14.452: Economic Growth
Problem Set 2

Due date: November 18, 2011 in recitation.

Exercise 1: Consider the Solow growth model in continuous time with constant saving rate s, depreciation rate δ, population growth rate n and labor-augmenting technological progress at rate $g > 0$, that is, suppose the production function function takes the form $Y(t) = F(K(t), A(t)L(t))$ where $\dot{A}(t)/A(t) = g > 0$. Recall that the effective capital-labor ratio $k(t) = K(t)/(A(t)L(t))$ is characterized by the differential equation

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (\delta + g + n), \tag{1}$$

where $f(k(t)) = F(k(t), 1)$ is the production function normalized by effective labor, and the output per-capita is given by

$$y(t) = A(t) f(k(t)). \tag{2}$$

This exercises concerns an approximation for the growth rate of output per-capita around the steady-state.

1. Let $y^*(t) = A(t) f(k^*)$ denote the steady-state level of output per capita, that is, the level of output per capita that would apply if the effective capital-labor ratio were at its steady-state level and technology were at its time t level. Show that, in a neighborhood of the steady state, output per capita $y(t)$ can be approximated by the following differential equation:

$$\frac{\dot{y}(t)}{y(t)} \simeq g - (1 - \varepsilon_k(k^*))(\delta + g + n)(\log y(t) - \log y^*(t)), \tag{3}$$

where $\varepsilon_k(k^*) = \frac{f'(k^*)k^*}{f(k^*)}$ is the elasticity of the production function $f(k)$ evaluated at k^*.

(Hint: First, consider the Taylor expansion of the right-hand side of Eq. (1) with respect to $\log(k(t))$ around $\log k^*$ and derive an approximation for $\dot{k}(t)/k(t)$. Second use this approximation along with Eq. (2) to derive an approximation for $\dot{y}(t)/y(t)$ as a function of the distance of effective capital-labor ratio from its steady-state, $\log k(t) - \log k^*$. Third, consider the Taylor expansion of Eq. (2) with respect to $\log(k(t))$ around $\log k^*$ to derive an approximation for the distance of output per-capita, $\log y(t) - \log y^*(t)$, in terms of the distance of effective capital-labor ratio, $\log k(t) - \log k^*$. Combine the last two steps to derive Eq. (3).)

2. Interpret Eq. (3). In particular, what does this equation imply about the growth rate of countries away from their steady-states? What happens to the growth rate as the countries approach their steady-states? Explain what determines the speed of convergence to the steady-state and provide an intuition.

Exercise 2: Consider an economy with $N < \infty$ goods, denoted by $j \in \{1, .., N\}$, and a set $\mathcal{H}$ of households. Suppose each household $h \in \mathcal{H}$ has total income w^h and CES preferences for the goods given by

$$U^h\left(x_1^h, ..., x_N^h\right) = \left[\sum_{j=1}^{N} \left(x_j^h - \xi_j^h\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

where $\sigma \in (0, \infty)$ and $\xi_j^h \in [-\xi, \xi]$.

1. Derive the utility maximizing demand and the indirect utility function for each household. Show that Theorem 5.2 of the textbook applies to this economy, and derive the indirect utility function of the representative household.

2. Consider a household with total income $w \equiv \int_{\mathcal{H}} w^h dh$ and CES preferences given by

$$U(x_1, ..., x_N) = \left[\sum_{j=1}^{N} \left(x_j - \xi_j\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \quad (4)$$

where $\xi_j = \int_{\mathcal{H}} \xi_j^h dh$. Derive the indirect utility function for this household and show that it agrees with the indirect utility function of the representative household obtained in part 1.

3. Now suppose that $U^h\left(x_1^h, ..., x_N^h\right) = \sum_{j=1}^{N} \left(x_j^h - \xi_j^h\right)^{\frac{\sigma-1}{\sigma}}$ with $\sigma > 1$. Repeat the same computations and verify that the resulting indirect utility function is homogeneous of degree 0 in p and w^h, but does not satisfy the Gorman form. Show, however, that a monotone transformation of the indirect utility function satisfies the Gorman form. Is this sufficient to ensure that the economy admits a representative household?

Exercise 3: Consider an economy consisting of N households each with utility function at time $t = 0$ given by

$$\sum_{t=0}^{\infty} \beta^t u\left(c^h\left(t\right)\right),$$

with $\beta \in (0, 1)$, where $c^h\left(t\right)$ denotes the consumption of household h at time t. Suppose that $u\left(0\right) = 0$. The economy starts with an endowment of $y > 0$ units of the final good and has access to no production technology. This endowment can be saved without depreciating or gaining interest rate between periods.

1. What are the Arrow-Debreu commodities in this economy?
2. Characterize the set of Pareto optimal allocations of this economy.
3. Explain why the Second Welfare Theorem can be applied to this economy.
4. Now consider an allocation of y units to the households, $\left\{y^h\right\}_{h=1}^N$, such that $\sum_{h=1}^N y^h = y$. Given this allocation, find the unique competitive equilibrium price vector and the corresponding consumption allocations.
5. Are all competitive equilibria Pareto optimal?
6. Now derive a redistribution scheme for decentralizing the entire set of Pareto optimal allocations?