Measuring Strategic Uncertainty*

Stephen Morris
Cowles Foundation,
Yale University,
P.O.Box 208281,
New Haven CT 06520, U. S. A.
stephen.morris@yale.edu

Hyun Song Shin
London School of Economics,
Houghton Street,
London WC2A 2AE
U. K.
h.s.shin@lse.ac.uk

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1. Introduction

In his invited lecture at the American Economic Association meeting in January 2000, Larry Summers asks the audience to imagine that he is an emerging market borrower, and that they have all lent money to him. He then poses the following problem.

> "Imagine that everyone who has invested $10 with me can expect to earn $1, assuming that I stay solvent. Suppose that if I go bankrupt, investors who remain lose their whole $10 investment, but that an investor who withdraws today neither gains nor loses. What would you do? ... Suppose, first, that my foreign reserves, ability to mobilize resources, and economic strengths are so limited that if any investor withdraws, I will go bankrupt. It would be a Nash equilibrium (indeed, a Pareto-dominant one) for everyone to remain, but (I expect) not an attainable one. Someone would reason that someone else would decide to be cautious and withdraw, or at least that someone would

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> reason that someone would reason that someone would withdraw, and so forth. ...
>
> Now suppose that my fundamental situation were such that everyone would be paid off as long as no more than one-third of the investors chose to withdraw. What would you do then? Again, there are multiple equilibria: everyone should stay if everyone else does, and everyone should pull out if every else does, but the more favorable equilibrium seems much more robust.
>
> I think that this thought experiment captures something real. On the one hand, bank runs or their international analogues do happen. On the other hand, they are not driven by sunspots: their likelihood is driven and determined by the extent of fundamental weakness." [Lawrence Summers (2000, p. 7)].

The thought experiment urged on us by Summers helps to throw into sharper relief the issue of strategic uncertainty - that is, uncertainty concerning the actions and beliefs (and beliefs about the beliefs) of others. Even if the underlying fundamentals of the problem were known for sure, the strategic uncertainty is still all-pervasive. Adam Brandenburger (1996) similarly draws a distinction between strategic uncertainty and 'structural' uncertainty, the latter having to do with the underlying fundamentals.

Douglas Hofstadter (1985, pp.752 - 3) coined the term "reverberant doubt" to describe this type of uncertainty. The idea is that even a small seed of doubt concerning the ability of the players to close ranks to achieve the good outcome will start to undermine the resolve of an individual player to stick to the cooperative strategy. The small seed of doubt "reverberates" to become a much larger doubt, and when the players catch themselves thinking this way, it becomes a compelling reason to act unilaterally, and opt out. Our task in this paper will be to pin down what could be meant by "reverberant doubt", and to see whether we can gain a foothold in beginning to quantify such uncertainty.

As well as the purely conceptual interest in this exercise, it is also worth noting in passing that many examples from economics and finance conform to this stylized setting, especially when an element of formal voting is invoked. Collective action clauses (CACs) in bond convenants stipulate a critical majority in the creditors' vote to restructure the claims. More generally, for the restructuring or recapitalization of distressed firms (whether under formal bankruptcy proceedings or informal offers), the success of the restructuring is determined by the strength of support for the injection of new funds to tide the firm over its current difficulties.

At the time of writing (summer of 2002), such distressed reorganizations have become depressingly familiar in the corporate world in the U.S. and elsewhere[1].

2. Contribution Game

The decision problem posed by Summers is an example of a voluntary contribution game for a public good, where each individual decides whether to contribute toward the public good, or to opt out. Contributing toward the public good is a risky action since the successful provision of the public good requires contribution from a critical mass of the players. If the contribution falls short of this critical mass, the provision of the public good fails, resulting in poor payoffs for those who decided to contribute. Opting out is a safe action for an individual agent, but leads to a socially sub-optimal outcome. Van Huyck, Battalio and Beil (1990, 1991) have conducted experiments on similar games, and Crawford (1991) gives an evolutionary rationalization of their results. Carlsson and van Damme (1993) have examined general stag hunt games, of which our game can be seen as a special case.

We will first deal with the case where there is a continuum of players. The continuum assumption simplifies some steps in the argument, but the essential feature of the problem is unaffected if we have a finite number of players, instead. The finite player case entails solving some combinatorial problems which is of independent interest. We will deal with the finite players case later in this paper.

Each player has to choose between two actions - to contribute to the public good, or to opt out. Denote by κ the proportion of agents who have chosen to contribute. The public good is successfully provided when κ is larger than some critical threshold $\hat{\kappa}$. The consumption value of the public good is 1, but player i faces a cost c_i in contributing to the provision of the public good. Thus, the payoff to player i from contributing to the public good is

$$\begin{cases} 1 - c_i & \text{if } \kappa \geq \hat{\kappa} \\ -c_i & \text{if } \kappa < \hat{\kappa} \end{cases} \tag{2.1}$$

The payoff to opting out is zero, and does not depend on κ.

[1] On a fairly typical day (July 12th 2002) the *Financial Times* carries the story of the coordination problem faced by the bank creditors of Energis (page 20) and how Worldcom's bond holders are in conflict with its bankers (page 26). The lead story is the new aggressive downgrading strategy of the credit rating agencies.

If $c_i < 0$, then the decision is trivial, since contributing is a dominant action. Similarly, if $c_i > 1$, then opting out is dominant. However, for c_i between zero and one, the optimal choice depends on the probability that player i attaches to κ exceeding the threshold $\hat{\kappa}$. The focus of our paper will be on how this belief is determined in the game.

Before getting into the detailed analysis, it is worthwhile taking a step backwards and considering what a reasonable outcome would be in such a game. In the simplest setting, such as the one outlined by Summers, there is no uncertainty over the fundamentals of the problem in terms of the cost c_i. The players are told that everyone has the same cost of provision c and each player knows his own cost. Hence the players ought to infer that everyone else's cost is given by c also. And yet, Summers's intuition (which seems eminently reasonable) is that successful coordination is achieved only when $\hat{\kappa}$ is sufficiently low relative to the cost c. For any given cost c, the reluctance to contribute would be increasing in the critical threshold $\hat{\kappa}$. Conversely, for any given critical threshold $\hat{\kappa}$, the reluctance to contribute would be increasing in cost c. The experiments by Van Huyck et al. (1990, 1991) reveal that coordination is, indeed, more difficult to achieve when the condition for successful coordination is more stringent.

Fixing the critical mass $\hat{\kappa}$, we may conjecture that there is some critical threshold for costs c^* such that, if c were larger than this, the players would choose to opt out, while if c were lower than c^*, they would choose to contribute. The evidence from experiments, such as from the paper by Frank Heinemann, Rosemarie Nagel and Peter Ockenfels (2002) suggest that such threshold strategies arise fairly commonly in games of this kind even when the fundemantals are ostensibly common knowledge among the players. We say "ostensibly", since the cautious actions of the players betray a lack of confidence in the proposition that c is common knowledge.

Pushing one's intuition a little further, it would be reasonable to expect that the threshold cost c^* depends on the critical threshold $\hat{\kappa}$. The cost of contribution determines the balance of the net gain from the public good and the costs of failure, while the critical mass $\hat{\kappa}$ determines the margin for error in failing to coordinate. We would expect that the threshold cost level c^* is higher (thereby increasing the likelihood of contribution) when the critical mass $\hat{\kappa}$ is lower. Thus, when we view c^* as a function of $\hat{\kappa}$, we would expect c^* to be a decreasing function of $\hat{\kappa}$. Experiments may uncover precisely what this decreasing function looks like. What would such a function tell us about the problem?

Being able to identify the switching point c^* reveals a lot about the strategic uncertainty that the players face in the game. When $c_i = c^*$, player i is indifferent between contributing and opting out. Denoting by $F(c^*)$ the probability that $\kappa < \hat{\kappa}$ conditional on $c_i = c^*$, the expected payoff to contributing is given by

$$\begin{aligned} & -c^* F(c^*) + (1 - c^*)(1 - F(c^*)) \\ = \; & (1 - c^*) - F(c^*) \end{aligned}$$

When player i is indifferent between contributing and opting out, we have

$$F(c^*) = 1 - c^* \tag{2.2}$$

Thus, being able to observe the threshold point c^* gives us direct information on the strategic uncertainty facing player i in terms of $F(c^*)$ - the conditional probability that $\kappa < \hat{\kappa}$ - and hence the perceived danger that the attempt at coordination will fail.

More ambitiously, we can think about the beliefs that players hold about κ - the proportion of players who contribute. κ is a random variable which takes values in the unit interval $[0, 1]$. We can ask what information is revealed about the subjective density of κ from the actions of the players. Denoting by

$$g(\kappa | c^*)$$

the subjective density over κ conditional on c^*, equation (2.2) tells us that

$$\int_0^{\hat{\kappa}} g(\kappa | c^*) \, d\kappa = F(c^*) = 1 - c^*$$

Thus, observation of c^* gives us one fix on the "reverberant doubt" that faces the players in this game in terms of the value of the cumulative distribution function of κ at the point $\hat{\kappa}$. In this way, we can begin to reconstruct the beliefs that underlie the players' actions.

The most challenging task is to ask whether there is a way of rationalizing such outcomes in terms of the conventional apparatus of game theory. This would entail showing how, even in the absence of any uncertainty on the costs of contribution c, there is nevertheless non-trivial strategic uncertainty in equilibrium in terms of a non-degenerate density over κ. We will explore how the tools of global games can be used to answer some of these questions.

3. Global Games

We examine the case where the costs of contribution $\{c_i\}$ across players are determined both by a common element, and by small idiosyncratic elements that introduce small differences in costs around the central tendency. We begin with a very simple case. Suppose that player i's cost c_i satisfies

$$c_i = \theta + s_i \tag{3.1}$$

where θ is the common element in the costs of all players, while s_i is the idiosyncratic element for player i. The idiosyncratic element s_i is uniformly distributed over the interval $[-\varepsilon, \varepsilon]$, where ε is a small positive number. For any two distinct individuals $i \neq j$, s_i is independent of s_j. Finally, let us suppose that θ itself has a uniform ex ante distribution.

On observing his own cost, player i reasons his way towards the probability density over κ. As a working hypothesis, player i assumes that all other players are using the switching strategy around c^*, so that anyone who has cost below c^* will contribute, while anyone with cost above c^* will opt out. In particular, suppose that player i's cost happens to be exactly c^*. Player i then asks himself what the cumulative distribution function over κ is, conditional on c^*. For this, he needs to answer the following question.

"My cost is c^*. What is the probability that κ is less than z?" (Q)

This question is the key to our task, since the answer to question (Q) yields the value of the cumulative distribution function $G(\cdot|c^*)$ evaluated at z. The density over κ is then obtained by differentiating this function. The steps to answering question (Q) are illustrated in figure 3.1.

When the common element of cost is θ, the individual costs are distributed uniformly over the interval $[\theta - \varepsilon, \theta + \varepsilon]$. The players who contribute are those whose costs are below c^*. Hence,

$$\kappa = \frac{c^* - (\theta - \varepsilon)}{2\varepsilon}$$

When do we have $\kappa < z$? This happens when θ is high enough, so that the area under the density to the left of c^* is squeezed out. There is a value of θ at which κ is precisely equal to z. This is when $\theta = \theta^*$, where

$$\theta^* = c^* + \varepsilon - 2\varepsilon z$$

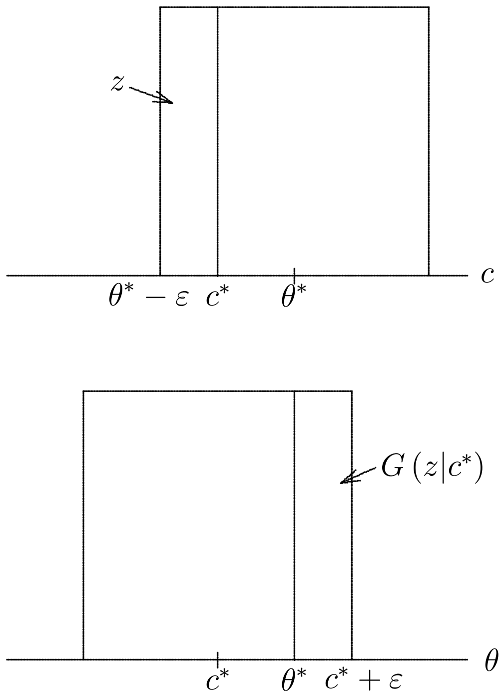


Figure 3.1: Deriving cumulative distribution function $G(z|c^*)$

See the top panel of figure 3.1. We have $\kappa < z$ if and only if $\theta > \theta^*$. Thus, we can answer question (Q) if we can find the probability that $\theta > \theta^*$.

For this, we must turn to player i's posterior density over θ conditional on his cost being c^*. This posterior density is uniform over the interval $[c_i - \varepsilon, c_i + \varepsilon]$. This is because the ex ante distribution over θ is uniform and the idiosyncratic element of cost is uniformly distributed around θ. The bottom panel of figure 3.1 depicts this posterior density over θ. The probability that $\theta > \theta^*$ is then the area under the density to the right of θ^*. This is,

$$\begin{aligned} & \frac{(c^* + \varepsilon) - \theta^*}{2\varepsilon} \\ = & \frac{(c^* + \varepsilon) - (c^* + \varepsilon - 2\varepsilon z)}{2\varepsilon} \\ = & z \end{aligned} \tag{3.2}$$

In other words, the probability that $\kappa < z$ conditional on cost level c^* is exactly

z. The cumulative distribution function $G(z|c^*)$ is the identity function:

$$G(z|c^*) = z \tag{3.3}$$

The density over κ is then obtained by differentiation.

$$g(\kappa|c^*) = 1 \quad \text{for all } \kappa \tag{3.4}$$

The density over κ is uniform. The noteworthy feature of this result that the constant ε does not enter into the expression for the density over κ. No matter how small or large is the dispersion of costs, κ has the uniform density over the unit interval $[0, 1]$. Figure 3.1 reveals the intuition for why ε does not matter. As ε shrinks, the dispersion of costs shrinks with it, but so does the support of the posterior density over θ. The region on the top panel corresponding to z is the mirror image of the region on the bottom panel corresponding to $G(z|c^*)$. Changing ε stretches or squeezes these regions, but it does not alter the fact that the two regions are equal in size. This identity is the key to our result.

In the limit as $\varepsilon \to 0$, every player's cost converges to θ. Thus, fundamental uncertainty disappears. Everyone's cost converges to the common element θ, and everyone knows this fact. And yet, even as fundamental uncertainty disappears, the strategic uncertainty is unchanged. The subjective density over κ is invariant. The "reverberant doubt" remains intact.

Let us now see what the equilibrium of the contribution game looks like in the limit as $\varepsilon \to 0$. Since the density over κ is uniform at the switching point c^*, the probability that the contributions will fail to produce the public good is

$$F(c^*) = \int_0^{\hat{\kappa}} \kappa d\kappa = \hat{\kappa}$$

Thus, from (2.2), the switching point c^* satisfies:

$$c^* = 1 - \hat{\kappa} \tag{3.5}$$

Figure 3.2 illustrates. As we conjectured in the informal discussion, the switching point c^* is, indeed, a decreasing function of the critical mass $\hat{\kappa}$, and takes a particularly simple form. We should also verify that the switching strategy around c^* is the optimal strategy for an individual player when everyone else uses it. If $c_i > c^*$, then from (3.2), the conditional probability that $\kappa < \hat{\kappa}$ is greater than $F(c^*)$, so that it is optimal to opt out. Conversely, if $c_i < c^*$, then the conditional

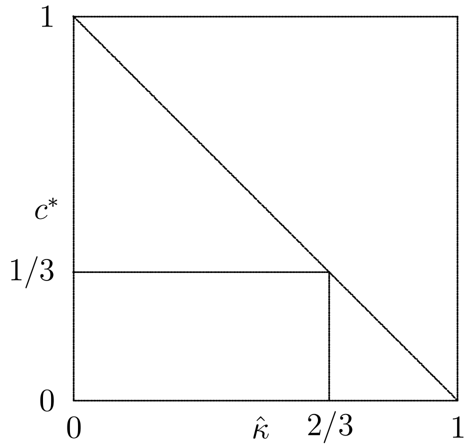


Figure 3.2: Switching point c^* as a function of $\hat{\kappa}$

probability that $\kappa < \hat{\kappa}$ is less than $F(c^*)$, so that it is optimal to contribute. This shows that everyone following the switching strategy around c^* is an equilibrium.

It is somewhat ironic that Summers's conjecture is actually false in the context of this example. Summers considered the case where $\hat{\kappa} = 2/3$, so that a two third critical mass is required for the good outcome. Each investor stakes 10 dollars, and either gets 11 dollars or nothing. Normalizing the payoffs, we have $c_i = 10/11$ for all i in the limit as $\varepsilon \to 0$. However, the equilibrium switching point is 1/3. This means that the cost of contribution is too high to sustain contribution to the public good. For the critical mass of two thirds, the costs must be lower than one third in order for contribution to take place.

4. A More General Framework

Suppose that the common element of cost θ now has a normal distribution with mean y and precision α (i.e. with variance $1/\alpha$). Let player i's cost be given by

$$c_i = \theta + s_i$$

where the idiosyncratic component of cost s_i is drawn from the normal density with zero mean and precision β. We will derive the strategic uncertainty facing the players by deriving the subjective density over κ of a typical player.

4.1. Density over κ

Let us begin with the working hypothesis that all players are following the switching strategy around the point c^*. Suppose further that player i's cost happens to be exactly c^*. We will derive this player's subjective density over κ - the proportion of players who contribute - by following the analogous steps to the discussion above. The cumulative distribution function over κ can be obtained from the answer to the following question.

> "My signal is c^*. What is the probability that κ is less than z?"

The answer to this question will yield $G(z|c^*)$ - the probability that the proportion of players who contribute is at most z, conditional on being at the switching point c^*. Given the common cost element θ, the proportion of players who contribute is

$$\Phi\left(\sqrt{\beta}\,(c^* - \theta)\right) \tag{4.1}$$

where $\Phi(\cdot)$ is the cumulative distribution function for the standard normal. Let θ^* be the common cost element at which this proportion is exactly z. Thus,

$$\Phi\left(\sqrt{\beta}\,(c^* - \theta^*)\right) = z \tag{4.2}$$

When $\theta \geq \theta^*$, the proportion of players that contribute is z or less. So, the question of whether $\kappa \leq z$ boils down to the question of whether $\theta \geq \theta^*$. Conditional on c^*, the density over θ is normal with mean

$$\frac{\alpha y + \beta c^*}{\alpha + \beta}$$

and precision $\alpha + \beta$. Thus, the probability that $\theta \geq \theta^*$ is the area under this density to the right of θ^*, namely

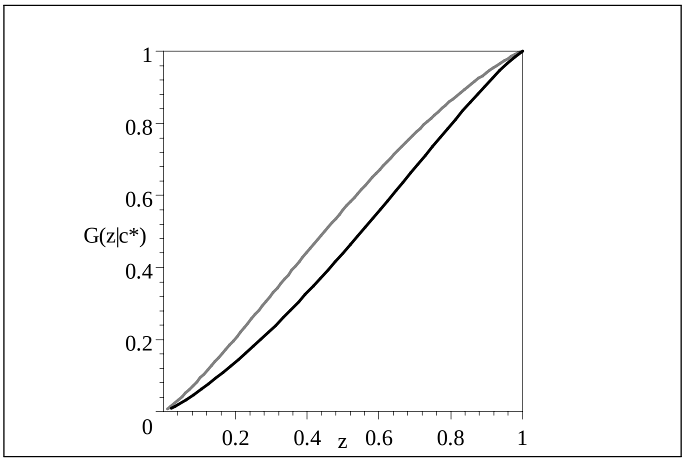
$$1 - \Phi\left(\sqrt{\alpha + \beta}\left(\theta^* - \tfrac{\alpha y + \beta c^*}{\alpha + \beta}\right)\right) \tag{4.3}$$

This expression gives $G(z|c^*)$. Substituting out θ^* by using (4.2) and re-arranging, we can re-write (4.3) to give:

$$G(z|c^*) = \Phi\left(\tfrac{\alpha}{\sqrt{\alpha+\beta}}\,(y - c^*) + \sqrt{\tfrac{\alpha+\beta}{\beta}}\,\Phi^{-1}(z)\right) \tag{4.4}$$

Differentiation of this expression with respect to z will give us the subjective density over κ. We note an important difference between (4.4) and the uniform example discussed in the previous section. The distribution over κ as given by (4.4) depends, in general, on c^*. So, the density over κ will shift around as we consider alternative switching points c^*. The strategic uncertainty thus depends on the switching point. This is in contrast to the uniform case discussed in the previous section. There, the density over κ was shown to be uniform, irrespective of the switching point.

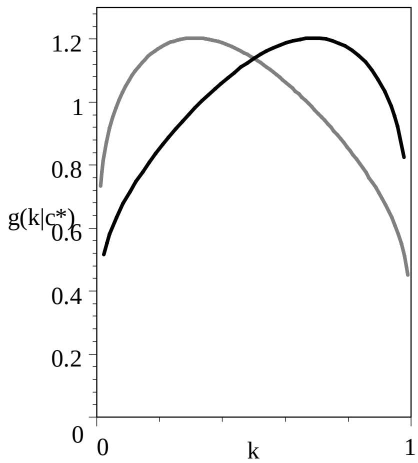
Below, we plot $G(z|c^*)$ for $c^* = 0.5$ for two alternative values of y. The dark line is the plot for $y = 0.2$, while the faint line is the plot for $y = 0.8$. We have set $\alpha = 1$ and $\beta = 3$.



Plot of $G(z|0.5) : y = 0.2$ (dark), $y = 0.8$ (faint)

When y is low, so that the players' costs are drawn from a density with lower costs, the players are more optimistic about the successful provision of the public good. The probability that the public good fails to be provided is the value of G evaluated at $z = \hat{\kappa}$. We can see that for any threshold $\hat{\kappa}$, the probability that the public good provision fails is lower when y is lower. The intuition for this lies in the inference that a player makes about the costs of others. When player i draws cost $c_i = 0.5$ for himself but the ex ante mean of the costs is $y = 0.2$, he reasons that others are likely to have costs lower than himself, which makes him more optimistic about successful provision of the public good. However, if y were 0.8 instead, he would reason that others are likely to have costs higher than himself, making him less optimistic.

We can see this more clearly from the corresponding density functions over κ - the proportion of players who contribute to the public good. When y is low, the density over κ puts more weight on higher values of κ, thereby making the successful provision of the public good more likely.



Densities over κ for $y = 0.2$ (dark), $y = 0.8$ (faint)

We can see that, in general, the densities over κ are not uniform. Nor are they invariant to the value of the switching point c^* or the ex ante mean of the costs y. There is, however, one special case where the density is invariant to both c^* and y. This is the special case where $\beta \to \infty$. In this limiting case, dispersion of costs around θ shrinks to zero, so that everyone's cost is given by θ, and everyone knows this fact. In this sense, the fundamental uncertainty disappears from the problem. We are just left with the strategic uncertainty. The limiting case of (4.4) when β becomes large is given by:

$$G(z|c^*) \to \Phi\left(\Phi^{-1}(z)\right) = z$$

so that G is the identity function, and we retrive the case discussed in the last section in which the density over κ is uniform. In this limit, the density over κ is uniform and invariant over both c^* and y.

4.2. Observable Implications of Equilibrium

What are the observable implications of these beliefs over κ? Can we use the particularly simple form of the contribution game to reconstruct the beliefs of the players, and thereby to measure the "reverberant doubt" of the players? There are a number of pitfalls along the way, but we will see that it is possible to go some way toward retrieving the players' beliefs from their observed behaviour.

For the moment, let us continue with the working hypothesis that all players follow the switching strategy around some point c^*. We will see below that for some parameter ranges, this is not always consistent with equilibrium behaviour. We will be careful to point out the conditions under which the switching strategy is the unique equilibrium strategy.

Recall that the indifference condition for the player with cost c^* is

$$1 - c^* = F(c^*) \tag{4.5}$$

where $F(c^*)$ is the probability that the provision of the public good fails, conditional on c^*. That is, $F(c^*) = G(\hat{\kappa}|c^*)$. Equation (4.5) defines the switching point c^* as a function of the critical mass $\hat{\kappa}$. This is a mapping which is, in principle, observable by the experimental economist. We will see what mapping is implied by our theory for a variety of parameter values.

To proceed further with the analysis while maintaining tractability, it is useful to make use of the logistic approximation

$$\Phi(x) \simeq \frac{1}{1 + e^{-mx}}$$

Setting the constant $m = \pi/\sqrt{3}$ ensures that the standard deviation is 1, but Amemiya (1981) recommends $m = 1.6$ as a better approximation for the overall shape of the density[2]. We have $\Phi^{-1}(w) \simeq \frac{1}{m}\log\left(\frac{w}{1-w}\right)$ so that we can write (4.4) as

$$G(z|c^*) \simeq \frac{1}{1 + \left(\frac{1-z}{z}\right)^{\sqrt{\frac{\alpha+\beta}{\beta}}} \exp\left(\frac{m\alpha}{\sqrt{\alpha+\beta}}(c^* - y)\right)} \tag{4.6}$$

Indifference at the switching point c^* then implies that

$$1 - c^* = G(\hat{\kappa}|c^*) \tag{4.7}$$

We can then solve this equation to obtain our mapping for the switching point c^* as a function of the critical mass $\hat{\kappa}$. Although this equation cannot be solved

[2]This is due to the fatter tails of the logistic, as compared to the normal.

explicitly for c^*, it is possible to solve for the inverse function - that is, we can solve for $\hat{\kappa}$ as a function of c^*. We have

$$\hat{\kappa} \simeq \frac{1}{1 + \left(\frac{c^*}{1-c^*} \exp\left(\frac{m\alpha}{\sqrt{\alpha+\beta}} (y - c^*) \right) \right)^{\sqrt{\frac{\beta}{\alpha+\beta}}}} \tag{4.8}$$

We can invert (4.8) to obtain the main observable implication of our game in terms of the plot of the switching point c^* as a function of the critical mass $\hat{\kappa}$. Below, we plot $c^*(\hat{\kappa})$ for the two values of y that we have dealt with above - namely, $y = 0.2$ (the dark line) and $y = 0.8$ (the faint line). We have set $m = 1.6$, as recommended by Amemiya, and we have set $\alpha = 1$ and $\beta = 3$. As we would expect, the lower y is more conducive toward contribution by the players.

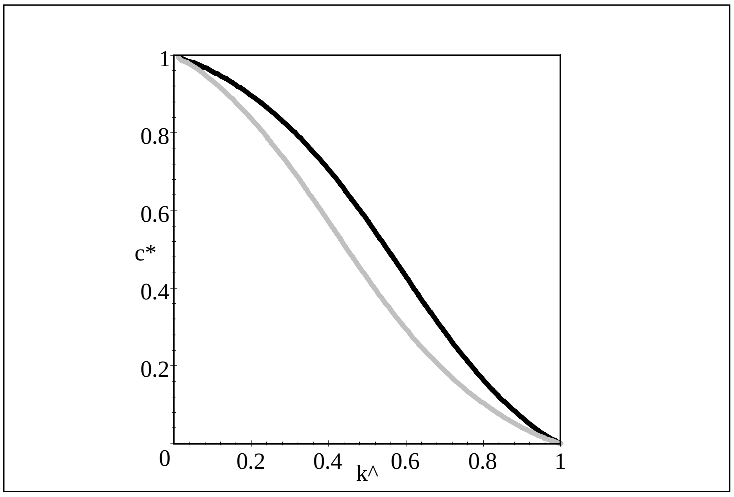


Figure 4.1: Plot of $c^*(\hat{\kappa})$ for $y = 0.2$ (dark), $y = 0.8$ (faint)

4.3. Retrieving Beliefs from Actions

The plot of c^* as a function of $\hat{\kappa}$ is something that is (at least in principle) observable by the experimental economist. There is a close relationship between this function and the beliefs of the players concerning the strategic uncertainty face, and this gives us an intriguing re-interpretation that goes to the heart of the issue of measuring the strategic uncertainty.

Suppose we have a downward sloping function $c^*(\hat{\kappa})$ with the additional pair of restrictions that $c^*(0) = 1$ and $c^*(1) = 0$. Then $1 - c^*$ can be seen as a cumulative distribution function over κ. Denote this cumulative distribution function as $G(\kappa)$. Consider an individual player whose subjective distribution over κ is given by $G(\kappa)$ irrespective of his own cost level. Then, this player's optimal switching point is given by

$$1 - c^* = G(\hat{\kappa})$$

which exactly coincides with the observed switching point $c^*(\hat{\kappa})$. In this way, the decision of this individual player will mimic the observed plot of c^*.

We should add immediately that the beliefs $G(\kappa)$ that are reconstructed in this way are not the true beliefs of the player in the game. We have seen already that the true density over κ will shift around as a function of c^* and y. However, the observable features of the problem in terms of the *decisions* of a player will coincide exactly with the decisions of our hypothetical player in equilibrium. Players in the actual game will act 'as if' they held the invariant beliefs $G(\kappa)$ and acted in accordance with them. For this reason, the density function over κ implied by $G(\kappa)$ takes on great significance. This density over κ can be obtained from implicit differentation of the function $c^*(\hat{\kappa})$ and is given by

$$-\frac{dc^*}{d\hat{\kappa}}$$

This density gives us the behavioural counterpart to the strategic uncertainty faced by the players.

4.4. Case of Independent Costs

So far, we have taken as a working hypothesis that the switching strategy around c^* is the only equilibrium strategy in the game. However, for some parameters, there is no such unique switching point. In particular, the case where players costs are indepedent is particularly badly behaved. Not only is there more than one equilibrium, there is no equilibrium in which players switch at some point c^* in the interior of the unit interval $[0, 1]$.

We can accommodate the case of independent costs by letting $\alpha \to \infty$ in our framework. Then the common element in cost θ is known to everyone, and the only source of uncertainty is the (independent) idiosyncratic variations in costs given by s_i.

When players have independent costs, there is no equilibrium where the players use a switching point c^* strictly between zero and one. To see this, suppose for the sake of argument that the players are using a switching strategy around $c^* \in (0, 1)$. Then, the proportion of players that contribute is given by

$$\kappa = \Phi\left(\sqrt{\beta}\left(c^* - \theta\right)\right) \tag{4.9}$$

Since θ is known to all players, so is the fact that κ is given as above. Now, either $\kappa \geq \hat{\kappa}$ or $\kappa < \hat{\kappa}$. If it is the former, than the provision of the public good is successful, in which case those players with costs between c^* and 1 are playing sub-optimally, since they are opting out even though they would get a higher payoff by contributing. Conversely, if $\kappa < \hat{\kappa}$, then those players with costs between 0 and c^* are playing sub-optimally, since the provision of the public good fails, and they would be better off by opting out rather than contributing.

This argument shows that there can be no non-trivial switching point c^*. The only equilibrium switching points are at $c^* = 0$ and at $c^* = 1$, and either can be supported as an equilibrium. Thus, there is more than one equilibrium[3].

4.5. Uniqeness of Equilibrium

In general, the question of the uniqueness of equilibrium can be settled by the rate of change of the perceived failure probability for the provision of the public good. This perceived probability is given by (4.4), and the slope with respect to c^* is given by

$$-\phi\left(...\right)\frac{\alpha}{\sqrt{\alpha+\beta}} \tag{4.10}$$

where $\phi\left(...\right)$ indicates the value of the standard normal density evaluated at $\frac{\alpha}{\sqrt{\alpha+\beta}}\left(y - c^*\right) + \sqrt{\frac{\alpha+\beta}{\beta}}\Phi^{-1}\left(\hat{\kappa}\right)$. Since $\phi \leq 1/\sqrt{2\pi}$, a sufficient condition for the absolute value of the slope to be less than one is $\frac{\alpha}{\sqrt{\alpha+\beta}} \leq \sqrt{2\pi}$.

5. Finite Number of Players

So far in the paper, we have examined the continuum player case, where the law of large numbers arguments simplify the analysis. However, the analysis remains

[3] This result stands in contrast to other examples of global games that have been examined in the literature (McKelvey and Palfrey (1995), Baliga and Sjostrom (2001) and Morris and Shin (2002)) where independent types are compatible with unique equilibrium.

unchanged in spirit when we suppose that the number of players is finite. Let us begin with the uniform-uniform example of section 3. Suppose there are N players, where the cost of player i is given by

$$c_i = \theta + s_i$$

where θ is the common cost element, and s_i is the idiosyncratic cost element for player i. We maintain the assumption that θ has a uniform density, and s_i has uniform density over $[-\varepsilon, \varepsilon]$ for small $\varepsilon > 0$. In addition, s_i and s_j are independent for $i \neq j$.

Just as for the continuum case, let us maintain the working hypothesis that all players follow the switching strategy around c^*, so that any player with cost below c^* contribute, and any player with cost above c^* opt out. Then, let us consider the following analogue to question (Q).

$$\text{My cost is } c^*. \text{ What is the probability that exactly } n \text{ players contribute?} \quad (\mathrm{Q}')$$

The answer to question (Q′) turns out to be $1/(N+1)$, irrespective of n. In other words, the probability mass function over the number of players who contribute is uniform with support $\{0, 1, 2, \cdots, N\}$. Thus, the spirit of the continuum player case holds for the finite player case. Hans Carlsson and Eric van Damme (1993) and Young Se Kim (1996) have presented solutions of many player coordination games that rely on this result.

Let us demonstrate that the answer to question (Q′) is $1/(N+1)$. Figure 3.1 for the continuum case is still useful for illustrating the reasoning. When the common cost element is θ, the probability that player i's cost is below c^* (and hence contributes) is the area under the density to the left of c^* in the top panel of figure 3.1. This is given by

$$z = \frac{c^* - \theta + \varepsilon}{2\varepsilon} \tag{5.1}$$

Thus, the probability that exactly n players contribute out of the total population of N players is the binomial probability:

$$\begin{aligned} & \binom{N}{n} z^n (1-z)^{N-n} \\ = & \binom{N}{n} \frac{(c^* - \theta + \varepsilon)^n (\theta - c^* + \varepsilon)^{N-n}}{(2\varepsilon)^N} \end{aligned} \tag{5.2}$$

Conditional on c^*, the density of θ is uniform, with support $[c^* - \varepsilon, c^* + \varepsilon]$. Thus, the answer to question (Q$'$) is given by the expectation of (5.2) as θ takes values in the interval $[c^* - \varepsilon, c^* + \varepsilon]$ with a uniform density. In other words,

$$\frac{\binom{N}{n}}{2\varepsilon} \int_{c^*-\varepsilon}^{c^*+\varepsilon} \frac{(c^* - \theta + \varepsilon)^n (\theta - c^* + \varepsilon)^{N-n}}{(2\varepsilon)^N} d\theta \tag{5.3}$$

We can simplify this expression by taking the change of variables (5.1). Then $d\theta = 2\varepsilon dz$, while the limits of the integral are from 0 to 1. Then, (5.3) can be written as

$$\binom{N}{n} \int_0^1 z^n (1 - z)^{N-n} dz \tag{5.4}$$

Our claim is that this integral does not depend on n, and is equal to $1/(N+1)$.

Lemma 1. $\binom{N}{n} \int_0^1 z^n (1 - z)^{N-n} dz = \frac{1}{N+1}$*, for all* n*.*

The proof of lemma 1 is given in the appendix. This result tells us that the answer to question (Q$'$) is $1/(N+1)$. The probability mass function over the number of players who contribute is uniform.

Just as in the continuum case, the uniform density over the number of players who contribute is one facet of a more general result. When the variation in the idiosyncratic portion of the players' costs is small relative to the variation of the common cost element, the uniform mass function over the number of players that contribute is a good approximation to the true mass function. We will give an informal sketch of the argument here. A more precise argument can be constructed using the technique shown in the survey of global games by Morris and Shin (2000, section 2).

Let θ be uniformly distributed, and let the idiosyncratic element in cost s_i be i.i.d. across players, with density function $f(.)$. For the purpose of this illustration, we will suppose that the density $f(.)$ is symmetric around zero[4]. The interpretation of the uniform density for θ is that the information contained in the prior density over θ is swamped by the information contained in the realization of c_i. So, this situation is an approximation to the case where θ has some general (possbly non-uniform) density, but where the idiosyncratic element in costs s_i is very small, so that the information contained in the prior density over θ is swamped by the observation of one's own cost.

[4]Symmetry of the density around zero is not necessary for the general argument. See Morris and Shin (2000, section 2).

What we need to show is that the probability z that any particular player contributes is itself a random variable that has a uniform density over the unit interval $[0, 1]$. Then, the probability that exactly n players contribute will be given by the average $\binom{N}{n} \int_0^1 z^n (1-z)^{N-n} dz$. Suppose that the players follow the switching strategy around c^*. Denote by z the probability that any particular player contributes. Let

$$G(w|c^*)$$

be the probability that z is w or lower, conditional on c^*. In other words, $G(\cdot|c^*)$ is the cumulative distribution function over z. When the common element of cost is θ, the probability that any particular player contributes is given by

$$z = \int_{-\infty}^{c^*-\theta} f(c-\theta)\, dc$$

We know that $z \leq w$ if and only if $\theta \geq \theta^*$ where θ^* is defined in terms of

$$w = \int_{-\infty}^{c^*-\theta^*} f(c-\theta^*)\, dc$$

Thus, $G(w|c^*)$ is given by the probability that $\theta \geq \theta^*$ conditional on c^*. Since the prior density over θ is uniform and $f(.)$ is symmetric around zero, we have

$$\begin{aligned} G(w|c^*) &= \int_{\theta^*-c^*}^{\infty} f(\theta - c^*)\, d\theta \\ &= \int_{-\infty}^{c^*-\theta^*} f(c-\theta^*)\, dc \\ &= w \end{aligned}$$

so that the cumulative distribution function $G(\cdot|c^*)$ for z is the identity function, implying that the density function over z is uniform. This was what we wanted to show.

6. Concluding Remarks

The public good contribution game examined in this paper has turned out to be a useful vehicle in piecing together the beliefs of the players from their observed choices. In particular, we have seen two themes emerge from the analysis. First, the case where the strategic uncertainty is given by the uniform density over the proportion of players who contribute has particular significance. Not only is this the shape of the strategic uncertainty in the uniform-uniform case, it is also a good approximation to the shape of the strategic uncertainty in the case where the variation in idiosyncratic costs is small relative to the variation in the common cost. Second, it is possible to reconstruct the beliefs of an individual whose behaviour exactly mimics the equilibrium behaviour of the players in the game. When, in addition, the strategic uncertainty is uniform, the behaviour conforms to the linear rule in which the switching point c^* is given by $1 - \hat{\kappa}$.

Appendix

In this appendix, we prove lemma 1. Begin by noting that

$$\begin{aligned} z^n (1-z)^{N-n} &= z^n \sum_{i=0}^{N-n} \tbinom{N-n}{i} (-1)^i z^i \\ &= \sum_{i=0}^{N-n} \tbinom{N-n}{i} (-1)^i z^{n+i} \end{aligned}$$

Thus

$$\int z^n (1-z)^{N-n} dz = \sum_{i=0}^{N-n} \tbinom{N-n}{i} (-1)^i \frac{z^{n+i+1}}{n+i+1}$$

so that

$$\int_0^1 z^n (1-z)^{N-n} dz = \sum_{i=0}^{N-n} \frac{\tbinom{N-n}{i} (-1)^i}{n+i+1}$$

However,

$$\begin{aligned} \sum_{i=0}^{N-n} \frac{\tbinom{N-n}{i} (-1)^i}{n+i+1} &= \frac{n!\,(N-n)!}{(N+1)!} \\ &= \frac{1}{(N+1)\tbinom{N}{n}} \end{aligned}$$

Thus

$$\tbinom{N}{n} \int_0^1 z^n (1-z)^{N-n} dz = \frac{1}{N+1}$$

which is the statement of lemma 1.

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