

## 14.452: Problem Set 1

Due date: October 31, 2014.

Please only hand in Question 1, which will be graded. The rest will be reviewed in the recitation but should not be handed in.

**Question 1:** Consider a modified version of the continuous-time Solow growth model where the aggregate production function is

$$F(K, L, Z) = L^\beta K^\alpha Z^{1-\alpha-\beta},$$

where  $Z$  is land, available in fixed inelastic supply. Assume that  $\alpha + \beta < 1$ , capital depreciates at the rate  $\delta$ , and there is an exogenous saving rate of  $s$ .

1. First suppose that there is no population growth. Find the steady-state capital-labor ratio and the steady-state output level. Prove that the steady state is unique and globally stable.
2. Show that, in the steady-state equilibrium, there is a monotonic relationship between the interest rate and the saving rate of the economy. Using this result, show that there exists a saving rate  $s^*$  such that above this, the interest rate is negative. Show that when the interest rate is negative, starting from the steady-state equilibrium, it is possible to reallocate resources so that consumption increases at all points in time. Explain what this means and why such a possibility is present in this model.
3. Now suppose that there is population growth at the rate  $n$ , that is,  $\dot{L}/L = n$ . Does a steady-state equilibrium exist? What happens to the capital-labor ratio and output level as  $t \rightarrow \infty$ ? What happens to returns to land and the wage rate as  $t \rightarrow \infty$ ?
4. Would you expect the population growth rate  $n$  or the saving rate  $s$  to change over time in this economy? If so, how? What other adjustments might you expect in this economy as  $t \rightarrow \infty$ ?

**Question 2:** Consider the discrete-time Solow growth model with constant population growth at the rate  $n$ , no technological change and depreciation rate of capital equal to  $\delta$ . Assume that the saving rate is a function of the capital-labor ratio, thus given by  $s(k)$ .

1. Suppose that  $f(k) = Ak$  and  $s(k) = s_0 k^{-1} - 1$ . Show that if  $A + \delta - n = 2$ , then for any  $k(0) \in (0, As_0/(1+n))$ , the economy immediately settles into an asymptotic cycle and continuously fluctuates between  $k(0)$  and  $As_0/(1+n) - k(0)$ . [Suppose that  $k(0)$  and the parameters are given such that  $s(k) \in (0, 1)$  for both  $k = k(0)$  and  $k = As_0/(1+n) - k(0)$ ].

2. Now consider more general continuous production function  $f(k)$  and saving function  $s(k)$ , such that there exist  $k_1, k_2 \in R_+$  with  $k_1 \neq k_2$  and

$$\begin{aligned} k_2 &= \frac{s(k_1) f(k_1) + (1 - \delta) k_1}{1 + n} \\ k_1 &= \frac{s(k_2) f(k_2) + (1 - \delta) k_2}{1 + n}. \end{aligned}$$

Show that when such  $(k_1, k_2)$  exist, there may also exist a stable steady state.

3. Prove that such cycles are not possible in the continuous-time Solow growth model for any (possibly non-neoclassical) continuous production function  $f(k)$  and continuous  $s(k)$ .
4. What does the result in parts 1-3 imply for the approximations of discrete time by continuous time in the Solow model (suggested in Section 2.4 of the textbook)? What does this imply for the cycles in parts 1 and 2?
5. Show that if  $f(k)$  is nondecreasing in  $k$  and  $s(k) = k$ , cycles as in parts 1 and 2 are not possible in discrete-time either.

**Question 3:** Consider the Solow growth model with constant saving rate  $s$  and depreciation rate of capital equal to  $\delta$ . Assume that population is constant and the aggregate output is given by the CES production function

$$F(A_K(t)K(t), A_L(t)L) = \left[ \gamma (A_K(t)K(t))^{\frac{\sigma-1}{\sigma}} + (1-\gamma) (A_L(t)L)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}$$

where  $\dot{A}_L(t)/A_L(t) = g_L > 0$  and  $\dot{A}_K(t)/A_K(t) = g_K > 0$ . Suppose the elasticity of substitution between capital and labor is less than one,  $\sigma < 1$ , and capital-augmenting technological progress is faster than labor-augmenting progress,  $g_K \geq g_L$ . Show that as  $t \rightarrow \infty$ , the economy converges to a BGP where the share of labor in national income is equal to 1, and capital, output and consumption all grow at the rate  $g_L$ . In light of this result, discuss the claims in the literature that capital-augmenting technological change is inconsistent with balanced growth.

**Question 4:** Consider the basic Solow model in continuous time and suppose that  $A(t) = A$ , so that there is no technological progress of the usual kind. However, assume that the relationship between investment and capital accumulation is modified to

$$\dot{K}(t) = q(t)I(t) - \delta K(t),$$

where  $[q(t)]_{t=0}^{\infty}$  is an exogenously given time-varying process. Intuitively, when  $q(t)$  is high, the same investment expenditure translates into a greater increase in the capital stock. Therefore, we can think of  $q(t)$  as the inverse of the relative prices of machinery to output. When  $q(t)$  is high, machinery is relatively cheaper, and thus suppose that  $\dot{q}(t) > 0$ .

1. Suppose that  $\dot{q}(t)/q(t) = \gamma_K > 0$ . Show that for a general production function,  $F(K, L)$ , there exists no steady-state equilibrium.

2. Now suppose that the production function is Cobb-Douglas,  $F(K, L) = K^\alpha L^{1-\alpha}$ , and characterize the unique steady-state equilibrium.
3. Show that this steady-state equilibrium does not satisfy the Kaldor fact of constant  $K/Y$ . Is this a problem? [Hint: how is “ $K$ ” measured in practice? How is it measured in this model?].