Regime Change Global Games with Heterogeneous Players and Continuous Actions

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A continuum population must invest or not invest. The return to investing is 0. The cost of investing is $c \in (0,1)$. The return to investing is 1 if the proportion investing is at least θ, 0 otherwise. If θ were common knowledge, there would be multiple Nash equilibria (all invest, all not invest) as long as θ were between zero and one. But suppose instead we assume that θ is normally distributed with mean y and variance τ^2, and that each agent i observes a private signal $x_i = \theta + \varepsilon_i$, where the noise terms ε_i are distributed normally in the population with mean 0 and variance σ^2. If and only if

$$\sigma \leq \tau^2 \sqrt{2\pi}, \tag{0.1}$$

there is a unique equilibrium.[1]

This simple parameterization of a binary action "global game" has been used in a number of applications. In Guimarães and Morris (2005), we show how this uniqueness condition can be extended to model of exchange rate crises with continuum actions. In this note, we report the more general result underlying the result in that paper.

1. Model

There are I types of players and a continuum of players of each type. The proportion of players of type i is λ_i and the total mass of players is normalized to 1. A player of type i must choose an action $a_i \in [\underline{z}_i, \overline{z}_i]$. A player's payoff depends on his own action, the average actions

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[1] This model and result first appeared in our 1999 working paper on "Coordination Risk and The Price of Debt," eventually published as Morris and Shin (2004). Angeletos, Hellwig and Pavan (2003) dubbed this particular class of games "regime change games". A similar condition on precisions is necessary and sufficient for uniqueness in global games where payoffs are linear in the unknown parameter θ, the class of global games used in our expository pieces, Morris and Shin (1999, 2000, 2003). Related conditions can be derived for more general global games, although in this case there is some gap between the known necessary and sufficient conditions, see Morris and Shin (2003, 2005). Appendix B of Carlsson and van Damme (1993) already worked out explicit uniqueness conditions for a normal signal "private value global game," where private shocks represent idiosyncratic payoffs. Morris and Shin (2005) analyze the differences between the uniqueness conditions for common value and private value global games.

of others and an unknown state θ. But payoffs depend on the average action $\overline{a}$ and state in a special way:

$$u_i(a_i, \overline{a}, \theta) = \begin{cases} \overline{v}_i(a_i), \text{ if } \theta \geq f(\overline{a}) \\ \underline{v}_i(a_i), \text{ if } \theta < f(\overline{a}) \end{cases}$$

We make the following assumptions on payoffs:

1. The threshold function f is (a) continuous; and (b) strictly decreasing in $\overline{a}$
2. $a_i > a_i' \Rightarrow \overline{v}_i(a_i) - \overline{v}_i(a_i') > \underline{v}_i(a_i) - \underline{v}_i(a_i')$.

Assumptions 1a ensures the limit uniqueness property of global games: at a high (low) enough values of θ, it is dominant strategy to choose a certain high (low) action - i.e., $\underset{a_i}{argmax} \overline{v}_i(a_i)$ ($\underset{a_i}{argmax} \underline{v}_i(a_i)$) respectively.

Assumptions 1b and 2 ensure strategic complementarities between players' actions and also state monotonicity (higher states give higher incentive to choose higher actions, for any fixed actions of the opponents).

We make the following assumptions on the information structure. We assume that θ is distributed with mean y and that $\frac{\theta - y}{\tau}$ has a standard normal distribution. We assume that each player observes a private signal x, and that within each type of player, their private signals are such that $\frac{x-\theta}{\sigma_i}$ has a standard normal distribution. Thus τ and σ_i are parameters measuring the standard deviation of public and private signals respectively.

2. Sufficient Condition for Uniqueness

Proposition 2.1. *There is a unique equilibrium if f is differentiable and*

$$\left(\sum_{i=1}^{I} \lambda_i (\overline{z}_i - \underline{z}_i) \sigma_i \right) \max_{\theta} f'(\theta) < \tau^2 \sqrt{2\pi}. \tag{2.1}$$

PROOF. First, let

$$a_i^*(\pi) = \arg\max_{a_i} \pi \overline{v}_i(a_i) + (1 - \pi) \overline{v}_i(a_i)$$

Assumption 2 ensures that each a_i^* is single valued for almost every value of π and strictly increasing in π when $\pi \neq 0$ and $\pi \neq 1$. We will assume that it is single valued in what follows: since it is almost always true, it will not effect the argument. One could make it true always by adding strict concavity of the $\overline{v}_i$ and $\overline{v}_i$, which is true in our currency crisis application.

Also observe that an individual of type i observing signal x thinks that θ is normally distribution with mean

$$\frac{\sigma_i^2 y + \tau^2 x}{\sigma_i^2 + \tau^2}$$

and standard deviation

$$\sqrt{\frac{\sigma_i^2 \tau^2}{\sigma_i^2 + \tau^2}}.$$

We say that there is a threshold equilibrium if it is common knowledge that that $\theta \geq f(\overline{a})$ only if $\theta \geq \theta^*$. In such an equilibrium, each player of type i would following the strategy

$$s_i(x|\theta^*) = a_i^* \left(1 - \Phi\left(\theta^* - \frac{\sigma_i^2 y + \tau^2 x}{\sigma_i^2 + \tau^2}\right)\right)$$

Thus at any θ, the average action would be

$$z(\theta, \theta^*) = \sum_{i=1}^{I} \lambda_i \int_{\varepsilon=-\infty}^{\infty} a_i^* \left(1 - \Phi\left(\theta^* - \frac{\sigma_i^2 y + \tau^2(\theta + \sigma\varepsilon)}{\sigma_i^2 + \tau^2}\right)\right) d\varepsilon$$

which is increasing in θ. Since f is strictly decreasing in θ, there is unique value of θ solving $f(z(\theta, \theta^*)) = \theta$. We write $\beta(\theta^*)$ for that unique solution. Observe since $z(\theta, \theta^*)$ is decreasing in θ^*, β is increasing in θ^*. Now a threshold equilibrium with threshold θ^* exists if and only if $\beta(\theta^*) = \theta^*$. Thus we have

CLAIM 1. There exists a threshold equilibrium with threshold θ^* if and only if

$$\theta^* = f\left(\sum_{i=1}^{I} \lambda_i \int_{\varepsilon=-\infty}^{\infty} a_i^* \left(1 - \Phi\left(\theta^* - \frac{\sigma_i^2 y + \tau^2(\theta^* + \sigma\varepsilon)}{\sigma_i^2 + \tau^2}\right)\right) d\varepsilon\right). \tag{2.2}$$

CLAIM 2. If f is differentiable with bounded derivatives and (2.1) holds, then there is a unique threshold equilibrium.

PROOF of claim 2. An individual of type i observing signal x attaches probability

$$\pi = 1 - \Phi\left(\sqrt{\frac{\sigma_i^2 + \tau^2}{\sigma_i^2 \tau^2}}\left(\theta^* - \frac{\sigma_i^2 p + \tau^2 x}{\sigma_i^2 + \tau^2}\right)\right)$$

to $\theta \geq \theta^*$. Making x the subject of the above equation, we have

$$x(\pi) = \left(1 + \frac{\sigma_i^2}{\tau^2}\right)\theta^* - \frac{\sigma_i}{\tau}\left(\sqrt{\sigma_i^2 + \tau^2}\right)\Phi^{-1}(1 - \pi) - \frac{\sigma_i^2}{\tau^2} y$$

Now anyone observing a signal less than x_i will assign probability π or less to the peg being maintained. Thus the proportion of players of type i assigning probability less than π to $\theta \geq \theta^*$ will be

$$\begin{aligned}
\Gamma_i(\pi|\theta^*) &= \Phi\left(\frac{1}{\sigma_i}(x_i(\pi) - \theta^*)\right) \\
&= \Phi\left(\frac{1}{\sigma_i}\left(\left(1 + \frac{\sigma_i^2}{\tau^2}\right)\theta^* - \frac{\sigma_i}{\tau}\sqrt{\sigma_i^2 + \tau^2}\Phi^{-1}(1 - \pi) - \frac{\sigma_i^2}{\tau^2} y - \theta^*\right)\right) \\
&= \Phi\left(\frac{\sigma_i}{\tau^2}\theta^* - \frac{1}{\tau}\sqrt{\sigma_i^2 + \tau^2}\Phi^{-1}(1 - \pi) - \frac{\sigma_i}{\tau^2} y\right)
\end{aligned}$$

Observe that $\frac{d}{d\theta^*}\Gamma_i(\pi|\theta^*) = \phi(\cdot)\frac{\sigma_i}{\tau^2} \leq \frac{\sigma_i}{\sqrt{2\pi\tau^2}}$. Now

$$\begin{aligned}
\frac{d}{d\theta^*}\int_{\varepsilon=-\infty}^{\infty} a_i^* \left(1 - \Phi\left(\theta^* - \frac{\sigma_i^2 y + \tau^2(\theta^* + \sigma\varepsilon)}{\sigma_i^2 + \tau^2}\right)\right) d\varepsilon &\leq (\overline{z}_i - \underline{z}_i)\frac{d}{d\theta^*}\Gamma_i(\pi|\theta^*) \\
&\leq (\overline{z}_i - \underline{z}_i)\frac{\sigma_i}{\sqrt{2\pi\tau^2}}
\end{aligned}$$

Thus

$$\frac{d}{d\theta^*} f\left(\sum_{i=1}^{I} \lambda_i \int_{\varepsilon=-\infty}^{\infty} a_i^* \left(1 - \Phi\left(\theta^* - \frac{\sigma_i^2 y + \tau^2 (\theta^* + \sigma\varepsilon)}{\sigma_i^2 + \tau^2}\right)\right) d\varepsilon\right)$$

$$\leq \quad f'(\cdot) \sum_{i=1}^{I} \lambda_i (\overline{z}_i - \underline{z}_i) \frac{\sigma_i}{\sqrt{2\pi\tau^2}}.$$

But the condition of the lemma requires that the right hand side be strictly less than one. But this implies that the right hand side of (2.2) has a derivative strictly less than 1, guaranteeing uniqueness.

CLAIM 3. Let $\underline{\theta}^0 = -\infty$, $\overline{\theta}^0 = \infty$, $s_i\left(x|\overline{\theta}^{k-1}\right) = \beta\left(\underline{\theta}^k\right)$ and $\overline{\theta}^{k+1} = \beta\left(\overline{\theta}^k\right)$ for each $k = 0, 1, ...$ Then a strategy for a type i player survives k rounds of iterated deletion of strictly dominated strategies if and only if

$$s_i(x) \in \left[s_i\left(x|\overline{\theta}^{k-1}\right), s_i\left(x|\underline{\theta}^{k-1}\right)\right].$$

Observe that since β is an increasing function with bounded range, $\underline{\theta}^k$ must be an increasing sequence and $\overline{\theta}^k$ must be a decreasing sequence.

PROOF of claim 3. By induction on k. True by definition for $k = 0$. Suppose it is true for k. By supermodularity of payoffs, the highest action surviving $k + 1$ rounds will be a best response to the highest strategy profile surviving k rounds. This will be $s_i\left(x|\overline{\theta}^k\right)$.

CLAIM 4. Let $\overline{\theta}^* = \lim_{k\to\infty} \overline{\theta}^k$ and $\underline{\theta}^* = \lim_{k\to\infty} \underline{\theta}^k$. There exist threshold equilibria with thresholds $\overline{\theta}^* = \underline{\theta}^*$.

PROOF of claim 4. The limits exist as they are limits of bounded monotonic sequences on the real line. By continuity, each will be a fixed point of β.

COMPLETION OF PROOF OF PROPOSITION. Now if condition (2.1) is satisfied, then (by claim 2) there is a unique threshold equilibrium. Thus the $\overline{\theta}^*$ and $\underline{\theta}^*$ in claim 4 must be equal. Thus by claim 3, the strategy profile of the unique threshold equilibrium is the unique profile surviving iterated deletion of strictly dominated actions.

3. Tightness of the Sufficient Condition

The sufficient condition is clearly not necessary. However, a well known example attains the bound. Suppose $I = 1$, $\underline{z}_1 = 0$, $\overline{z}_1 = 1$ and $f(\overline{a}) = 1 - \overline{a}$, $\overline{v}_i(a_i) = -t(1 - a_i)$ and $\underline{v}_i(a_i) = (1 - t)(1 - a_i)$. The interpretation is that action 0 is "attack the currency", action 1 is "defend the currency", there is a per unit cost t of attacking the currency and a per unit gain 1 if the attack is successful. The attack is successful if and only if the proportion attacking is at least θ. In this special case the sufficient condition for uniqueness becomes

$$\sigma_1 < \tau^2\sqrt{2\pi}. \tag{3.1}$$

It is well known that if $\sigma_1 > \tau^2\sqrt{2\pi}$, there are multiple equilibria.

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