)^\tau \operatorname{Rg}(x) \\
&+ (1-\varepsilon)\,\beta\sum_{\tau=0}^{\hat{t}-1}(\beta\varepsilon)^\tau \operatorname{Rg}(x_\beta) \\
&= (1-\beta)\frac{1-(\beta\varepsilon)^{\hat{t}}}{1-\beta\varepsilon}\operatorname{Rg}(x) + (\beta\varepsilon)^{\hat{t}}\operatorname{Rg}\left(W^{\hat{t}}x_\beta\right) + (1-\varepsilon)\,\beta\frac{1-(\beta\varepsilon)^{\hat{t}}}{1-\beta\varepsilon}\operatorname{Rg}(x_\beta) \\
&\leq (1-\beta)\frac{1-(\beta\varepsilon)^{\hat{t}}}{1-\beta\varepsilon}\operatorname{Rg}(x) + (\beta\varepsilon)^{\hat{t}}\left(1-\delta^{\hat{t}}\right)\operatorname{Rg}(x_\beta) + (1-\varepsilon)\,\beta\frac{1-(\beta\varepsilon)^{\hat{t}}}{1-\beta\varepsilon}\operatorname{Rg}(x_\beta) \\
&= (1-\beta)\frac{1-(\beta\varepsilon)^{\hat{t}}}{1-\beta\varepsilon}\operatorname{Rg}(x) + \left(\frac{\beta-\beta\varepsilon-\beta(\beta\varepsilon)^{\hat{t}}+(\beta\varepsilon)^{\hat{t}}-\delta^{\hat{t}}(\beta\varepsilon)^{\hat{t}}+\delta^{\hat{t}}(\beta\varepsilon)^{\hat{t}+1}}{1-\beta\varepsilon}\right)\operatorname{Rg}(x_\beta).
\end{aligned}$$

Since $(\beta\varepsilon)^{\hat{t}}\left(1-\delta^{\hat{t}}\right) + (1-\varepsilon)\,\beta\frac{1-(\beta\varepsilon)^{\hat{t}}}{1-\beta\varepsilon} \in (0,1)$, this implies that

$$\operatorname{Rg}(x_\beta) \leq \frac{(1-\beta)\left(1-(\beta\varepsilon)^{\hat{t}}\right)}{(1-\beta)\left(1-(\beta\varepsilon)^{\hat{t}}\right)+\delta^{\hat{t}}(\beta\varepsilon)^{\hat{t}}(1-\beta\varepsilon)}\operatorname{Rg}(x) \qquad \forall x \in \mathbb{R}^k, \forall \beta \in (0,1),$$

proving the statement. ■

From the previous lemma, we obtain an estimate on the range of x_β, provided T admits a decomposition as in (40) and W has eventually a strictly positive column. This latter property is achieved whenever $A(W)$ not only is regular, but also "aperiodic". As for the former property, by [12, Proposition 5], we have that if T is normalized, monotone, and translation invariant and $\underline{A}(T)$ is nontrivial, then there there exist a stochastic matrix W and $\varepsilon \in (0,1]$ such that

$$T(y) = \varepsilon W y + (1-\varepsilon) S(y) \qquad \forall y \in \mathbb{R}^k$$

where $S : \mathbb{R}^k \to \mathbb{R}^k$ is normalized, monotone, and translation invariant. Moreover, W can be chosen to be such that $A(W) = \underline{A}(T)$. Thus, if $\underline{A}(T)$ is also regular and aperiodic, so is $A(W)$. Since in our statements we have the property of regularity, but not aperiodicity, we consider an auxiliary operator closely related to T and which will satisfy the property of aperiodicity.

Given $T : \mathbb{R}^k \to \mathbb{R}^k$, it will be thus useful to consider the averaged operator $T_\lambda = \lambda I + (1-\lambda) T$ with $\lambda \in (0,1)$ where I is the identity.[21] Note that $\underline{A}(T_\lambda) \geq \underline{A}(T)$ and, in particular, the only difference between $\underline{A}(T_\lambda)$ and $\underline{A}(T)$ consists in the entries of the diagonal of $\underline{A}(T_\lambda)$ which are all 1, while those of $\underline{A}(T)$ might be 0.

Building on Lemma 14, the next result provides a result on the convergence of $\mathrm{Rg}(x_{\beta,\lambda})$ where for each $x \in \mathbb{R}^k$ and for each $\beta, \lambda \in (0,1)$, $x_{\beta,\lambda}$ is the unique point satisfying

$$T_\lambda((1-\beta)x + \beta x_{\beta,\lambda}) = x_{\beta,\lambda}.$$

Lemma 15 *Let $T : \mathbb{R}^k \to \mathbb{R}^k$ be normalized, monotone, and translation invariant. If $\underline{A}(T)$ is regular, then*

$$\mathrm{Rg}(x_{\beta,\lambda}) \leq \frac{1}{1 + \frac{\hat{\delta}^{\hat{t}}\left(\beta\hat{\delta}\right)^{\hat{t}}\left(1-\beta\hat{\delta}\right)}{(1-\beta)\left(1-\left(\beta\hat{\delta}\right)^{\hat{t}}\right)}} \mathrm{Rg}(x) \quad \forall x \in \mathbb{R}^k, \forall \beta, \lambda \in (0,1), \tag{41}$$

where $\hat{\delta} = \min\{\lambda, (1-\lambda)\delta\}$, $\delta = \min_{i,j:\underline{a}_{ij}=1} \inf_{x \in \mathcal{D}} \frac{\partial T_i}{\partial x_j}(x)$, and $\hat{t} \in \mathbb{N}$ is the smallest natural number such that $\underline{A}(T_\lambda)^{\hat{t}}$ has one column with all positive entries.

Proof. Since T is normalized, monotone, translation invariant and $\underline{A}(T)$ is nontrivial, we have that $\delta \in (0,1]$. Since $\underline{A}(T_\lambda) \geq \underline{A}(T)$ and $\underline{A}(T)$ is regular, we have that $\underline{A}(T_\lambda)$ is regular. By [12, Proposition 5], we have that there exist a stochastic matrix W and $\varepsilon \in (0,1]$ such that

$$T_\lambda(y) = \varepsilon W y + (1-\varepsilon) S(y) \qquad \forall y \in \mathbb{R}^k$$

where $S : \mathbb{R}^k \to \mathbb{R}^k$ is normalized, monotone, and translation invariant. Moreover, W can be chosen to be such that $A(W) = \underline{A}(T_\lambda)$. By the proof of Proposition 5 in [12], it follows that ε can be chosen to be equal to $\hat{\delta}$ and all the strictly positive entries of W are greater than or equal to $\hat{\delta}$. This implies that $A(W)$ is regular and $A(W) \geq I$. In particular (see, e.g., [39, Exercise 4.13]), the set of natural numbers $t \in \mathbb{N}$ such that $A(W)^{\hat{t}} = \underline{A}(T_\lambda)^{\hat{t}}$ has one column with all positive entries is nonempty and $\hat{t}$ is well defined. By Lemma 14, the statement follows. ■

[21] With a small abuse of notation, we denote with I both the identity matrix and the identity operator.

Remark 2 Note that

$$\frac{1}{1+\frac{\hat{\delta}^{\hat{t}}\left(\beta\hat{\delta}\right)^{\hat{t}}\left(1-\beta\hat{\delta}\right)}{(1-\beta)\left(1-\left(\beta\hat{\delta}\right)^{\hat{t}}\right)}}=\frac{(1-\beta)\left(1-\left(\beta\hat{\delta}\right)^{\hat{t}}\right)}{(1-\beta)\left(1-\left(\beta\hat{\delta}\right)^{\hat{t}}\right)+\hat{\delta}^{\hat{t}}\left(\beta\hat{\delta}\right)^{\hat{t}}\left(1-\beta\hat{\delta}\right)}.$$

Consider the quantity at the denominator of the fraction on the right-hand side. It is immediate to see that it is equal to

$$\begin{aligned}\left(1-\beta\hat{\delta}\right)\left((1-\beta)\frac{1-\left(\beta\hat{\delta}\right)^{\hat{t}}}{1-\beta\hat{\delta}}+\hat{\delta}^{\hat{t}}\left(\beta\hat{\delta}\right)^{\hat{t}}\right)&=\left(1-\beta\hat{\delta}\right)\left((1-\beta)\sum_{\tau=0}^{\hat{t}-1}\left(\beta\hat{\delta}\right)^{\tau}+\hat{\delta}^{\hat{t}}\left(\beta\hat{\delta}\right)^{\hat{t}}\right)\\&\geq\left(1-\beta\hat{\delta}\right)\left(1-\beta+\hat{\delta}^{\hat{t}}\left(\beta\hat{\delta}\right)^{\hat{t}}\right).\end{aligned}$$

Consider now the function $h:\mathbb{R}\to\mathbb{R}$ defined by $h(\beta)=1-\beta+\hat{\delta}^{2\hat{t}}\beta^{\hat{t}}$ for all $\beta\in\mathbb{R}$. Since $\hat{\delta}\in(0,1)$ and $\hat{t}\in\mathbb{N}$, h is convex and differentiable on $\mathbb{R}$ with derivative $h'(\beta)=-1+\hat{t}\hat{\delta}^{2\hat{t}}\beta^{\hat{t}-1}$ for all $\beta\in\mathbb{R}$. Clearly, h' is negative in a neighborhood of 0. We thus have two cases:

1. $h'(\beta)\leq 0$ for all $\beta\in[0,1]$. This happens if and only if $\hat{\delta}\leq\left(\frac{1}{\hat{t}}\right)^{\frac{1}{2\hat{t}}}$ and, in this case, $h(\beta)\geq\hat{\delta}^{2\hat{t}}>0$ for all $\beta\in[0,1]$.

2. $h'(\beta)>0$ for some $\beta\in[0,1]$. Since $h'(\beta)>0$ for some $\beta\in[0,1]$, we have that $\hat{\delta}>\left(\frac{1}{\hat{t}}\right)^{\frac{1}{2\hat{t}}}$. Since h is convex and $\hat{\delta}>\left(\frac{1}{\hat{t}}\right)^{\frac{1}{2\hat{t}}}$, that is $1>1/\hat{t}\hat{\delta}^{2\hat{t}}>0$, this implies that h is minimized at $\beta_\star\in(0,1)$ where $\beta_\star=\sqrt[\hat{t}-1]{1/\hat{t}\hat{\delta}^{2\hat{t}}}\in(0,1)$ and

$$\begin{aligned}h(\beta_\star)&=1-\left(\frac{1}{\hat{t}\hat{\delta}^{2\hat{t}}}\right)^{\frac{1}{\hat{t}-1}}+\hat{\delta}^{2\hat{t}}\left(\frac{1}{\hat{t}\hat{\delta}^{2\hat{t}}}\right)^{\frac{\hat{t}}{\hat{t}-1}}\\&=1-\left(\frac{1}{\hat{t}\hat{\delta}^{2\hat{t}}}\right)^{\frac{1}{\hat{t}-1}}+\hat{\delta}^{2\hat{t}}\left(\frac{1}{\hat{t}\hat{\delta}^{2\hat{t}}}\right)\left(\frac{1}{\hat{t}\hat{\delta}^{2\hat{t}}}\right)^{\frac{1}{\hat{t}-1}}\\&=1-\left(\frac{1}{\hat{t}\hat{\delta}^{2\hat{t}}}\right)^{\frac{1}{\hat{t}-1}}\left(1-\frac{1}{\hat{t}}\right)\geq 1-\left(1-\frac{1}{\hat{t}}\right)\geq\frac{1}{\hat{t}}>0.\end{aligned}$$

We can conclude that

$$\frac{1}{1+\frac{\hat{\delta}^{\hat{t}}\left(\beta\hat{\delta}\right)^{\hat{t}}\left(1-\beta\hat{\delta}\right)}{(1-\beta)\left(1-\left(\beta\hat{\delta}\right)^{\hat{t}}\right)}}\leq(1-\beta)\frac{1}{\left(1-\hat{\delta}\right)\min\left\{\frac{1}{\hat{t}},\hat{\delta}^{2\hat{t}}\right\}}.\tag{42}$$

Finally, since $\hat{\delta}=\min\{\lambda,(1-\lambda)\delta\}$ and λ can be arbitrarily chosen, $\hat{\delta}$ is maximized for $\lambda=\delta/(1+\delta)$. In this case, $\hat{\delta}=\delta/(1+\delta)$. We will use (42) with this choice of $\hat{\delta}$ later on. ▲

Lemma 15, paired with Remark 2, is instrumental in proving Theorem 4. In fact, it only provides an estimate for the range of the fixed points of the averaged operator T_λ with $\lambda = \delta/(1+\delta)$. The next formula describes the relation between the points $\tilde{x}_\beta$ which solve (2) for the operator T and the points $\tilde{x}_{\beta,\lambda}$ which solve the same equation, but for the operator T_λ. In turn, this provides a relation between $\mathrm{Rg}(\tilde{x}_\beta)$ and $\mathrm{Rg}(\tilde{x}_{\beta,\lambda})$, and via (37) and (39), between $\mathrm{Rg}(x_\beta)$ and $\mathrm{Rg}(x_{\beta,\lambda})$.

Lemma 16 *If T is normalized, monotone, and translation invariant, then*

$$\tilde{x}_\beta = \tilde{x}_{\frac{\beta}{(1-\lambda)+\lambda\beta},\lambda} \qquad \forall x \in \mathbb{R}^k, \forall \beta, \lambda \in (0,1).$$

Moreover, for each $\lambda \in (0,1)$ the function $f_\lambda : (0,1) \to (0,1)$, defined by $f_\lambda(\beta) = \beta/[(1-\lambda)+\lambda\beta]$ for all $\beta \in (0,1)$, is strictly increasing and $\lim_{\beta\to 1} f_\lambda(\beta) = 1$.

Proof. Define the averaged operator $T_\lambda = \lambda I + (1-\lambda)T$ with $\lambda \in (0,1)$. By definition of $\tilde{x}_{\beta,\lambda}$, note that

$$\begin{aligned}
&(1-\beta)x + \beta\lambda\tilde{x}_{\beta,\lambda} + \beta(1-\lambda)T(\tilde{x}_{\beta,\lambda}) \\
&= (1-\beta)x + \beta[\lambda\tilde{x}_{\beta,\lambda} + (1-\lambda)T(\tilde{x}_{\beta,\lambda})] \\
&= (1-\beta)x + \beta T_\lambda(\tilde{x}_{\beta,\lambda}) = \tilde{x}_{\beta,\lambda} \quad \forall x \in \mathbb{R}^k, \forall \beta, \lambda \in (0,1),
\end{aligned}$$

that is,

$$\frac{1-\beta}{1-\beta\lambda}x + \frac{\beta(1-\lambda)}{1-\beta\lambda}T(\tilde{x}_{\beta,\lambda}) = \tilde{x}_{\beta,\lambda} \quad \forall \beta, \lambda \in (0,1), \forall x \in \mathbb{R}^k,$$

yielding that $\tilde{x}_{\beta,\lambda}$ solves equation (2) for the operator T with weight $\frac{\beta(1-\lambda)}{1-\beta\lambda}$. By the uniqueness of the solution, we can conclude that $\tilde{x}_{\frac{\beta(1-\lambda)}{1-\beta\lambda}} = \tilde{x}_{\beta,\lambda}$ for all $x \in \mathbb{R}^k$ and for all $\beta, \lambda \in (0,1)$. Fix $\lambda \in (0,1)$. If we define $g_\lambda : (0,1) \to (0,1)$ by $g_\lambda(\beta) = \beta(1-\lambda)/(1-\beta\lambda)$ for all $\beta \in (0,1)$, then g_λ is well defined and $g'_\lambda > 0$. The inverse of g_λ is f_λ and shares the same properties and, in particular, $\lim_{\beta\to 1} f_\lambda(\beta) = 1$. Since λ was arbitrarily chosen, it follows that

$$\tilde{x}_{f_\lambda(\beta),\lambda} = \tilde{x}_{g_\lambda(f_\lambda(\beta))} = \tilde{x}_\beta \qquad \forall x \in \mathbb{R}^k, \forall \lambda, \beta \in (0,1),$$

proving the statement. ■

Proof of Theorem 4. Set $\bar{\lambda} = \delta/(1+\delta) \in (0,1)$. By Lemma 15 and Remark 2 and since $\underline{A}(T_{\bar{\lambda}}) = \underline{A}(T) \vee I$, we have that for each $x \in \mathbb{R}^k$ and for each $\beta \in (0,1)$

$$\mathrm{Rg}\left(x_{\beta,\bar{\lambda}}\right) \leq (1-\beta)\frac{1}{\left(1-\frac{\delta}{1+\delta}\right)\min\left\{\frac{1}{\hat{t}}, \left(\frac{\delta}{1+\delta}\right)^{2\hat{t}}\right\}}\mathrm{Rg}(x) \leq (1-\beta)\kappa_T \mathrm{Rg}(x).$$

By (37), we have that

$$\mathrm{Rg}\left(\tilde{x}_{\beta,\bar{\lambda}}\right) \leq (1-\beta)(1+\beta\kappa_T)\mathrm{Rg}(x) \quad \forall x \in \mathbb{R}^k, \forall \beta \in (0,1).$$

By Lemma 16, recall that

$$\tilde{x}_\beta = \tilde{x}_{\frac{\beta}{(1-\bar{\lambda})+\bar{\lambda}\beta},\bar{\lambda}} = \tilde{x}_{f_{\bar{\lambda}}(\beta),\bar{\lambda}} \qquad \forall \beta \in (0,1), \forall x \in \mathbb{R}^k.$$

We can conclude that

$$\mathrm{Rg}\left(\tilde{x}_\beta\right) = \mathrm{Rg}\left(\tilde{x}_{f_{\bar{\lambda}}(\beta),\bar{\lambda}}\right) \leq \left(1 - f_{\bar{\lambda}}(\beta)\right)\left(1 + f_{\bar{\lambda}}(\beta)\kappa_T\right)\mathrm{Rg}(x) \quad \forall x \in \mathbb{R}^k, \forall \beta \in (0,1).$$

By (39), we have that

$$\mathrm{Rg}\left(x_\beta\right) \leq \mathrm{Rg}\left(\tilde{x}_\beta\right) \leq \left(1 - f_{\bar{\lambda}}(\beta)\right)\left(1 + f_{\bar{\lambda}}(\beta)\kappa_T\right)\mathrm{Rg}(x) \quad \forall x \in \mathbb{R}^k, \forall \beta \in (0,1).$$

Finally, observe that $\left(1-\bar{\lambda}\right) + \bar{\lambda}\beta \in (\beta, 1)$, that is, $1 > \frac{\beta}{(1-\delta)+\delta\beta} > \beta$ for all $\beta \in (0,1)$. This implies that $1 > f_{\bar{\lambda}}(\beta) > \beta > 0$ and $0 < 1 - f_{\bar{\lambda}}(\beta) < 1 - \beta$ for all $\beta \in (0,1)$. Since $\kappa_T > 0$, we can conclude that

$$\left(1 - f_{\bar{\lambda}}(\beta)\right)\left(1 + f_{\bar{\lambda}}(\beta)\kappa_T\right) \leq (1-\beta)(1+\kappa_T) \quad \forall \beta \in (0,1),$$

yielding that

$$\mathrm{Rg}\left(x_\beta\right) \leq \mathrm{Rg}\left(\tilde{x}_\beta\right) \leq (1-\beta)(1+\kappa_T)\mathrm{Rg}(x) \quad \forall x \in \mathbb{R}^k, \forall \beta \in (0,1),$$

proving the statement. ■

E.3 Rate of convergence

Proof of Theorem 5. Consider $x \in \mathbb{R}^k$ and $\beta \in (0,1)$. As usual, we have that $\tilde{x}_\beta = (1-\beta)x + \beta x_\beta$. By point 2 of Proposition 12 and since T is continuously differentiable and $\underline{A}(T)$ is regular, $\Gamma\left(\partial_C T(0)\right)$ consists of only one element, denoted by γ. Set $\bar{h} = \langle \gamma, x \rangle$. By definition of $W_{\tilde{x}_\beta,\bar{h}}$ (see proof of Proposition 13), we have that

$$W_{\tilde{x}_\beta,\bar{h}}\left((1-\beta)x + \beta x_\beta\right) = W_{\tilde{x}_\beta,\bar{h}}\tilde{x}_\beta = T\left(\tilde{x}_\beta\right) = T\left((1-\beta)x + \beta x_\beta\right) = x_\beta.$$

By (3), we have that $x_\beta = x_{\beta, W_{\tilde{x}_\beta,\bar{h}}}$. By point 1 of Proposition 12 and since $\underline{A}(T)$ is regular, $\Gamma\left(W_{\tilde{x}_\beta,\bar{h}}\right)$ consists of only one element which we denote by γ_β. It follows that

$$\begin{aligned}
\left\| x_\beta - \langle \gamma, x \rangle e \right\|_\infty &\leq \left\| x_\beta - \langle \gamma_\beta, x \rangle e \right\|_\infty + \left\| \langle \gamma_\beta, x \rangle e - \langle \gamma, x \rangle e \right\|_\infty \\
&= \left\| x_{\beta, W_{\tilde{x}_\beta,\bar{h}}} - \langle \gamma_\beta, x \rangle e \right\|_\infty + \left| \langle \gamma_\beta - \gamma, x \rangle \right| \\
&\leq \left\| x_{\beta, W_{\tilde{x}_\beta,\bar{h}}} - \langle \gamma_\beta, x \rangle e \right\|_\infty + \left\| \gamma_\beta - \gamma \right\|_1 \left\| x \right\|_\infty.
\end{aligned}$$

We next bound the two terms on the right-hand side.

a By Example 1, we have that

$$\left\| x_{\beta,W_{\tilde{x}_\beta,\bar{h}}} - \langle \gamma_\beta, x \rangle e \right\|_\infty \leq \mathrm{Rg}\left(x_{\beta,W_{\tilde{x}_\beta,\bar{h}}} \right) = \mathrm{Rg}\left(x_\beta \right).$$

b We next bound $\|\gamma_\beta - \gamma\|_1 = \|\gamma - \gamma_\beta\|_1$. Consider $W = J_T(0)$. By Proposition 12, we have that $\min_{i,j:\underline{a}_{ij}=1} w_{ij} \geq \delta$. Set $\tilde{W} = \lambda I + (1-\lambda) W$ and $\hat{W} = \lambda I + (1-\lambda) W_{\tilde{x}_\beta,\bar{h}}$ where λ can be arbitrarily chosen in $(0,1)$. It is immediate to see that $\gamma^{\mathrm{T}} \tilde{W}^t = \gamma^{\mathrm{T}}$ and $\gamma_\beta^{\mathrm{T}} \hat{W}^t = \gamma_\beta^{\mathrm{T}}$ for all $t \in \mathbb{N}$. Note that $A\left(\tilde{W}\right)$ and $A\left(\hat{W}\right)$ are both regular and such that $A\left(\tilde{W}\right), A\left(\hat{W}\right) \geq \underline{A}(T) \vee I$. It follows that there exist $h \in \{1, ..., k\}$ and $\hat{t} \in \mathbb{N}$ such that $\tilde{w}_{ih}^{(\hat{t})} > 0$ for all $i \in \{1, ..., k\}$ and $\hat{t}$ can be chosen to be t_T. Since $\min_{i,j:\underline{a}_{ij}=1} w_{ij} \geq \delta$, $\min_{i\in\{1,...,k\}} \tilde{w}_{ih}^{(t_T)} \geq \tilde{\delta}^{t_T}$ where $\tilde{\delta} = \min\{\lambda, (1-\lambda)\delta\}$. By Seneta [40], this implies that

$$\|\gamma - \gamma_\beta\|_1 \leq \frac{1}{\tilde{\delta}^{t_T}} \left\| \tilde{W}^{t_T} - \hat{W}^{t_T} \right\|_\infty.$$

Since Since λ can be arbitrarily chosen, we then choose $\lambda = \delta/(1+\delta)$ so to maximize $\tilde{\delta}$, yielding that

$$\|\gamma - \gamma_\beta\|_1 \leq \frac{(1+\delta)^{t_T}}{\delta^{t_T}} \left\| \tilde{W}^{t_T} - \hat{W}^{t_T} \right\|_\infty. \tag{43}$$

Next, by induction and since the space of matrices endowed with $\| \|_\infty$ is a Banach algebra and $\left\|\bar{W}\right\|_\infty = 1$ for all stochastic matrices $\bar{W}$,[22] we have that

$$\left\| \tilde{W}^{t_T} - \hat{W}^{t_T} \right\|_\infty \leq t_T \left\| \tilde{W} - \hat{W} \right\|_\infty = \left(1 - \frac{\delta}{1+\delta}\right) t_T \left\| W - W_{\tilde{x}_\beta,\bar{h}} \right\|_\infty. \tag{44}$$

[22] First, recall that, given a $k \times k$ matrix E,

$$\|E\|_\infty = \max_{i\in\{1,...k\}} \sum_{j=1}^{k} |e_{ij}|.$$

In other words, $\|E\|_\infty$ is the dual norm of the operator $x \mapsto Ex$ when $\mathbb{R}^k$ is endowed with $\| \|_\infty$. For this reason, it can be computed by calculating the $\| \|_1$ of each row of E and then take the maximum of these values. By induction, we prove that

$$\left\| \tilde{W}^t - \hat{W}^t \right\|_\infty \leq t \left\| \tilde{W} - \hat{W} \right\|_\infty \quad \forall t \in \mathbb{N}.$$

The statement is trivial for $t = 1$. Assume it holds for t, we show it holds for $t+1$. Observe that

$$\begin{aligned}
\left\| \tilde{W}^{t+1} - \hat{W}^{t+1} \right\|_\infty &= \left\| \tilde{W}\left(\tilde{W}^t - \hat{W}^t\right) + \left(\tilde{W} - \hat{W}\right)\hat{W}^t \right\|_\infty \\
&\leq \left\| \tilde{W}\left(\tilde{W}^t - \hat{W}^t\right) \right\|_\infty + \left\| \left(\tilde{W} - \hat{W}\right)\hat{W}^t \right\|_\infty \\
&\leq \left\| \tilde{W} \right\|_\infty \left\| \tilde{W}^t - \hat{W}^t \right\|_\infty + \left\| \tilde{W} - \hat{W} \right\|_\infty \left\| \hat{W}^t \right\|_\infty \\
&= \left\| \tilde{W}^t - \hat{W}^t \right\|_\infty + \left\| \tilde{W} - \hat{W} \right\|_\infty \\
&\leq t \left\| \tilde{W} - \hat{W} \right\|_\infty + \left\| \tilde{W} - \hat{W} \right\|_\infty = (t+1) \left\| \tilde{W} - \hat{W} \right\|_\infty,
\end{aligned}$$

the statement follows by induction.

Consider the i-th row of $W - W_{\tilde{x}_\beta,\bar{h}}$. By definition of $W_{\tilde{x}_\beta,\bar{h}}$ and since $\nabla T_i(he) = \nabla T_i(0)$ for all $h \in \mathbb{R}$, we have that the i-th row of $W - W_{\tilde{x}_\beta,\bar{h}}$ is equal to $\nabla T_i\left(\hat{h}e\right) - \nabla T_i\left(\lambda_{i,\tilde{x}_\beta,\bar{h}}\tilde{x}_\beta + \left(1 - \lambda_{i,\tilde{x}_\beta,\bar{h}}\right)\bar{h}e\right)$ where $\lambda_{i,\tilde{x}_\beta,\bar{h}} \in [0,1]$ and $\hat{h}$ we chose it to be an element of

$$\left[\min\left(\lambda_{i,\tilde{x}_\beta,\bar{h}}\tilde{x}_\beta + \left(1 - \lambda_{i,\tilde{x}_\beta,\bar{h}}\right)\bar{h}e\right), \max\left(\lambda_{i,\tilde{x}_\beta,\bar{h}}\tilde{x}_\beta + \left(1 - \lambda_{i,\tilde{x}_\beta,\bar{h}}\right)\bar{h}e\right)\right].$$

Since the Jacobian of T is Lipschitz continuous, we also have that

$$\begin{aligned}
&\left\|\nabla T_i\left(\hat{h}e\right) - \nabla T_i\left(\lambda_{i,\tilde{x}_\beta,\bar{h}}\tilde{x}_\beta + \left(1 - \lambda_{i,\tilde{x}_\beta,\bar{h}}\right)\bar{h}e\right)\right\|_1 \\
&\leq L\left\|\hat{h}e - \left(\lambda_{i,\tilde{x}_\beta,\bar{h}}\tilde{x}_\beta + \left(1 - \lambda_{i,\tilde{x}_\beta,\bar{h}}\right)\bar{h}e\right)\right\|_\infty.
\end{aligned}$$

By (36) and given our choice of $\hat{h}$, we can conclude that

$$\begin{aligned}
\left\|\hat{h}e - \left(\lambda_{i,\tilde{x}_\beta,\bar{h}}\tilde{x}_\beta + \left(1 - \lambda_{i,\tilde{x}_\beta,\bar{h}}\right)\bar{h}e\right)\right\|_\infty &= \left\|\left(\lambda_{i,\tilde{x}_\beta,\bar{h}}\tilde{x}_\beta + \left(1 - \lambda_{i,\tilde{x}_\beta,\bar{h}}\right)\bar{h}e\right) - \hat{h}e\right\|_\infty \\
&\leq \mathrm{Rg}\left(\lambda_{i,\tilde{x}_\beta,\bar{h}}\tilde{x}_\beta + \left(1 - \lambda_{i,\tilde{x}_\beta,\bar{h}}\right)\bar{h}e\right) \\
&\leq \lambda_{i,\tilde{x}_\beta,\bar{h}}\,\mathrm{Rg}\left(\tilde{x}_\beta\right) + \left(1 - \lambda_{i,\tilde{x}_\beta,\bar{h}}\right)\mathrm{Rg}\left(\bar{h}e\right) \\
&\leq \mathrm{Rg}\left(\tilde{x}_\beta\right)
\end{aligned}$$

By definition of $\left\|W - W_{\tilde{x}_\beta,\bar{h}}\right\|_\infty$ and (43) and (44) and since i was arbitrarily chosen, this implies that

$$\begin{aligned}
\|\gamma - \gamma_\beta\|_1 &\leq \frac{(1+\delta)^{t_T}}{\delta^{t_T}}\left\|\tilde{W}^{t_T} - \hat{W}^{t_T}\right\|_\infty \leq \frac{(1+\delta)^{t_T}}{\delta^{t_T}}\left(1 - \frac{\delta}{1+\delta}\right)t_T\left\|W - W_{\tilde{x}_\beta,\bar{h}}\right\|_\infty \\
&\leq \frac{(1+\delta)^{t_T}}{\delta^{t_T}}\left(1 - \frac{\delta}{1+\delta}\right)t_T L\,\mathrm{Rg}\left(\tilde{x}_\beta\right).
\end{aligned}$$

By points a and b and Theorem 4 an since $\mathrm{Rg}\left(\tilde{x}_\beta\right) \leq \mathrm{Rg}\left(x_\beta\right)$, we can conclude that

$$\begin{aligned}
\|x_\beta - \langle\gamma, x\rangle e\|_\infty &\leq \mathrm{Rg}\left(x_\beta\right) + \frac{(1+\delta)^{t_T}}{\delta^{t_T}}\left(1 - \frac{\delta}{1+\delta}\right)t_T L\|x\|_\infty\,\mathrm{Rg}\left(\tilde{x}_\beta\right) \\
&\leq (1-\beta)(1+\kappa_T)\left(1 + \frac{(1+\delta)^{t_T}}{\delta^{t_T}}\left(1 - \frac{\delta}{1+\delta}\right)t_T L\|x\|_\infty\right)\mathrm{Rg}(x),
\end{aligned}$$

proving the statement. ■

Given a vector $y \in \mathbb{R}^k$, we set $\max y = \max_{i\in\{1,\ldots,k\}} y_i$ and $\min y = \min_{i\in\{1,\ldots,k\}} y_i$.

Lemma 17 *If $\{S_\alpha\}_{\alpha\in\mathcal{A}}$ is nice and such that S_α is positively homogeneous for all $\alpha \in \mathcal{A}$, then for each $x \in \mathbb{R}^k$*

$$\max x_\beta \geq \sup_{\alpha\in\mathcal{A}} \varphi_\alpha(x) \geq \min x_\beta \qquad \forall\beta \in (0,1).$$

Proof. Recall that $T : \mathbb{R}^k \to \mathbb{R}^k$ is defined by $T(y) = \sup_{\alpha \in \mathcal{A}} S_\alpha(y)$ for all $y \in \mathbb{R}^k$ and for each $y \in \mathbb{R}^k$ there exists $\alpha_y \in \mathcal{A}$ such that $T(y) = S_{\alpha_y}(y)$. Fix $x \in \mathbb{R}^k$ and $\beta \in (0,1)$. Recall that $x_\beta = T((1-\beta)x + \beta x_\beta)$ and $\varphi_\alpha(x) = \min_{\gamma \in \Gamma(\partial_C S_\alpha(0))} \langle \gamma, x \rangle$ for all $\alpha \in \mathcal{A}$. It follows that there exists $\bar{\alpha} \in \mathcal{A}$ such that

$$S_{\bar{\alpha}}((1-\beta)x + \beta x_\beta) = T((1-\beta)x + \beta x_\beta) = x_\beta.$$

This implies that $x_\beta = x_{\beta,\bar{\alpha}}$. By Proposition 14 and since $S_{\bar{\alpha}}$ is normalized, monotone, translation invariant, concave, and such that $E(S_{\bar{\alpha}}) = D$ as well as $\Gamma(\partial_C S_{\bar{\alpha}}(0)) \subseteq \Delta$, we have that $\sup_{\alpha \in \mathcal{A}} \varphi_\alpha(x) \geq \varphi_{\bar{\alpha}}(x) \geq \varphi_{\bar{\alpha}}(x_{\beta,\bar{\alpha}}) = \varphi_{\bar{\alpha}}(x_\beta) \geq \min x_\beta$, yielding one inequality. For each $n \in \mathbb{N}$ define $\alpha_n \in \mathcal{A}$ to be such that $\varphi_{\alpha_n}(x) \geq \sup_{\alpha \in \mathcal{A}} \varphi_\alpha(x) - 1/n$. Fix $n \in \mathbb{N}$. By construction, we have that

$$S_{\alpha_n}((1-\beta)x + \beta x_\beta) \leq T((1-\beta)x + \beta x_\beta) = x_\beta.$$

Fix $\hat{h} \in \mathbb{R}$. By Proposition 11, there exists a stochastic matrix $W_{\tilde{x}_\beta,\hat{h}}$ such that

$$x_\beta \geq S_{\alpha_n}((1-\beta)x + \beta x_\beta) = S_{\alpha_n}(\tilde{x}_\beta) = W_{\tilde{x}_\beta,\hat{h}}\tilde{x}_\beta = W_{\tilde{x}_\beta,\hat{h}}((1-\beta)x + \beta x_\beta)$$

where $w^i_{\tilde{x}_\beta,\hat{h}} \in \partial_C S_{\alpha_n,i}\left(\lambda_{i,\tilde{x}_\beta,\hat{h}}\tilde{x}_\beta + \left(1 - \lambda_{i,\tilde{x}_\beta,\hat{h}}\right)\hat{h}e\right)$ where $\lambda_{i,\tilde{x}_\beta,\hat{h}} \in [0,1]$ for all $i \in \{1,...,k\}$. By [23, Proposition A.3] and since $S_{\alpha_n,i}$ is positively homogeneous, $w^i_{\tilde{x}_\beta,\hat{h}} \in \partial_C S_{\alpha_n,i}(0)$ for all $i \in \{1,...,k\}$, yielding that $W_{\tilde{x}_\beta,\hat{h}} \in \partial_C S_{\alpha_n}(0)$. Since $W_{\tilde{x}_\beta,\hat{h}} \in \partial_C S_{\alpha_n}(0)$ and is a stochastic matrix, we have that there exists $\bar{\gamma} \in \Delta$ such that $\bar{\gamma}^{\mathrm{T}} W_{\tilde{x}_\beta,\hat{h}} = \bar{\gamma}^{\mathrm{T}}$ and $\bar{\gamma} \in \Gamma(\partial_C S_{\alpha_n}(0))$. It follows that

$$\langle \bar{\gamma}, x_\beta \rangle = \bar{\gamma}^{\mathrm{T}} x_\beta \geq \bar{\gamma}^{\mathrm{T}} W_{\tilde{x}_\beta,\hat{h}}((1-\beta)x + \beta x_\beta) = \bar{\gamma}^{\mathrm{T}}((1-\beta)x + \beta x_\beta) = (1-\beta)\langle \bar{\gamma}, x \rangle + \beta \langle \bar{\gamma}, x_\beta \rangle,$$

yielding that $\langle \bar{\gamma}, x_\beta \rangle \geq \langle \bar{\gamma}, x \rangle$ and $\max x_\beta \geq \langle \bar{\gamma}, x_\beta \rangle \geq \langle \bar{\gamma}, x \rangle \geq \min_{\gamma \in \Gamma(\partial_C S_{\alpha_n}(0))} \langle \gamma, x \rangle = \varphi_{\alpha_n}(x) \geq \sup_{\alpha \in \mathcal{A}} \varphi_\alpha(x) - 1/n$. By passing to the limit and since n was arbitrarily chosen, we have that $\max x_\beta \geq \sup_{\alpha \in \mathcal{A}} \varphi_\alpha(x)$, proving the statement. ■

Proof of Proposition 6. By the same techniques of Castagnoli, Cattelan, Maccheroni, Tebaldi, and Wang [11, Theorem 2] or Chandrasekher, Frick, Iijima, and Le Yaouanq [15], there exists a collection $\{S_\alpha\}_{\alpha \in \mathcal{A}}$ which is nice, represents T, and such that $\underline{A}(S_\alpha)$ is regular and S_α is positively homogeneous for all $\alpha \in \mathcal{A}$.[23] Fix $x \in \mathbb{R}^k$. By Proposition 1 and Lemma 17, $\bar{x} = \lim_{\beta \to 1} x_\beta = \sup_{\alpha \in \mathcal{A}} \varphi_\alpha(x) e$ where $\varphi_\alpha(x) = \min_{\gamma \in \Gamma(\partial_C S_\alpha(0))} \langle \gamma, x \rangle$ for all $\alpha \in \mathcal{A}$ and

$$\max x_\beta \geq \sup_{\alpha \in \mathcal{A}} \varphi_\alpha(x) \geq \min x_\beta \quad \forall \beta \in (0,1),$$

yielding that $\|\bar{x} - x_\beta\|_\infty \leq \mathrm{Rg}(x_\beta)$ for all $\beta \in (0,1)$. By Theorem 4, the statement follows. ■

[23] A proof is available upon request.

Proof of Proposition 7 Fix $\beta \in (0,1)$ and observe that if

$$\|x_\beta - \langle \gamma, x\rangle e\|_\infty \leq \hat{\xi}(\beta, c, x)$$

then

$$\begin{aligned}
|U_i(b(\langle \gamma, x\rangle e_{-i}), \langle \gamma, x\rangle e_{-i}) - U_i(\langle \gamma, x\rangle e)| &= \left|U_i^{\star,\beta}(\langle \gamma, x\rangle e_{-i}) - U_i(\langle \gamma, x\rangle e)\right| \\
&\leq \left|U_i^{\star,\beta}(\langle \gamma, x\rangle e_{-i}) - U_i^{\star,\beta}(x_{\beta,-i})\right| + |U_i(x_\beta) - U_i(\langle \gamma, x\rangle e)| \\
&\leq (L^\star(\beta) + L(\beta))\,\hat{\xi}(\beta, c, x)
\end{aligned}$$

$$\max_{i\in N} |U_i(x_\beta) - U_i(\langle \gamma, x\rangle e)| \leq \bar{L}\,\|x_\beta - \langle \gamma, x\rangle e\|_\infty \leq \bar{L}\hat{\xi}(\beta, c, x)$$

■

E.4 Computing the fixed point

Consider a nonexpansive operator $T : \mathbb{R}^k \to \mathbb{R}^k$. Given $\beta \in (0,1)$ and $x \in \mathbb{R}^k$, define $T_{\beta,x}, \tilde{T}_{\beta,x} : \mathbb{R}^k \to \mathbb{R}^k$ by

$$T_{\beta,x}(y) = T((1-\beta)x + \beta y) \text{ and } \tilde{T}_{\beta,x}(y) = (1-\beta)x + \beta T(y) \quad \forall y \in \mathbb{R}^k.$$

Lemma 18 *Let T be nonexpansive. If $\beta \in (0,1)$ and $x \in \mathbb{R}^k$, then $T_{\beta,x}$ and $\tilde{T}_{\beta,x}$ are β-contractions. In particular, for each $\beta \in (0,1)$ and for each $x \in \mathbb{R}^k$, there exist unique $x_\beta, \tilde{x}_\beta \in \mathbb{R}^k$ such that*

$$T^t_{\beta,x}(y) \to x_\beta \quad \forall y \in \mathbb{R}^k,\ T_{\beta,x}(x_\beta) = x_\beta$$

and

$$\tilde{T}^t_{\beta,x}(y) \to \tilde{x}_\beta \quad \forall y \in \mathbb{R}^k,\ \tilde{T}_{\beta,x}(\tilde{x}_\beta) = \tilde{x}_\beta.$$

Proof. Fix $\beta \in (0,1)$ and $x \in \mathbb{R}^k$. We prove that $T_{\beta,x}$ is a β-contraction. A similar argument holds for $\tilde{T}_{\beta,x}$. Since T is nonexpansive, we have that for each $y, z \in \mathbb{R}^k$

$$\begin{aligned}
\|T_{\beta,x}(y) - T_{\beta,x}(z)\|_\infty &= \|T((1-\beta)x + \beta y) - T((1-\beta)x + \beta z)\|_\infty \\
&\leq \|(1-\beta)x + \beta y - (1-\beta)x - \beta z\|_\infty = \beta\|y - z\|_\infty,
\end{aligned}$$

proving that $T_{\beta,x}$ is a β-contraction. By the Banach contraction principle, for each $y \in \mathbb{R}^k$ we have that $T^t_{\beta,x}(y) \to x_\beta$ as well as $T_{\beta,x}(x_\beta) = x_\beta$ where x_β is the unique fixed point of $T_{\beta,x}$. ■

Consider two nonexpansive operators $S, T : \mathbb{R}^k \to \mathbb{R}^k$.If for each $\beta \in (0,1)$ and for each $x \in \mathbb{R}^k$ we define $x_{\beta,S}$ and $x_{\beta,T}$ to be such that

$$x_{\beta,S} = S((1-\beta)x + \beta x_{\beta,S}) \text{ and } x_{\beta,T} = T((1-\beta)x + \beta x_{\beta,T}),$$

then we have the following simple monotonicity result.

Lemma 19 *Let S and T be nonexpansive. If S is monotone and $S \geq T$, then*

$$x_{\beta,S} \geq x_{\beta,T} \qquad \forall \beta \in (0,1), \forall x \in \mathbb{R}^k.$$

Proof of Proposition 8. Fix $\beta \in (0,1)$ and $x \in \mathbb{R}^k$. By Lemma 19, $x_{\beta,T} \leq x_{\beta,S_\alpha}$ for all $\alpha \in \mathcal{A}$, proving that $x_{\beta,T} \leq \inf_{\alpha \in \mathcal{A}} x_{\beta,S_\alpha}$. Since for each $x \in \mathbb{R}^k$ there exists $\alpha_x \in \mathcal{A}$ such that $S_{\alpha_x}(x) = T(x)$, we have that there exists $\bar{\alpha} \in \mathcal{A}$ such that $S_{\bar{\alpha}}((1-\beta)x + \beta x_{\beta,T}) = T((1-\beta)x + \beta x_{\beta,T}) = x_{\beta,T}$, proving that $\inf_{\alpha \in \mathcal{A}} x_{\beta,S_\alpha} \geq x_{\beta,T} = x_{\beta,S_{\bar{\alpha}}} \geq \inf_{\alpha \in \mathcal{A}} x_{\beta,S_\alpha}$, proving the statement. ■

E.5 Beyond Normalization

Proof of Lemma 3. Fix $\beta \in (0,1)$ and $x \in \mathbb{R}^k$. We have

$$\begin{aligned}
(x-x^*)^*_\beta &= T^*\left((1-\beta)(x-x^*) + \beta (x-x^*)^*_\beta\right) \\
&= T\left((1-\beta)x + \beta\left((x-x^*)^*_\beta + x^*\right)\right) - x^* \\
&\implies (x-x^*)^*_\beta + x^* = T\left((1-\beta)x + \beta\left((x-x^*)^*_\beta + x^*\right)\right) \\
&\implies x_\beta = (x-x^*)^*_\beta + x^*.
\end{aligned}$$

■

Proof of Lemma 4. 1. Fix $k \in \mathbb{R}$ and observe that

$$T^*(ke) = T(x^* + ke) - x^* = T(x^*) - x^* + ke = ke$$

proving that T^* is normalized and that $D \subseteq E(T^*)$. Assume that there exists $y \in E(T^*) \setminus D$. It follows that

$$\begin{aligned}
y = T^*(y) &= T(x^* + y) - x^* \\
&\implies x^* + y = T(x^* + y),
\end{aligned}$$

that is $x^* + y \in E(T) \subseteq \left\{x^* + ce \in \mathbb{R}^k : c \in \mathbb{R}\right\}$ yielding a contradiction with $y \notin D$.

2. Let $B_\varepsilon(x^*)$ be neighborhood of x^* where T is continuously differentiable. Observing that for all y such that $||y|| \leq \frac{\varepsilon}{2}$,

$$\lim_{h \to 0} \frac{T^*(y+hv) - T^*(y)}{h} = \lim_{h \to 0} \frac{T(x^* + y + hv) - T(x^* + y)}{h}$$

and the right hand side by construction exists and is a continuous function of y, proving the statement.

3. Let T be concave and fix $x, x' \in \mathbb{R}^k$ and $\lambda \in [0,1]$. We have

$$\begin{aligned}
T^*\left((1-\lambda)\, x' + \lambda x\right) &= T\left(x^* + (1-\lambda)\, x' + \lambda x\right) - x \\
&= T\left((1-\lambda)\left(x' + x^*\right) + \lambda\left(x + x^*\right)\right) - x^* \\
&\geq (1-\lambda)\, T\left(x' + x^*\right) + \lambda T\left(x + x^*\right) - x^* \\
&= (1-\lambda)\left(T\left(x' + x^*\right) - x^*\right) + \lambda\left(T\left(x + x^*\right) - x^*\right) \\
&= (1-\lambda)\, T^*\left(x'\right) + \lambda T^*\left(x\right).
\end{aligned}$$

This proves that T^* is concave.

4. Fix $x \in \mathbb{R}^k$ and $\lambda \in (0,1)$. We have

$$\begin{aligned}
T^*\left(\lambda x\right) &= T\left(x^* + \lambda x\right) - x^* = T\left((1-\lambda)\, x^* + \lambda\left(x + x^*\right)\right) - x^* \\
&\geq (1-\lambda)\, T\left(x^*\right) + \lambda T\left(x + x^*\right) - x^* \\
&= \lambda\left(T\left(x + x^*\right) - x^*\right) \\
&= \lambda T^*\left(x\right).
\end{aligned}$$

This proves that T^* is star-shaped. ■

Proof of Lemma 5. Consider $\boldsymbol{\beta}, \boldsymbol{\gamma} \in (0,1)^k$. Since $\left\|x_{\boldsymbol{\beta}}\right\|_\infty \leq \|x\|_\infty$ and T is nonexpansive, note that

$$\begin{aligned}
\left\|x_{\boldsymbol{\beta}} - x_{\boldsymbol{\gamma}}\right\|_\infty &\leq \left\|(1-\boldsymbol{\beta})\, x + \boldsymbol{\beta} x_{\boldsymbol{\beta}} - (1-\boldsymbol{\gamma})\, x - \boldsymbol{\gamma} x_{\boldsymbol{\gamma}}\right\|_\infty \\
&= \left\|(\boldsymbol{\gamma} - \boldsymbol{\beta})\, x + \boldsymbol{\beta} x_{\boldsymbol{\beta}} - \boldsymbol{\gamma} x_{\boldsymbol{\beta}} + \boldsymbol{\gamma} x_{\boldsymbol{\beta}} - \boldsymbol{\gamma} x_{\boldsymbol{\gamma}}\right\|_\infty \\
&= \left\|(\boldsymbol{\gamma} - \boldsymbol{\beta})\left(x - x_{\boldsymbol{\beta}}\right) + \boldsymbol{\gamma}\left(x_{\boldsymbol{\beta}} - x_{\boldsymbol{\gamma}}\right)\right\|_\infty \\
&\leq \left\|\boldsymbol{\gamma} - \boldsymbol{\beta}\right\|_\infty \left\|x - x_{\boldsymbol{\beta}}\right\|_\infty + \left\|\boldsymbol{\gamma}\right\|_\infty \left\|x_{\boldsymbol{\beta}} - x_{\boldsymbol{\gamma}}\right\|_\infty,
\end{aligned}$$

proving the statement. ■

Proof of Lemma 6. Define $\boldsymbol{\gamma}_\alpha = \overline{\boldsymbol{\beta}_\alpha} e$ for all $\alpha \in A$. By the previous lemma and since $\lim_{\beta \to 1} x_\beta = \bar{x}$, we have that

$$\begin{aligned}
\left\|\bar{x} - x_{\boldsymbol{\beta}_\alpha}\right\|_\infty &\leq \left\|\bar{x} - x_{\boldsymbol{\gamma}_\alpha}\right\|_\infty + \left\|x_{\boldsymbol{\gamma}_\alpha} - x_{\boldsymbol{\beta}_\alpha}\right\|_\infty \leq \left\|\bar{x} - x_{\overline{\boldsymbol{\beta}_\alpha}}\right\|_\infty + 2\frac{\left\|\boldsymbol{\beta}_\alpha - \boldsymbol{\gamma}_\alpha\right\|_\infty}{1 - \left\|\boldsymbol{\gamma}_\alpha\right\|_\infty} \|x\|_\infty \\
&= \left\|\bar{x} - x_{\overline{\boldsymbol{\beta}_\alpha}}\right\| + 2\frac{\overline{\boldsymbol{\beta}_\alpha} - \underline{\boldsymbol{\beta}_\alpha}}{1 - \overline{\boldsymbol{\beta}_\alpha}} \|x\|_\infty \to 0,
\end{aligned}$$

proving the statement. ■

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F Online appendix: omitted proofs

Proof of Lemma 8. Consider $\bar{x} \in L$. By definition of L, there exists $\{x_{\beta_n}\}_{n\in\mathbb{N}} \subseteq \{x_\beta\}_{\beta\in(0,1)}$ such that $\beta_n \to 1$ and $x_{\beta_n} \to \bar{x}$. By definition of x_β and since T is Lipschitz continuous and $\lim_n [(1-\beta_n)x + \beta_n x_{\beta_n}] = \bar{x}$, we have that

$$\bar{x} = \lim_n x_{\beta_n} = \lim_n T\left((1-\beta_n)x + \beta_n x_{\beta_n}\right) = T(\bar{x}),$$

proving that $\bar{x} \in E(T)$, that is, $L \subseteq E(T)$. Next, assume that $E(T) = D$. By the previous part of the proof, we have that $L \subseteq E(T) = D$. This implies that there exists a set $H \subseteq \mathbb{R}$ such that $\{he\}_{h\in H} = L$ and, in particular, $\liminf_{\beta\to 1} x_\beta = (\inf H)e$ as well as $\limsup_{\beta\to 1} x_\beta = (\sup H)e$. Since L is closed and bounded, it follows that $(\inf H)e, (\sup H)e \in L \subseteq E(T)$, proving the second part of the statement. ■

Proof of Lemma ??. We start with a preliminary observation. By induction and since T is nonexpansive and normalized, it is obvious that $\|x^t\|_\infty \le \|x\|_\infty$ for all $t \in \mathbb{N}$. Note that for each $l, t \in \mathbb{N}$

$$\begin{aligned}
\left\|x^{t+1} - x_{\beta_l}\right\|_\infty &= \left\|T\left((1-\beta_{t+1})x + \beta_{t+1}x^t\right) - T\left((1-\beta_l)x + \beta_l x_{\beta_l}\right)\right\|_\infty \\
&\le \left\|T\left((1-\beta_{t+1})x + \beta_{t+1}x^t\right) - T\left((1-\beta_l)x + \beta_l x^t\right)\right\|_\infty \\
&+ \left\|T\left((1-\beta_l)x + \beta_l x^t\right) - T\left((1-\beta_l)x + \beta_l x_{\beta_l}\right)\right\|_\infty \\
&\le \left\|(1-\beta_{t+1})x + \beta_{t+1}x^t - (1-\beta_l)x - \beta_l x^t\right\|_\infty \\
&+ \left\|(1-\beta_l)x + \beta_l x^t - (1-\beta_l)x - \beta_l x_{\beta_l}\right\|_\infty \\
&= \left\|(\beta_l - \beta_{t+1})\left(x - x^t\right)\right\|_\infty + \beta_l\left\|x^t - x_{\beta_l}\right\|_\infty \\
&\le \beta_l\left\|x^t - x_{\beta_l}\right\|_\infty + |\beta_{t+1} - \beta_l|\left\|x - x^t\right\|_\infty \\
&\le \beta_l\left\|x^t - x_{\beta_l}\right\|_\infty + 2\|x\|_\infty|\beta_{t+1} - \beta_l|.
\end{aligned}$$

Since $\{\beta_t\}_{t\in\mathbb{N}}$ is an increasing sequence, we have that for each $t \ge l$

$$\left\|x^{t+1} - x_{\beta_l}\right\|_\infty \le \beta_l\left\|x^t - x_{\beta_l}\right\|_\infty + 2\|x\|_\infty(\beta_{t+1} - \beta_l). \tag{45}$$

We next prove (**??**) by induction. By (45) and setting $s = t$, the statement is true for $m = 1$. Assume (**??**) holds for m. We show it holds for $m+1$. By (45) and inductive hypothesis, we have that

$$\begin{aligned}
\left\|x^{s+m+1} - x_{\beta_l}\right\|_\infty &\le \beta_l\left\|x^{s+m} - x_{\beta_l}\right\|_\infty + 2\|x\|_\infty(\beta_{s+m+1} - \beta_l) \\
&\le \beta_l^{m+1}\|x^s - x_{\beta_l}\|_\infty + 2\|x\|_\infty\sum_{r=1}^{m}\beta_l^{m+1-r}(\beta_{s+r} - \beta_l) + 2\|x\|_\infty(\beta_{s+m+1} - \beta_l) \\
&= \beta_l^{m+1}\|x^s - x_{\beta_l}\|_\infty + 2\|x\|_\infty\sum_{r=1}^{m+1}\beta_l^{m+1-r}(\beta_{s+r} - \beta_l),
\end{aligned}$$

proving the inductive step. ■

Proof of Lemma 19. Fix $\beta \in (0,1)$ and $x \in \mathbb{R}^k$. We prove by induction that $S^t_{\beta,x}(x) \geq T^t_{\beta,x}(x)$ for all $t \in \mathbb{N}$. For $t = 1$, note that $S^1_{\beta,x}(x) = S((1-\beta)x + \beta x) = S(x) \geq T(x) = T((1-\beta)x + \beta x) = T^1_{\beta,x}(x)$. Next, we assume the statement is true for t and we prove it holds for $t+1$. Since S is monotone and $S \geq T$, we have that

$$
\begin{aligned}
S^{t+1}_{\beta,x}(x) = S_{\beta,x}\left(S^t_{\beta,x}(x)\right) = S\left((1-\beta)x + \beta S^t_{\beta,x}(x)\right) \geq S\left((1-\beta)x + \beta T^t_{\beta,x}(x)\right) \\
\geq T\left((1-\beta)x + \beta T^t_{\beta,x}(x)\right) = T_{\beta,x}\left(T^t_{\beta,x}(x)\right) = T^{t+1}_{\beta,x}(x),
\end{aligned}
$$

proving the statement. By Lemma 18 and passing to the limit, $x_{\beta,S} = \lim_t S^t_{\beta,x}(x) \geq \lim_t T^t_{\beta,x}(x) = x_{\beta,T}$. ■

F.1 Computations for Section 3.1

The first-order conditions for the cost-minimization problem given $w_i \in \Delta$ read as

$$
\begin{aligned}
Q_{i0} &= \mu(1-\beta) F_i(Q_i, w_i) \qquad (46)\\
Q_{ij} &= \frac{1}{P_j}\mu\beta w_{ij} F_i(Q_i, w_i) \qquad \forall j \in \{1, ..., k\}
\end{aligned}
$$

where μ is the Lagrange multiplier of the only constraint. Given that $F_i(Q_i, w_i) = 1$ in the optimum, by plugging the previous conditions back in the production function we have:

$$
\begin{aligned}
1 &= S_i(w_i)\,\xi_i(\beta, w_i)\,(Z_i\mu(1-\beta) F_i(Q_i, w_i))^{1-\beta} \prod_{j=1}^{k} \left(\frac{1}{P_j}\mu\beta w_{ij} F_i(Q_i, w_i)\right)^{\beta w_{ij}} \\
&= \mu S_i(w_i)\,\xi_i(\beta, w_i)\,(Z_i(1-\beta))^{1-\beta} \prod_{j=1}^{k} \left(\frac{1}{P_j}\beta w_{ij}\right)^{\beta w_{ij}} \\
&= \mu S_i(w_i)\,(Z_i)^{1-\beta} \prod_{j=1}^{k} \left(\frac{1}{P_j}\right)^{\beta w_{ij}}
\end{aligned}
$$

which implies that

$$
\mu = \left(\frac{1}{Z_i}\right)^{1-\beta} \prod_{j=1}^{k} P_j^{\beta w_{ij}} \frac{1}{S_i(w_i)}.
$$

Next, observe that, in the optimum, for every i we have

$$
\begin{aligned}
Q_{i0} &= (1-\beta)\left(\frac{1}{Z_i}\right)^{1-\beta} \prod_{j=1}^{k} P_j^{\beta w_{ij}} \frac{1}{S_i(w_i)} \qquad (47)\\
Q_{ij} &= \frac{1}{P_j}\beta w_{ij}\left(\frac{1}{Z_i}\right)^{1-\beta} \prod_{j=1}^{k} P_j^{\beta w_{ij}} \frac{1}{S_i(w_i)} \qquad \forall j \in \{1, ..., k\}
\end{aligned}
$$

as well as

$$K_i(P, w_i) = Q_{i0} + \sum_{j=1}^{k} P_j Q_{ij} = \mu(1-\beta) F_i(Q_i, w_i) + \sum_{j=1}^{k} P_j \left(\frac{1}{P_j} \mu \beta w_{ij} F_i(Q_i, w_i) \right)$$

$$= \mu(1-\beta) + \mu\beta \sum_{j=1}^{k} w_{ij} = \mu = \left(\frac{1}{Z_i} \right)^{1-\beta} \prod_{j=1}^{k} P_j^{\beta w_{ij}} \frac{1}{S_i(w_i)}.$$

Given that each firm can pick its technology so to minimize their unitary cost, he zero-profit condition for every $i \in \{1, ..., k\}$ reads

$$P_i = \min_{w_i \in \Delta_i} K_i(P, w_i) = \min_{w_i \in \Delta} \left\{ \left(\frac{1}{Z_i} \right)^{1-\beta} \prod_{j=1}^{k} P_j^{\beta w_{ij}} \frac{1}{S_i(w_i)} \right\}$$

Taking the logarithms on both sides we get

$$\ln(P_i) = \min_{w_i \in \Delta} \left\{ (1-\beta) \ln\left(\frac{1}{Z_i} \right) + \beta \sum_{j=1}^{k} w_{ij} \ln(P_j) + \ln\left(\frac{1}{S_i(w_i)} \right) \right\}.$$

Next, by defining $p_i = \ln(P_i)$, $x_i = \ln\left(\frac{1}{Z_i}\right)$, and $c_i(w_i) = \ln\left(\frac{1}{S_i(w_i)}\right)$, we finally get

$$p_i = (1-\beta) x_i + \beta \min_{w_i \in \Delta} \left\{ \sum_{j=1}^{k} w_{ij} p_j + \frac{1}{\beta} c_i(w_i) \right\}.$$

Similarly, defining $q_{i0} = \ln(Q_{i0})$ and $q_{ij} = \ln(Q_{ij})$ we get

$$q_{i0}(\beta) = \ln(1-\beta) + (1-\beta) x_i + \beta \sum_{j=1}^{k} w_{ij}(\beta) p_j(\beta) + c_i(w_{ij}(\beta))$$

$$q_{ij}(\beta) = \ln(\beta w_{ij}(\beta)) - p_i^{\beta} + (1-\beta) x_i + \beta \sum_{j=1}^{k} w_{ij}(\beta) p_j(\beta) + c_i(w_{ij}(\beta))$$