Abstract

Collective Action under Uncertainty
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I examine collective action under uncertainty in public good games and social networks. The first chapter analyzes discrete public good games in which the threshold number of contributions needed for provision is randomly selected from a known probability distribution function. Simultaneous voluntary-contribution equilibria are often inefficient under binary contribution decisions. Wider uncertainty about the threshold can reduce inefficiencies, however, because contributions are higher when there is greater uncertainty about the threshold (in terms of second-order stochastic dominance) if the value of the public good is sufficiently high. Games with continuous contribution decisions and sequential contribution decisions are also considered.

The second chapter presents results from public good experiments specifically designed to test the main qualitative predictions of the first chapter. As predicted for within-session changes, contributions are higher under wider uncertainty when the value of the public good is high, and contributions are lower when the value of the public good is low. I use a proper scoring rule to elicit data on beliefs, and these data exhibit qualitative features of standard learning models. Using these data to proxy for actual beliefs, I show both parametrically and non-parametrically that aggregate decisions are not consistent with expected payoff maximization. Decisions are more consistent with a game-theoretic decision rule that accounts for risk aversion and innate cooperativeness.

The third chapter examines equilibrium network structures under uncertainty. Bala and Goyal (*Econometrica* 2000) model network formation as a non-cooperative game under complete information. I extend their approach to include uncertainty about the network structure and uncertainty about the benefits of network participation. I define an incomplete-information imperfect-monitoring equilibrium concept, called Generalized Conjectural Equilibrium. I apply this concept to find strict equilibrium network structures in twenty different informational environments. I find that the unique equilibrium architecture of complete information games—the center-sponsored star—is still unique even under large decreases in information. I show exactly how the type of equilibrium structure depends on the informational environment. A main contribution of this paper is the characterization of network uncertainty. Another contribution is the definition and use of a new non-Nash equilibrium concept.

Collective Action under Uncertainty

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INTRODUCTION

This collection of three chapters examines collective action under uncertainty in public good games and social networks. The first two chapters consider discrete public good games in which the threshold number of contributions needed to provide the good is not known. For example, peasants who could participate in a revolution might not know how big the peasant army needs to be in order to successfully defeat the dictator. The first chapter contains theoretical analysis to determine if and when this uncertainty can ever be good in the sense of efficiency or in the sense of contributions. The second chapter presents results from a series of laboratory experiments that support the main theoretical predictions of the theoretical analysis. The third chapter examines social networks in which there is uncertainty about both the benefits of participating in and the structure of the network. For example, an individual might know the benefits of the network of which he is a member, but he might not know the benefits of another network of which he is not a member.

The first chapter analyzes discrete public good games where the threshold is randomly selected from a publicly known probability distribution. I examine situations in which individuals make a binary choice to participate or not participate. I consider the full set of equilibria, including mixed equilibria, under different threshold distributions. Simultaneous voluntary-contribution equilibria are often inefficient. Greater uncertainty about the threshold, however, can reduce inefficiencies. The reason is that contributions are higher when there is greater uncertainty about the threshold (in terms of second-order stochastic dominance) if the value of the public good is sufficiently high. This result arises because of the increased probability of being pivotal at higher contribution strategy profiles when the value of the public good is high. Binary contribution decisions are contrasted with situations in which individuals choose contribution levels from a continuous set, like monetary contribution levels. Such continuous contribution games are less inefficient than binary contribution games. These results are unchanged when considering sequential equilibria.

The second chapter presents results from a series of public good experiments specially designed to test the main qualitative predictions of the above theoretical work. As predicted for within-session changes, actual contributions are higher when the uncertainty is increased if the value of the public good is high, and contributions are lower when the value of the public good is low. I also elicit data on agents' beliefs using a proper scoring rule. These data exhibit qualitative features of standard learning models. Using these data to proxy for actual beliefs, I show that aggregate decisions are not consistent with expected payoff maximization. This result is confirmed both parametrically and non-parametrically. However, I show that decisions in later rounds become more consistent with a game-theoretic decision rule as I account for risk aversion and innate cooperativeness.

The third chapter examines equilibrium network structures under uncertainty. Bala and Goyal (Econometrica 2000) model network formation as a non-cooperative game under complete information. I extend their approach to include two types of uncertainty: uncertainty about the network structure and uncertainty about the benefits of network participation. I also consider five levels of uncertainty: (1) each individual receives information about all others in the game (i.e., the Bala and Goyal (2000) case); (2) each individual only receives information about others in his own network; (3) each individual only receives information about all others directly connected to his own direct neighbors; (4) each individual only receives information about those to whom his own neighbors have initiated connections; (5) each individual receives no information. To study these environments, I define an incomplete-information imperfect-monitoring equilibrium concept, called Generalized Conjectural Equilibrium. I use this concept to solve for the strict equilibria in each of the twenty different cases. I find that the unique equilibrium architecture of complete information games—the center-sponsored star—is still unique even under large decreases in information. The crucial aspect in determining network equilibrium structures is whether or not an agent knows aspects of the network that are beyond his direct links. Furthermore, as uncertainty increases, the set of equilibrium structures increases in interesting ways. For example, the set of equilibria in a game where players know the benefits of network participation but not the network architecture can be very different from the set of equilibria when the players know the architecture but not the benefits. A main contribution of this paper is the characterization of network uncertainty. Another contribution is the definition and use of a new non-Nash equilibrium concept.

CHAPTER 1

DISCRETE PUBLIC GOODS UNDER THRESHOLD UNCERTAINTY

1.1 INTRODUCTION

Potential participants in a collective action often have uncertainty about the threshold number of contributions needed to provide a discrete public good. For example, peasants will not know how big their revolutionary army needs to be to overthrow the incumbent dictator; neighborhood residents will not know how many individual requests to the city will be required to install a signal at a local intersection; local citizens might not know the amount of money required to build a public project. This chapter gives gives necessary and sufficient conditions for wider uncertainty to increase voluntary contributions. More specifically, for a large class of threshold probability distributions, contributions will increase after a widening of uncertainty when the players sufficiently value the public good, while contributions will decrease when the players' valuations of the public good are sufficiently low.

Palfrey and Rosenthal (1984) show that in a discrete public good game with complete information, an efficient simultaneous-contribution equilibrium always exists, although inefficient equilibria can also exist. My paper adds uncertainty about the threshold to their basic model and then focuses on how contribution levels change under different levels of that uncertainty. In these games, a player contributes when her probability of being pivotal in providing the public good is greater than some critical number. After an increase in uncertainty by 2nd-order stochastic dominance, a non-contributor's probability of being pivotal can increase, thus driving up contributions if that probability increases above the critical number. However, the probability of being pivotal can also decrease for those already contributing, thus decreasing contributions if the probability drops below the critical number. Final contributions in both pure and mixed equilibria will thus be determined by this critical number, which in equilibrium, is inversely related to the value of the public good. Furthermore, in games with binary contributions, wider uncertainty can also lead to increased efficiency. Efficiency will rise when the probability of public good provision significantly increases due to the larger number of contributions.

Various types of uncertainty have been added to discrete public good games.[1] Nitzan and Romano (1990) add a type of uncertainty similar to the threshold uncertainty that I add. Assuming that individuals choose contributions from a continuous set and that the threshold is chosen from a commonly known continuous distribution over $[a, b]$, Nitzan and Romano derive necessary and sufficient conditions for the efficiency of a Pareto-undominated equilibrium. By assuming continuous contributions, they are effectively assuming that contributions are monetary in nature.

[1]Previous studies have examined discrete public good games with uncertainty about other strategic variables. These studies can be grouped into those that examine binary (or discrete) contributions and those that allow continuous contributions. Bagnoli and Lipman (1989) allow for continuous contributions and show that, under complete information, if refunds are allowed then undominated equilibria are efficient and also that sequential equilibria can be efficient. Palfrey and Rosenthal (1988) model discrete contributions where there is uncertainty about other players' altruism. Palfrey and Rosenthal (1991) model discrete contributions where there is uncertainty about individual contribution costs. Menezes, Monteiro, and Temimi (2001) show that with continuous contributions and thresholds that require more than one persion, uncertainty about others' valuations of the public good generally results in inefficient equilibria.

The binary contributions considered in my paper are more akin to participation or "in/out" contributions. After presenting my main theoretical findings, I explain how the equilibrium properties under these different contribution assumptions will result in greater efficiency when contributions are continuous.

The literature on collective action spans many different methodological approaches,[2] and there are many criticisms of the public good analysis used in this paper. A prominent set of challenges argues that the rational choice methodology misrepresents the complicated social environments in which actual behavior occurs. From this perspective, successful collective action depends on political opportunities in societies that are characterized by social, economic, and political groups, and these groups operate within complicated cultural and institutional contexts.

In this chapter, I take the middle of the road perspective presented by Lichbach (1998). On the one hand, while other approaches (e.g., Structuralist) provide valuable insights into the social context of collective action, they are less precise on the individual and strategic considerations of social behavior. The individual-level approach provides important pieces to the collective action puzzle, and even though this rational choice methodology does not account for all the richness of social context, it provides key insights into the minutest aspects of social behavior. For the purposes of this study, I take it for granted that uncertainty about the threshold is a

[2] There are many strands of collective action research. See Olson (1989) and Sandler (1992) for general discussions of collective action from the economics perspective. Another formal approach (apart from discrete public good analysis) used often by sociologists to explain riots and revolutions is the threshold or tipping model, as in Granovetter (1978), Yin (1998), and Chwe (1999, 2000). A totally different approach to collective action is one that focuses on authority and structure (Lichbach 1998).

real and important aspect of many collective action scenarios. In addition to being an important benchmark and framework for studying aspects of collective action,[3] the discrete public good game framework (e.g., Palfrey and Rosenthal 1984) allows us to isolate of the effects of changes in threshold uncertainty.

Section 1.2 describes the basic model with binary contributions and threshold uncertainty. Section 1.3 contains the analysis for the main theoretical predictions. Section 1.4 compares binary contribution games with continuous contribution games and briefly discusses the similarities between simultaneous and sequential contribution games. The results in this chapter yield inights into collective action, but I will postpone discussion of these insights until after presenting the experimental results in Chapter 2. This discussion of insights into collective action will consider both my theoretical and experimental results, and so the discussion will be presented at the end of Chapter 2.

1.2 MODEL

The discrete public good game consists of the following. The set of expected payoff maximizing players is $N = \{1, ..., n\}$, $2 < n < \infty$. Players have identical binary action sets $A_i = \{0, 1\}$, with actions labeled {don't contribute, contribute}. When

[3] Here are some reasons that justify this benchmark. First, although it is well-established that certain modeling implications of expected utlity theory do not exist in reality, the expected utility framework can be a useful predictor of behavior (Rabin 1999). This is the case in the experiments reported here. Second, recent work using evolutionary models gives justification for looking at equilibria because they can arise even from boundedly rational individuals. This suggests the existence of scenarios in which predicted outcomes can arise even if the modeled motives exactly represent the individuals' true motives. Third, from a theoretical perspective, a precise benchmark allows for both departure and comparison.

players mix over those actions, let α_i be the probability that $a_i = 1$ (i contributes). The cost of contributing c and the value of a provided public good v are the same for all individuals. The contribution threshold t to provide the public good is chosen from a publicly known distribution cdf F with pdf f s.t. $F(0) = 0$. Given C realized contributions, the payoffs are:

$$\text{payoff for } i = \begin{cases} v & \text{if } C \geq t \text{ and } a_i = 0 \\ v - c & \text{if } C \geq t \text{ and } a_i = 1 \\ 0 & \text{if } C < t \text{ and } a_i = 0 \\ -c & \text{if } C < t \text{ and } a_i = 1 \end{cases}.$$

For most of this chapter, I use discrete threshold distributions, but assuming an underlying continuous contribution will be necessary to consider what happens when the binary action set assumption is relaxed in Section 1.4.[4]

The timing of the game is as follows: (1) n, v, c, F, and the game set-up are commonly known; (2) the players simultaneously choose whether or not to contribute; (3) payoffs are received. The analysis focuses on Nash equilibria.

1.3 ANALYSIS

The number of contributions in equilibrium is of greater interest than which players contribute in equilibrium. I will treat two equilibria with the same number of contributions as a unique equilibrium.

Section 1.3 will assume that the threshold distribution is strictly quasi-concave. Formally, call a cdf F with pdf f *strictly quasi-concave* if the pdf is single-peaked and $f(x) \neq f(x+1)$ for any x in its range. The pdf cannot be flat at any x with

[4]The analysis is unchanged if the discrete F is actually a discrete representation of an underlying continuous threshold distribution. For example, consider a continuous cdf $H(x)$ with pdf $h(x)$ over $(0, \infty)$. Then F the discrete version of H (with pdf $h(x)$) is calculated s.t. $F(x) = \int_0^x h(x)dx$ for all $x = \{1, 2, ...\}$.

$f(x) > 0$, but it can be flat at x where $f(x) = 0$.[5] The main results in this paper can be obtained without this condition, as discussed in Section 1.4. This condition is not unrealistic, however, and it greatly simplifies the intution for the analysis.

1.3.1 Pure Equilibria

For now, the focus is on pure equilibria. An agent's decision in equilibrium will depend on the probability he is pivotal in providing the public good. Denote C_{-i} to be the set of contributing agents besides i. The payoff matrix is

	$C_{-i} < t-1$ (lost cause)	$C_{-i} = t-1$ (pivotal)	$C_{-i} > t-1$ (redundant)
spend	$-c$	$v-c$	$v-c$
keep	0	0	v

Let $a_{-i} = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$. Further denote $\Pr[piv|a_{-i}, F]$ to be the probability that i is pivotal given a_{-i} and F, $\Pr[lost|a_{-i}, F]$ to be the probability of a lost cause, and $\Pr[red|a_{-i}, F]$ to be the probability of being redundant.[6] Given a_{-i} and F, a player is willing to contribute if

$$\begin{aligned} & \Pr[lost|a_{-i}, F](-c) + \Pr[piv|a_{-i}, F](v-c) + \Pr[red|a_{-i}, F](v-c) \\ \geq\ & \Pr[red|a_{-i}, F]\, v \\ \Rightarrow\ & \Pr[piv|a_{-i}, F] \geq \frac{c}{v}. \end{aligned}$$

[5] To illustrate, a uniform pdf over $\{3,4\}$ is single-peaked, but it is not strictly quasi-concave because it is flat at $f(3)$ and $f(4)$.

[6] The calculation of these is straightforward: $\Pr[piv|a_{-i}, F] = \sum_{x=1}^{\infty}(\Pr[C_{-i} = x-1|a_{-i}]\, f(x))$, $\Pr[lost|a_{-i}, F] = \sum_{x=1}^{\infty}(\Pr[C_{-i} < x-1|a_{-i}]\, f(x))$, $\Pr[red|a_{-i}, F] = \sum_{x=1}^{\infty}(\Pr[C_{-i} > x-1|a_{-i}]\, f(x))$.

It follows that for each i in a pure Nash equilibrium:

$$a_i = \begin{cases} 0 & \text{if} \quad \Pr\left[piv|a_{-i}, F\right] < \frac{c}{v} \\ a' \in \{0,1\} & \text{if} \quad \Pr\left[piv|a_{-i}, F\right] = \frac{c}{v} \\ 1 & \text{if} \quad \Pr\left[piv|a_{-i}, F\right] > \frac{c}{v} \end{cases} . \tag{1}$$

Denote a pure equilibrium by C, which now signifies the number of pure contributions in equilibrium. In a pure equilibrium C a contributing player believes with probability one that exactly $C-1$ others are contributing. That player is pivotal with probability $f(C)$. A non-contributing player in the proposed equilibrium C is pivotal with probability $f(C+1)$. It follows that conditions for existence of an equilibrium C are:

$$C = \begin{cases} 0 & \text{if} \quad f(1) \leq \frac{c}{v} \\ x \in \{1, .., n-1\} & \text{if} \quad f(x) \geq \frac{c}{v} \text{ and } f(x+1) \leq \frac{c}{v} \\ n & \text{if} \quad f(n) \geq \frac{c}{v} \end{cases} . \tag{2}$$

If there is more than one pure equilibrium in a game, one of the equilibria is the trivial equilibrium with zero contributions. Proposition 1 contains some preliminary conditions for uniqueness of equilibria. In this proposition and throughout the rest of the paper, the *feasible mode* is the mode of the distribution from 1 to n. More formally, $x \in N$ is the feasible mode where $f(x) > f(x')$ for all $x' \in N$, $x' \neq x$ ("$>$" by strict quasi-concavity).

Proposition 1: *(Uniqueness of Pure Equilibria) Fix the players N, the contribution cost c, the value of the value of the public good v, and the threshold distribution F.*

(a) The unique equilibrium is $C = 0$ if and only if the cost-value ratio $\frac{c}{v}$ is strictly greater than the mass at the feasible mode. The unique

equilibrium is $C = n$ *if and only if the cost-value ratio* $\frac{c}{v}$ *is weakly less than* $f(x)$ *for all feasible contributions* x*.*

(b) If the threshold distribution F *is strictly quasi-concave, then there is at most one non-trivial equilibrium with* $C > 0$*. Furthermore, if there is more than one equilibrium then there are exactly two equilibria: one is the trivial equilibrium* $C = 0$*, and the other is a non-trivial equilibrium with* $C > 0$*.*

(c) If the threshold distribution F *is strictly quasi-concave, then any non-trivial equilibrium with* $C > 0$ *has* C *(weakly) to the right of the feasible mode.*

Proof: (a) Follows directly from the conditions in (2).

(b) For there to be more than one non-trivial equilibrium, there must be $x < x' < x''$, all in N, s.t. $f(x) \geq \frac{c}{v}$, $f(x') \leq \frac{c}{v}$, and $f(x) \geq \frac{c}{v}$. But the single-peaked nature of strict quasi-concavity prevents this from being true. Because there is only one non-trivial equilibrium, if there is more than one equilibrium, then the only other possible equilibrium is the trivial equilibrium at $C = 0$.

(c) By (2), any interior equilibrium $C > 0$ must have $f(C) \geq \frac{c}{v} \geq f(C+1)$. By strict quasi-concavity, C must be to the right of the feasible mode where the pdf is "downward sloping." If $C = n$ then C is trivially to the right since it is not left of any feasible $x \in N$. ■

These pure equilibria can be represented graphically. Figure 1.1(a) shows a typical pdf. As shown, there is a trivial equilibrium $C = 0$, and with $n = 5$ there is a non-trivial equilibrium at $C = 4$. Notice that for a strictly quasi-concave distribution, the internal/non-trivial equilibrium must be to the right of the feasible mode and on the downward-sloping side of the pdf. As stated in Proposition 1, this fact comes straight from the nature of equilibrium. Each contributor must be willing to contribute $\left(f(C) \geq \frac{c}{v}\right)$, and each non-contributor must not want to contribute $\left(f(C+1) \leq \frac{c}{v}\right)$.

Hereafter, we focus on the *non-trivial equilibrium* C^*. The non-trivial equilibrium is $C^* = 0$ if the cost-value ratio $\frac{c}{v}$ is higher than the feasible mode (by Proposition 1). Otherwise, the non-trivial equilibrium is $C^* > 0$. As shown later, this non-trivial equilibrium is the Pareto-undominated equilibrium, although it can be inefficient.

I now state the two main propositions of this paper. The first of these will consider a threshold distribution that is totally feasible. Say that a distribution F is *totally feasible* when $F(n) = 1$.

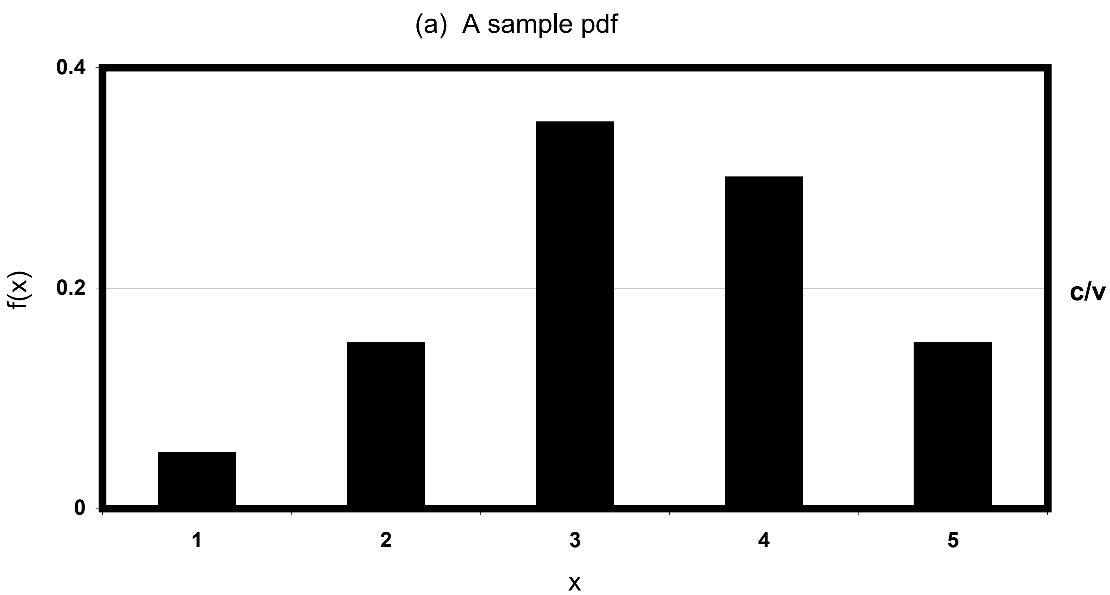
Proposition 2: *(Higher Contributions under 2nd-order Stochastic Dominance) Consider two games that are identical except for their strictly quasi-concave threshold distributions* F *and* $\widehat{F}$, $F \neq \widehat{F}$. *Denote* C^* *and* $\widehat{C}^*$ *their respective non-trivial equilibria. If* F *2nd-order stochastically dominates* $\widehat{F}$, F *and* $\widehat{F}$ *have the same mean, and* F *and* $\widehat{F}$ *are both totally feasible, then there exists a scalar* $k > 0$ *such that* $\widehat{C}^* \geq C^*$ *if the* *cost-value ratio* $\frac{c}{v} \leq k$. *Furthermore, if it is also true that the feasible mode of* F *is strictly greater than the feasible mode of* $\widehat{F}$*, then there exists a second scalar* $k' \geq k$ *such that* $C^* \geq \widehat{C}^*$ *if the cost-value ratio* $\frac{c}{v} > k'$.

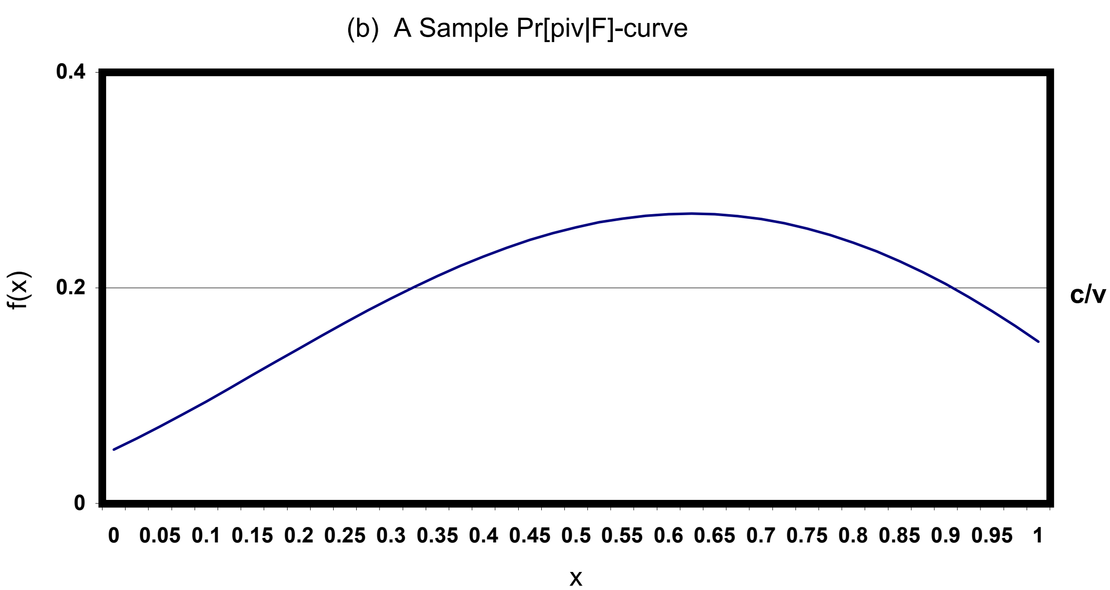
Proposition 3: *(Higher Contributions under a Single-Crossing Condition) Consider two games that are identical except for their strictly quasi-concave threshold distributions* F *and* $\widehat{F}$, $F \neq \widehat{F}$. *Denote* C^* *and* $\widehat{C}^*$ *their respective non-trivial equilibria, and assume that the feasible mode of* F *is higher than the feasible mode of* $\widehat{F}$. *If the pdf's* f *and* $\widehat{f}$ *cross exactly once on the right of the feasible mode of* F*, then there exists a scalar* $k > 0$ *such that* $\widehat{C}^* \geq C^*$ *if the cost-value ratio* $\frac{c}{v} \leq k$*, and* $C^* \geq \widehat{C}^*$ *if the cost-value ratio* $\frac{c}{v} > k$.

The proofs of Proposition 2 and 3 will follow directly from a more general result about the comparison of non-trivial equilibria in games that differ only in their threshold distributions. This more general result is Lemma 1. In the rest of Section 1.3.1, we will first work towards establishing Lemma 1, after which Propositions 2 and 3 will be proven.

Because the non-trivial equilibrium $C^* > 0$ is to the right of the feasible mode, we can restrict our attention to that part of the threshold pdf that is between the feasible mode and the maximum number of contributions n. We can also go one step further when comparing the non-trivial equilibria of otherwise identical games

Figure 1.1: An Example for Finding Equilibria Graphically





with different threshold distributions. The following corollary to Proposition 1 says that when looking for the equilibrium with higher contributions of the two games, we can restrict our attention to that part of the two distributions that is between the feasible mode with higher mass and n. In other words, if the mass at F's feasible mode is higher than the mass at $\widehat{F}$'s feasible mode, then we need only be concerned with the area to the right of F's feasible mode.

Corollary 1: *(Comparing Non-trivial Equilibria) Consider two games that are identical except for strictly quasi-concave threshold distributions F and $\widehat{F}$, $F \neq \widehat{F}$. Denote C^* and $\widehat{C}^*$ their respective non-trivial equilibria.*

(a) $\widehat{C}^* > C^*$ if and only if there exists some level of contributions x, $C^* < x \leq n$, such that $f(x)$ is weakly greater than the cost-value ratio $\frac{c}{v}$.

(b) Consider the feasible modes of F and $\widehat{F}$. If $\widehat{C}^* > C^*$ then $\widehat{C}^*$ must be to the right of the feasible mode with higher mass.

With our attention now restricted to the right of the feasible mode with higher mass, we now look more closely at the behavior of the two distributions F and $\widehat{F}$ from that feasible mode to n. I will often call this specific area the *interior* and denote it by I.

One key condition of interest is when one of the pdf's has a higher interior-right tail, that is, one pdf is greater than the other pdf for all contribution levels from some number in the interior I to n. The "interior-right" means that we are looking at the right tail in this interior I. Another key condition is the analog for the interior-left, but this condition will also be defined by the height of the pdf's to the right of the interior-left. After formally defining these conditions, I will illustrate them graphically.

Interior Tails Conditions: *Consider two strictly quasi-concave distributions* F *and* $\widehat{F}$, $F \neq \widehat{F}$, *with respective pdf's* f *and* $\widehat{f}$. *Denote* I *to be the interior, that is, the set of contribution levels between the feasible mode with higher mass* m *and* n, $I = \{m, ..., n\}$.

(a) Say that $\widehat{f}$ *has a fatter interior-right tail than* f *if there exists an* $x \in I$, *such that* $\widehat{f}(x') \geq f(x')$ *for all* $x \leq x' \leq n$.

(b) Say that f *has a fatter interior-left tail than* $\widehat{f}$ *if there exists an* $x \in I$, *such that* $f(x') \geq \widehat{f}(x')$ *for all* $m \leq x' \leq x$, *and, if* $x < n$, $f(x') > f(x'')$ *for all* $m \leq x' \leq x$, $x < x'' \leq n$.

Figure 1.2 illustrates these conditions. Smooth pdf's are drawn for clarity. Figure 1.2(a) shows the case where both a fatter interior-right tail I_R and a fatter interior-left tail I_L exist. Notice that these tails do not necessarily meet. Figure 1.2(b) shows when the two tails meet on the vertical axis. Figure 1.2(c) shows when one pdf is always above the other in the interior I.

The reason for the second condition in the definition of the fatter interior-left tail is that we want to know when the non-trivial equilibrium C^* will be in that interior-left tail *and* when $C^* \geq \widehat{C}^*$. This idea is illustrated on Figure 1.2(b). Notice that if $\frac{c}{v} = k_1$, then $\widehat{C}^*$ is higher than $C^* = x_1$ even though $f(x_1) > \widehat{f}(x_1)$. This is because x_1 is to the left of the feasible mode of $\widehat{F}$.

We can use Figure 1.2(a)-(b) to demonstrate the two main propositions of the paper and lead us closer to Lemma 1. Notice that the k and k' in Figure 1.2(a) satisfy the k and k' in Proposition 2. In this graph, F 2nd-order stochastically dominates $\widehat{F}$, and F has a higher feasible mode. We see that if $\frac{c}{v} \leq k$ then $\widehat{C}^* \geq C^*$, whereas if $\frac{c}{v} > k'$ then $C^* \geq \widehat{C}^*$. Figure 1.2(b) illustrates Proposition 2. The horizontal line at k is such that if $\frac{c}{v} \geq k$ then $\widehat{C}^* \geq C^*$, whereas if $\frac{c}{v} > k$ then $C^* \geq \widehat{C}^*$.

These illustrations lead us to the main lemma from which Propositions 2 and 3 are derived.

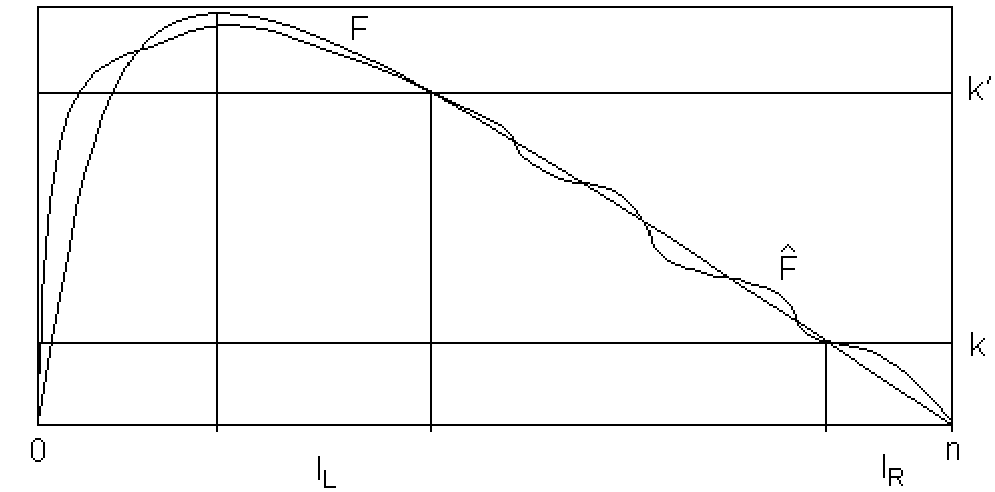
Lemma 1: *(Fatter Interior Tails and Pure Equilibria) Consider two games that are identical except for their strictly quasi-concave threshold distributions F and $\widehat{F}$, $F \neq \widehat{F}$. Denote C^* and $\widehat{C}^*$ their respective non-trivial equilibria.*

(a) If $\widehat{f}$ has a fatter interior-right tail than f, then there exists a scalar k, $0 < k < 1$, such that $\widehat{C}^* \geq C^*$ if the cost-value ratio $\frac{c}{v} \leq k$.

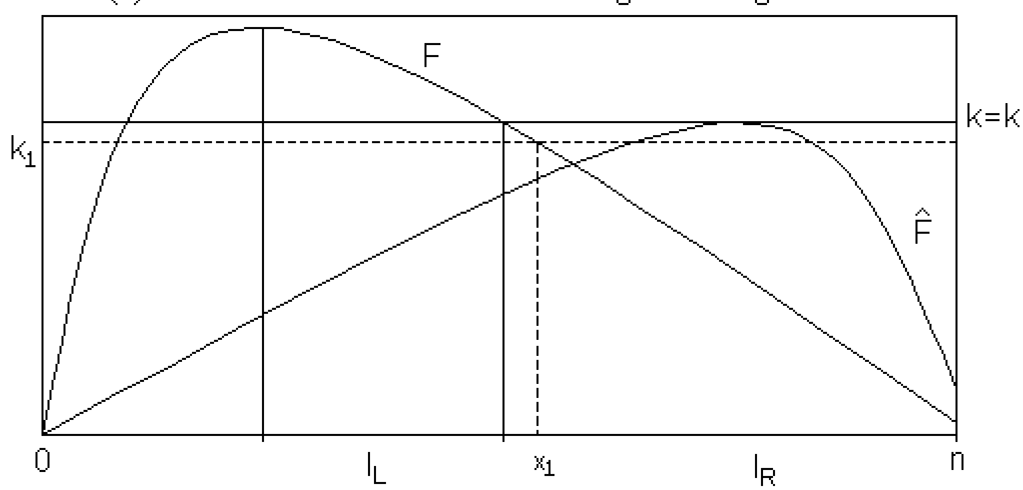
(b) If f has a fatter interior-left tail than $\widehat{f}$, then there exists a scalar k', $0 < k' < 1$, such that $C^* \geq \widehat{C}^*$ if the cost-value ratio $\frac{c}{v} > k'$.

Graph 1.2: Illustration of Fatter Interior Tails

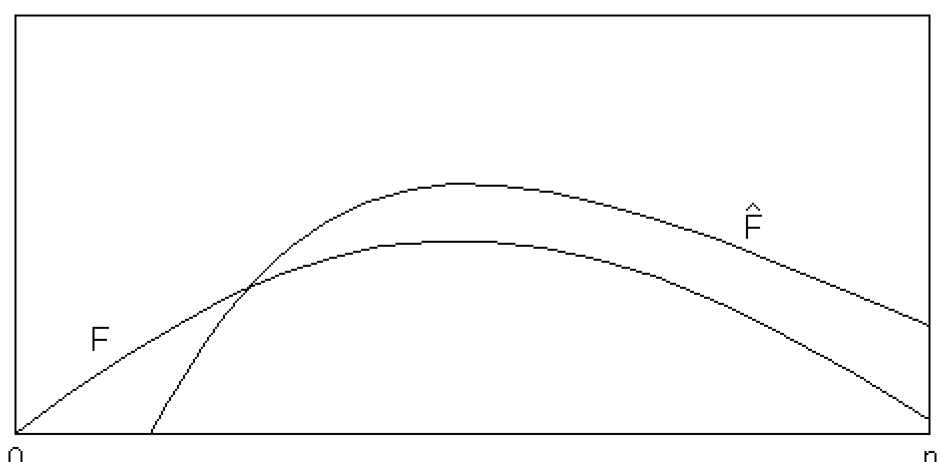
(a) Both Interior Tails Exist but do not Meet



(b) Both Interior Tails Exist under the Single-Crossing Condition



(c) One pdf is Always above the Other



Proof: (a) Suppose the contrary, that $\widehat{f}$ has the fatter interior-right tail but $C^* > \widehat{C}^*$ for some $\frac{c}{v} \leq k$. Choose $k = f(m_R)$, where m_R is the mode of $\widehat{f}$ over the fatter interior-right tail $\{x, ..., n\}$. It follows that $\frac{c}{v} \leq k$ implies that $C^* \geq x$, where x is the beginning of the fatter interior-right tail. For $C^* > \widehat{C}^*$, it must be that $f(x') \geq \frac{c}{v} > \widehat{f}(x')$ for some x', $x \leq x' \leq n$, but this contradicts the fact that $\widehat{f}$ has a fatter interior-right tail. Similar logic will show that any $k \in \left[0, \widehat{f}(m_R)\right]$ will satisfy the claim.

(b) Follows from logic similar to that used in part (a). ■

With Lemma 1 established, we can now prove Propositions 2 and 3. I first prove and comment on Proposition 2 before addressing Proposition 3.

Proof of Proposition 2: F 2nd-order stochastically dominates $\widehat{F}$ implies that $\sum_{x=0}^{x'} \widehat{F}(x) \geq \sum_{x=0}^{x'} F(x)$ for all $x' \in N$. Total feasibility and same means together imply that $\sum_{x=0}^{n} \widehat{F}(x) = \sum_{x=0}^{n} F(x)$ (see Laffont 1989). Subtracting the first condition from the second condition yields

$$\begin{aligned}\sum_{x=0}^{n} \widehat{F}(x) - \sum_{x=0}^{x'} \widehat{F}(x) &\leq \sum_{x=0}^{n} F(x) - \sum_{x=0}^{x'} F(x) \\ \sum_{x=x'+1}^{n} \widehat{F}(x) &\leq \sum_{x=x'+1}^{n} F(x).\end{aligned}$$

This last equation says that, starting from n and moving to the left on the graph of the cdf's, when F and $\widehat{F}$ first separate, $\widehat{F}$ must be below

F. This implies that $\widehat{f}$ must have a fatter interior-right tail than f. (Notice that if F and $\widehat{F}$ do not separate in interior I, then $\widehat{f}$ and f have identical interiors, which means that $\widehat{f}$ has a fatter interior-right tail by the weakness.) Since $\widehat{f}$ has a fatter interior-right tail, we use Lemma 1 to show that there exists a k that satisfies the first claim in Proposition 2.

If the feasible mode of f has mass strictly greater than the feasible mode of $\widehat{f}$ then it follows that f has a fatter interior-left tail. We use Lemma 1 to show that there exists a k' as in the second claim in Proposition 2. ■

The intuition for Proposition 2 is straightforward. The spread will push probability mass to the right part of the tail, and the total feasibility restriction means that this mass will stay in the feasible region. With more mass in the interior-right tail, the probability of being pivotal is higher for high contribution levels. This alone is not enough to ensure that contributions will be higher in the game with the spread probability. If the cost-value ratio $\frac{c}{v}$ is too high, then the mass increase on the right will not be enough, and there might be a drop in contributions. This is seen in Figure 1.2(a) when $\frac{c}{v} > k'$. If the cost is low enough (below k) then contributions are higher.

Notice that total feasibility is sufficient, but not necessary. What is necessary in this case of a mean-preserving spread is that enough mass is spread to the interior-right. In other words, all I need for the Proposition 2 is a fatter interior-right tail. Proposition 3, which I now prove, demonstrates this point (because it does not assume total feasibility) while making another claim about an implication of the single-crossing property.

Proof of Proposition 3: Denote m_F and $m_{\widehat{F}}$ the feasible modes of F and $\widehat{F}$, respectively. The claim assumes that $f(m_F) > \widehat{f}\left(m_{\widehat{F}}\right)$ and that the pdf's cross exactly once over $\{m_F, ..., n\}$. Suppose they cross at $x \in \{m_F, ..., n\}$, so that $f(x') \geq \widehat{f}(x')$ for all $m_F \leq x' < x$, and $f(x'') \leq f(x'')$ for all $x \leq x'' \leq n$. It follows then that $\widehat{f}$ has a fatter interior-right tail and f has a fatter interior-left tail. It remains to show that $k = k'$ in Lemma 1.

If $f\left(m_{\widehat{F}}\right) \geq \widehat{f}\left(m_{\widehat{F}}\right)$ then by strict quasi-concavity, f has a fatter interior-left tail from m_F to $x+1$ and $\widehat{f}$ has a fatter interior-right tail from x to n. These two tails meet each other, so $k = k'$ in Lemma 1, thus satisfying the claim. Now suppose that $f\left(m_{\widehat{F}}\right) < \widehat{f}\left(m_{\widehat{F}}\right)$. (This is akin to Graph 3.2(b).) Set $k = \widehat{f}\left(m_{\widehat{F}}\right)$, and then find where f crosses k. For any $\frac{c}{v} \leq k$, we satisfy Lemma 1(a), and for any $\frac{c}{v} > k$, we satisfy Lemma 1(b). Thus $k = k'$ in Lemma 1. ■

Proposition 3 gives a very clean result that is applicable in a wide variety of threshold distributions. For example, many monotone mean-preserving spreads will meet this single-crossing condition. Also, the class of uniform threshold distributions meets this single-crossing condition. I take advantage of this last fact in the experiments I conducted (see Chapter 2).

I now turn to discussing mixed equilibria before looking at the efficiency considerations.

1.3.2 Symmetric/mixed Equilibria

Similar logic is used in examining the mixed equilibria, but there is one important difference. While the results for pure equilibria come from looking at fatter interior tails of the probability distributions, the results for the mixed equilibria come from looking at fatter interior tails of *transformations* of the probability distributions.

For reasons given below, when looking at mixed equilibria, we can restrict our attention to symmetric equilibria. With α_i the probability that player i contributes, we now let $\alpha = \alpha_i = \alpha_j$, for all $i, j \in N$, be the rate at which every player mixes.

From (1), we see that the conditions for a symmetric equilibrium are

$$\alpha_i = \begin{cases} 0 & \text{if} \quad \Pr\left[piv|\alpha = 0, F\right] < \frac{c}{v} \\ \alpha' \in [0,1] & \text{if} \quad \Pr\left[piv|\alpha = \alpha', F\right] = \frac{c}{v} \\ 1 & \text{if} \quad \Pr\left[piv|\alpha = 1, F\right] > \frac{c}{v} \end{cases} . \tag{3}$$

The transformation of the probability distribution that we are interested in is what we will call the $\Pr[piv|F]$-curve. This curve maps the probability player i is pivotal given that all others are mixing at rate $\alpha \in [0,1]$. Figure 1.1(b) illustrates this $\Pr[piv|F]$-curve for the pdf in Figure 1.1(a). This curve is derived as follows:

$$\Pr\left[piv|\alpha, F\right] = \sum_{x=1}^{n} \binom{n-1}{x-1} \alpha^{x-1} (1-\alpha)^{n-x} f(x) .$$

With five players in the game, there are three symmetric equilibria in Figure 1.1(b): $\alpha = 0$, $\alpha = 0.32$, and $\alpha = 0.91$. From the conditions in (3), it follows that symmetric equilibria can only occur at three places in the figure: at the origin if the $\Pr[piv|F]$-curve crosses the vertical axis at a place above $\frac{c}{v}$, at a place where the $\Pr[piv|F]$-curve intersects the $\frac{c}{v}$-line, and at the $\alpha = 1$ line if the $\Pr[piv|F]$-curve crosses it above $\frac{c}{v}$. This last possibility would happen in Figure 1.1(b) if $\frac{c}{v} \leq 0.15$.

Equilibria at $\alpha = 0$ and $\alpha = 1$ have a nice stability property: an ε increase in α from 0 would drive contributions back down to zero, and an ε decrease in α from 1 would drive contributions back to one. Strictly mixing equilibria only share this property if the slope of the $\Pr[piv|F]$-curve is downward sloping where it crosses the $\frac{c}{v}$-line. In Figure 1.1(b), the 0.32 equilibrium is not stable, but the 0.91 one is stable.

Thus we see that the stable symmetric equilibria have qualitative properties similar to the pure equilibria: they occur where the distribution (whether F for pure or it's $\Pr[piv|F]$-curve transformation for symmetric) crosses the $\frac{c}{v}$-line from above. We take advantage of this fact in analyzing the symmetric equilibria.

This stability notion coincides with the concept of evolutionarily stable strategies[7] (ESS): i is at least as better off playing α than playing the perturbed strategy given that the others play α, and if i is indifferent to playing the perturbed strategy given the other play α, then i is strictly better off playing α than playing the perturbed strategy when all others play the perturbed strategy.

[7] See Gintis (2000) for a discussion of ESS.

With this concept of stability, we state Propositions 1A. This proposition is the $\Pr[piv|F]$-curve analog to Proposition 1 for symmetric/mixed equilibria but with the addition of part (0).

Proposition 1A: *(Uniqueness of Symmetric Equilibria) Fix the players N, the contribution cost c, the value of the value of the public good v, and the threshold distribution F.*

(0) If there is a stable equilibrium in which at least two players i and j are strictly mixing $\alpha_i, \alpha_j \in (0,1)$, then it must be that $\alpha_i = \alpha_j$ (generically).

(a) The unique equilibrium is $\alpha = 0$ if and only if the cost-value ratio $\frac{c}{v}$ is strictly greater than the maximum of the $\Pr[piv|F]$-curve. The unique equilibrium is $\alpha = 1$ if and only if the cost-value ratio $\frac{c}{v}$ is weakly less than $\Pr[piv|F]$-curve for all $\alpha \in [0,1]$.

(b) If the $\Pr[piv|F]$-curve is strictly quasi-concave, then there is at most one stable equilibrium with $\alpha > 0$. Furthermore, if there is more than one stable equilibrium then there are exactly two stable equilibria: one is the trivial equilibrium $\alpha = 0$, and the other is a non-trivial equilibrium with $\alpha > 0$.

(c) If the $\Pr[piv|F]$-curve is strictly quasi-concave, then any stable equilibrium with $\alpha > 0$ has α (weakly) to the right of the mode of the $\Pr[piv|F]$-curve.

Proof: (0) Suppose there is a mixed equilibrium with two players i and j such that α_i and α_j are both in $(0, 1)$. Without loss of generality let $\alpha_i < \alpha_j$. Because of the symmetry, both players strictly mixing implies that each has a probability of being pivotal equal to $\frac{c}{v}$ by (3). Since j is mixing at a higher rate than i, i's expected number of contributors other than himself must be higher than j's expected number of contributors other than himself. However, this means that i and j do not have equal probabilities of being pivotal (generically), which means that both cannot have probabilities of being pivotal equal to $\frac{c}{v}$ which is a contradiction.

(a)-(c) Follows directly from analysis similar to that used in proving Proposition 1. ■

From now on, the focus is on the non-trivial equilibria that are stable and symmetric. Part (0) provides justification for looking only at symmetric equilibria. Any strict mixers must mix at the same rate, so if a mixed equilibrium is asymmetric, the asymmetry is in who mixes and not the rate at which they mix. In fact, the mixing rate in one of these asymmetric equilibria is the mixing rate in a symmetric equilibrium of a transformed game.[8] As a result, we are examining the main strategic aspects[9] of all mixed equilibria when considering symmetric equilibria. We can justify looking at stable equilibria, too. First, the ESS concept has nice stability properties

[8] More precisely, if N_{mix} is the set of mixers and N_1 is the set of pure contributors, then the rate at which the mixers is mixing is equal to the mixing rate in game G' with $f'(x) = f(x + |N_1|)$ for all $x > 0$.

[9] The only aspect missing is the possibility of a different number of strict mixers.

that suggest that such strategies are more likely to be observed. Second, ESS can arise out of many dynamic processes which again suggests they are more likely to be observed.[10] Third, symmetric mixed ESS will exhibit comparative static properties that are qualitatively similar to the asymmetric pure equilibria thereby giving added justification to the comparative static predictions of these equilibria.

The analog to the non-trivial pure equilibrium C^* is the non-trivial stable and symmetric equilibrium α^*. Lemma 1 can be restated as Lemma 1A in terms of the fatter interior tails of the $\Pr[piv|F]$-curves.

Lemma 1A: *(Fatter Interior Tails and Symmetric Equilibria) Consider two games that are identical except for their threshold distributions F and $\widehat{F}$, $F \neq \widehat{F}$. Denote α^* and $\widehat{\alpha}^*$ their respective symmetric and stable non-trivial equilibria.*

[10]Here are two examples of dynamic processes that lead to an ESS being reached. The first example is a restricted best-response dynamic process. Suppose in period t, each player chooses a best response to the strategies of the previous period with the restriction that $BR_{i,t} \in [a_{i,t-1} - \delta, a_{i,t-1} + \delta] \subset [0,1]$. In other words, each player is restricted to making only small deviations from his previous period's strategy. Without this restriction, players would jump back and forth between contributing and not contributing, and no mixed equilibrium would be reached. In these dynamics, only ESS will be reached if the system starts out of equilibrium. The second example of a dynamic process is based on the interpretation of mixed strategies in terms of large population, random interaction models in which players only play pure strategies. Consider a large population of players that is randomly divided into n-sized groups over a long period of time. If players that receive higher payoffs reproduce at faster rates, convergence will occur to a population where a fraction of the population that contributes towards the public good will be equal to the mixed strategy equilibrium. Each of these two examples have dynamics that will lead to convergence to identical equilibria, either mixed in the first model or pure but in the mixed proportions in the second model. Equilibria that are not ESS cannot be sustained with pertubations under any evolutionary model. We therefore expect to see actual behavior fall more in line with these ESS.

(a) If the $\Pr\left[piv|\widehat{F}\right]$*-curve has a fatter interior-right tail than the* $\Pr[piv|F]$*-curve, then there exists a scalar* k, $0 < k < 1$, *such that* $\widehat{\alpha}^* \geq \alpha^*$ *if the cost-value ratio* $\frac{c}{v} \leq k$.

(b) If the $\Pr[piv|F]$*-curve has a fatter interior-left tail than the* $\Pr\left[piv|\widehat{F}\right]$*-curve, then there exists a scalar* k', $0 < k' < 1$, *such that* $\alpha^* \geq \widehat{\alpha}^*$ *if the cost-value ratio* $\frac{c}{v} > k'$.

A 2nd-order stochastic dominance relationship between threshold distributions F and $\widehat{F}$ does not necessarily imply a 2nd-order stochastic dominance between the $\Pr[piv|F]$- and $\Pr\left[piv|\widehat{F}\right]$-curves. However, that dominance relationship between F and $\widehat{F}$ will generally imply that the $\Pr\left[piv|\widehat{F}\right]$-curve has a fatter interior-right tail, and it is the fatter interior-right tail that really matters when comparing the equilibrium contribution rates. This means that there will generally exist the k that yields $\widehat{\alpha}^* \geq \alpha^*$ whenever $\frac{c}{v} \leq k$, which is the symmetric equilibrium analog to Proposition 2.

Similarly, when the feasible mode of F is strictly higher than the feasible mode of $\widehat{F}$, the mode of the $\Pr[piv|F]$-curve will often have a higher mass than the mode of the $\Pr\left[piv|\widehat{F}\right]$-curve. This will result in a fatter interior-left tail for the $\Pr[piv|F]$-curve, which is sufficient for the existence of the k' that yields $\alpha^* \geq \widehat{\alpha}^*$ whenever $\frac{c}{v} > k'$. This fact, which completes the symmetric analog to Proposition 2, follows from logic similar to that used to establish the last claim in Proposition 2.

1.3.3 Efficiency

We are interested in the efficiency of equilibria when there is threshold uncertainty and also in the comparative efficiency of equilibria under different threshold distributions. The welfare criterion used here is the sum of expected utilities

$$W(C) = nF(C)\,v - Cc.$$

An increase in contributions does not imply an increase in expected welfare. For an increase in welfare to follow from an increase in contributions, we must have a sufficient increase in the probability of provision. Before coming back to the discussion of welfare changes, we will first discuss Proposition 4, which makes some basic claims about efficiency.

Proposition 4: *(Efficiency) Fix the players N, the contribution cost c, the value of the value of the public good v, and the threshold distribution F. Assume F is strictly quasi-concave.*

(a) The non-trivial pure equilibrium C^* is the Pareto-undominated equilibrium in the class of pure equilibria. This equilibrium C^*, $0 \leq C^* < n$, is inefficient when $\frac{c}{vn} < f(C^* + 1) < \frac{c}{v}$.

(b) The symmetric and stable non-trivial equilibrium α^* is generically inefficient, but it can yield higher expected welfare than the non-trivial pure equilibrium C^*.

Proof: (a) Suppose the contrary, that there exists some equilibrium $C < C^*$, where $nF(C)v - Cc > nF(C^*)v - C^*c$. Some algebra yields $\frac{n(F(C^*)-F(C))}{C^*-C} < \frac{c}{v}$. This inequality is a contradiction for all values of the LHS. In particular, since $f(C^*) \geq \frac{c}{v}$, the lowest the LHS can be is $\frac{n\left(\frac{c}{v}\right)}{C^*}$, so $\frac{n\left(\frac{c}{v}\right)}{C^*} < \frac{c}{v}$ implies $n < C^*$ which is a contradiction. It must similarly be a contradiction for any LHS greater than $\frac{n\left(\frac{c}{v}\right)}{C^*}$.

We prove the case for when $C^* \in \{1, ..., n-1\}$. By Proposition 1, C^* is to the right of the feasible mode. By strict quasi-concavity, $f(C^*) \geq \frac{c}{v} > f(C^*+1) \geq f(C^*+k)$ for all $1 < k \leq n - C^*$. This means that the largest marginal welfare gain to be had by an increase in one contribution is from C^* to C^*+1. Welfare is higher under C^*+1 when $W(C^*+1) > W(C^*)$. Doing the algebra shows this to be equivalent to $f(C^*+1) > \frac{c}{vn}$. It follows that the C^* is inefficient when $\frac{c}{vn} < f(C^*+1) < \frac{c}{v}$.

(b) That mixed equilibria are generically inefficient is trivial. That the symmetric equilibrium can yield higher expected welfare than the pure equilibrium is illustrated by an example. Suppose $n = 5$, $f(1) = 0.54$, $f(2) = 0.13$, $f(3) = 0.12$, $f(4) = 0.11$, $f(5) = 0.10$, and $\frac{c}{v} = 0.14$. Then we can find that $C^* = 1$, $\alpha^* \simeq 0.5$, $W(C^*) = 18.29$, and $W(\alpha^*) \simeq 23$. ■

As is common in public good games, inefficiencies arise because the marginal gain to an individual from contributing is different from the marginal social gain from that same contribution. This difference comes from the welfare function accounting for all players' marginal gain's instead of just one individual's marginal gain. This inefficiency does not arise when $f(C^*+1) < \frac{c}{v}$, $C^* = n$, or when $F(C^*) = 1$. Notice that this implies that the non-trivial equilibrium C^* is efficient when the threshold is known with certainty—a fact already established by Palfrey and Rosenthal (1984). Their result is thus a special case of the more general result in Proposition 4(a).

As shown in the example in the proof of part (b), the symmetric equilibrium can have higher expected welfare when the pdf has a tail to the right of C^* that is close to but under $\frac{c}{v}$. In this example, the efficient outcome is $C^* = n$ (or $\alpha^* = 1$), and each additional contribution increases welfare. In this example, the symmetric equilibrium has higher welfare because expected contributions are higher, and this increase in contributions more than offsets the decline in welfare due to greater total contribution cost.

Because contributions can increase due to an increase in uncertainty (i.e., by 2nd-order stochastic dominance), it turns out that welfare can be higher under an increase in uncertainty. Again, suppose that the initial distribution has a right tail above C^* that below $\frac{c}{v}$ but above $\frac{c}{vn}$ from C^*+1 to n or close to n. A widening of uncertainty that drives up the right tail will increase contributions, and if the increase in the probability of provision is sufficient then there will be an increase in expected welfare.

1.4 Other Modeling Considerations

This section discusses the robustness of the main theoretical findings to alterations in certain assumptions made in the model. First, we consider allowing for threshold distributions that are not strictly quasi-concave. Second, we consider allowing individuals to make contributions from a continuous contribution set. Third, we briefly consider how the analysis changes when there is risk aversion. Finally, we briefly discuss sequential contributions.

General Threshold Distributions. I have worked out the analogs to the main claims for when the threshold distributions are not restricted to be strictly quasi-concave. The added complication when the distribution is not strictly quasi-concave is that there might be more than one non-trivial equilibrium. The way around this complication is to look at the equilibrium with the highest level of expected contributions. Doing so allows us to do the same analysis as before on this high-contribution equilibrium. For pure equilibria, this high-contribution equilibrium is the Pareto-undominated equilibrium, and Lemma 1 and Propositions 2 and 3 can be restated exactly word for word substituting only "high-contribution equilibrium" in place of "non-trivial equilibrium." The analysis will also be similar for symmetric equilibria.

Continuous Contributions. Nitzan and Romano (1990) allow individuals to make continuous contributions. These continuous contributions can be likened to monetary contributions, whereas binary contributions can be likened to participation decisions. Leaving the binary case means we must consider the underlying threshold distribution. Proposition 5 assumes that the underlying threshold distribution is a continuous and strictly quasi-concave threshold distribution function H from which the discrete transformations F and F' are derived so as to assign mass over A_i and A_i', respectively.

> **Proposition 5:** *(Continuous Contributions, partly from Nitzan and Romano (1990)) Consider two games that are identical except for their contribution sets A_i and A_i'. Assume that the threshold distribution function H is continuous. Suppose binary contributions $A_i = \{0, 1\}$ for all i in the first game, and assume continuous contributions $A_i' = [0, 1]$ for all i in the second game. Then expected welfare is always (weakly) higher in the second game with continuous contributions.*

The proof of this proposition combines a result from Nitzan and Romano (1990) with my Proposition 4. Nitzan and Romano (1990) show that in continuous contribution games we need to consider the maximum number of contributions that players can make. Because we restrict players to contribute at most 1, the total number of contributions players can make is n. In their notation, H is continuous over $[a, b]$, $0 < a < n$. If $n \geq b$ in the continuous contribution game, then $C^* = b$,[11] the public good is provided with probability 1, and the equilibrium is efficient. If

[11] Here C^* is the total contribution but not necessarily the number of contributors because C^* is not necessarily an integer.

$n < b$, then $C^* = n$ and the public good is provided with probability strictly less than 1. My Proposition 4 says that things can be very different when contributions are binary. First, if the public good is totally feasible ($n \geq b$) then the equilibrium is only efficient if $f(b) \geq \frac{c}{v}$. Second, if the public good is not totally feasible, then equilibrium is only efficient if $f(n) \geq \frac{c}{v}$. Put succinctly, the equilibrium under continuous contributions will always be efficient, but the equilibrium is not always efficient under binary contributions.

The logic is straightforward. Consider a proposed equilibrium $C < b$. In the binary case, a non-contributor considers if $f(C+1)$ is greater than $\frac{c}{v}$. Now in the continuous case, if the player considers a $\frac{1}{2}$ contribution, then he compares $f\left(C + \frac{1}{2}\right)$ with $\frac{c}{2v}$ in his decision rule. More generally, it can be shown that for a $\frac{1}{2m}$ contribution, the player's decision rule will compare $f\left(C + \frac{1}{2m}\right)$ with $\frac{c}{2mv}$. As $m \to \infty$, the $\frac{c}{2mv}$-line converges to 0 and $f\left(C + \frac{1}{2m}\right)$ converges to $f(C)$. In the limit, the player he will contribute an ε amount whenever $f(C) > 0$. Thus we see that contributions will cover the whole feasible domain of the threshold distribution. For symmetric equilibria, similar reasoning will show that with continuous contributions, $\alpha^* = \frac{b}{n}$ when $n \geq b$.

This logic implies that wider threshold uncertainty can only decrease efficiency for the continuous contribution case (when b increases past n), while wider uncertainty can increase efficiency under binary contributions (even if b goes past n). While this is a strikingly different result, the underlying behavior and analysis in each case is the same. The difference lies in the fact that we do not always have complete provision of the public good in the binary case due to the $\frac{c}{v}$-line above the horizontal axis.

Risk Aversion. If the players are risk averse then the free-rider effect (the worry about donating a redundant contribution) is relatively diminished while the lost-cause effect (the worry about contributing to a lost-cause) is relatively enlarged. Risk aversion will likely be present in the experiments, and will be discussed later. For now, I make the point that a qualitative result similar to Lemma 1 will hold, but there is an important difference. The decision rule (1) will not compare $\Pr[piv|a_{-i}, F]$ with $\frac{c}{v}$. Instead of drawing a horizontal $\frac{c}{v}$-line, there will be a "$\frac{c}{v}$-curve" that varies by contribution level. On the graph of the pdf, this curve will be decreasing over the domain of contribution levels with $f(x) > 0$, and its slope and shape will depend on the size of the risk aversion. Under extreme amounts of risk aversion, the slope becomes more negative and the whole curve shifts up. With a change in uncertainty from F to $\widehat{F}$, the curve will also change. For the analog to Lemma 1(a), our definition of the fatter interior-right tail will have to consider not just the comparison pdf's but also the comparison of these curves.

Sequential Contributions. Since there is no private information in this game, there are not the normal information issues involved in comparing simultaneous and sequential equilibria. Sequential moves in this game only allow players to condition on observed behavior. This observation will matter when comparing a mixed equilibrium to a sequential equilibrium, but any pure equilibrium of a the simultaneous game is an equilibrium of any sequential move game.[12] The main results from Section 3.1 will thus still apply in the sequential move game. Hence, the focus on simultaneous contributions in this paper is not missing other important strategic issues (other than timing) that would arise in a sequential move game.

[12] Dekel and Piccione (2000) have a similar finding for voting games in symmetric binary elections.

CHAPTER 2

THRESHOLD UNCERTAINTY IN DISCRETE PUBLIC GOOD EXPERIMENTS

2.1 INTRODUCTION

This chapter presents new data from laboratory experiments specially designed to test the main theoretical predictions of the model presented in the first chapter. The main predictions of interest are that an increase in threshold uncertainty will increase contributions when the value of the public good is sufficiently high, and that an increase in threshold uncertainty will decrease contributions when the value of the public good is sufficiently low. My experimental data verify these qualitative predictions, and they also support the underlying behavioral logic of the model.

In these experiments, subjects play multiple rounds of a public good game in which they know all relevant information except the actual threshold. After being told the distribution from which the threshold will be randomly selected, the subjects privately make contributions. To induce proper incentives, subjects receive payments according to the actual outcomes of play. As predicted, after an increase in uncertainty, contribution rates increase (in later rounds) when the value of the public good is high, and contribution rates decrease when the value of the public good is low. I also elicit the subjects' beliefs about other players' contribution decisions using a proper scoring rule. These reported beliefs show evidence of learning. Using the reported beliefs to proxy for true beliefs, I conduct parametric and non-parametric analysis to study whether the subjects' behavior is consistent with the behavioral assumptions of the model. Although behavior is inconsistent with expected payoff maximization, the subjects' behavior becomes more consistent with a game-theoretic decision rule once when I allow for both risk-aversion and innate cooperativeness.

Section 2.2 describes the experiment design in detail. Section 2.3 presents five main preliminary results from an initial analysis of the data. In these two sections, I explain why the experiment design is correct for testing the model's predictions and what potential difficulties are to be expected in verifying those predictions. Section 2.4 summarizes the experimental findings and discusses insights into collective action that follow from the findings. In particular, there can be a status-quo bias towards wide threshold uncertainty, and communication is likely to be vital for successful collective action in one-shot provision games. Section 2.5 is an appendix that contains the dialogue of the experiment.

2.2 EXPERIMENT DESIGN

Experiments were conducted at the California Social Science Experimental Laboratory (CASSEL) at the University of California—Los Angeles (UCLA). Subjects were drawn from the UCLA summer 2001 student population. Each experiment session consisted of 4 practice rounds and 30 real rounds (the exception being the experiment on 8/21 which ended after 26 real rounds), and each session had either 25 or 30 students. All decisions were made over a computer network in a computer currency called "tokens." Subjects amassed tokens depending on the decisions and the factors determined by the computer. At the end of the session, subjects were paid U.S. dollars according to a pre-announced token/dollar exchange rate. The upper half of Table 2.1 lists basic session information and some basic session statistics.

The dialogue from the instructional period is included in the appendix (Section 2.5) of this chapter. After an instructional period, the students participated in the practice rounds to become familiar with the computer interface. In each round, the computer randomly and anonymously assigned the subjects in the room into groups of five, and each student was then given one computer token. Each subject's computer then displayed the public good value and the threshold range. Subjects were told that the threshold range is a range $\left\{\underline{t}, ..., \overline{t}\right\}$ from which the computer will randomly and uniformly select the true threshold. Subjects are told that all displayed information is the same for all individuals and groups in the room.

Before deciding whether to keep (do not contribute) or spend (contribute) the one given token towards the public good, each subject is asked to assign percentage probabilities to what the others in his or her group will do.[1] Since five students are in each group, a student assigns probabilities to the following five events: exactly 0 others in the group spend, exactly 1 other spends, exactly 2 others spend, exactly 3 others spend, and exactly 4 others spend. Once the assigned percentages add up to 100%, the student then makes the decision to keep or spend his or her token. Subjects are not allowed to communicate with any other subjects in the room during the practice or real rounds.

[1]See Nyarko and Schotter (2000) for discussion of the validity of using elicited beliefs.

Table 2.1: Experiment Session Descriptions (8/21-9/7)

	Session 1	2	3	4	5
(a) Session Information					
Date	8/21	8/22	8/29	9/6	9/7
Number of Subjects	30	25	25	30	25
Exchange Rate (tokens/dollar)	15	15	7.5	15	15
Parameter order	ordered	ordered	ordered	ordered	random
Threshold range order of uncertainty	increasing	increasing	increasing	decreasing	random
Parameters					
Rounds, Obs. by Parameter Profile					
6, {3}	15, 450	15, 375		15, 450	10, 250
6, {2,3,4}		15, 375		15, 450	10, 250
6, {1,2,3,4,5}	11, 330				10, 250
3, {3}			15, 375		
3, {1,2,3,4,5}			15, 375		
(b) Basic Statistics by Session					
(i) Sym. Expected Payoff Max Eqbm.					
6, {3}	**0.79**	**0.79**		**0.79**	**0.79**
6, {2,3,4}		**0.84**		**0.84**	**0.84**
6, {1,2,3,4,5}	**1.00**				**1.00**
3, {3}			**0.62**		
3, {1,2,3,4,5}			**0.00**		
(ii) % Contributed Overall					
6, {3}	73.1%	62.7%		67.3%	70.0%
6, {2,3,4}		65.3%		73.1%	70.8%
6, {1,2,3,4,5}	70.0%				67.2%
3, {3}			53.9%		
3, {1,2,3,4,5}			49.9%		
(iii) % Contributed Rounds 8+					
6, {3}	**73.3%**	**60.5%**		**67.5%**	**74.7%**
6, {2,3,4}		**65.5%**		**72.1%**	**73.3%**
6, {1,2,3,4,5}	**75.0%**				**66.7%**
3, {3}			**53.5%**		
3, {1,2,3,4,5}			**46.0%**		
t-statistic for H_0 of no difference	**-0.34**	**-1.04**	**1.50**	**-1.09**	
p-value (one-sided)	**0.367**	**0.149**	**0.067**	**0.138**	
(iv) % Public Goods Provided Rounds 8+					
6, {3}	85.4%	75.0%		77.1%	93.3%
6, {2,3,4}		70.0%		81.3%	100.0%
6, {1,2,3,4,5}	100.0%				86.7%
3, {3}			52.5%		
3, {1,2,3,4,5}			42.5%		
(c) Movement of Reported Beliefs by Type					
Beliefs avg. moved toward last actual	77.4%	75.3%	75.1%	78.9%	64.9%
Beliefs last actual (weakly) increased	82.9%	85.3%	79.1%	85.2%	73.2%
Avg. Beliefs Error	-0.060	-0.017	0.138	-0.053	-0.140

Source: Experiment sessions conducted by author at CASSEL in the summer of 2001.

A subject's payment for a given round has two parts: the payment from spending or keeping and meeting or not meeting the threshold, and the payment based on the accuracy of the reported beliefs. The first payment is described in the matrix

	$C_{-i} < t-1$	$C_{-i} = t-1$	$C_{-i} > t-1$
spend	0	v	v
keep	1	1	$v+1$

,

where C_{-i} is the contributions made by others in the group, and t is the true threshold chosen by the computer from the threshold range $\{\underline{t}, ..., \overline{t}\}$. A proper scoring rule was used to elicit true beliefs. Let $bpay_t$ be the payment for that round t's beliefs report, let $\frac{v}{2}$ be the amount for a perfect reporting, let $b_{it}(e)$ be the probability assigned by i in round t to event that exactly e others in his group contribute, and let $actual_t$ be what was actually contributed by others in the group in t:

$$bpay_t = \frac{v}{2}\left(b_{it}(actual_t) - \sum_{e=0}^{4} b_{it}(e)\right) - \frac{v}{4}.$$

The lowest possible total payment for any given round is zero tokens while the highest is $v + 1 + \frac{v}{2}$.

As seen in Table 2.1, the sessions could vary in three ways: the parameters (v and $\{\underline{t}, ..., \overline{t}\}$), the order in which the parameters changed from round to round, and the direction of change in uncertainty. The five different parameter settings allow for high v or low v, 6 or 3, respectively, and for varying levels of uncertainty, $\{3\}$, $\{2, 3, 4\}$, $\{1, 2, 3, 4, 5\}$. The order of parameters can be *ordered* or *random*. Ordered means that the parameters were the same for 15 rounds, and then the parameters changed in the 16th round but were the same thereafter. For example, in Session

2, $v = 6$ and $\{\underline{t}, ..., \bar{t}\} = \{3\}$ in rounds 1-15, but thereafter the parameters were $v = 6$ and $\{\underline{t}, ..., \bar{t}\} = \{2, 3, 4\}$. The direction of change in uncertainty can be either *increasing*, *decreasing*, or *random*. Increasing uncertainty is illustrated in Session 2 where the threshold range is smaller in the early rounds and larger in the later rounds. Decreasing uncertainty is illustrated in Session 4, where rounds 1-15 were under range $\{2, 3, 4\}$ and rounds 16-30 were under range $\{3\}$. Random uncertainty means that the range varied randomly from round to round.

This design is the correct design to test the effects of threshold uncertainty on contributions in public good games. Here are some of the main justifications for the set-up. (1) This set-up only differs from standard public good games in two ways: the unknown threshold and the eliciting of beliefs. Keeping the set-up similar to established experimental procedure allows for comparison with other results from other public good experiments. (2) The uniform threshold range $\{\underline{t}, ..., \bar{t}\}$ is the best way to model the threshold distribution because subjects can easily understand a uniform distribution. The uniform threshold range also implies single-crossing for both pure and symmetric equilibria, and this single-crossing implies nice qualitative predictions of contribution movements with changes in uncertainty (Proposition 3 from Chapter 1). (3) The chosen parameters profiles will allow for high and low v and for high and low uncertainty. Data for all these scenarios are needed to compare with the predictions. (4) Since there are legitimate reasons to expect the results to not perfectly match the predictions (see below), eliciting beliefs will allow for more direct testing of the underlying behavior of the subjects. Providing incentives to report true beliefs adds credibility to the beliefs data. (5) Groups always have exactly five students so that we can ignore the effects of group size. (6) No communication is allowed so that there are no social pressures or social comparisons that might affect behavior. (7) The maximum payment for beliefs is half as much as the payment from the keep/spend decision. This should remove the motive for players to play a game that maximizes the beliefs payment.

It also worth noting why the laboratory is the correct place to test the predictions. The experiments are unique because I specifically control for both levels of and changes in threshold uncertainty. Because the threshold uncertainty can be controlled so precisely in the laboratory, the laboratory is the ideal place to conduct the first test of the theoretical predictions.

Although this study is not about learning, we expect the subjects to be engaged in some form of learning, and this reality must be considered in interpreting results. For this reason, a number of rounds are done for a given set of parameters. Since the predictions are equilibrium predictions, looking at the later rounds for a given parameter setting is likely to be more appropriate for assessing the validity of the predictions. On the same note, varying the order in which subjects experience parameters allows us to compare how learning might be affected by the ordering. This is important since if learning is slowed then close-to-equilibrium behavior might not be reached in the rounds for which we have data.

Careful wording was used during the experiment. Words like “game,” “contribute,” “win,” “lose,” “reward,” and “punishment” are not used when speaking to the subjects, since such words carry subtle meanings that can affect behavior. Instead, words and phrases such as "decision making environment," "keep," "spend," and "payment" were used.

2.3 RESULTS

The literature on past experiments suggests that we should be concerned about uncontrollable factors that can lead to the non-verification of the theory. While presenting the five main preliminary results, I discuss the relevance of many of these concerns.[2] I organize the five results into three categories: results about contribution changes, results about the movements of reported beliefs, and results about the consistency of players actions with their beliefs. The first two results are concerned with verification of the theoretical predictions. The third result demonstrates the meaningfulness of the elicited beliefs data. The fourth and fifth results describe in what manner the subjects' behavior matches the behavioral assumptions of the game-theoretic model.

[2]Consider the following possible reasons for non-verification of the predictions. (1) Subjects might have extreme attitudes towards risk that lead to qualitative movements in contributions that differ from what is predicted. (2) Convergence to equilibrium might not have yet occurred in the limited rounds of the experiment. Since the analysis examines equilibrium behavior, if the players are far from equilibrium then the behavior might not coincide with the predicted equilibrium behavior. (3) Some subjects might have other qualitative differences like innate tendencies to cooperate or not cooperate, and a small session might have such players in proportions not equal to true population proportions. Such differences across sessions can make comparisons of contribution levels across sessions problematic. (4) Subjects might be playing more complicated strategies than a simple single-period best-response decision rule. (5) Subjects might not fully understand the game.

When presenting the results, I discuss the relevance of concerns (1)-(4). As for concern (5), I asked the subjects informal questions after the experiment to assess their understanding of the experiment's environment. It is impossible to show that all subjects had a perfect understand, but the subjects' answers to the informal questions indicated general understanding of the game.

2.3.1 Contribution Changes

Result 1: *Qualitative predictions are moderately verified for in-session uncertainty changes.*

Contribution percentages are listed in Table 2.1(b). The numbers listed as the Expected Payoff Maximizing Equilibrium are computed contribution rates of the symmetric equilibrium in the game under the given parameters with expected payoff maximizing players. As seen in the table, $v = 6$ is sufficiently high to lead to higher contributions when the threshold range is increased from $\{3\}$ to $\{2, 3, 4\}$ and also to $\{1, 2, 3, 4, 5\}$, and $v = 3$ is sufficiently low to decrease contributions when the range goes from $\{3\}$ to $\{1, 2, 3, 4, 5\}$. Contribution movements would be similar for pure equilibria. The symmetric equilibria are listed only to illustrate the expected direction of change in contributions.

Looking at part (ii), we see that contribution movements match the predictions for Sessions 2-4 but not for Sessions 1 and 5. As mentioned above, because of potential learning effects, it is best to consider the contributions from the later rounds. Part (iii) lists the contributions after round 7 (Rounds 8+) in each session. The qualitative prediction is verified within the ordered parameter sessions (Sessions 1-4), but the statistical analysis is not totally conclusive.

For Sessions 1, 2, and 4, I test the null hypothesis that contributions under range $\{3\}$ are greater than contributions under wider uncertainty. The t-statistic for testing the equality of two means p_x and p_y is

$$Z = \frac{\widehat{p}_x - \widehat{p}_y}{\sqrt{p_0 \left(1 - p_0\right) \left(\frac{n_x + n_y}{n_x n_y}\right)}},$$

where $p_0 = \frac{p_x n_x + p_y n_y}{n_x + n_y}$ (Newbold 1995, 360). I want to reject this null hypothesis, and corresponding t-statistics and p-values are mixed. Risk aversion can possibly account for the very low power in Session 1 since 6 might not be sufficiently high to induce higher contributions in the $\{1, 2, 3, 4, 5\}$ range under extreme risk aversion.[3] The t-statistics for Sessions 2 and 4 give us much more confidence, and they should even in the face of risk aversion. For Session 4, I test the null hypothesis that contributions are lower under less uncertainty, and the corresponding t-statistic again gives moderate support for rejecting this null.

Two points are worth mentioning here. Since there exist many possible equilibria (asymmetric mixed, symmetric mixed, pure), nothing can be said about which equilibrium is played. However, since the change in contributions for any of the equilibria (whether asymmetric, symmetric, or pure) should move up if $v = 6$ (and down if $v = 3$) for expected payoff maximizers, we do not need to specify which equilibrium might be being played. As such, trying to compare the observed behavior with the quantitative prediction of a particular equilibrium is problematic, and such comparisons should not be a standard to use to test the validity of the theory. Instead, comparing qualitative movements in contributions is a better way to assess the theoretical predictions than trying to match observed behavior with

[3] See Section 2.2.3 for more discussion on risk aversion.

certain quantitative predictions because this avoids the problem of selecting which equilibrium to quantitatively test.

Another point is that the predictions do not hold in Session 5. As will be discussed below (see Result 3), there is evidence that the random ordering of the parameters slows down the learning which then in turn prevents behavior from approaching the Nash equilibrium behavior. This suggests that the ordered parameter sessions offer the best test of the theory which assumes correct beliefs. Although this apparent non-verification has a possible explanation that does not invalidate the theory, it does illustrate the sensitivity of the result to the environment (more on this below).

Result 2: *Comparisons across sessions are problematic, so comparisons within session provide a better test of the theory.*

Notice that the 65.3% contributions for range $\{2, 3, 4\}$ in Session 2 is lower than the 73.3% contributions for range $\{3\}$ in Session 1, which the qualitative prediction that contributions in range $\{2, 3, 4\}$ should be higher than contributions under range $\{3\}$. Evidence suggests there are important differences in subjects across sessions (see Result 5). For example, one session might consist of many risk-lovers, while another session might consist of extreme risk averters. Another example could be that subjects in one session have stronger innate tendencies to cooperate. The presence of these differences across sessions suggests that looking at within-session contribution changes is most appropriate for testing the theory because it would hold fixed changes in unobservable subject characteristics (Camerer 1995, 633).

2.3.2 Movements of Reported Beliefs

Result 3: *The movements of reported beliefs respond to past history, thereby suggesting the beliefs data are meaningful.*

Denote i's mean beliefs in t to be $\overline{b_{it}} = \sum_{e=0}^{n-1} eb_{it}(e)$. We ask if $\overline{b_{it}} - actual_{t-1} \leq \overline{b_{it-1}} - actual_{t-1}$. If yes then the average of i's reported beliefs for round t moved towards what actually happened in round $t-1$. Such movement might be expected if the subjects are using a beliefs-updating mechanism. Another belief movement that can be expected is if the probability assigned to what happened in the previous round is increased in the current round. So we also consider if $b_{it}(actual_{t-1}) \geq b_{it-1}(actual_{t-1})$.

Table 2.1(c) lists the percent of belief movements that coincide with these two notions. The first round of a particular parameter profile is not included in the calculation of this percentage. For example, round 16 in Session 1 is the first round of range $\{1, 2, 3, 4, 5\}$, so comparing what is believed in that round with what happened under range $\{3\}$ in round 15 is not useful. For Sessions 1-4, movements of the beliefs average toward the last actual event occurs at least 75% of the time, while movements of the absolute reported belief occur about 80% or more. These numbers strongly suggest that an individual updates his beliefs while learning about the behavior of the other subjects. These numbers also suggest that the data on beliefs do have meaningful information.

Because Session 5 has random parameter ordering, the calculation of these percentages was done by ordering the data by parameter profile and then by round. So the percentage in Session 5 is for comparing what is believed in round t with the actual event from the last time that parameter profile occurred. The movements occur substantially less in Session 5, and this suggests that the beliefs updating and learning occurs at a slower rate in this session. The slower rate is probably due to the fact that the subjects are learning about three different parameter settings almost simultaneously, and this will slow down their learning. Because of this, we suspect that the contribution rates in Session 5 might still be far from equilibrium contribution rates. Therefore, Session 5 is not the best session with which to test a prediction based on Nash equilibrium behavior.

Below these percentages in Table 2.1 is the average difference between the mean beliefs and the actual event for each session:

$$\frac{1}{(rounds)\,(subjects)}\sum_{t=1}^{rounds}\sum_{i=1}^{subjects}\left(\overline{b_{it}}-actual_t\right).$$

It in interesting that this number is an order of magnitude higher in Sessions 3 and 5 than in Sessions 1, 2, and 4. This is interesting because the large errors corresponds with the largest deviations from the expected payoff symmetric equilibria. Under $v = 3$ and range $\{1,2,3,4,5\}$ in Session 3 we should really expect very low contributions according to the model (unless, of course there are other unobservable characteristics), yet contributions are still almost 50%. Contribution movements also deviate significantly from the expected payoff symmetric equilibria. In fact, the qualitiative movements do not even match the prediction.

Table 2.2 lists results from regressions of mean beliefs $\overline{b_{it}}$ on control variables. Again, data from the first round of a given parameter profile is dropped because lags of the dependent variable are control variables. A simple OLS regression of $\overline{b_{it}}$ on $\overline{b_{it-1}}$, $\left(actual_{t-1} - \overline{b_{it-1}}\right)$, and $\left(actual_t - \overline{b_{it-1}}\right)^2$ gives the results shown in the table. OLS does not account for possible autocorrelation, and the standard Durbin-Watson test indicates the presence of negative autocorrelation, as evidenced by a test statistic significantly different than 2. Autocorrelation is detected even though this statistic should be biased towards 2 because of the lagged dependent variable. The Durbin h test that accounts for the lagged dependent variable also indicates negative autocorrelation.[4]

Results from two different 1st-degree autoregressions are in Table 2.2. The first AR(1) gives results similar to the OLS results. As expected, the current mean depends on the previous mean, the difference between the last actual and last mean, and that difference squared. The signs on all of these are expected. The coefficient on the squared difference indicates a second order effect due to larger differences. The second AR(1) includes more control variables that try to capture how beliefs-updating might be different in later rounds of a particular parameter profile. The R^2 values over 50% indicate that a significant amount of the mean beliefs can be explained by the regressors used.

[4]See Chapter 13 in Greene (1997) for a discussion of autocorrelation and autocorrelation tests.

Table 2.2: Beliefs Data Regression Results

Regression	OLS	1st AR(1)	2nd AR(1)
Estimates			
intercept	0.4897 (0.0346)	0.4147 (0.0332)	0.4996 (0.0743)
mean belief $_{t-1}$	0.8066 (0.0127)	0.8357 (0.0122)	0.7905 (0.0271)
actual $_{t-1}$ - mean belief $_{t-1}$	0.1507 (0.0070)	0.1510 (0.0069)	0.2378 (0.0155)
(actual $_{t-1}$ - mean belief $_{t-1}$)^2	0.0116 (0.0044)	0.0116 (0.0043)	0.0170 (0.0092)
Parameter Round	--	--	-0.0125 (0.0085)
(mean belief $_{t-1}$)*(Parameter Round)	--	--	-0.0065 (0.0031)
(actual $_{t-1}$ - mean belief $_{t-1}$)*(Parameter Round)	--	--	-0.0116 (0.0018)
(actual $_{t-1}$ - mean belief $_{t-1}$)^2*(Parameter Round)	--	--	-0.0013 (0.0011)
R^2	0.5345	0.5376	0.5464
Durbin-Watson	2.1338	--	--
Durbin-h	-4.5605	--	--

Source: Experiment sessions conducted by author at CASSEL in the summer 2001.

2.3.3 Consistency of Actions with Reported Beliefs

The beliefs movements in Table 2.1(c) and the regression results in Table 2.2 suggest that the beliefs data do carry information. This likelihood justifies using the beliefs data to proxy for true beliefs. Results 4 and 5 describe how the subjects' decisions are or are not consistent with game-theoretic decision rules.

Result 4: *Actions are not consistent with expected payoff maximization.*

As shown in Section 3, expected payoff (EP) maximization yields the following EP decision rule: spend if $\Pr[piv|b_{it}, F] > \frac{c}{v}$; keep or spend if $\Pr[piv|b_{it}, F] = \frac{c}{v}$, keep if $\Pr[piv|b_{it}, F] < \frac{c}{v}$. Figure 2.1(a) contains non-parametric fits of the EP decision rule using the subjects decisions and reported beliefs. I use the Epanechnikov kernel in the Nadaray-Watson kernel estimator under three different smoothing bandwidth parameters $h = 0.025$, 0.1, and 0.15 (Härdle 1990). Denoting $x = \Pr[piv|b_{it}, F] - \frac{c}{v}$, this estimator is

$$m_h(X_i, h) = \frac{\frac{1}{(h)(\#obs)} \sum_{obs} \frac{3}{4} \left(1 - \left(\frac{x_{obs} - X_i}{h}\right)^2\right) I\left(\frac{x - X_i}{h} \leq 1\right) a_{obs}}{\frac{1}{(h)(\#obs)} \sum_{obs} \frac{3}{4} \left(1 - \left(\frac{x_{obs} - X_i}{h}\right)^2\right) I\left(\frac{x - X_i}{h} \leq 1\right)}.$$

The curve labeled "EP Perfect" is only for comparison. That curve depicts what we would expect the estimate to look like if the behavior is consistent with EP maximization. As we should expect, the estimated curves have positive slopes which suggests that a subject is more likely to increase the higher the probability of being pivotal. However, the curves are generally above 0.5—even for negative differences—which suggests that subjects contribute much more often than predicted by EP maximization. The curves for the 8+ rounds (not shown) are only marginally closer to EP maximization. Figure 2.1(b) plots approximate 95% confidence intervals[5] around the estimates using bandwidth $h = 0.1$. As seen on the graph, we must go well outside the confidence interval to get consistencyd, which supports the rejection of consistency of actions with EP maximization. The evidence also suggests a bias towards contributing.

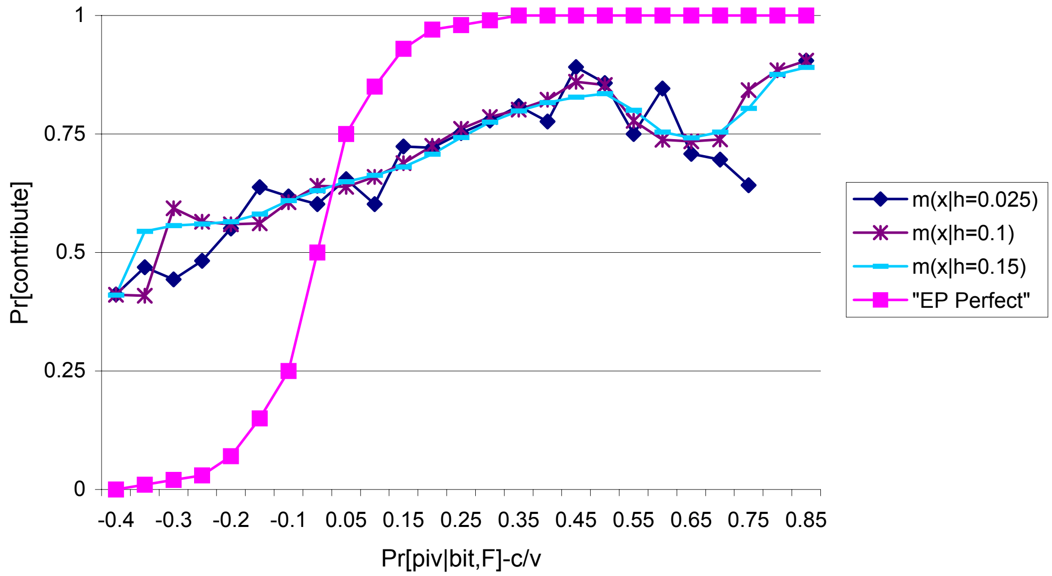
Table 2.3(a) shows a breakdown of decisions using the reported beliefs as a proxy for true beliefs and assuming the players are expected payoff maximizers. "Should Go & Did Go" means that the reported beliefs yield $\Pr[piv|b_{it}, F] > \frac{c}{v}$ and the player did spend his token. "Indifferent" means that $\Pr[piv|b_{it}, F] = \frac{c}{v}$, so that both keeping and spending are optimal. As seen in Table 2.3(a), when $v = 6$ (Sessions 1, 2, 4, 5) about 60-69% of the decisions are consistent with the EP decision rule. When $v = 3$ (Session 3), about 55% of the decisions are EP consistent. Further breaking down of these decisions into earlier and later rounds (not in table) reveals that in each session more decisions in rounds 8+ are EP consistent than decisions in rounds 1-7 for all Sessions, although the increased percentage is not very large.

The percentages in Table 2.3(a) suggest that the EP decision rule, although not a perfect approximation, might still be a decent first approximation for the

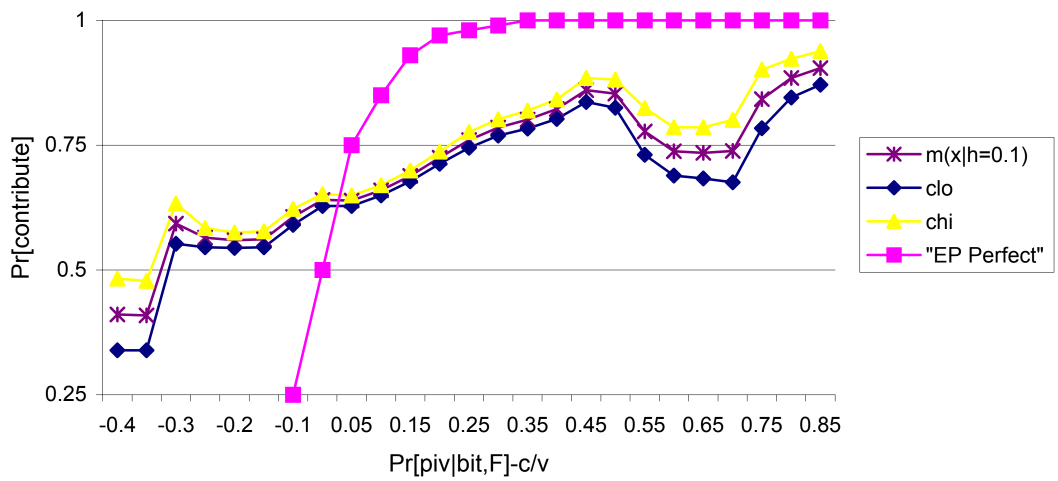
[5]To obtain better confidence intervals, I should compute bootstrap interval estimates. For statistical ease, however, I use the approximate confidence interval described by Härdle (1990, 100-101). The interval is $m_h(x) \pm \left(c_\alpha c_K^{1/2} \widehat{\sigma}(x)\right) / \sqrt{\left(nh\widehat{f}(x)\right)}$, where c_α is the $100 - \alpha$ quantile of the normal distribution, $c_K^{1/2}$ is a kernel constant, $\widehat{\sigma}(x)$ is the estimate of the standard deviation, and $\widehat{f}(x)$ is the estimate of the density. This confidence interval is hampered by a bias, but if the bias is negligible then these confidence intervals are good approximates. As we see from the graph, there would have to be a huge bias for consistency with EP maximization to be a legitimate possibility.

Figure 2.1: Non-parametric Regressions for EP Decision Rule

(a) Non-parametric EP Regressions with h=0.025, 0.1, 0.15



(b) Non-parametric EP Regression with 95% Confidence Interval for h=0.1



Source: Experiment sessions conducted by author at CASSEL in the summer of 2001.

Table 2.3: Results from Comparison of Behavior with Expected Payoff Maximization

(a) Percentage of Contributions Consistent with EP Maximization

	Session				
	1	2	3	4	5
All Observations					
1. Should Go & Did Go	55.8%	55.1%	18.0%	58.4%	62.7%
2. Should Go but Did Not Go	23.6%	31.9%	11.3%	20.9%	25.5%
3. Should Not Go but Did Go	16.0%	7.7%	33.9%	10.6%	6.3%
4. Should Not Go & Did Not Go	4.6%	3.9%	36.8%	8.0%	4.8%
5. Indifferent	0.0%	1.4%	0.0%	2.1%	0.7%
% Consis. with EP Max. (sum of 1, 4, & 5)	60.4%	60.4%	54.8%	68.5%	68.2%
Rounds 8+					
1. Should Go & Did Go	56.7%	55.5%	19.8%	57.1%	64.9%
2. Should Go but Did Not Go	20.3%	32.3%	12.0%	20.4%	24.0%
3. Should Not Go but Did Go	17.2%	6.0%	30.0%	10.6%	6.7%
4. Should Not Go & Did Not Go	5.8%	4.8%	38.3%	8.3%	4.4%
5. Indifferent	0.0%	1.5%	0.0%	3.5%	0.0%
% Consis. with EP Max. (sum of 1, 4, & 5)	62.5%	61.8%	58.0%	69.0%	69.3%

(b) Results from Probit Regressions Testing Consistency with EP Decision Rule

	Probit			Logit		
Regression	**(1)**	**(2)**	**(3)**	**(1a)**	**(2a)**	**(3a)**
Rounds	**All Rds**	**Rds 1-7**	**Rds 8+**	**All Rds**	**Rds 1-7**	**Rds 8+**
Coefficients						
$Pr[piv\|b_{it},F]$	1.5488	1.2331	1.9025	2.5856	2.0479	3.1869
	(0.1234)	(0.1676)	(0.1833)	(0.2117)	(0.2852)	(0.3166)
c/v	-0.3088	0.2107	-0.8600	-0.5789	0.2904	-1.4982
	(0.1755)	(0.2452)	(0.2532)	(0.2908)	(0.4059)	(0.4201)
H_0: Consistency*						
Wald-stat: (2)=(3), (2a)=(3a)		9.3274			9.4261	
Hausman: (1)=(2), (1a)=(2a)		9.4580			9.6797	
Hausman: (1)=(3), (1a)=(3a)		9.1461			9.1957	

* The Chi(2) critical values at 97.5% and 99% are 7.38 and 9.21, respectively.

Source: Experiment sessions conducted by author at CASSEL in the summer of 2001.

decision rule used by subjects. This claim would be especially true if most of the inconsistent decisions are made when the difference between $\Pr[piv|b_{it}, F]$ and $\frac{c}{v}$ is very small. A more formal test of consistency with EP maximization is by regressing the decision (keep or spend) on $\Pr[piv|b_{it}, F]$ and $\frac{c}{v}$. The probit and logit regression procedures (without an intercept) are appropriate given the discrete nature of the dependent variable. If the decisions are EP consistent then the coefficient on $\Pr[piv|b_{it}, F]$ should be 1 and the coefficient on $\frac{c}{v}$ should be -1.

Regressions 1-3 on Table 2.3(b) show these probit estimates for all rounds, rounds 1-7, and rounds 8+, respectively. In all three regressions the null hypothesis of EP consistency is rejected since the coefficients are never simultaneously equal to 1 and -1. I perform a Wald test for structural change in the parameters from the early rounds to the later rounds. This test is calculated as

$$W = \left(\widehat{\theta}_{1-7} - \widehat{\theta}_{8+}\right)' \left(\widehat{V}_{1-7} + \widehat{V}_{8+}\right)^{-1} \left(\widehat{\theta}_{1-7} - \widehat{\theta}_{8+}\right),$$

where $\widehat{\theta}$ and $\widehat{V}$ are the respective coefficient vectors and variance matrices. This test statistic is distributed with a chi-squared distribution, and we reject the null when W is large. The test statistic for structural change from the early rounds to the later rounds is 9.3274. This Chi(2) statistic rejects the null hypothesis of no change at 1% levels thus indicating that the decision rule used in later rounds is different than behavior in the early rounds.

The Hausman test statistic tests if the parameters are the same in all rounds as they are for a subset of rounds. Under the null for testing rounds 1-7 against all rounds, $\widehat{\theta}_{all}$ is consistent and efficient while $\widehat{\theta}_{1-7}$ is consistent and inefficient. The statistic is calculated as

$$H = \left(\widehat{\theta}_{all} - \widehat{\theta}_{1-7}\right)' \left(\widehat{V}_{1-7} - \widehat{V}_{all}\right)^{-1} \left(\widehat{\theta}_{all} - \widehat{\theta}_{1-7}\right),$$

which is distributed with a chi-squared distribution. We reject the null when H is large. Again, we find that the Chi(2) statistics show difference high confidence levels, again indicating that the decision rules differ in later versus early rounds. Regressions 4-6 show that logit regressions yield similar conclusions. These results in Table 2.3 and on Figure 2.1 provide strong evidence against the EP hypothesis as the best explanation for the observed behavior.

Result 5: *Behavior is significantly more consistent with expected utility maximization that accounts for both risk aversion and innate cooperativeness than with expected payoff maximization.*

Previous work has established that subjects' behavior is better understood in terms of expected utility (EU) maximization instead of EP maximization (Camerer 1995). Evidence of three particular variations on EP appears in many public good experiments. First, players show signs of risk aversion even in small payoff games where expected utility theory suggests players should approximately risk neutral. Even though this ultimately illustrates the non-validity of expected utility theory, expected utility theory can have some predictive power (Camerer 1995, Rabin 1999). Second, subjects can be grouped into types by innate tendencies to cooperate even at costs to themselves, and these innate cooperators are a substantial subset of the subject population (Ledyard 1995, Offerman 1997, Ostrom 2000). Third, empirical best response functions are better understood as probabilistic than deterministic, and this notion is captured in McKelvey and Palfrey's (1995) Quantal Response Equilibrium (QRE). This concept has two parts. First, each player's calculations of expected utility are unbiased (correct on average) but prone to i.i.d. error. As a result, individuals' best-response functions are probabilistic instead of deterministic. Second, each player assumes that the other players make similarly error-prone but unbiased calculations. This random process has a fixed point called a QRE.

I model risk aversion as a simple power function in the monetary payoff $(payoff)^{\gamma_i}$ where γ_i is the risk aversion coefficient. I model innate cooperativeness as a simple additive utility term labelled B_i for contribution bias.[6] These modifications yield the following utility matrix:

	$C_{-i} < t-1$	$C_{-i} = t-1$	$C_{-i} > t-1$
spend	B_i	$v^{\gamma_i} + B_i$	$v^{\gamma_i} + B_i$
keep	1	1	$(v_i+1)^{\gamma}$

Expected utility maximization yields the following decision rule:

$$\begin{cases} \text{spend if } \Pr\left[piv|b_{it},F\right] > \frac{\Pr[not\ needed|b_{it},F]((v+1)^{\gamma_i}-v^{\gamma_i})+\Pr[lost\ cause|b_{it},F]}{(v^{\gamma_i}-1)} - \frac{B_i}{(v^{\gamma_i}-1)} \\ \text{keep if } \Pr\left[piv|b_{it},F\right] < \frac{\Pr[not\ needed|b_{it},F]((v+1)^{\gamma_i}-v^{\gamma_i})+\Pr[lost\ cause|b_{it},F]}{(v^{\gamma_i}-1)} - \frac{B_i}{(v^{\gamma_i}-1)} \\ \text{spend or keep otherwise} \end{cases} . \tag{1}$$

The separability assumption between the payoff and B allows for a distinct contribution bias term in this decision rule, and this makes estimation much easier. For convenience, denote the first and second terms on the RHS to be RHS and $BIAS$, respectively.

[6] A more realistic (but complicated) notion of contribution bias has players modeled as "conditional cooperators." Such an agents has a contribution bias in early rounds, but the bias decreases if he or she perceives no reciprocation by other subjects (Ostrom 2000).

This decision rule in the QRE framework yields a probabilistic best-response function. Assuming normal errors, the probability that player i contributes is

$$p_{it} = \int_{-\infty}^{\lambda_i(\Pr[piv|b_{it},F]-RHS(b_{it},\gamma_i)+BIAS(B_i))} \phi(z)\,dz, \qquad (2)$$

where λ_i is a measure of i's error in decision making and $\phi(z)$ is the normal pdf. Note that $\lambda_i \to 0$ implies complete randomness in the decision making while $\lambda_i \to \infty$ means there is no error. A QRE converges to a Nash Equilibrium if $\lambda_i \to \infty$ for all i.

To examine global behavior, I assume identical risk aversion, contribution bias, and calculation error across individuals: $\gamma_i = \gamma$, $B_i = B$, and $\lambda_i = \lambda$ for all i. Table 2.4 shows the maximum-likelihood results from a sequence of regressions that add these new components to the decision making process. When not separately estimated, λ is fixed at 1, γ is fixed at 1 (risk neutrality imposed), and B is fixed at 0 (no contribution bias). In this table, we see that when estimated alone or jointly with other coefficients, there is always risk aversion (γ significantly less than 1) and there is always a contribution bias (B is significantly greater than 0). Also note that I always estimate less calculation error (higher λ) in later rounds than in earlier rounds. McKelvey and Palfrey (1995) discuss how this can be interpreted as reflecting learning through experience that yields more accurate expected utility calculations. These findings are robust to different starting values in the estimation algorithm.

The regressions in Table 2.4 impose the consistency with the decision rule, and we are interested in whether or not it is reasonable to make this consistency assumption. To find out, I re-ran Regressions (1)-(15) from Table 2.4 with the addition of coefficients on the $\Pr[piv|b_{it}, F]$ and RHS terms. (This estimation is similar to regression in Table 2.3 except I allow for risk aversion, the contribution bias, and decision error.) Under the joint H_0, the respective coefficients on $\Pr[piv|b_{it}, F]$ and RHS are simultaneously 1 and -1. We can then calculate an F-statistic to test the joint null that the coefficients are 1 and -1, respectively.[7]

The results from these regressions are not shown, but I will summarize the findings. In most of these modified (1)-(15) regressions, the maximum likelihood procedure produced multiple solutions. These multiple convergences are due to relative flat spots or non-convexities in the likelihood function. These function characteristics are probably due to the complicated nature of the non-linear estimation. For example, at small γ, the denominator $(v^{\gamma} - 1)$ approaches 0 as γ decreases.[8]

It turns out that for any of the modified (1)-(15) regressions, I can obtain solution estimates that reject the hypothesis of consistency. However, when I include both risk aversion γ and bias B, some of the solutions do not reject the hypothesis of consistency. In short, if I do not allow for risk aversion and the bias, then I always reject the null hypothesis of consistency, but if I allow for both then I do not always reject the null. Of course, the multiple convergences demonstrate that these tests have low power in preventing Type II errors (non-rejections of a false hypothesis).

[7]The F-statistic is calculated as

$$F\left[2, \#obs - K\right] = \frac{1}{2}\left(\widehat{\theta} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)' \left(\widehat{V}\right)^{-1} \left(\widehat{\theta} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right).$$

[8]One way to get around this problem of multiple convergences is to obtain maximum likelihood estimates using a grid search procedure.

Table 2.4: QRE Probit Estimates with Imposed Consistency

Regression Rounds	(1) 1-7	(2) 8+	(3) 1-7	(4) 8+	(5) 1-7	(6) 8+	(7) 1-7	(8) 8+	(9) 1-7	(10) 8+	(11) 1-7	(12) 8+	(13) all	(14) 1-7	(15) 8+
Coefficients															
LAMBDA	--	--	--	--	1.259	1.687	1.048	1.566	--	--	1.300	1.686	0.972	0.652	1.311
					(0.1269)	(0.1398)	(0.1322)	(0.4602)			(0.1266)	(0.1388)	(0.1300)	(0.1837)	(0.1865)
GAMMA	--	--	0.756	0.810	--	--	--	--	0.332	0.401	0.776	0.855	0.356	0.236	0.502
			(0.0647)	(0.0798)					(0.0262)	(0.0380)	(0.0569)	(0.0603)	(0.0460)	(0.0616)	(0.0754)
B	1.018	0.769	--	--	--	--	0.974	0.460	0.356	0.334	--	--	0.353	0.400	0.310
	(0.1029)	(0.1018)					(0.1704)	(0.0819)	(0.0244)	(0.0301)			(0.0268)	(0.0381)	(0.0404)

Source: Experiment sessions conducted by author at CASSEL in the summer of 2001.

Nonetheless, the finding is clear: if the behavior is globally consistent with the modified decision rule in (4), then it must be true that the there is both risk aversion and a cooperation bias.

Ideally, we could take advantage of the panel structure of the data to find evidence of risk aversion and contribution biases at the individual level. I use a maximum score method to obtain some preliminary results along these lines, and as the results in Table 2.5 show, allowing for risk aversion and contribution bias can explain up to 90% of the choices. In the bottom half of Table 2.5, I use the following grid: $(\gamma_i, B_i) \in \{0.01, 0.02, ..., 3.00\} \times \{-6.00, -5.99, ...6.00\}$. At each point in this grid, I calculate the number of observed decisions that, given those parameters, match what the decision rule says the individual should do. Because multiple parameter profiles can lead to the same number of decision matches, I choose as my estimate the profile whose distance is closest to the EP profile. That is, among the set of profiles that maximize the number of matches, I choose the profile $\left(\widehat{\gamma}_i, \widehat{B}_i\right)$ that minimizes $\sqrt{(\widehat{\gamma}_i - 1)^2 + \left(\widehat{B}_i\right)^2}$.[9]

The average listed in the table is the average of the individual estimates, and the percent of matches is the percent of matches using the individual estimates. The standard deviation and "min,max" ranges show that there is wide variation across individuals within sessions. There are substantial differences across sessions, too. For all rounds in Session 1, the averages of the individual risk coefficients and the individual contribution biases are 0.86 and 0.15, respectively. For session 2, the

[9]Only rarely does this procedure produce more than one estimate. If multiple estimates remain, I take the average those estimates.

averages are 0.73 and -0.07. These numbers suggest that there are qualitative differences across sessions, and these differences make comparing absolute contribution levels across sessions problematic in terms of testing our qualitative predictions (see Result 2).

The top half of Table 2.5 shows the results from solving for only γ_i and B_i. When solving for the risk coefficient, I held the bias fixed at 0 and examined the matches for γ_i in $\{0.01, 0.02, ..., 3.00\}$. When solving for the bias, I held the risk coefficient fixed at 1 and examined the matches for B_i in $\{-6.00, -5.99, ...6.00\}$. Table 2.3 shows that about 60% of the decisions can be explained by EP maximization. Table 2.5 shows that if we allow for heterogeneity in only the risk preferences, we can explain 75% of the decisions. Allowing for heterogeneity in only the contribution biases, we can explain even more of the decisions—about 85%. Allowing simultaneously for both risk preferences and contribution biases, we can explain almost 90% of the decisions.

These estimates are not perfect measures. The grids used were discrete and bounded. The bounded nature of the grid means that the size of the grid can affect the estimates. Increasing the upper grid boundary will increase the estimates. Also, the fact that my procedure chooses the parameter profile (within the set of score maximizing profiles) that is closest to EP maximization will obviously bias the results towards the EP estimates, and this can explain why the estimated biases are fairly low (near 0 on average for many sessions). Lastly, we cannot perform any formal statistical significance tests.

Table 2.5: Maximum Score Estimates by Session and Overall

	Gamma chosen from [0.01,3], Bias=0				Bias chosen from [-6,6], Gamma=1			
	Average	Std. Dev	min,max	match%	Average	Std. Dev	min,max	match%
All Rounds								
1	**1.14**	0.59	0.38,2.78	**73.8%**	**0.18**	1.05	-2,1	**87.4%**
2	**1.04**	0.89	0.02,2.99	**80.8%**	**-0.33**	1.55	-4.1,1	**83.5%**
3	**0.80**	0.45	0.1,1.73	**62.8%**	**-0.06**	0.69	-2,1	**75.9%**
4	**1.19**	0.76	0.18,2.99	**81.8%**	**0.16**	0.67	-1,1	**88.0%**
5	**1.18**	0.84	0.13,2.97	**82.9%**	**0.00**	0.95	-2,1	**87.9%**
Overall	**1.08**	0.72	0.02,2.99	**76.5%**	**0.00**	1.02	-4.1,1	**84.8%**
Rds 1-7								
1	**1.10**	0.62	0.13,2.78	**74.0%**	**0.33**	0.84	-2,1	**86.4%**
2	**1.02**	0.84	0.02,2.99	**80.9%**	**-0.47**	1.53	-4.1,1	**84.0%**
3	**0.80**	0.31	0.18,1.26	**58.0%**	**-0.06**	0.63	-2,1	**77.1%**
4	**1.03**	0.78	0.03,2.91	**85.5%**	**-0.06**	0.91	-3.8,1	**89.5%**
5	**1.07**	0.78	0.13,2.97	**83.0%**	**-0.08**	1.17	-3.5,1	**88.0%**
Overall	**1.01**	0.69	0.02,2.99	**76.5%**	**-0.05**	1.06	-4.1,1	**85.2%**
Rds 8+								
1	**1.10**	0.58	0.13,2.38	**76.9%**	**0.25**	0.87	-1.88,1	**90.8%**
2	**1.12**	0.82	0.02,2.78	**84.3%**	**-0.39**	1.03	-3.21,1	**85.5%**
3	**0.85**	0.45	0.13,1.78	**69.0%**	**-0.03**	0.47	-1.11,1	**79.3%**
4	**1.04**	0.73	0.12,2.93	**83.1%**	**0.09**	0.66	-1.7,1	**89.2%**
5	**1.15**	0.71	0.07,2.81	**86.7%**	**-0.06**	0.56	-1.41,1	**91.1%**
Overall	**1.05**	0.67	0.02,2.93	**80.0%**	**-0.01**	0.77	-3.21,1	**87.4%**

	Gamma chosen from [0.01,3] and Bias chosen from [-6,6] simultaneously						
	Average	Std. Dev	min,max	match%	Average	Std. Dev	min,max
All Rounds							
1	**0.86**	0.43	0.31,2	**89.4%**	**0.15**	0.59	-1.4,1
2	**0.73**	0.56	0.01,2.59	**88.5%**	**-0.07**	0.41	-1.4,0.37
3	**0.78**	0.55	0.01,2.52	**80.3%**	**0.34**	0.86	-0.35,4.2
4	**0.89**	0.38	0.29,2	**89.0%**	**0.02**	0.60	-1.4,1
5	**0.75**	0.35	0.22,2	**90.9%**	**-0.11**	0.64	-2.6,0.34
Overall	**0.81**	0.46	0.01,2.59	**87.7%**	**0.07**	0.64	-2.6,4.2
Rds 1-7							
1	**0.78**	0.32	0.31,2	**89.5%**	**0.17**	0.50	-1.4,1
2	**0.70**	0.41	0.07,2	**91.4%**	**-0.12**	0.43	-1.4,0.34
3	**0.77**	0.30	0.01,1.05	**81.1%**	**0.16**	0.34	-0.5,1
4	**0.91**	0.55	0.13,2.89	**92.6%**	**0.03**	0.58	-1.4,2.2
5	**0.68**	0.23	0.24,1.01	**90.9%**	**-0.03**	0.37	-1.02,0.34
Overall	**0.77**	0.39	0.01,2.89	**89.3%**	**0.05**	0.47	-1.4,2.2
Rds 8+							
1	**0.78**	0.22	0.31,1.07	**93.1%**	**0.16**	0.44	-0.5,1
2	**0.93**	0.66	0.06,3	**89.5%**	**-0.10**	0.34	-1.07,0.37
3	**0.92**	0.44	0.18,2.2	**84.3%**	**0.27**	0.63	-0.39,2.86
4	**0.90**	0.36	0.14,2	**90.6%**	**0.02**	0.82	-1.7,3.4
5	**0.83**	0.37	0.07,2	**94.2%**	**-0.07**	0.35	-1.4,0.34
Overall	**0.87**	0.42	0.06,3	**90.4%**	**0.06**	0.56	-1.7,3.4

Source: Experiment sessions conducted by author at CASSEL in the summer of 2001.

Despite the above complications with these initial estimates, allowing for heterogeneity in risk aversion and innate cooperativeness can explain a significant amount of the behavior. This fact provides further support for the conclusion that an expected utility model that accounts for both risk preferences and contribution biases is a very good model for predicting the behavior in these experiments.

2.3.4 Comments on Beliefs, Learning, and Strategy

The different behavior between early and later rounds in some sessions suggests the presence of learning, dynamic strategies, both, and other factors. Examining the beliefs updating in the early versus later rounds indicates that beliefs are being updated in a very similar manner in all rounds. This suggests that if there is learning going on, it is not about the rules of the game. Instead, subjects are learning to make more accurate best response calculations and learning about other subjects' behavior.

Let us look closer at the beliefs. In a symmetric mixed equilibrium, contributors and non-contributors should have identical beliefs, and in an asymmetric equilibrium, the non-contributors should expect more contributions by others. But this is opposite of what occurs in Sessions 1, 2, 3, and 5, where contributors always expect more contributions by others than do non-contributors. This fact is more striking if we look separately at the early and later rounds of the different parameter profiles. In the early rounds under a given parameter profile, the difference between what the contributors and non-contributors expect others to spend is much larger than the difference in later rounds.

This convergence of beliefs indicates that the players are learning about each others' behavior and updating their beliefs accordingly. A convergence of beliefs is expected if players behavior is approaching a symmetric equilibrium, but how do we reconcile the initial divergence of beliefs with equilibrium behavior since the divergence appears on the surface to be opposite of what it would be in equilibrium? I suggest two explanations: dynamic strategies and risk aversion. (Future research should include the development of methods to separately identify the presence of these two factors in the experimental data.)

The notion that players use dynamic strategies is supported by statements made by subjects after being asked informal questions at the end of the experiment. Some subjects described early spending as attempts to ensure future contributions by others by getting them accustomed to high payoffs from provided public goods. It is as if these players want to take advantage of the presence of *conditional cooperators* (Ostrom 2000), i.e., players that are more willing to contribute when they perceive others to be cooperative. This notion is captured in the modified decision rule (4) as a bias term that increases when others are expected to cooperate and that decreases when others are expected to not cooperate. The existence of such individuals is one possible reason why it is the contributors in Session 1, 2, 3, and 5 that expect others to be spending more often. A subject who sees others as being more cooperative is, therefore, more willing to cooperate, and a subject who believes that there are fewer cooperators will be less likely to spend. The presence of conditional cooperators can thus explain why contributors reported higher mean beliefs in Session 1, 2, 3 and 5.

A second explanation is that players are risk averse. According to the modified decision rule (4) that incorporates risk aversion, the probability a player contributes goes down when the probability of a lost cause increases relative to the probability of being redundant (all else equal). In rounds 1-7 under range $\{1,2,3,4,5\}$ in Session 1, the contributing subjects reported higher mean beliefs than non-contributing subjects. In fact, the mean beliefs for non-contributing subjects in rounds 1-7 of range $\{1,2,3,4,5\}$ is much lower than the mean beliefs for the contributing and non-contributing subjects the rounds 8+ of range $\{3\}$ in that same session. The modified decision rule (4) suggests that the subjects did not contribute because they believed there was a greater chance of a lost cause due to a drop in expected spending by others. The presence of risk aversion can thus explain why the non-contributors reported lower mean beliefs.

Each of these two explanations provides a story for why contributions did not increase as predicted in the early rounds under range $\{1,2,3,4,5\}$ in Session 1. With the divergence of beliefs and certain players' lowered expectations of others' spending, the presence of risk aversion and/or conditional cooperators can lead to a decrease in contributions. This decrease is opposite of what the Nash equilibrium-based theory predicted. It was only when the beliefs converged in the later rounds under range $\{1,2,3,4,5\}$ that the behavior matched the predictions. The implication is that the Nash-equilibrium predictions based on converged beliefs are not going to do a good job predicting behavior when beliefs are not converged.

2.4 SUMMARY AND DISCUSSION OF CHAPTERS 1 AND 2

The theoretical work suggests that there can be advantages to having wider threshold uncertainty, while the experiments help clarify when these advantages are more likely to exist. More precisely, when the value of the public good is high, wider uncertainty can lead to higher contributions and higher efficiency, but contributions and efficiency can decrease when the value of the public good is low (Propositions 2 and 3, Lemmas 1 and 1A). The experiments show that the game-theoretic logic underlying these main findings does a good job of explaining behavior once it accounts for risk aversion and innate cooperativeness (Results 4 and 5). The experiments also demonstrate, however, that the theoretical predictions are sensitive to the equilibrium assumption of the convergence of beliefs (Results 1-3, Section 2.3.4).

These theoretical and experimental findings yield insights into understanding individuals' behavior in collective action environments marked by threshold uncertainty. I will briefly discuss two particular insights. The first insight is that there can be a status-quo bias towards initially high levels of threshold uncertainty. The second insight is that communication is vital to collective action success in one-shot provision environments.

Status-quo Bias Towards Initially Wide Uncertainty. Suppose all individuals observe the same public signal $t_{pub} = t + e_{pub}$, where $t > 0$ is the actual threshold and e_{pub} is drawn from a known error distribution with mean 0 and variance σ^2. This signal and error structure imply a threshold distribution F, which will then imply an equilibrium C^*. Now suppose that before the players make their contributions but after observing the signal, the players can generate and observe another noisy public signal. Will the individuals ever prefer to not generate this second signal, even if generating the signal is cost-less? The answer is yes.

For simplicity, assume that players are randomly selected to be the contributing agents (see the next footnote). The first signal and the error structure imply a probability distribution over the possible second signals, and this distribution implies another distribution over the possible new equilibria that will result from observing the additional signal. Suppose the initial equilibrium results from an inferred F with a fat interior-right tail so that the probability of provision in equilibrium C^* is at or near 1. If players expect the distribution after observing the second signal to have an interior-right tail that is thick but below $\frac{c}{v}$, then they will expect the new equilibrium to have much fewer contributions and a much lower probability of provision. If they expect a sufficient drop in the probability of provision (i.e., a drop that leads to lower expected utilities despite the drop in the probability of having to contribute), then they will prefer to not observe that signal. In other words, the players will prefer the status-quo uncertainty to a new state with lower uncertainty. The origin of this *status-quo bias* is in the worry that the additional information will lead to an equilibrium which will provide the good with a much lower probability than the original equilibrium C^*.[10]

[10] The story is slightly different when contributors and non-contributors in the initial equilibrium are known because contributors and non-contributors will have different desires for new information. Obviously, this will also depend on who contributes in the new equilibrium. This variation is closely connected to the idea of Fernandez and Rodrik (1991), who explain how uncertainty about who gains and benefits from a reform can prevent that reform even if that reform is welfare-enhancing. Even though their idea can be present in my example, it is incorrect to think their idea is driving the status-quo bias I mention here. The reason is that the same aversion to additional information can

There is another environment in which a status-quo bias towards wide uncertainty can exist. Up to this point we have talked about identical players in an abstract public good setting, but many collective action settings have other strategic elements. For example, suppose that the public good provider (e.g., public television) cares about maximizing the difference between contributions and the threshold ($C^* - t$) because any contributions over the threshold are a transfer from the contributors to the provider. In this case, the provider will prefer wider uncertainty when a highly valued public good implies a larger amount of contributions under that wider uncertainty. Furthermore, if the provider can act to reduce the public's uncertainty about the threshold, the provider has the incentive to not undertake such an action.

Further strategic issues follow from this logic (for example, if I know the provider has those incentives, then the fact that the provider does not act might reveal information to me about the threshold, thereby affecting my decision to contribute, thereby affecting the provider's decision to act, and so on), but for the purposes of this paper, it is sufficient to note that diametric incentives can result in a bias towards the wide uncertainty of the status-quo.

Communication in One-shot Provision Environments. In Section 2.3.4, I discussed how the divergence of beliefs led to changes in contribution levels opposite of those that were predicted in the initial rounds following a change in uncertainty. We expect and observe a convergence in beliefs in the experiments as

---

result even when there is no individual-specific uncertainty. The real reason for the result is the fear of F' having a right boundary that is relatively fat but also underneath the $\frac{c}{v}$-line, which translates into a low probability of public good provision.

the rounds advance, but in early rounds, players do not have much of a past history to use in guiding their beliefs. If beliefs and behavior in these early rounds are indicative of beliefs and behavior in one-shot games, then there is reason to believe that the equilibrium predictions will not be accurate in one-shot games because of the beliefs divergence.

In one-shot coordination games, communication can lead to the convergence of beliefs. Communication could also allow for coordination on an asymmetric equilibrium in which risk averse subjects would be less concerned with lost causes (as a reminder, the lost cause is the event that the public good is not provided even when you contribute). In short, the communication makes the convergence of beliefs—and, hence, the Nash equilibrium prediction—more likely. This reasoning implies that, for groups facing threshold uncertainty in one-shot provision scenarios, communication or formal coordination of contributions might be essential for public good provision. This claim should be especially true when there is wide uncertainty. Under wide uncertainty, extreme risk averters will only contribute when they believe that the probability of a lost cause is very small. Formal coordination of contributions might be essential for those fears of a lost cause to be diminished.[11]

[11] The social psychology literature points out that for simple problems (e.g, the coordination of contributions), highly centralized communication networks are more efficient. See Chapter 3 of Brown (2000) for a discussion of these and similar findings. My findings suggest that such communication networks probably matter more for one-shot scenarios than for repeated interaction.

2.5 EXPERIMENT DIALOGUE

Introduction. Hello, and welcome to the Social Science Experimental Laboratory. Thank you for coming. I am Michael McBride, a graduate student at Yale University. Today you will be placed in a decision making environment. In participating you receive a show-up payment of $__ but you will also receive additional monetary payments. Your actual total payment will depend on your decisions, the decisions of others, and factors determined randomly by the computer.

We should finish in approximately 100 minutes. During this time, you will be interacting with the others in this room over a computer network. At any time during the study, you are free to end your participation if you feel the need. Should you decide to stop your participation, you will still receive the show-up payment. To stop your participation at any time, please raise your hand to notify me.

All decisions you make today will be made in a computer currency which we call "tokens." As you make your decisions, you will amass a large number of these tokens. When we are finished, you will receive actual US dollars according to a token/dollar exchange rate. The more tokens you amass, the more US currency you receive. The exchange rate is 15 tokens for 1 dollar. Each student in the room is likely to take home a different amount. You are under no obligation to tell your amount to anyone.

I will be giving you verbal instructions. You must follow my instructions. Do not take any initiative by typing or mouse-clicking before I tell you. When you have a question, please raise your hand. Please be careful around the computers. If one computer loses its connection then the software will freeze. So please do not kick any wires or bump the computer under your desk. Also, please turn off your cell phones to prevent any interruptions.

Consent. In order to participate, you must first give your consent to participate. By registering on the CASSEL web page for today's session, you acknowledged that your participation is voluntary. Is there anyone who did not give this consent? [WAIT] You are again reminded that your participation is voluntary. If you wish to end your participation, please notify me by raising your hand.

Begin Public Good Game. The computer software places each of you in a decision making environment and records your decisions. First, I will briefly explain this decision making environment. Then you will participate in 4 practice rounds to familiarize yourself with the computer interface. After the practice rounds, you will participate in 30 real rounds. The tokens you amass during the real rounds are the ones you will exchange for US dollars. The practice rounds are only to familiarize yourself with the decision making environment and the computer interface.

In each round, both practice and real, you will make choices using the mouse only. During these rounds, do not speak or communicate with any other students in the room. In other words, you are never allowed to speak to anyone else. All of your decisions must be made privately. If you have questions, you can raise you hand, and I will come answer your question.

Begin Worded Explanation. Let me now give you a brief explanation in words of the decision making environment. In each round, the computer will randomly assign you and four other students in this room into a five-student group. You will not know who else is in your group. You will also be given one ”computer token.” You will ultimately decide to either “keep” or “spend” that one token. If the number of tokens spent by students in your group including yourself is greater than or equal to a certain amount then everyone in your group receives a certain number of tokens back whether or not they spent tokens. Otherwise, no one gets any tokens back.

On the board is an example. Suppose the computer tells you that at least 3 tokens must be spent in order for everyone in the group to get back 5 tokens. The 3 token requirement is called the “threshold,” and the 5 token payment is called the “threshold-met-value.” Suppose four of the students in your group spent their tokens and one kept her token. Then the threshold was met since four spent tokens is greater than or equal to threshold 3. This means that each person in your group gets 5 additional tokens. The students who spent their tokens therefore receive 5 tokens for that round. But notice that one student kept her token. She has 6 tokens for that round: five from the threshold-met-value, and 1 from the kept token. Suppose in another group, two students spent their tokens and three kept their tokens. The threshold was not met. The two students who spent their tokens have a total of 0 tokens for that round. The other three students each have a total of 1 token for that round-the token that was not spent.

You will always be told the threshold-met-value, and it is the same for everyone in your group. However, you will generally not be told the actual threshold. Instead, the computer will tell you a range of values from which the threshold will be chosen. For example, the computer might tell you that the "threshold range" is 3-5. This means that the computer will randomly select either 3, 4, or 5 to be the actual threshold. Each of these is equally likely to be chosen by the computer. There are many other possible ranges: 1-5, 2-4, 2-3, 3-3, and so on. If the range is 3-3 then you know for sure that 3 is the threshold.

In addition to making a keep/spend decision, the computer will ask what you think the others in your group will do. Since there are four others in your group, there are five possible outcomes: first, none of those four others spend; second, one of those four spends; third, two of those four spends; fourth, three of those four spend; and fifth, four of those four spend. The computer will ask you what you think are the chances of each of those five occurrences. You will be shown how the computer asks you this in a minute. In addition to the payment you receive from your group meeting or not meeting the threshold, you will also receive a payment according to the your answers to these questions about the likelihoods of these occurrences. We will describe this payment in a minute, too.

Begin Practice Rounds. Now I will walk you through doing this on the computer. Please pull out the dividers that separate you from your neighbors. Please close any windows that might be open on your computer. To start the software, please use your mouse to double-click on the PUBGOOD_MCBRIDE icon on your screen. A window will come up asking you to enter your name. Once you have entered your name, please click CONFIRM. You must click CONFIRM instead of pressing the ENTER key. Once everyone has clicked CONFIRM, your screen will show the layout. Please raise your hand if you do not see the layout on your screen at this moment. [WAIT.]

There are two parts to the layout. The first of these is the main window. This is what you see on your monitor right now. The second part is currently hidden and will be explained in a minute. Look at the main window. Near the top-left of your screen you will see your ID number in blue. After you entered your name, you were assigned an ID number by the computer. Below the ID number on the main screen, you should see ten different column headings. The left-most column heading is the match or round number. The second column is the threshold-met-value for that round. The third column is the threshold range. At the beginning of a new round, you will be given the information in these three left columns.

The other columns record your keep/spend decision, the total tokens spent by the others in your group, the actual threshold chosen by the computer from the threshold range, your payment from the keep/spend decisions in your group, your payment from your probability assignments, the sum of these two payments for that round, and the overall total payment through all rounds. These right columns will show their information after all choices have been made in a particular round.

The bottom half of the main window is a message box that will prompt you at times with messages. Right now it says, "Please listen carefully to the instructions given by the experimenter, and so on" At the very bottom of the screen are the KEEP, SPEND, and PROCEED buttons. The KEEP and SPEND buttons are used when deciding whether to keep or spend your token. The PROCEED button is mainly used to advance to new rounds.

Please click this PROCEED button now. Your message box now tells you we are about to begin the practice rounds. Click PROCEED again. We have now started the first practice round. At the very top left of your screen, the probability window has just popped-up. Please grab this window with your mouse and move it to the bottom right of your screen so that you can see the left columns on the main window. You should see that the match or round number is 1, the threshold-met-value is 5, and the threshold range is 3-3. This means that you know for sure that the threshold is 3. To remind you, this means that at least 3 tokens must be spent in your group for everyone to get the threshold-met-value back. The threshold range and threshold-met-values are the same for all groups in the room. Further notice that the right columns have questions marks indicating that your decisions have not yet been made.

In each round, both practice and for real, you do two things. First, you assign your likelihoods to the different occurrences of no others going, one other going, etc. Second, you make your spend/keep decision. You can only do these two things in this order. I will walk you through doing this now.

Look at the probability window. There are five pull-down lists-one corresponding to each of the outcomes of the decisions of the others in your group. In blue are the possible outcomes: 0 through 4 others spending. Immediately beneath each blue outcome is the corresponding pull down list, each of which currently says 0% and has a black down arrow. Go to the pull-down list for 0 and click and hold your mouse button. Notice that you are allowed to select the percentages in 1% increments. Move the mouse to 50% and release the mouse button. Please raise your hand if you have not been able to assign 50% as asked. [WAIT.] You have just assigned a 50% chance to 0 others in your group spending their tokens. Notice that on the very right of this window, the total probability you have assigned says 50%. Now use the pull-down list under 4 to select 45% that four others spend. Your total should now say 95%.

Once your assigned probabilities add up to 100%, you are allowed to press the white CLICK TO PROCEED button that runs the length of the probability window. If you press CLICK TO PROCEED and the probabilities do not add up to 100%, then a message will pop up right above your CLICK TO PROCEED button in the probability window telling you that the probabilities do not add up to 100%. In that event you must adjust your assigned probabilities until they add to 100% before you can proceed to making your keep/spend decision. Once you have pressed CLICK TO PROCEED after making probability assignments that add to 100%, the probability window disappears. At this point you are not allowed to change those probability assignments for that round. So be sure to not press CLICK TO PROCEED until you have entered and/or adjusted your desired probability assignments.

Let's now do the first practice round. First make your probability assignments in the probability window using the pull-down lists. Do this now. Of course, if you do not want to select 50% for the 0 event and 45% for the 4 event, please select other percentages using the pull-down lists. Once you have assigned your probabilities, press CLICK TO PROCEED. The probability window will disappear. Next you make your keep/spend decision by pressing either KEEP or SPEND at the bottom of the main window. Once everyone in the room has made the keep/spend decision, the right columns on your main screen should tell you the results of this first round. [WAIT.]

Does everyone see the results for your group for the first round in the right columns on your main screen? Please raise your hand if you do not see the results. [WAIT.] Look to see if your group spent enough tokens to meet the threshold. Look in the decision payment column. If your group met the threshold, the decision payment should say either 5 or 6, depending on whether you spent or kept your token. If your group did not meet the threshold, the decision payment should be either 0 or 1, again depending on whether you spent or kept your token. Also look in the probability payment column. This column tells you how many tokens you received from your probability assignments. [WAIT.]

The highest probability payment you can receive in a given round is 3 tokens. For example, if you assign 100% to the occurrence that exactly 3 tokens would be spent and exactly 3 others spent, then you receive a probability payment of 3 tokens. If you care to know the exact formula used to calculate this payment, it is on the board. The more accurate your probability assignments, the more tokens you receive. The least you can receive in a given round is 0 tokens.

After briefly examining the information in the right columns, press PROCEED at the bottom of the main window to advance to the next round. Your message box contains information about the decisions made, but it also has instructions that do not apply for this session. Please ignore the instruction to write down your choices. In fact, the message box contains no new information, so you can ignore it's instructions and press PROCEED. Once everyone has pressed PROCEED, the screen will advance to the next round. I repeat, the round will only advance once everyone has clicked PROCEED, so please click PROCEED now if you have not clicked it yet.

Are there any questions? [WAIT.] Now try practice rounds 2, 3, and 4 on your own. Let me remind you that you that the computer randomly places you into a newly selected group in each round. For round 2, Notice that the threshold-met value and threshold range have changed. Remember to always look to see the threshold-met-value and threshold range at the start of every round. Also remember to not talk or communicate in any manner with any other students in the room. If you have any questions, please raise your hand. Go ahead and finish the rest of the practice rounds on your own. When the message area in the bottom half of the main windows prompts you to click PROCEED, please click PROCEED.

[WAIT AND WATCH UNTIL PRACTICE ROUNDS ARE OVER.] Now that the four practice rounds are over, do you have any questions? [ANSWER QUESTIONS.]

Begin Real Rounds. Let's now do the real rounds. Please press PROCEED to do the real rounds. The message box tells you that your payoffs in the real rounds will count towards your earnings. Press PROCEED one more time and start doing the real rounds now. Remember that you are not allowed to communicate with any other students in the room during these rounds. [WAIT FOR FINAL ROUND TO FINISH.]

Conclusion. Please follow my special directions in filling out the receipt. Fill out the date, the time of the experiment, your name, and your computer assigned ID number. Please enter the show-up fee of $5. On your screen should be a window that popped up after the last round was completed. Notice where your total amount is listed in dollars. On the line on your receipt for the decision payment, please enter your dollar amount from your screen rounded up to the nearest quarter. For example, if your dollar amount is $10.04, then round up to $10.25. When you have done this click QUIT to close that window. Do not sign the receipt yet. You will sign it when you receive your money.

Over the next few weeks I will be conducting more of these sessions. By participation today you are ineligible to participate in other sessions run by me. Please note in future emails if it says that your participation today makes you ineligible. If it says so, please do not sign up to participate. Thank you for coming. Please form a line starting at the lab office to receive your payments.

CHAPTER 3

NON-COOPERATIVE NETWORK EQUILIBRIA UNDER HETEROGENEITY AND UNCERTAINTY

3.1 INTRODUCTION

Social networks are increasing used in both theoretical and empirical research in economics.[1] The basic idea is that individuals make their choices in environments that are either constrained, enhanced, or defined by some sort of social interaction, and the concept of the social network is used to precisely define that social interaction. Bala and Goyal (2000) (BG hereafter) made an important stride forward in the theoretical work on network formation by modeling network formation as a non-cooperative game. By so doing, they can take advantage of the wide variety of concepts and tools of non-cooperative game theory. They look at the statics and dynamics of network formation, provide a characterization of strict Nash equilibria networks, and show that under some simple best-response dynamics, individual efforts to access the benefits of a network can lead quickly to the emergence of equilibrium networks. These limiting networks often have simple structures and will likely be socially efficient.

In BG's model, the benefits of network participation and the structure of the network are commonly known. The full information assumption is strong and begs the question of how network formation and limiting networks might change

[1] See Bala and Goyal (2000) for a list of theoretical and empirical work.

when there is uncertainty about the benefits of the network and the structure of the network. Because we would expect that individuals in many social networks might not have the full information that BG assume, examining network formation under uncertainty can yield important insights into the types of actual networks that social scientists should expect to find. In this study, I add two types of uncertainty to the BG model and examine the implications of the uncertainty concerning the values of other agents' network goods and uncertainty about the structure of the network. Table 3.1.1 summarizes the differences between the model in BG and the model of this paper.

In the network games in this paper, each agent i has a unique, non-rival good that is excluded from any agent j not directly or indirectly connected to i through the network. Each network good has a certain value which is received by any connected agent. The game is non-cooperative because the cost c of each network connection—called a link—is paid only by the link-initiating agent. BG show that when all agents have goods valued at $v > c$ and the flow is two-way, the unique strict Nash equilibrium is a connected, center-sponsored star, as show in Figure 3.1.1(a). A circle on a link near an agent signifies which agent initiated and pays for the link.

This study shows that the qualitative feature of the center-sponsored star is robust to large decreases in information. Although under low levels of information we may have networks that are not connected (a network is connected when all agents are directly or indirectly connected to one another), subnetworks retain the center-sponsored architecture except in the most extreme times of uncertainty. For example,

Table 3.1.1: Comparison of BG and McBride

	BG	McBride
Flow direction	one-way flow, two-way flow	two-way flow
Flow decay	decay, no decay	no decay
Network Good values	all agents have same values	randomly assigned values
Information	full information about network good values and structure	varying levels of information about network good values and structure

Figure 3.1.1: Some Equilibrium Networks

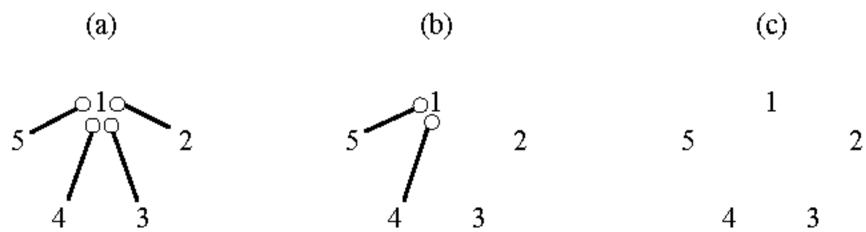
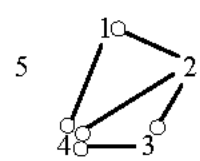


Figure 3.2.1: Illustration of Example 3.2.1



consider a network where the whole network structure and the values of the network goods are not know *a priori* by the agents. Further consider that the agents do know all j with whom they are directly connected and they know all k directly connected to those j. If all agents have goods valued higher than c, the network in Figure 3.1.1(b), one in which not everyone is connected, can be sustained as an equilibrium. This network, although not connected, does have a subnetwork with a center-sponsored star. To have an equilibrium that does not have a center-sponsored star subnetwork, we must remove the ability of the agents to see everything beyond their direct links.

As expected, as uncertainty increases, the set of equilibrium structures increases (weakly), but this increase depends on the type of uncertainty, the actual values of the agents' network goods, and the belief-updating procedure used by agents. For example, in a setting where the agents know the values of others' goods but do not know the structure, equilibrium network structures can be different than when agents know the structure but do not know the agents' good values. Again assuming all agents have goods valued at $v > c$, the network in Figure 3.1.1(b) is one that can be sustained as an equilibrium when the structure is known but the values are not known. This same network, however, cannot be sustained as an equilibrium when the values are known but the structure is not known.

To obtain these results, I had to overcome many hurdles. First, how do we account for *flows of information* about the network structure and about the other agents' network goods? Second, what is the correct *concept of equilibrium* to use? Third, how should agents use the information they receive to form *probabilities over* *the possible network states*? To overcome this first hurdle, I suppose agents assign probabilities to different states of the world, and these probabilities can be updated according to the information that the agents receive. To overcome the second hurdle, I propose and apply a generalized version of the *conjectural equilibrium* concept (Gilli (1999)). This generalized concept allows for incomplete-information, imperfect-monitoring, and the relaxing of the common knowledge of rationality assumption. I do not completely clear the third hurdle, but I illustrate certain aspects of beliefs-updating to which equilibria are particularly sensitive. The ultimate goal is to clear all hurdles and develop a dynamic model of network formation under uncertainty. This paper examines the static equilibria of this network game, but it only offers a brief discussion on aspects of the dynamics of the formation.

Section 3.2 describes the model and the notation. This section describes how I deal with the first two of these hurdles. Since the actual equilibria depend on the realized network good values, we must describe the network equilibria accordingly. Section 3.3 uses examples to illustrate many of our results for the case when all agents have goods valued higher than the cost of forming a link. Section 3.4 describes the results for the other cases and discusses how the results are robust to adding common knowledge of rationality and removing a key assumption (called the overflow assumption). In Sections 3.3 and 3.4, I characterize the set of equilibrium structures that can be sustained as equilibria when the beliefs are chosen arbitrarily by the game theorist. In Section 3.5, I discuss when the equilibria have beliefs that are justified by some dynamic process.

The general contribution of this paper is that it takes a first step towards understanding network formation under certain types of uncertainty. A more specific contribution of this paper is its approach used to describe and account for uncertainty in the network game, i.e., the characterization of information and beliefs and how these relate to the decision making processes of the agents. This characterization is easily extended to other types of non-cooperative network games. For example, in otherwise identical games but with one-way flow, we can obtain similar qualitative results as those found here.[2] The reason is that, whether flows are one-way or two-way, the stability of a network depends on the ability of an agent to see beyond his own direct links.

This paper also contributes to the literature on non-Nash equilibrium concepts. Gilli (1999) remarked, "to study the notion of equilibrium in game theory, we should study the role of players' information in strategic situations" (185). The network game is a natural setting in which to examine how players' actions determine what they can learn, and how what they learn will affect how they act. I pay particular attention to the information received by the individuals and show how the equilibria are or are not sensitive to such information. To do this, I propose a generalized version of an existing non-Nash equilibrium concept, and apply this concept appropriately. This paper is thus a detailed application of an original non-Nash concept.

[2]The difference between the one-way and two-way games is that the strict Nash equilibrium network in the one-way game is a wheel network. Just as the center-sponsored star architecture is robust to large informational decreases in the two-way game, the wheel architecture is robust to large decreases in information in the one-way game. This paper only examines the two-way game.

3.2 MODEL AND NOTATION

3.2.1 Basics of the Network Game

Consider a game with set of agents $N = \{1, 2, \ldots, n\}$, where $3 < n < \infty$[3] and each player $i \in N$ must decide with which agents he wants to initiate a link. Agent i will want to link with other agents in order to access the network goods that those agents have. Each agent has a non-rivalrous network good that can only be transferred through connections in the network. Agent i will pay a marginal link cost c for each link he initiates. I will always assume that network goods flow in both directions, and there is no flow decay of the network good.

The value of other agents' network goods and the other agents' actions might not be known. As such, each i will have beliefs concerning certain possible states of the world. Denote v' to be an $n \times 1$ matrix containing values of network goods associated with N, and denote s' an $n \times n$ matrix describing the structure of the network. Row j in that matrix is the action of player j. A *state of the world* (v', s') is an network good value matrix (vector) v' and a network structure matrix s' combination. Denote the *true state of the world* (v^t, s^t). Example 3.2.1 illustrates a possible state of the world. The network structure of this example is illustrated in Figure 3.2.1.

[3]We will generally assume $n > 3$, yet in some pictures we use $n = 3$ to illustrate simple ideas.

Example 3.2.1: The following is a state of the world. Figure 3.2.1 shows how s^t in this example is illustrated in pictorially.

$$\left(v^t, s^t\right) = \left(\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right).$$

The state of the world might not be known will certainty, so player i might believe there to be many possible states of the world. Let V be the set of possible combinations of values of agents' goods and S be the set of possible network structures. Let π_i be a probability distribution that i has over the possible states of the world (V, S). Let Π_i be the set of probability assignments that i has over states of the world, and $\Pi = \times_{i \in N} \Pi_i$.

Let a_i be i's action, which is a $1 \times n$ matrix of zeros and ones, with a 1 in the jth place signifying i is initiating a link with j. While it costs c to pay for a link, i is always costlessly linked with himself. Let i's strategy set be $\{0, 1\}_{1 \times n}$, where the ith place must be 1 (because linked to oneself), so that the diagonal of s^t is ones.

I will assume u_i equals the sum of the values obtained by i (including his own good) through the network minus the cost of any links. In Example 3.2.1, if $c = 0.5$ then $u_1 = 6.5$ and $u_5 = 1$.

The *network game* G is a combination of utility maximizing agents N, sets of actions A, sets of states of the world (V, S), and for each $i \in N$, corresponding utility functions u_i, beliefs π_i over the states of the world, and belief updating functions h_i (to be described below). In abbreviated notation, $G = \left\langle N, A, (V, S), \Pi, (u_i)_{i \in N}, (h_i)_{i \in N} \right\rangle$.

3.2.2 Generalized Conjectural Equilibrium (GCE)

The approach used to find equilibria in this paper is as follows: pick a network structure and see if we can choose beliefs that sustain this structure as an equilibrium and are consistent with what the agents know about the network. The equilibrium concept closest to what I need is the *Conjectural Equilibrium* concept described by Battigalli, Gilli, and Molinari (1992) and Gilli (1999), yet this concept does not allow for incomplete information. I propose a new concept that will fit the needs of this paper. After defining a general incomplete-information, imperfect monitoring game, I define the Generalized Conjectural Equilibrium.

Definition of General Incomplete-information, Imperfect-monitoring Game (IIG): *The general incomplete-information, imperfect monitoring game is a combination*

$$\left\langle N, \Theta, A, \Pi, (u_i)_{i \in N}, (m_i)_{i \in N}, (h_i)_{i \in N} \right\rangle$$

where: N is a set of players; Θ is a set of states; A_i is the set of actions for $i \in N$ and $A = \times_{i \in N} A_i$; Π_i is i's set of probability distributions over $\{\Theta \times A\}$ and $\Pi = \times_{i \in N} \Pi_i$; $u_i : \{\Theta \times A\} \rightarrow R$ is i's utility function; $m_i : \{\Theta \times A\} \rightarrow M_i$ is i's message (or signal) function; and $h_i : \{\Pi_i \times M_i\} \rightarrow \Pi_i$ is i's beliefs-updating function or rule.

In general, it might be the case that $m_i(\theta, a) = m_i(\theta', a')$ for some (θ, a), $(\theta', a') \in \{\Theta \times A\}$. When the player's message is the same under two different states of the world, he will generally not be able to separately distinguish which of the two is the true state of the world. In this case of imperfect monitoring, each player then has an information partition $I_i(m_i(\theta, a)) = \{(\theta', a') \in \{\Theta \times A\} | m_i(\theta, a) = m_i(\theta', a')\}$. In words, given his signal m_i, i knows that the true state of the world is some $(\theta, a) \in I_i$, and I_t will generally be a strict subset of $\{\Theta \times A\}$.

Definition of Generalized Conjectural Equilibrium (GCE): *Fix $\theta \in \Theta$. A GCE of IIG is a $\langle (a_i^*, \pi_i^*)_{i \in N} \rangle \in \{A \times \Pi\}$ combination such that for each $i \in N$:*

(1) for all $a_i \neq a_i^*$, $a_i \in A_i$, it is true that

$$\sum_{(\theta', a') \in \{\Theta \times A\}} \pi_i^*(\theta', a') \, u(a_i^* | \theta', a') \geq \sum_{(\theta', a') \in \{\Theta \times A\}} \pi_i^*(\theta', a') \, u(a_i^* | \theta', a');$$

(2) $\pi_i^* = h_i(\pi_i^*, m_i(\theta, a^*))$, where $a^* = \times_{j \in N} a_j^*$;

(3) for any $(\theta', a') \in \{\Theta \times A\}$ that is assigned strictly positive probability by π_i^*, it must be that $Pr[(\theta', a') \in I_i(m_i(\theta, a^*))] = 1$.

Condition 1 of GCE states that the player is playing a best response to his conjecture concerning the true state of the world. Condition 2 states that the player's posterior beliefs equal his prior beliefs given his signal and his beliefs-updating rule. Condition 3 states that equilibrium beliefs must assign probability 1 to the observed signal. This third condition restricts $h_i(\cdot)$ to assign probability 0 to any $(\theta'', a'') \notin I_i(m_i(\theta, a^*))$. This h_i function updates i's beliefs given his message and knowledge of the game. There can be many methods that this function uses to assign probabilities, e.g., Bayes rule or some other rule. Notice that this function does not assume common knowledge of rationality. As is usual with solution concepts, there is no explanation why the equilibrium is reached but instead the concept says that if these actions and beliefs are played for some reason then there is no incentive to change behavior.

GCE is the appropriate concept to study networks under uncertainty. In general, an agent might not know the values of the other agents' network goods (incomplete-information). Agents will not generally observe the actions of all other players (imperfect-monitoring), and what the agents know about the network is likely to depend on with whom they are connected (they receive messages). And since we are motivating this study by a dynamic network formation game, our equilibrium concept need not impose equilibrium deductions (no imposed common knowledge of rationality).

The conjectural equilibrium from the literature is a GCE with only one possible state of the world $\Theta = \{\theta\}$. If $\Theta = \{\theta\}$ and common knowledge of rationality is added, we have Gilli's (1999) Strong Rationalizable Conjectural Equilibrium. In general, i's action will affect the messages and rational play of the others, and, hence, his own message and rational play. To get to Rubinstein and Wolinsky's (1994)

Rationalizable Conjectural Equilibrium, we must further impose that i's message be independent of his own action. Fudenburg and Levine's (1993) Self-confirming Equilibrium is an extensive form version of rationalizable conjectural equilibrium with the restriction that i's signal is the strategies that others play at all information sets that are reached with positive probability. Player i's beliefs are correct along the equilibrium path but not necessarily correct off that path.[4] A GCE is a Nash equilibrium when it is common knowledge that m_i contains the true θ and a for each i.

I explain how I apply the GCE to networks in Section 3.2.6.

3.2.3 Assumptions on Network Goods and Flows

The network goods are assigned randomly before actions are taken. Network good values are drawn randomly from $\{d, 2d, ..., D\}$. Agent i's good is non-excludable to anyone that is linked to i, but his good is excluded from any agent not connected to him either directly or indirectly agent. I assume that player i knows with certainty the value of his own network good and his action, but i might not know the values of the goods of other agents even though he might be connected with some subset of them. Further assume that $d < c < D$.

Notice, however, that i only receives the sum total of values so that there is no "double-counting" of goods. For example, in Figure 3.2.2(b), player 1 receives agent 3's network good through player 5 and player 2, but 1 does not receive double

[4] Although we use different notation than that used in the definitions of conjectural equilibrium, rationalizable conjectural equilibrium, and self-confirming equilibrium, the relationships are clear once the role of the signal is understood.

utility.[5] The agent knows his own utility at a given point in time. This utility, as stated above and known by the agent, is equal to the total flow of network good values received through the network.

I place a restriction on an agent's beliefs about the flows that he would receive in other networks: an agent knows what his total flow utility would be for each subset of his current direct links (whether or not he initiated the link), holding the rest of the network fixed. I call this restriction the *overflow assumption* because, as shown below, it allows the players to detect redundant links by the flows. A consequence of this restriction is that (i) the agent knows the utility flow from one of his direct links when removing all others (and the rest of the network fixed) and (ii) the agent knows the utility flow from any network with all but two of his direct links removed (again, holding the rest of the network fixed).

Consider Figure 3.2.2, where $v^t = (1, 1, 1, 1, 1)'$. Let f_1 be 1's utility flow (this is the flow of utility the player receives from other agents without subtracting the cost of paying for any links). In Figure 3.2.2(a), agent 1 knows that $f_1 = 4$. By the overflow assumption, he also knows that his total utility flow in this network if all links but his link with 2 were removed is $f_{1|2} = 2$. Similarly, $f_{1|5} = 2$. Since $f_{1|2} + f_{1|5} = f_1$, agent 1 knows he has no redundant links. In Figure 3.2.2(b), $f_1 = 3$, $f_{1|2} = 3$, and $f_{1|5} = 3$. A consequence of this 1 knows that his link with 5 can be removed without decreasing his total flow utility f_1. In Figure 3.2.2(c), $f_1 = 4$,

[5] For example, suppose player A is in Albania, B is in Bermuda, and C is in Canada. Each knows the weather in his country, and utility is obtained by knowing the weather in the states. Assume A and B both have direct links with C. That means that A, B, and C each know the weather in all three countries. Suppose now we add another link between A and B. Since A already knows the weather in Bermuda and Canada, this extra link with B provides no new utility.

Figure 3.2.2: Illustration of Link Flows

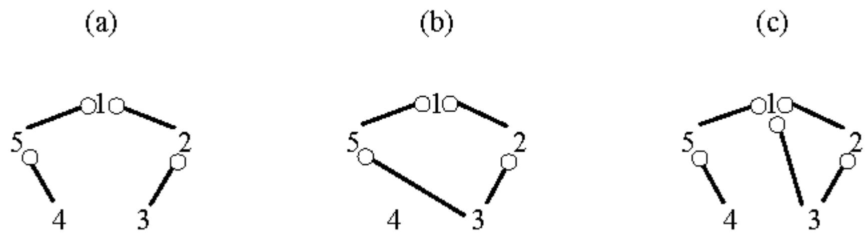


Table 3.2.1: Types of 1-link Revelation

Revelation	What i learns
G_{1AA}	if i initiated a link to j, then i learns of links that j has initiated
G_{1AC}	if i initiated a link to j, then i learns all j's connections
G_{1CA}	i learns j's link initiations
G_{1CC}	i learns j's link connections

Figure 3.2.3: Some Examples of Connectivity

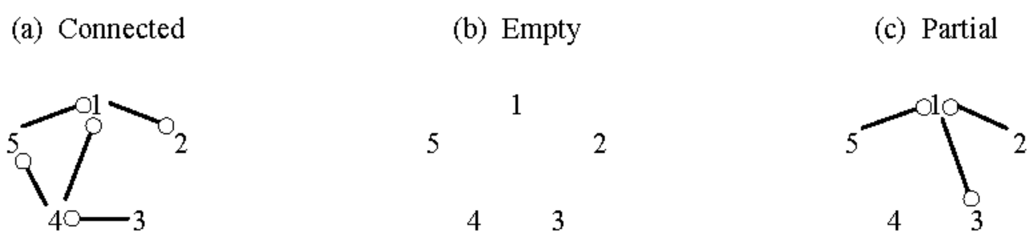
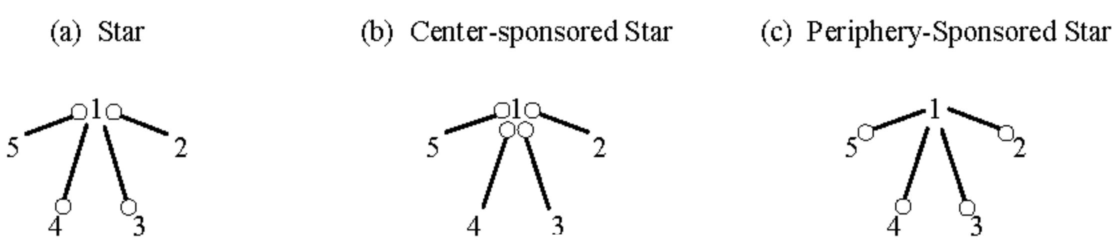


Figure 3.2.4: Center-sponsored, Periphery-sponsored, and Empty Network



$f_{1|2} = 2$, $f_{1|3} = 2$, and $f_{1|5} = 3$. By the overflow assumption 1 also knows $f_{1|2,3} = 2$, $uf_{1|2,5} = 4$, and $uf_{1|3,5} = 4$, where $f_{1|j,k}$ is 1's utility flow when all links but those with j and k are removed. He can then know by these flows that his link with either 2 or 3 can be removed without decreasing his total flow. Note that the agent not only knows if there is a redundant link, but he also knows which link is redundant.

3.2.4 Information Regimes and Revelation

An information regime is the information setting in which the game takes place, and we will consider four different information regimes in this paper. One information regime is the *full information regime*, where the true state of the world is known with certainty by each player. Denote this game G^{VS}, where the superscript says that the values and the structure are known with certainty. If s^t is known with certainty but v^t is not necessarily known, we say we are in the *structure regime*, denoted G^S. If v^t is known with certainty but s^t is not necessarily known, we say we are in the *value regime*, denoted G^V. If neither v^t nor s^t are known then we are in the *empty* or *null information regime*, denoted $G^{\emptyset}$.

While the information regime describes the setting in which the game takes place, we must also consider how agents learn about the network by actually being a member of the network. I use the term *revelation* to describe the information about the true state of the world that is revealed to the agent through his links. I say the network game has (i) ∞-*link revelation,* denoted G_{∞}, when to each i is revealed the value and action every j with whom i is either directly or indirectly connected (in a sense, to i is revealed everything about his own subnetwork but nothing about agents not connected with him); (ii) *1-link revelation,* denoted G_1, when to each i is revealed certain information concerning agents one link away (more said on this below); (iii) *0-link revelation,* denoted G_0, when to each i is only revealed with whom he is connected (whether or not he initiated that link).

Assume that with 1-link revelation, i always learns the network good values for all agents with whom i is directly connected (whether or not he initiated that link). But whether or not i knows more will depend on the type of 1-link revelation. I characterize four types of 1-link revelation corresponding to Table 3.2.1. Of these types of 1-link revelation, G_{1CC} has the (weakly) most information revealed to agent i. In G_{1CC}, for each i directly connected to some j, i learns all which k are directly connected to j. Notice that anything revealed in G_{1CA} is also revealed in G_{1CC} since any link initiation by j is a connection. But revealed in G_{1CC} and not in G_{1CA} are the links to j which j did not initiate. Similarly, G_{1AA} reveals the (weakly) least information since agent i has to initiate the link with j to learn his actions. Again, G_{1AC} yields (weakly) more information than G_{1AA} since it reveals the links which j did not initiate. Below I will denote $G_{1 \cdot A}$ to mean G_{1AA} or G_{1CA}.

3.2.5 Structure Features and Terminology

One "welfare" feature of a network is the network's connectedness. A network is *connected* if all every agent is either directly or indirectly connected with all others. A network is *empty* if there are no links. A network is *partial* if it is not connected and not empty. When I say a network is *non-connected*, it can be either partial or empty. A network is *minimal* if the removal of a link by some i results in the removal of access to i for some j's network good where $i \neq j$. An agent is *unconnected* or *separate* if he is not linked with any agents. The idea behind minimal networks is that "redundant" links are removed. Figure 3.2.3 illustrates some network types. A minimal network might be connected, partial, or empty. Figures 3.2.3(b) and 3.2.3(c) are minimal, but 3.2.3(a) is not minimal.

Star networks, as in Figure 3.2.4, are important networks. The center of a star is the sole means whereby the other agents in the network receive access to the other agents' network goods. Such an agent is the channel through with another agent to receive other agents' network goods. The center-sponsored and periphery-sponsored stars are one-channel networks. The non-channel agents in those stars that have only one direct connection will be called stems. Note that a network that is not connected can have a star subnetwork, as is the case in Figure 3.2.3(c).

More precisely: an agent i is a *channel* if there is some agent j that receives some agent k's network good only through i; an agent i is a *strict channel* if he is a channel to all agents with whom he is directly connected; an agent i is a *stem* if he has only one direct connection.

3.2.6 Applying GCE in the Network Game: Strict Network Equilibrium (SNE)

I apply the GCE concept in the network game with the additional restrictions and assumptions from Sections 3.2.2 and 3.2.3. An equilibrium of the network game will comprise actions and beliefs such that no agent has the incentives to change actions or alter beliefs. Each agent's action is a best-response to his beliefs about the values and structure of the network, and each agent's beliefs must be consistent with his message.

> **Definition of Strict Network Equilibrium (SNE):** *A SNE is a pure strategy GCE of the network game where (1) the best-response must be a strict best-response, (2) beliefs are restricted according to the overflow assumption, and (3) each individual's signal carries information entitled by the information regime and the revelation.*

The strictness stated in condition (1) is used for two reasons. First, strictness has nice refining power, as shown by BG. The number of possible network structures increases exponentially as n increases, and strictness nicely reduces the set of equilibria to be examined. Second, in a dynamic framework, strict equilibria are stable, absorbing points. What is meant in condition (3) is that the message will contain all available information that is permitted under the information regime and type of revelation. For example, if the information regime is that the values v^t are known and the revelation is ∞-link revelation, each agent's message contains v^t and the actions of all agents in his own subnetwork but nothing about the actions of agents not in his subnetwork. These characterizations will become clear in the next section through some examples.

In Sections 3.3, 3.4, and 3.5, we will distinguish between two types of SNE. Sections 3.3 and 3.4 looks at the structures of SNE with "arbitrary" but "consistent" beliefs, the set of which is denoted $\mathcal{S}$. The beliefs of these SNE are arbitrary in that they are chosen by the game theorist. They are, of course, consistent in that the beliefs and actions are correspond with the message and strict best-response optimization. In terms of our notation, a *SNE with arbitrary beliefs* can be thought of as a SNE where the priors are chosen by the game theorist and where the $h_i(\cdot)$ function is arbitrarily chosen by the game theorist.

Section 3.3 works through an extended example to illustrate the basic logic behind the characterization of $\mathcal{S}$ for a particular case. Section 3.4 works through the logic for the other cases. Section 3.5 examines SNE structures that can arise out of the dynamic process, denoted $\mathcal{S}^*$. Obviously, we are really interested in $\mathcal{S}^*$, but we look for $\mathcal{S}$ first for three reasons. First, $\mathcal{S}$ is easier to derive. Second, it turns out that $\mathcal{S}^* \subseteq \mathcal{S}$ (in fact, $\mathcal{S}^* = \mathcal{S}$ in many instances), so finding $\mathcal{S}$ can help us find $\mathcal{S}^*$. Third, further restrictions on h_i turn out to affect the SNE in complicated ways, so finding $\mathcal{S}$ might in some ways and instances be the best description of equilibrium structures (more on this below).

Our search for equilibria is as follows. The game type (e.g., G^V), type of revelation (e.g., $1CA$-link revelation), value range $\{d, ..., D\}$, c, and N are common knowledge. Each agent is assigned a network good value from $\{d, ..., D\}$, and each agent is assigned an action randomly to define an initial network structure. Given these values and actions, we ask if the structure can be sustained as an equilibrium.

If any agent (given his message which has his revelation) believes that he can strictly improve his expected utility by changing his action, then the network is not an equilibrium. Otherwise, the network structure is a SNE structure.

Note that agent i does not have beliefs over the other agents' beliefs, which rules out deductions based on equilibrium logic (which would further refine the set of equilibria in some cases). One reason for this is that network formation arises in an out-of-equilibrium dynamic process. At any point during a dynamic network game, a person may have just played what he perceived to be a best-response, but then learn that it was an action that made him strictly worse off. His action was not an equilibrium action, so he should not conclude that other players' current actions are equilibrium actions. Also, in the dynamic game, for i to try and use some equilibrium logic, he must keep track of all other players' beliefs—not just their actions. It was decided to focus more on the static equilibria that would correspond to the out-of-equilibrium concept and dynamics used in BG. Also, although the solution is static, I have the dynamic game in mind. We reconcile these by supposing that the agents are concerned with their instantaneous payoffs, that is, they are not concerned about the effect their actions might have on the other players' actions or beliefs.

The focus is on SNE structures because, even though beliefs determine the behavior, it is the resulting structure that determines the payoffs. An infinite number of beliefs might be able to sustain a single structure as an equilibrium, so instead of focusing on the specific beliefs, we characterize necessary conditions of the beliefs that are consistent and sustain the structure as an equilibrium.

3.3 EXAMPLES TO ILLUSTRATE FINDING SNE FOR CASE I

As illustrated in Figure 3.2.2, when link overflow occurs, an agent will know if one of his immediate links is unnecessary. It turns out that this assumption means that any SNE will be minimal.

Lemma L1: *Any SNE is minimal.*

Proof: Suppose a SNE network is not minimal. By the definition of a minimal network, non-minimal means that there is some i that has initiated a link with some k such that i and k are still connected without the link initiated by i. Either (1) k has also initiated a link with i or (2) there is some j through which both i and k can access each other's good without the direct link by i. In each case, i is strictly better off by removing the link with k, and i knows this fact by his ability to determine link overflow. This contradicts the strictness. ■

Lemma L2: *Each agent in a SNE is either a strict channel, a stem, or unconnected.*

Proof: In a SNE, a player can have either 0, 1, or more than 1 direct links. By definition, he is unconnected if he has 0 links, or he is a stem if he has 1 link. Consider i who has more than 1 link, say with j and k. Suppose i is not a strict channel. Since i is not a strict channel, there must be some agent k directly connected with i who receives access to some j's network good by some path other than through i. Then the network is not minimal, which violates SNE by L1. Any agent that has more than two links must therefore be a strict channel. ■

L1 and L2 imply that to find the set of SNE for any network game we need only look at minimal networks with only stems, strict channels, and unconnected agents.[6] The set of equilibria will depend on the actual values of the agents network goods, so we break down our description into four cases. Case I is when all agents have actual values greater than or equal to c. Case II is when exactly one agent has value less than c and all others have high-valued goods. Case III is when at least two but less than n agents have low-valued goods (i.e., there are between 1 and $n-2$ agents with high-valued goods). Case IV is when all agents are low-valued.

In the rest of this section, I will illustrate the process of finding the set of equilibrium structures for Case I under varying types of uncertainty and revelation. Consider Example 3.3.1, which will be used in this section to illustrate the SNE sets.

Example 3.3.1. Suppose the following network, represented in Figure 3.3.1, where $n = 5$, $d = 1$, $D = 2$, $c = \frac{3}{2}$, and:

$$\left(v^t, s^t\right) = \left(\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right).$$

[6] Strictness is not enough ensure minimality, as illustrated in Example 3.5.1.

Figure 3.3.1: Illustration of Example 3.3.1

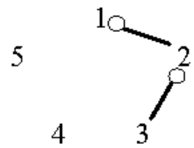


Figure 3.3.2: Modification of Example 3.3.1

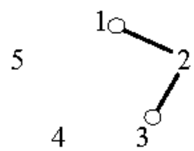


Figure 3.3.3: Illustration of Example 3.3.2

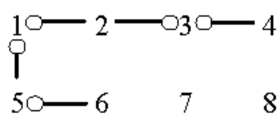
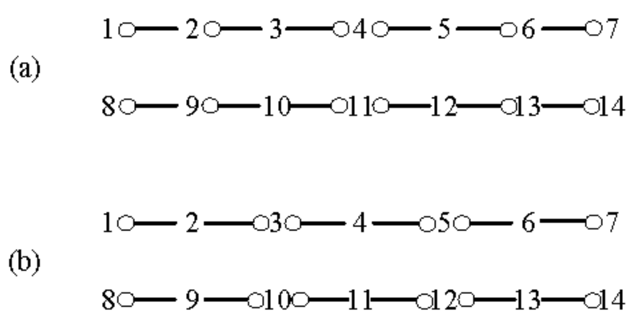


Figure 3.3.4: Illustration for P3.7



3.3.1 Full Information Regime Case I: G^{VS}

Consider Example 3.3.1 in the full information case G^{VS}. Since (v^t, s^t) is known by all, agent 4 knows that he is better off by initiating a link to $j \in \{1, 2, 3\}$ and 5 because he knows that the flow he receives will be greater than the cost. So this network is not in $\mathcal{S}^{VS}$. Notice also that the agents in the subnetwork will also want to initiate with 4 and 5, so the connected agents also have the incentive to deviate from the existing a. This illustrates that in this full information setting with all agents high-valued, we expect the network to be connected. This intuition is correct.

It turns out that the only structures that can be sustained as equilibria in this case are center-sponsored stars. There are n of them since each agent can be the center. In each of these, $\pi_i = (v^t, s^t)$ with probability 1. I use the notation "$i \to j$" to mean "i is the link-initiator in a direct link with j." N_i will be used to denote all agents linked, either directly or indirectly, to i, i.e., N_i is the set of agents in i's subnetwork.

Proposition P3.1: *For Case I, S^{VS} is the set of connected, center-sponsored stars.*

Proof: *(Necessity.)* Suppose you have a SNE in $\mathcal{S}^{VS}$ that is not a connected, center-sponsored star. Either the network is not connected, or the network is connected but is not a center-sponsored star. If the network is not connected then any member of a subnetwork is strictly better off when initiating a link with someone not in his subnetwork. This is known by full revelation, hence the network must be connected.

Suppose the network is connected but not a center-sponsored star. By L1, the network must be minimal. By L2, the network must only have stems and channels since there are no unconnected agents in a connected network. If the network has only one channel, then there must be a link-initiating stem since it is not a center-sponsored star. This stem knows he can remove his link with the channel and link with one other stem and receive the same payoff. This violates the strictness. If the network has more than one channel, say i and j such that $i \to j$, then i can remove his link with j and link with some k directly connected with j and receive the same payoff. This violates the strictness. Hence, you cannot have a SNE in $\mathcal{S}^{VS}$ which is not a connected, center-sponsored star.

(Sufficiency.) A connected, center-sponsored star is a SNE since each player is playing a strict best-response given his knowledge. ■

3.3.2 Structure Regime for Case I: G^S

Suppose in Example 3.3.1 we now are in G^S where the structure is revealed to everyone. Consider agent 1. He knows that $2 \to 3$ because s^t is common knowledge, so 1 knows that if he switches his link from 2 to 3 he will be no worse off, which violates the strictness condition for SNE. This holds for any link-initiating stem in the subnetwork (as long as the subnetwork has more than two agents) which suggests that any network with a link-initiating stem cannot be a SNE. But we can say even more. In fact, any subnetwork must be an center-sponsored star in order for no agent in that subnetwork to meet the strictness condition of SNE. The intuition comes from P3.1. Notice that this holds for 0-, 1-, and ∞-link revelation.

Consider agent 4. He knows that 1, 2, and 3 are connected by G^S, but he does not know the summed valued of their subnetwork. He does know, however, that the lowest the sum could be is $3d > c$, so 4 knows that the least flow he can receive from linking with, say, player 1 is greater than the cost of initiating that link. He will then link with $i \in \{1, 2, 3\}$.

Your intuition might tell you then that $\mathcal{S}^S$ is the set of connected, center-sponsored stars, but that would be incorrect for two reasons. First, consider the empty network. If everybody believed with probability one that everyone else had values less than c, then the network is a SNE. For example, if

$$\pi_1 = \left\{ \left(\begin{bmatrix} 2 \\ d \\ d \\ d \\ d \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \text{ with probability 1} \right\},$$

then player 1 would not initiate any links. If players 2, 3, 4, and 5 had similar beliefs then the empty network in this example is a SNE. Hence, the empty network is in $\mathcal{S}^S$ so long as $d < c$.

Second, what if instead $d = \frac{1}{4}$ but the rest of the network was the same? Player 1 could still switch from 2 to 3 so the subnetwork must still be a center-sponsored star. But player 4 might not want to initiate with the 1-2-3 subnetwork if his expected flow from linking with a player in that subnetwork is less than c, which is true if

$$\pi_4 = \left\{ \left(\begin{bmatrix} d \\ d \\ d \\ 2 \\ d \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \text{ with probability } 1 \right\}.$$

In this case, the expected benefit from linking with a member of that subnetwork is $3d < c$. Since 4 receives no revelation, he has no new information to contradict his beliefs. Given these beliefs, he will not initiate any links.

Proposition P3.2: *For Case I, S^S is contains (a) the empty network, (b) all partial networks where any subnetwork is a center-sponsored star with fewer than $\frac{c}{d}$ agents, and (c) connected, center-sponsored stars.*

Proof: *(Sketch.)* (a) In our discussion above, we illustrated how we can choose beliefs for each agent such that the empty network is a SNE. (b) In our discussion above, we illustrated how a partial network in which the subnetwork is a center-sponsored star with fewer than $\frac{c}{d}$ agents is a SNE, so long as the beliefs of any agent i outside subnetwork N_j are such that i believes that all $j \in N_j$ are unconnected and low-valued. A two-person, minimal subnetwork is a center-sponsored star with one stem. (c) Obvious. ■

Notice that a non-connected network with a center-sponsored star subnetwork might not exist. In our example with $d = 1$, $D = 2$, and $c = \frac{3}{2}$, any subnetwork will have more than $\frac{c}{d}$ players since $\frac{c}{d} < 2$. Also notice that if such a network does exist, there might be more than one center-sponsored star subnetwork.

Corollary C3.1: $\mathcal{S}^S \supset \mathcal{S}^{VS}$.

Proof: *(Right to Left.)* Any SNE in $\mathcal{S}^{VS}$ is in $\mathcal{S}^S$ because any connected, center-sponsored star is in $\mathcal{S}^S$. *(Left to Right.)* From PI-2, $\mathcal{S}^S$ contains the empty network which is not in $\mathcal{S}^{VS}$, so $\mathcal{S}^S \supset \mathcal{S}^{VS}$. $\square$

Notice that the set of SNE structures for G^S is the same under 0-, 1-, and ∞-link revelation. Knowing your own subnetwork's structure means that you can always switch unless it is a center-sponsored star. To do such switching, you do not need to know the actual values of agent j in N_i. All you need to know is that if you switch a link from l to j you can receive the same payoff. Also, it matters that i does not receive any revelation about the values of agents who are not in his network.

3.3.3 Null Regime for Case I: $G^{\emptyset}$

Equilibrium Structures for $G^{\emptyset}$ and $G^{\emptyset}_{1 \cdot C}$. Let us now add some more uncertainty in Example 3.1 to get from G^S to $G^{\emptyset}_{\infty}$. Consider agent 4 with

$$\pi_4 = \left\{ \left(\begin{bmatrix} d \\ d \\ d \\ 2 \\ d \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \text{ with probability } 1 \right\}.$$

Since agent 4 receives no revelation, he believes he is strictly worse off by initiating links. So he will not initiate any links.

Consider agent 1. Suppose

$$\pi_1 = \left\{ \left(\begin{bmatrix} 2 \\ 2 \\ 2 \\ d \\ d \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right) \text{ with probability } 1 \right\}.$$

Although 1 does not want to initiate any new links with agents not in his subnetwork, he can switch his link from 2 to 3. This violates the strictness of SNE, so this network is not a SNE. We can deduce from this that any subnetwork must be a center-sponsored star. Similar logic ends up showing that $\mathcal{S}^{\emptyset}_{1\cdot C} = \mathcal{S}^{\emptyset}_{\infty}$.

Proposition P3.3: *For Case I, $S^{\emptyset}_{\infty}$ is the set of empty, partial, and connected minimal networks in which any subnetwork (including the connected network) is a center-sponsored star.*

Proof: *(Necessity.)* Suppose the SNE network has a subnetwork which is not a center-sponsored star. By L1, the network must be minimal. By L2, we need only look at unconnected, stems, and strict channels. If it is not a center-sponsored star, then either (i) there is a link-initiating stem in a subnetwork or (ii) there is more than one strict channel. (i) If the subnetwork has only two members then we are fine. If it has three or more then if there is a link-initiating stem $i \to l$, then i, by ∞-link revelation, i knows all j directly connected to l. Player i knows that he can switch from l to j and receive the same payoff. This violates strictness of SNE.

A contradiction. (ii) Suppose at least two strict-channels. By definition of strict channels, two of those strict channels, i and k, must be directly connected. By ∞-link revelation, i knows all j on the other side of k. He also knows that he can switch his link from k to j and receive the same payoff. This contradicts the strictness for SNE.

(Sufficiency.) The empty network is a minimal network without subnetworks, so it meets the criterion. It is a SNE if each i believes all other agents are unconnected with low-valued goods. This is so because no i receives revelation that contradicts his beliefs and is already playing a best-response given these beliefs. A partial network with center-sponsored star subnetworks is also a SNE if each connected agent believes any $j \notin N_i$ is unconnected and low-valued. Player i does not receive any contradictory revelation, so he does not want to initiate any new links. The connected, center-sponsored star is a SNE because each person is playing a best response given their knowledge of (v^t, s^t). ■

Proposition P3.4: $\mathcal{S}^{\emptyset}_{1 \cdot C} = \mathcal{S}^{\emptyset}_{\infty}$.

Proof: We need to show that $\mathcal{S}^{\emptyset}_{1 \cdot C}$ is the set of all empty, partial, and connected minimal networks in which any subnetwork (including the connected network) is a center-sponsored star. *(Necessity.)* The same logic as in P3.3 can be used. Unless the subnetwork structure is a center-sponsored star, by $1 \cdot C$-link revelation, some agent in a subnetwork will know that he can switch links to another person in his subnetwork. *(Sufficiency.)* Again, each agent, if he believes all others to be unconnected and low-valued, will not initiate new links. Center-sponsored star subnetworks will prevent link switching. ■

Corollary C3.2: *For Case I,* $S^{\emptyset}_{1 \cdot C} = S^{\emptyset}_{\infty} \supseteq S^{S}$.

Proof: *(Right to Left.)* Any SNE in $\mathcal{S}^S$ is i $\mathcal{S}^{\emptyset}_{\infty}$ because (i) the empty network is in $\mathcal{S}^{\emptyset}_{\infty}$, (ii) partial networks with center-pointing star subnetworks of size $\frac{c}{d}$ are in $\mathcal{S}^{\emptyset}_{\infty}$, and (iii) the connected, center-sponsored star is in $\mathcal{S}^{\emptyset}_{\infty}$. This also holds for $\mathcal{S}^{\emptyset}_{1 \cdot C}$ by P3.4. *(Left to Right.)* Depending on the actual c, d, D, and n, this relationship can be strict. If $\frac{c}{d} > n-1$ then $\mathcal{S}^{\emptyset}_{\infty} = \mathcal{S}^{S}$. If $\frac{c}{d} \leq n-1$ then there is a partial network with subnetwork of size $n-1$ which is in $\mathcal{S}^{\emptyset}_{\infty}$ but not in $\mathcal{S}^{S}$. ■

Equilibrium Structures for $G^{\emptyset}_{1 \cdot A}$. With $1 \cdot A$-link revelation, our SNE set will change again in Example 3.3.1. Player 4 still receives no revelation about the 1-2-3 subnetwork, so if he assigns full probability to 1, 2, 3,and 5 being low-valued and unconnected, player 4 will not want to initiate with any of them. Now think of player 1. By $1 \cdot A$-link revelation, 1 will know that $2 \rightarrow 3$ because $1 \rightarrow 2$. So 1 can switch his link from 2 to 3, which violates the strictness.

But what if the network in Example 3.3.1 were the following

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which is shown in Figure 3.3.2. Player 4 still might want to remain unconnected. Player 1 would know by his flow from 2, that 2 is connected with at least one other player (since the flow is higher than D). But by $1 \cdot A$-link revelation, 1 will not know if 2 is connected with 3, 4, or 5. Suppose π_1 assigns equal probability to the following states of the world:

$$\left(\begin{bmatrix} 2 \\ 2 \\ 2 \\ d \\ d \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right), \left(\begin{bmatrix} 2 \\ 2 \\ 2 \\ d \\ d \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right),$$
$$\left(\begin{bmatrix} 2 \\ 2 \\ d \\ 2 \\ d \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right), \left(\begin{bmatrix} 2 \\ 2 \\ d \\ 2 \\ d \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right),$$
$$\left(\begin{bmatrix} 2 \\ 2 \\ d \\ d \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right), \left(\begin{bmatrix} 2 \\ 2 \\ d \\ d \\ 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}\right),$$

and zero probability to all other states. With these beliefs, 1 is indifferent between linking with 2 and the player on the other side of 2, but he does not know which player is on the other side of 2. With π_1, if he removes his link with 2 and links to some other player $m \in \{3, 4, 5\}$ then his expected payoff is $\frac{2}{3}(d) + \frac{1}{3}(4)$, with the 4 being the flow he will receive from a successful switch. His expected cost is the flow of 4 he gives up with certainty. Since $d < 4$, he is strictly worse off in expectation by attempting the switch.

The issue illustrated here is whether two link-initiators are directly connected. If they are connected then one of them, by $1 \cdot A$-link revelation, knows with certainty of a successful switch which violates the strictness of SNE. So a necessary condition for a SNE in $\mathcal{S}^{\emptyset}_{1 \cdot A}$ is that there cannot be two directly connected link-initiators. In fact, in Case I, any non-connected network without two directly connected link-initiators is a SNE. If the network is not connected, then the flow is never high enough to convince i that the network must be connected. If i believes there is a positive probability that there is some agent $m \notin N_i$ and believes that m to be low-valued and unconnected, then i might be afraid to switch links.

Things are trickier when the network is connected. If the flows are sufficiently high, then by the flow i receives, he might be able to know that the network is connected. In our example, since all agents have goods valued at 2, if the network was connected, each i would know it was connected by the flow being so high. In this case, i knows the network is connected, so he might be able to make a successful switch. This is true under certain conditions of the connected network.

Condition C1′ defines these necessary aspects of the network. Part (a) of C1′ is met if the network is not connected or if the network is connected but the flows are sufficiently low as to prevent i from knowing that the network is connected. Part (b) considers when the network is connected *and* the agent knows it is connected. Then there are special conditions to be met to prevent link switching. In that condition,

f_l is the flow received by i through his link with l, v_l^t is the true value of l's good, L_i is the set of agents directly linked with i, and $|L_i|$ is the number of agents in L_i. Given C1$'$, I state our next proposition characterizing $\mathcal{S}^{\emptyset}_{1\cdot A}$. The proof of this is very detailed, but it is straightforward once we have proven the characterization of $\mathcal{S}^{\emptyset}_{0}$. Intuition is provided after explaining C1.

Condition C1$'$: *C1$'$ holds if for each i either (a) or (b) is true:*

(a) if $\sum_{l\in L_i}\left[up\left(\frac{f_l-v_l^t}{D}\right)\right] < n-1-|L_i|$.

(b) if $\sum_{l\in L_i}\left[up\left(\frac{f_l-v_l^t}{D}\right)\right] = n-1-|L_i|$, *then*

(i) if exactly one $l\in L_i$ *has* $up\left(\frac{f_l-v_l^t}{D}\right) > 0$, *it is also true that* l *initiated the link with* i, *or*

(ii) if more than one $l\in L_i$ *has* $up\left(\frac{f_l-v_l^t}{D}\right) > 0$ *and* $up\left(\frac{f_l-v_l^t}{D}\right) = 1$ *for each of those* l, *it is also true that at least one of those initiated the link with* i, *or*

(iii) if more than one $l\in L_i$ *has* $up\left(\frac{f_l-v_l^t}{D}\right) > 0$ *and at least one has* $up\left(\frac{f_l-v_l^t}{D}\right) > 1$.

The $up\left(\frac{f_l-v_l^t}{D}\right)$ term tells i the least number of agents that are beyond his link with l. If the sum of these is less than $n-1-|L_i|$, as for C1$'$(a), then it is possible that i's subnetwork is not connected. If this is the case then i might be afraid of switching or initiating a new link because he might believe any $j\notin N_i$ to be low-valued. On the other hand, if the sum equals $n-1-|L_i|$ then i knows that his subnetwork is connected, so in order for him to not want to switch, some other conditions must be met. C1$'$(b)-(i) describes when i has exactly one link with a j that is not a stem. If he switches then he will switch to someone on the other side of j, but if j initiated the link with i then any removal of a link means i is not getting someone's good. C1$'$(b)-(ii) describes when i has more than one $j \in L_i$ where j is a channel. If this condition is met then one of those j's initiated a link with i, so that i does not want to switch for fear of linking with that agent on the other side of that j, thereby becoming worse off. C1$'$(b)-(iii) describes when there is some channel j which has more than 1 agent on the side opposite i. Now if i switches more than one link, he might hit two or more on that other side and lose some utility.

In a randomly assigned structure, C1$'$(a) will generally be met. However, we must consider all cases, so C1$'$(b) considers the special case when C1$'$(a) is not met.

Proposition P3.5: *For Case I, $S^{\emptyset}_{1\cdot A}$ contains all minimal networks in which (i) no two link-initiators are directly connected and (ii) C1$'$ is satisfied.*

Corollary C3.3: *For Case I, $S^{\emptyset}_{1\cdot A} \supset S^{\emptyset}_{1\cdot C}$.*

Proof: *(Right to Left.)* Any minimal network with center-sponsored stars does not have two directly connected link-initiators, so any SNE in

$\mathcal{S}^{\emptyset}_{1\cdot C}$ is also in $\mathcal{S}^{\emptyset}_{1\cdot A}$. *(Left to Right.)* A partial network with periphery-sponsored star subnetworks is in $\mathcal{S}^{\emptyset}_{1\cdot A}$ but not in $\mathcal{S}^{\emptyset}_{1\cdot C}$. Such a network always exists. ■

Equilibrium Structures for $G^{\emptyset}_{0}$. Let us look at Example 3.3.1 with 0-link revelation. Agent 4 again will not initiate new links if he believes all others are unconnected and low-valued. Things are different for agent 1 as compared to our last case. Agent 1 receives flow of 4 through his link with 2, but he does not know with whom 2 is connected. He knows (since $4 > D$) that 2 must be connected to another person; maybe he is connected to all others or just one other. Let 1's beliefs be the same as those formally listed when looking at the equilibrium structures for $G^{\emptyset}_{1\bullet A}$. In this event, 1 will not want to switch because he is strictly worse off in expectation. It turns out that this network is a SNE in $\mathcal{S}^{\emptyset}_{0}$. In fact, any non-connected network that is minimal is in $\mathcal{S}^{\emptyset}_{0}$, and many connected networks are also in $\mathcal{S}^{\emptyset}_{0}$. I again state the formal condition and then give the proposition.

Condition C1: *C1 holds if for each i either (a) or (b) is true:*

(a) if $\sum_{l\in L_i}\left[up\left(\frac{f_l}{D}\right)-1\right] < n-1-|L_i|$.

(b) if $\sum_{l\in L_i}\left[up\left(\frac{f_l}{D}\right)-1\right] = n-1-|L_i|$, *then*

(i) if exactly one $l \in L_i$ has $up\left(\frac{f_l}{D}\right)-1>0$, it is also true that l initiated the link with i, or

(ii) if more than one $l \in L_i$ *has* $up\left(\frac{f_l}{D}\right) - 1 > 0$ *and* $up\left(\frac{f_l}{D}\right) - 1 = 1$ *for each of those* l*, it is also true that at least one of those initiated the link with* i*, or*

(iii) if more than one $l \in L_i$ *has* $up\left(\frac{f_l}{D}\right) - 1 > 0$ *and at least one has* $up\left(\frac{f_l}{D}\right) - 1 > 1$*.*

Proposition P3.6: *For Case I, any minimal network is in* $S_0^{\emptyset}$ *if and only if the network satisfies C1.*

Proof: *(Necessity.)* By L1, we know any SNE network must be minimal. Consider a minimal network where C1 is violated. This occurs when $\sum_{l \in L_i} \left[up\left(\frac{f_l}{D}\right) - 1\right] = n - 1 - |L_i|$ for any i, and either (i) $|L_i| = 1$ and $i \to l$ or (ii) $up\left(\frac{f_l}{D}\right) - 1 = 1$ and $i \to l$ for all $l \in L_i$.

(a) Suppose $|L_i| = 1$ and $i \to l$. Since i knows by the flow that the network is connected, he can remove his link with l and initiate the link with any $m \neq l$ and receive the same payoff while paying the same cost. This violates the strictness condition for equilibrium.

(b) Suppose $up\left(\frac{f_l}{D}\right) - 1 = 1$ and $i \to l$ for all $l \in L_i$. By the flow, i knows that each of his links has one and only one m on its other side. Knowing this and since i has initiated all links, he can remove all of his links and initiate links with all $m \notin (L_i \cup \{i\})$ and receive the same payoff while paying the same cost. This violates the strictness condition for equilibrium.

(Sufficiency.) Need to show that if C1 holds, we can choose $\left\langle (a_i^*, \pi_i^*)_{i \in N} \right\rangle$ that makes any minimal network a SNE.

Consider π_i such that i believes (i) that any unconnected agent has good valued less than c with probability 1 and (ii) that any agent not in his subnetwork is unconnected with probability 1, and (iii) there is no m on the other side of some $l \in L_i$ such that i knows with probability 1 that m is on the other side of l.

(C1(a).) Consider a minimal network where C1(a) is satisfied. By L2, i must either be a stem, a strict channel, or unconnected.

Assume i is a strict channel. Choose π_i such that (i), (ii), and (iii) from the previous paragraph are satisfied. By C1(a), $\sum_{l \in L_i} \left[up\left(\frac{f_l}{D}\right) - 1 \right] < n - 1 - |L_i|$, and since the $LHS < RHS$, i believes there is positive probability that his subnetwork is not connected. By 0-link revelation, i does not know who is on the other side of any $l \in L_i$, nor does i receive any information about who is unconnected. It follows that (i), (ii), and (iii) are not contradicted. Hence, $\pi_i = h_i\left(\pi_i^*; a^*, v^t\right)$. Player i is strictly worse off in expectation should he initiate a link with some $k \in N \backslash \left(L_i, \{i\}\right)$. This is so because if i links with someone new, he expects to receive value less than c, and if i links with someone already connected to him, he pays

c but gets no new value. So we need only be concerned about when i might want to remove a link from some $l \in L_i$ and link to someone on the other side of that l to receive the same payoff. First, notice that $f_l > c$ (Case I). Second, notice that player i does not know which m might be on the other side of that l, so removing the link with l and initiating a link to m yields strictly less than F_l in expectation since there is positive probability that i might instead link with some agent not on the other side of l. Since f_l with certainty is greater then the expected payoff by switching, i is playing a strict best response. Notice that this reasoning already includes the case where i is a stem with one link.

Consider unconnected agent i. Set π_i such that with probability 1 that all agents have good valued less than c and that the network is empty. By 0-link revelation, i receives no information to contradict π_i, so π_i is a fixed point of $h_i(\cdot)$. Given π_i, i's believes he is strictly worse off by forming any links, thus no links is a strict best response.

(C1(b).) Consider a minimal network where C1(b) holds. C2(b) implies that $\sum_{l \in L_i} \left[up\left(\frac{f_l}{D}\right) - 1 \right] = n - 1 - |L_i|$. Thus i knows the network is connected, and we need only consider when i wants to switch a link.

(i) If exactly one $l \in L_i$ then i is a stem where $l \to i$. In this case, i already receives the most possible flow and pays the least possible amount. Any link initiation by him would make him strictly worse off, so he is already playing a strict best response.

(ii) Consider now if at least two $l \in L_i$ have $up\left(\frac{f_l}{D}\right) - 1 > 1$, where each of those l is such that $up\left(\frac{F_l}{D}\right) - 1 = 1$, and one of them $l' \rightarrow i$. For any l where $up\left(\frac{f_l}{D}\right) - 1 = 1$, i knows that there is at least one other agent $m \in N \backslash (L_i, \{i\})$ on the other side of l, but he does not know which particular agent m is behind which particular link l because of 0-link revelation. If i were to remove a link with $l \neq l'$ and link with some $m \in N \backslash (L_i, \{i\})$, there is positive probability that i will link with k on the other side of l', in which case, i is strictly worse off because he pays c but gains no additional value. So i is already playing a strict best response. Choose π_i such that the network is connected, any player $l \in L_i$ is assigned value $\min\{D, f_l\}$, and any player $m \in N \backslash (L_i, \{i\})$ is not assigned connection with a particular l with probability 1. Under 0-link revelation these beliefs will not be contradicted, so $\pi_i^* = h_i(\pi_i^*, a_i^*)$.

(iii) Consider now if one or more $l \in L_i$ has $up\left(\frac{f_l}{D}\right) - 1 > 0$ and at least, say l' one has $up\left(\frac{f_l}{D}\right) - 1 > 1$. Player i knows that the network is connected, so he believes there are no unconnected agents for whom to initiate links. From his flow, he knows that there are at least two agents on the other side of l'. Choose π_i such that no particular two agents are believed to be on the other side of l' with probability 1.

If i were to remove x links he has initiated and initiate y links where $y > x$, he knows he is strictly worse off because he pays more than he currently pays and gets strictly less value (because the network is minimal).

If $y < x$, he is strictly worse off because, by Case I, any flow is greater than c and he would be losing some profitable flow (because the network is minimal). Consider $y = x$. If $x = y = 1$. If he removes his link with any l then there is positive probability that he links with some k not on the other side of l but instead on the other side of $l' \in L_i$ and receives zero while paying c. This makes i strictly worse off in expectation. The same holds for when he removes a link with l' and links with an agent on the other side of some l. If $x = y \geq 2$. When at least two links are switched, there is a positive probability that i links with both m on the other side of l'. In that case, i initiates a link that gains zeros but pays c. This makes him strictly worse off. Because he is strictly worse off in expectation, i's best response is to not change his links. With 0-link revelation, he receives no information that contradicts his beliefs, so $\pi_i^* = h_i(\pi_i^*, a_i^*)$. ■

Before showing $\mathcal{S}_0^{\emptyset} \supset \mathcal{S}_{1 \cdot A}^{\emptyset}$, I describe the relationship between C1 and C1′.

Lemma L3: *C1′ ⇒C1, but C1 ⇏C1′.*

Proof: (1) Let C1′ be satisfied. If $up\left(\frac{f_l}{D}\right) - 1 < up\left(\frac{f_l - v_l^t}{D}\right)$ for any l, then C1(a) is satisfied. If $up\left(\frac{f_l}{D}\right) - 1 = up\left(\frac{f_l - v_l^t}{D}\right)$, then either C1′(a) is satisfied or C1′(b) is satisfied. So we need only show that $up\left(\frac{f_l}{D}\right) - 1 \leq up\left(\frac{f_l - v_l^t}{D}\right)$. Since the RHS is lowest when $v_l^t = D$, we need only check that case:

$$up\left(\frac{f_l - v_l^t}{D}\right) = up\left(\frac{f_l - D}{D}\right) = up\left(\frac{f_l}{D} - 1\right) = up\left(\frac{f_l}{D}\right) - 1.$$

(2) The second statement is true if there exists some (v^t, s^t) that satisfies C1 but does not satisfy C1′. Consider the following example with $v^t = [2, 2, 2, 2]$, $D = 5$, and $s^t =$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

C1(b) part (ii) is satisfied for players 2 and 4, and C1(a) is satisfied for players 1 and 3. C1′(b) part (ii) is violated because player 1 has exactly one link (with player 3) where $up\left(\frac{f_l - v_l^t}{D}\right) > 0$, but player 1 initiates that link. ■

Corollary C3.4: *For Case I,* $S_0^{\emptyset} \supset S_{1 \cdot A}^{\emptyset}$.

Proof: *(Right to Left.)* Any network in $\mathcal{S}_{1 \cdot A}^{\emptyset}$ is in $\mathcal{S}_0^{\emptyset}$ because (i) any network in $\mathcal{S}_{1 \cdot A}^{\emptyset}$, meets C1 (by L3) and (ii) any minimal network without two directly connected link-initiators is minimal. *(Left to Right.)* We only need to show a network that always exists that is in $\mathcal{S}_0^{\emptyset}$ but not in $\mathcal{S}_{1 \cdot A}^{\emptyset}$. Any minimal network that has two directly connected link initiators and satisfies C1(a) is in $\mathcal{S}_0^{\emptyset}$ but is not in $\mathcal{S}_{1 \cdot A}^{\emptyset}$ because one of those two link initiators knows he can switch and receive the same payoff, thereby violating strictness. Such a network always exists. ■

3.3.4 Structure not Commonly Known for Case I: G^V

Things get trickier as we examine the relationships for when the values are known but the structure is not (G^V) because, depending on the exact values and structure of the network, i might be able to tell with whom he is connected even though he does not know the exact structure. Consider Example 3.3.2.

Example 3.3.2: Suppose a network with $v^t = [1, 3, 4, 5, 8, 10, 20.3, 22]$ and s^t according to Figure 3.3.3. Assume the game is G_0^V and $c = 0.9$. Consider agent 4. He receives flow of 26 through agent 3 and knows that agent 3's value is 4 (remember v^t is common knowledge in G^V). He knows then that the sum of values on the other side of agent 3 is 22. However, there are two agent combinations that add up to 22: agent 8 by himself and the sum of agents 1, 2, 5, and 6. Therefore agent 4 cannot deduce exactly which agents are connected to agent 3.

If all agents had the same problem identifying the connections, then we can have a SNE. This condition is not met in this example. Agent 1 receives a flow of 12 through agent 2 and knows that 2's value is 3, so agent 1 knows that agent 2 must be linked to at least one other agent. The sum of values of these agents must be 9 (since $12 - 3 = 9$), and since only the sum of agents 3 and 4's values is 9, agent 1 deduces that agents 3 and 4 must be on the other side of agent 2. Although he does not know exactly how they are connected, he knows who is connected. By similar logic, 1 knows that 6 must be on the other side of 5. Since agent 1 knows this, he can remove his link with 5 and initiate a link with 6 to receive the same payoff.

There is another reason why this cannot be a SNE. Agents 7 and 8 are unconnected and know that they can link with any other agents and receive higher utility.

Proposition P3.7: *For Case I, (a)* $S_0^{\emptyset} \supset S_0^V$ *and (b)* $S_0^V \supseteq S_{1\cdot A}^V \supseteq S_{1\cdot C}^V = S_{\infty}^V = S^{VS} \neq \emptyset$.

Proof: (a) If there is any unconnected agent, that agent knows that he can link with any other agent in increase his utility (because we are in Case I and v^t is known). Since $\mathcal{S}_0^{\emptyset}$ includes any minimal network that satisfies C1, we have shown the left to right. Right to left follows since $\mathcal{S}_0^{\emptyset}$ includes the empty network.

(b) With $\mathcal{S}_{1\cdot C}^V$ and $\mathcal{S}_{\infty}^V$, every agent will know his own subnetwork with certainty thus knowing which agents are not in the subnetwork. By Case I, they will initiate links with all those not in the subnetwork. The network must then be a connected, center-sponsored star. Depending on the actual values, it might be that $\mathcal{S}_0^V = \mathcal{S}_{1\cdot A}^V = \mathcal{S}_{1\cdot C}^V$. Let $v^t = [1, 3, 5, 7, 14]$ and $c = 0.9$. Because no combination of agents will have a sum equal to any other combination of agents, every i will know exactly the other agents in his subnetwork. The network must thus be a connected, center-sponsored star. No other structure can be sustained as a SNE.

To consider the "$\supset$" relation, set $v^t = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$ and s^t according to Figure 3.3.4(a). This network is in $\mathcal{S}_0^V$ but not in $\mathcal{S}_{1\cdot A}^V$.

It is in $\mathcal{S}_0^V$ because no agent is willing to take the risk to initiate a new link or switch a link. We can choose beliefs for agent 1 (agent 1 believes all agents not in his subnetwork are unconnected) such that his expected benefit from switching is $\frac{37}{12}$ (1 with probability $\frac{7}{12}$ and 6 with probability $\frac{5}{12}$) while his expected cost is the 6 that he gives up with certainty by removing the link with 2. Agent 1 will not switch. We can choose beliefs for agent 2 such that his expected benefit from switching his link with 1 is $\frac{7}{11}$ (1 with probability $\frac{7}{11}$ and 0 with probability $\frac{4}{11}$) which is less than his expected cost of 1. Similarly, if he switches his link with 3, his expected benefit is $\frac{27}{11}$ (1 with probability $\frac{7}{11}$ and 5 with probability $\frac{4}{11}$) which is less than his expected cost of 5. Similar logic is use to show that none of the other agents will switch or initiate new links.

That network is not in $\mathcal{S}_{1\cdot A}^V$ because there are two directly connected link-initiators. Consider the same v^t with s^t according to Figure 3.3.4(b). This network is in $\mathcal{S}_0^V$ and $\mathcal{S}_{1\cdot A}^V$ but not in $\mathcal{S}_{1\cdot C}^V$. We can pick beliefs (in the same manner as in the previous paragraph) to show that this network is in $\mathcal{S}_0^V$ and $\mathcal{S}_{1\cdot A}^V$. It is not in $\mathcal{S}_{1\cdot C}^V$ because it is not a connected, center-sponsored star. ■

In illustrating that it might be the case that $\mathcal{S}_0^V = \mathcal{S}_{1\cdot A}^V = \mathcal{S}_{1\cdot C}^V$, I stated that this was possible when each agent can tell by the flow exactly which other agents are in his subnetwork. If no combination of agents' values equals another combination of other agents' values then each agent will know exactly the other players in his subnetwork. This is a sufficient condition for $\mathcal{S}_0^V = \mathcal{S}_{1 \cdot A}^V = \mathcal{S}_{1 \cdot C}^V$. If we knew d and D before the values were actually assigned, we could calculate the probability that the one combination of values equaled another combination of values. Proposition P3.8 states what happens to this probability as $d \to 0$ or as $D \to \infty$.

Proposition P3.8: *For Case I,* $\Pr\left[\mathcal{S}^V = \mathcal{S}^{VS}\right] \to 1$ *as* $\frac{d}{D} \to 0$.

The proof is in the appendix, but the intuition is simple. As the number of possible agents' values gets very large, the likelihood that any combination of agents' values equals any other combination of agents' values goes to 0. Since this likelihood goes to 0, the probability that an agent will know exactly the other agents in his subnetwork will go to 1. An implication of this proposition is $\Pr\left[\mathcal{S}^V = \mathcal{S}^{VS}\right] \to 1$ if the values are i.i.d. random variables from a continuous distribution over $[d, D]$, where $0 \leq d < D < \infty$.

3.4 SNE WITH ARBITRARY BELIEFS

From Section 3.3, we can express the set relationships of the SNE structure sets for Case I. These relationships, along with the set relationships and descriptions for the other cases, are summarized in Table 3.4.1. Section 3.4.2 describes these cases. In Section 3.4.3 and 3.4.4, I discuss adding common knowledge of rationality and removing the overflow assumption. In Section 3.4.5, I comment on the results.

Table 3.4.1: Description of SNE Sets with Arbitrary Beliefs

(a) Set Relationships

Case I	$S_0^{\varnothing} \supset S_{1\cdot A}^{\varnothing} \supset S_{1\cdot C}^{\varnothing} = S_{\infty}^{\varnothing} = S^{S} \supset S^{VS} \neq \varnothing$
	$S_0^{\varnothing} \supset S_0^{V} \supseteq S_{1\cdot A}^{V} \supseteq S_{1\cdot C}^{V} = S_{\infty}^{V} = S^{VS} \neq \varnothing$
Case II	$S_0^{\varnothing} \supset S_{1\cdot A}^{\varnothing} \supset S_{1\cdot C}^{\varnothing} = S_{\infty}^{\varnothing} = S^{S} \supset S^{VS} \neq \varnothing$
	$S_0^{\varnothing} \supset S_0^{V} \supseteq S_{1\cdot A}^{V} \supseteq S_{1\cdot C}^{V} = S_{\infty}^{V} = S^{VS} \neq \varnothing$
Case III	$S_0^{\varnothing} \supset S_{1\cdot A}^{\varnothing} \supset S_{1\cdot C}^{\varnothing} = S_{\infty}^{\varnothing} = S^{S} \supset S^{VS} = \varnothing$
	$S_0^{\varnothing} \supset S_0^{V} \supseteq S_{1\cdot A}^{V} \supseteq S_{1\cdot C}^{V} = S_{\infty}^{V} = S^{VS} = \varnothing$
Case IV	$S_0^{\varnothing} \supset S_{1\cdot A}^{\varnothing} \supset S_{1\cdot C}^{\varnothing} = S_{\infty}^{\varnothing} = S^{S} = S^{VS} \neq \varnothing$
	$S_0^{\varnothing} \supseteq S_0^{V} \supseteq S_{1\cdot A}^{V} \supseteq S_{1\cdot C}^{V} = S_{\infty}^{V} = S^{VS} \neq \varnothing$

(b) Set Descriptions

Case I	$S_{1\cdot C}^{V}, S_{\infty}^{V}, S^{VS}$	connected, center-sponsored stars
	$S_{1\cdot C}^{\varnothing}, S_{\infty}^{\varnothing}, S^{S}$	empty; partial and connected center-sponsored stars
	$S_{1\cdot A}^{\varnothing}$	empty, partial, and connected with no two directly linked link-initiators and C1' satisfied
	$S_0^{\varnothing}$	empty, partial, and connected satisfying C1
	$S_0^{V}, S_{1\cdot A}^{V}$	no unconnected agents; may or may not be connected, center-sponsored stars depending on agents' identifiability by flows
Case II	for each information regime and revelation combination, it is the same for Case II except the low agent must be the center of the star or unconnected if permitted.	
Case III	$S_{1\cdot C}^{V}, S_{\infty}^{V}, S^{VS}$	there are no SNE
	$S_{1\cdot C}^{\varnothing}, S_{\infty}^{\varnothing}, S^{S}$	empty; partial networks where low agent is either unconnected or center
	$S_0^{\varnothing}, S_{1\cdot A}^{\varnothing}$ $S_0^{V}, S_{1\cdot A}^{V}$	same as Case I except low agent cannot be non-link-initiating stem
Case IV	$S_{1\cdot C}^{V}, S_{\infty}^{V}, S^{VS}$ $S_{1\cdot C}^{\varnothing}, S_{\infty}^{\varnothing}, S^{S}$	empty
	$S_0^{\varnothing}, S_{1\cdot A}^{\varnothing}$	same as Case I except no non-link-initating stems
	$S_0^{V}, S_{1\cdot A}^{V}$	unconnected agents possible; otherwise same as Case I except no non-link-initiating stems and flows from link-initiation must be higher than c

3.4.1 Cases II, III, and IV

First consider Case II when exactly one agent has a low-valued good. Formal proofs would follow as those for Case I with one crucial difference. Call agent L the low-valued agent. The equilibrium sets for Case II are the same as for Case I except that agent L must never be a non-link-initiating stem, since no agent would maintain a link to L only (since the value of L's good is less than c). Hence, L must be either a strict channel, unconnected, or a non-link-initiating stem. This refines the set of SNE structures. No formal proof is given since the logic is straightforward. Notice that for G^{VS}, L must be the channel of the center-sponsored star. That means that there is a unique SNE for G^{VS} in Case II (as opposed to n SNE in $\mathcal{S}^C$ in Case I).

The reasoning for Case III, when there are two or more but less than n low agents, is similar to that of Case II, except we have more low-valued agents. Again, a low-valued agent cannot be a non-link-initiating stem, but there are more of them so this refines the set of SNE structures. In fact, it refines so much that $\mathcal{S}^{VS}$ is the empty set. This can be understood using the logic from Case II. If a network is not a center-sponsored star then link-switching takes place, but we also know that the low-valued agents cannot be the stems of such a center-sponsored star since they are low-valued goods. Since the center-sponsored star can only have one low-valued center, our logic removes the possibility of having such a structure as an equilibrium. Therefore there is no equilibrium.

Now let us consider Case IV in which all agents are low-valued. Since all agents have low-valued goods, any stem in an equilibrium network must be a link initiator. Since Case IV is so different from the other cases, some formal descriptions and proofs of the relations will be given.

Proposition P4.1: *For Case IV,* $S_0^{\emptyset}$ *is the set of all minimal networks such that (i) C1 is satisfied and (ii) for each* i*,* $f_l > c$ *for any* $l \in L_i$ *where* $i \to l$*.*

Proof: *(Necessity.)* Using the same logic as in PI-6, the strictness condition of SNE is violated unless (i) is satisfied. If (ii) is not met, then there is some link-initiator $i \to l$ that is strictly better off by removing his link to l.

(Sufficiency—sketch.) Using the same logic as in PI-6, beliefs can be chosen to show that any minimal network satisfying (i) and (ii) can be sustained as a SNE for $\mathcal{S}_0^{\emptyset}$ for Case IV. ■

Corollary C4.1: *For Case IV, all stems are link initiators in any non-empty network in* $S_0^{\emptyset}$*.*

Proposition P4.2: *For Case IV, (a)* $S_0^{\emptyset} \supset S_{1 \cdot A}^{\emptyset}$ *and (b)* $S_{1 \cdot A}^{\emptyset}$ *is the set of all minimal networks such that (i) C1 is satisfied and (ii) for each* i*,*

$f_l > c$ *for any* $l \in L_i$ *where* $i \to l$, *and (iii) no two link-initiators are directly linked.*

Proposition P4.3: *For Case IV, (a)* $S^{\emptyset}_{1 \cdot A} \supset S^{\emptyset}_{1 \cdot C}$ *and (b)* $S^{\emptyset}_{1 \cdot C}$ *is comprised only of the empty network.*

Proof: (b) *(Necessity.)* Suppose there is a non-empty SNE network in $\mathcal{S}^{\emptyset}_{1 \cdot C}$. Since it is non-empty, there must be at least one stem i connected to j. If that stem is not a link-initiator, then it is not a SNE. If that stem is a link-initiator, then the subnetwork minus i's value must have value greater than c or else i will remove his link which makes the network not a SNE. If that value is greater than c, then j must be connected to some agent k since j's value is less than c (Case IV). By $1 \cdot C$-link revelation, i knows that j and k are linked and can remove his link with j and link with k to receive the same payoff. This violates strictness, and so a non-empty network cannot be a SNE.

(Sufficiency.) Suppose the network is empty. If all agents believe the network is empty, then no agent has an incentive to initiate a link since the value to be gained is less than the cost of the link. Therefore, the empty network is a SNE.

(a) *(Right to left.)* The empty network with correct beliefs is in both $\mathcal{S}^{\emptyset}_{1 \cdot A}$ and $\mathcal{S}^{\emptyset}_{1 \cdot C}$. Since only empty networks are in $\mathcal{S}^{\emptyset}_{1 \cdot C}$, the right to left is shown. *(Left to right.)* A non-empty network that meets (i), (ii), and (iii) of P1.2 is in $\mathcal{S}^{\emptyset}_{1\cdot A}$ but not in $\mathcal{S}^{\emptyset}_{1\cdot C}$. ■

Proposition P4.4: *For Case IV,* $S^{\emptyset}_{1\cdot C} = S^{\emptyset}_{\infty} = S^{S} = S^{VS}$.

Proposition P4.5: *For Case IV, (a)* $S^{\emptyset}_{0} \supseteq S^{V}_{0}$*, and (b)* $S^{V}_{0} \supseteq S^{V}_{1\cdot A} \supseteq S^{V}_{1\cdot C} = S^{V}_{\infty} = S^{VS} \neq \emptyset$.

Intuition: Part (b) holds by reasoning similar to that used in the proof for P3.7. Part (a) is not strict as it is for Case IV. First, if a member of a non-connected subnetwork does not know who else is in his network, he will not want to initiate new links. Second, an unconnected agent might not want to initiate any links if he believes the rest of the network is empty. This second condition is violated in Cases I. The first condition is not if, say $v^t = [1, 1, ..., 1]$. ■

3.4.2 Adding Common Knowledge of Rationality

Adding common knowledge of rationality changes nothing for $\mathcal{S}^{VS}$ since that is already the set of strict Nash equilibria. Nothing changes for $\mathcal{S}^{\emptyset}$ for an opposite reason. Although common knowledge of rationality will place restrictions on equilibrium beliefs, for each network in $\mathcal{S}^{\emptyset}$ (with some particular revelation) we can still find beliefs that sustain that network to still be a SNE in $\mathcal{R}^{\emptyset}$. The proofs above for $\mathcal{S}^{\emptyset}$ each give beliefs that are rationalizable. The main reason why $\mathcal{R}^{\emptyset} = \mathcal{S}^{\emptyset}$ is that because the only information is from the flows and the revelation, and being an $\mathcal{S}^{\emptyset}$ already ensures that each agent is playing a best-response to some beliefs. Knowing this, adding common knowledge of rationality imposes no restriction that has any bite. Adding common knowledge or rationality will only change our equilibrium sets when in the structure information regime and the value structure regime.

Consider $\mathcal{S}^{S}$ for Case I, which is made up of the empty network and partial and connected networks in which any subnetwork is a center-sponsored star. Any partial network is not rationalizable: a member of the subnetwork would only initiate a link if it was a best response, and if the value from that link is high enough to make linking a best response, then it also in the interest of any unconnected agent to link to that subnetwork. In other words, if I am unconnected and know of a subnetwork out there, I know that for that subnetwork to be there it must be worth it, so I will want to be a part of it. As such, the set of rationalizable structures for Case I in the structure regime, denote it $\mathcal{R}^{S}$ is the empty network and the connected, center-sponsored star. This set is still strictly larger than $\mathcal{S}^{VS}$ since it contains the empty network, but it is much closer to it. Since it does contain the empty network, $\mathcal{R}^{S}$ will still be larger than $\mathcal{S}^{VS}$ for Cases I, II, and III. For Case IV, common knowledge of rationality does not further refining since $\mathcal{S}^{S}$ already equals $\mathcal{S}^{VS}$.

Things are more complicated under the value regime. Consider Case I. It can be proven that when all agents have the same good value, $v_i^t = v_j^t \; \forall i, j$, $\mathcal{R}^{V} = \mathcal{S}^{VS}$

no matter the revelation, and this proof is in the appendix. Things are trickier when $v_i^t \neq v_j^t$. No formal proof has yet been obtained, but considering the proof in the appendix, it seems correct that $\mathcal{R}^V = \mathcal{S}^{VS}$ even when $v_i^t \neq v_j^t$. The idea is simple. The only time a SNE in $\mathcal{S}^V$ is not in $\mathcal{S}^{VS}$ is when the agents in a subnetwork cannot tell who else is in their subnetwork so that they cannot link switch or form new links. When agents can tell, they do so because of they use their knowledge of the flows and v^t to deduce which agents' goods sum up to the total flow. When all agents have the same value, the agents are least likely to make these deductions, but common knowledge of rationality is still enough to make $\mathcal{R}^V = \mathcal{S}^{VS}$ in this case. In times when $v_i^t \neq v_j^t$, an agent should have more accurate guesses as to which agents are in his subnetwork. This increase in information should not lead us away from the $\mathcal{S}^{VS}$ but should bring us closer since it should mean more link-switching and new link formation. In this sense, common knowledge of rationality should still mean that $\mathcal{R}^V = \mathcal{S}^{VS}$. Using similar logic, it would seem that $\mathcal{R}^V = \mathcal{S}^{VS}$ for Cases I-III For Case IV under low revelation, however, we can still have rationalizable equilibria that are not strict Nash.

3.4.3 Removing the Overflow Assumption

The overflow assumption was used immediately to show that any SNE must be minimal. It turns out that this assumption really only has bite when revelation is $1 \cdot A$-link or lower. $1 \cdot C$-link or higher revelation is enough alone to ensure that the equilibrium is minimal even without the overflow assumption. When revelation is $1 \cdot C$-link or higher, this high revelation alone means that link-switching occurs in any subnetwork unless it is a center-sponsored star, and this link-switching can occur without the overflow assumption. In this event, minimality can come not by knowing the flows from each link, but because it is an attribute of a center-sponsored star.

On the other hand, when revelation is $1{\cdot}A$-link or lower, we can have equilibria which are not minimal. This can even hold when the values are known. Consider the following example.

Example 3.4.1: Consider G_0^V with $v^t = [v, v, v, v, v, v]$, $v > c$, and the network structure given in Figure 3.4.1. This network is not a SNE, but it can be an equilibrium if we remove the overflow assumption. Agents 2, 3, 5, and 6, obviously are playing best responses, so we need only consider agents 1 and 4. Consider agent 1 first. We see that $u_1 = 6v - 3c$. His revelation does not tell him with whom 4 has linked, and without the overflow assumption, he does not know that he can remove his link with 5 and be strictly better off.

Suppose for some reason he knows that one of his links is redundant, but does not know which. Suppose he assigns equal likelihood to each of them being redundant. Then there is a $\frac{2}{3}$ probability that removing his link with $j \in \{4, 5, 6\}$ will yield him $6v - 2c$, but there is a $\frac{1}{3}$ chance that he gets $5v - 2v$. Notice that for agent 1 to not want to remove a link with j, it must be the case that his current action makes him strictly better than removing that link:

$$\frac{2}{3}(6v - 2c) + \frac{1}{3}(5v - 2v) < 6v - 3c.$$

Figure 3.4.1: Illustration of Example 3.4.1

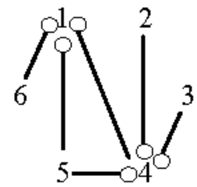


Table 3.5.1: Set Relationships S^* for Example 3.5.2

Case I	$S_0^{*\varnothing} \supset S_{1\cdot A}^{*\varnothing} \supset S_{1\cdot C0}^{*\varnothing} = S_\infty^{*\varnothing} = S^{*S} = S^{*VS} \neq \varnothing$ $S_0^{*\varnothing} \supset S_0^{*V} \supseteq S_{1\cdot A}^{*V} \supseteq S_{1\cdot C}^{*V} = S_\infty^{*V} = S^{*VS} \neq \varnothing$
Case II	$S_0^{*\varnothing} \supset S_{1\cdot A}^{*\varnothing} \supset S_{1\cdot C0}^{*\varnothing} = S_\infty^{*\varnothing} = S^{*S} = S^{*VS} \neq \varnothing$ $S_0^{*\varnothing} \supset S_0^{*V} \supseteq S_{1\cdot A}^{*V} \supseteq S_{1\cdot C}^{*V} = S_\infty^{*V} = S^{*VS} \neq \varnothing$
Case III	$S_0^{*\varnothing} \supset S_{1\cdot A}^{*\varnothing} \supset S_{1\cdot C0}^{*\varnothing} = S_\infty^{*\varnothing} = S^{*S} = S^{*VS} = \varnothing$ $S_0^{*\varnothing} \supset S_0^{*V} \supseteq S_{1\cdot A}^{*V} \supseteq S_{1\cdot C}^{*V} = S_\infty^{*V} = S^{*VS} = \varnothing$
Case IV	$S_0^{*\varnothing} \supset S_{1\cdot A}^{*\varnothing} \supset S_{1\cdot C0}^{*\varnothing} = S_\infty^{*\varnothing} = S^{*S} = S^{*VS} \neq \varnothing$ $S_0^{*\varnothing} \supseteq S_0^{*V} \supseteq S_{1\cdot A}^{*V} \supseteq S_{1\cdot C}^{*V} = S_\infty^{*V} = S^{*VS} \neq \varnothing$

This expression is true when $v > 3c$. The same condition must hold if agent 4 is to not remove a link under similar beliefs. If this expression holds, the non-minimal structure in Figure 3.4.1 can be sustained as an equilibrium.

This example illustrates that we can have a non-minimal structures if we remove the overflow assumption. We can come up with networks that can be equilibria under $1 \cdot A$-link revelation, too.

3.4.4 Comments on $\mathcal{S}$

I now summarize the main results and conclusions of the analysis to this point. In all information regimes and for all cases, $\mathcal{S}_{1 \cdot C} = \mathcal{S}_{\infty}$, and any subnetwork must be a center-sponsored star. This fact is interesting because there is a large informational difference between $1 \cdot C$- and ∞-link revelation. The equilibrium structures are the same in that seeing beyond your direct links allows link-switching if your subnetwork has at least two other agents. In equilibrium, if you can see beyond your direct links, there must be some reason why you would not switch links. In the center-sponsored stars of Cases I and II, the reason is that you are not a link-initiator. You are receiving benefits without paying costs so there is no reason to switch.

This reasoning shows why the center-sponsored star is so robust to large decreases in information—because it is immune to link-switching. We must have less than $1 \cdot C$-link revelation in order to not have the center-sponsored star structure in equilibrium. This center-sponsored star feature is even robust to removal of the overflow assumption. In fact, minimality of the SNE structure is assured in a center-sponsored star, so that we do not need the overflow assumption for minimality with $1 \cdot C$-link or higher revelation. So long as a link-initiator can see beyond his own direct links, he can make a successful switch, thereby violating strictness. The center-sponsored star is thereby very robust to decreases in information—both in terms of revelation and in terms of information about utility flows.

Another interesting fact is that uncertainty about the values and uncertainty about the structure have very different effects on the set of SNE. Even though in both G^V and G^S the agents can have information beyond their own direct links, this information is different in each regime and has different implications. Consider the following examples in Figure 3.4.2. Under any revelation, Figure 3.4.2(a) is in $\mathcal{S}^S$ but is never in $\mathcal{S}^V$. In 0-link revelation, Figure 3.4.2(b) is in $\mathcal{S}^V$ but is not in $\mathcal{S}^S$ for any revelation. One way to describe this difference is that knowing the structure has immediate implications about link-switching whereas knowing the values usually has implications about the network's connectivity.

So which is better if we want as many people linked together as possible in equilibrium? Knowing the values is generally better than knowing the structure because it can lead to link-initiation when all (or all but one) values are high and it can sustain non-center-sponsored star subnetworks when all values are low. For Cases I and II, if revelation is $1 \cdot C$-link or higher, then knowing the values assures a connected network in equilibrium. If revelation is $1 \cdot A$-link or lower but the agent's values are fairly different from one another then knowing the values is still better because agents can tell from their flows exactly which agents are in their networks.

This knowledge can lead to new link-initiation and a connected network. However, if the agents all have similar values then non-connected networks can exist as SNE. But knowing the structure does not necessarily save us here because there will still be no new link initiation even when the structure is not known. If the h_i function and initial priors are sufficiently optimistic (discussed below) will can get a connected network if the structure is known. For Cases III and IV, knowing the values instead of structure is definitely better. For $1 \cdot C$-link revelation or higher, it makes no difference, but for $1 \cdot A$-link revelation or lower, there is a chance of having a partial network in $\mathcal{S}^V$ but not in $\mathcal{S}^S$.

A similar question is: what is the minimal amount of information needed to assure us of having an equilibrium that is also a strict Nash? Having both common knowledge of v^t and $1 \cdot C$-link revelation is enough since $\mathcal{S}^V_{1 \cdot C} = \mathcal{S}^{VS}$. These criterion do not appear too strict, and notice that agents do not need to have beliefs about others' beliefs. If we do suppose agents to have beliefs on other agents' rationality, then we conjecture that common knowledge of rationality and v^t is sufficient even with 0-link revelation.

Another "welfare" criterion is minimality. What is the least needed to ensure minimality? Each of the following is sufficient alone to ensure that SNE structures are minimal: (1) $1 \cdot C$-link or higher revelation, or (2) the overflow assumption. The first of these is surprising since $1 \cdot C$-link revelation is not a strict requirement. At lower revelation, however, minimality is not assured in the null and value regimes without the overflow assumption.

We can also comment on other variations in the game. If $c < d$ then we can only be in Case I, a fact which is commonly known. $\mathcal{S}^{VS}$ is still connected, center-sponsored stars, but now $\mathcal{S}^{\emptyset}_{1\cdot C} = \mathcal{S}^{\emptyset}_{\infty} = \mathcal{S}^{S} = \mathcal{S}^{VS}$. $\mathcal{S}^{\emptyset}_{0}$ and $\mathcal{S}^{\emptyset}_{1\cdot A}$ no longer contain networks with unconnected agents. Even though there are no unconnected agents, the networks are not necessarily connected. $\mathcal{S}^{V}_{1\cdot C} = \mathcal{S}^{V}_{\infty} = \mathcal{S}^{VS}$ but $\mathcal{S}^{V}_{0}$ and $\mathcal{S}^{V}_{1\cdot A}$ are not necessarily connected. If $D < c$ then we can only be in Case IV. If c is sufficiently small, then the set relationships remain the same. If c is too large (e.g., $c > (n-1)D$) then the empty network is the only SNE for any game and revelation.

3.5 SNE STRUCTURES OF THE DYNAMIC GAME—DISCUSSION

As stated earlier, we are really interested in the sets of SNE structures that are not arbitrarily chosen but instead arise out of the dynamics of the network formation game. Denote these structures $\mathcal{S}^*$. The relationships of the SNE sets described in Table 3.4 illustrate nicely how incremental increases in uncertainty will increase the set of equilibria with unjustified priors, but what happens if we now turn to the BG dynamics? To what structures might the network converge when the beliefs arise out of the network formation dynamics? How do we characterize $\mathcal{S}^*$? The first thing to note is that the SNE that arise in the dynamic game will be contained by $\mathcal{S}$.

Lemma L4: $\mathcal{S}^* \subseteq \mathcal{S}$.

Proof: *(Left to right.)* Any beliefs that arise out of the dynamic process of the game can be arbitrarily chosen by the game theorist. *(Right to* *left.)* $\mathcal{S}^*$ places more restrictions on the beliefs so that some action-belief combinations that are SNE in $\mathcal{S}$ might not be in $\mathcal{S}^*$. ■

L4 is important. Unless we know the $h_i(\cdot)$ used by each agent, we might not know to which SNE in $\mathcal{S}$ the network will converge. But at least we do know that the SNE must be in $\mathcal{S}$ if the network does converge. Quite often, in fact, $\mathcal{S}$ is relatively small (i.e., when revelation is high), so knowing $\mathcal{S}$ is very informative in and of itself. Nevertheless, we should consider times when $\mathcal{S}$ will or will not be refined.

Consider the following dynamics. Assume v^t is determined at the beginning of the game and does not change—a fact commonly known. As stated earlier, each agent tries to maximize the immediate payoff, so there is no strategic connection between agents' link forming. The stage game is as follows. First, in period t, each agent i uses his revelation to update his prior beliefs $\pi_{i,t-1}$ to $\pi_{i,t}$ according to $h_i(\cdot)$. Second, each i chooses $a_{i,t}$, which is a best response given this $\pi_{i,t}$. This is called a "myopic" or "naive" best response. Third, there is a fixed probability that each player exhibits inertia in playing his previous period's strategy whether or not it is a best response. This rules out perpetual mis-coordination but can be justified as a behavioral property. Fourth, if there is more than one best response then one is chosen (uniformly) randomly from the set of best responses, which removes the possibility of settling into a non-strict equilibrium. Fifth, at the end of the period, the payoff to i is the sum of values of all agents with whom he is connected minus the cost of links he has initiated.

These steps are repeated until an equilibrium is reached, which occurs when for each i in time t, $\pi_{i,t-1} = \pi_{i,t}$ and $a_{i,t-1}$ is a strict best response given the updated beliefs $\pi_{i,t}$. In this instance, no agent will change his action, and the conditions for SNE are satisfied. BG show how these dynamics lead to convergence under full information when $c < \min\{v^t\}$ and $c > \max\{v^t\}$.

In our game, however, there is no guarantee that the network will converge to an equilibrium. For example, in the full information game in Case III described (Section 3.4.3) there is no SNE to which the system can converge. With this in mind, we offer some discussion about $\mathcal{S}^*$ instead of giving convergence proofs. Our discussion centers on two points: under certain $h_i(\cdot)$ functions, it might be the case that $\mathcal{S}^* = \mathcal{S}$,[7] and $\mathcal{S}^*$ gets refined as agents become more *optimistic* about the size of other possible subnetworks.

3.5.1 An Example of $\mathcal{S}^* = \mathcal{S}$

In our proofs characterizing $\mathcal{S}$, we arbitrarily chose beliefs that were not contradicted by revelation but would make each i least inclined to switch or initiate new links. Switching and new-link initiating are closely related. If an agent believes that initiating a link with some agent not in his network will reward him less than c, then the agent will not initiate that link. Even further, if i is afraid to initiate that new link, then he is generally more likely to not want to switch out of fear of accidently linking with that agent not in his network. These beliefs depend crucially on i's expected

[7]We mean by $\mathcal{S}^* = \mathcal{S}$ that $\mathcal{S}^* = \mathcal{S}$ for every game type and level of link revelation. That is, $\mathcal{S}_0^{*VS} = \mathcal{S}_0^{VS}$, $\mathcal{S}_{1\cdot A}^{*VS} = \mathcal{S}_{1\cdot A}^{VS}$, $\mathcal{S}_{1\cdot C}^{*VS} = \mathcal{S}_{1\cdot C}^{VS}, ..., \mathcal{S}^{*C} = \mathcal{S}^{C}$.

size of his own subnetwork ($\mathbf{E}\left|N_i\right|$) and his expected flow he will receive by linking to some agent j ($\mathbf{E}f_{i|j}$). For example, if $\mathbf{E}f_{i|j}$ is lower than c, i will not want to initiate a new link with j. Supposing i thinks that j is a member of some other subnetwork where $N_j \neq N_i$, i will be more inclined to link with j the larger j's subnetwork.

Call i *pessimistic about* j if, with no other revelation or knowledge of j, i believes $\mathbf{E}f_{i|j} < c$. Similarly say i is *optimistic about* j if i believes $\mathbf{E}f_{i|j} \geq c$. For example, i is pessimistic about all $j \notin N_i$ if he believes $\mathbf{E}f_{i|j} < c$ for each $j \notin N_i$. This can be the case if i believes every $j \notin N_i$ is unconnected and low-valued. It turns out that if $h_i(\cdot)$ assigns beliefs such that (i) i becomes inherently pessimistic about all $j \notin N_i$ and (ii) believes his own subnetwork is as small as cannot be contradicted by his information, then $\mathcal{S}^* = \mathcal{S}$ no matter the link revelation or game type. We will illustrate this for $G_0^{\emptyset}$.

Denote $h_i'(\cdot)$ to be the updating function that updates according to the manner described in the previous paragraph. Player i needs to consider all the states of the world to which he assigns probability greater than one. By 0-link revelation, condition (ii) suggests that i will assign positive probability to states of the world that have N_i's that are of size $\sum_{l \in L_i} up\left(\frac{f_l}{D}\right)$ in which all $l \in L_i$ are connected according to i's revelation. By 0-link revelation, i is not going to know which other agents are in N_i, so i assigns equal likelihood to each of those states of the world. In each instance, any agent not in his network is assigned a low value and believed to be unconnected. If our initial network is minimal and satisfies C1, this network will be in $\mathcal{S}_0^{*\emptyset}$ for Case I. These are exactly the characteristics of a network in $\mathcal{S}_0^{\emptyset}$ for Case I. For the other

Cases, it also turns out that $\mathcal{S}_0^{*\emptyset} = \mathcal{S}_0^{\emptyset}$. It also holds if we increase the link revelation or change the game type.

The main point of this example is that it is possible to have an updating function that does not further refine the set of SNE structures.

3.5.2 Refining $\mathcal{S}^*$

It might seem unrealistic to assume that each i is pessimistic about other agents. For example, with expected value of a network good $\frac{D+d}{2} = \overline{v} > c$, then we would suspect that i is inherently optimistic about other players. Player i will also be optimistic if he believes that j is in a relatively large subnetwork Since the condition for i to form a new link with some j is

$$\mathbf{E}f_{i|j}\left(1 - \Pr\left[i \text{ already connected to } j\right]\right) - c \geq 0,$$

we see that a higher $\mathbf{E}f_{i|j}$ definitely increases the chances that i will initiate the link with j. In fact, if the only agents that i knows are in his subnetwork with certainty are his direct connections (as might be the case in G_0^{VS}) then this condition becomes

$$\left(\mathbf{E}f_{i|j}\right)\left(1 - \frac{\mathbf{E}\left|N_i\right| - \left|L_i\right|}{\left|M_i\right|}\right) - c \geq 0, \tag{1}$$

where M_i is the set of agents which i does not know if the $j \in M_i$ is in his subnetwork or not.

If we now choose $h_i''(\cdot)$ such that (i) each i is inherently optimistic about the others and (ii) same as above, then we can see that $\mathcal{S}_0^{*VS} \subset \mathcal{S}_0^{VS}$ for Case I. This is easily verified because if any unconnected agent is optimistic, then that agent will initiate a link. In Case I, he will maintain that link because $f_{i|j} \geq v_j^t > c$. In this case, $\mathcal{S}_0^{*VS}$ is refined substantially. The general conclusion is that if all else is equal, optimism about flows will refine $\mathcal{S}^*$. We can illustrate this by an example. The summarized set characteristics of this example are listed in Table 3.5.1.

EXAMPLE 3.5.2. Suppose each agent sets $\mathbf{E}f_{i|j} = \overline{v} > c$ for all $j \notin N_i$. This is akin to assuming that i believes any agent not in his network is unconnected by has good valued at $\overline{v}$.

First, lets examine $\mathcal{S}^{*V}$, $\mathcal{S}^{*S}$, $\mathcal{S}_{\infty}^{*\emptyset}$, and $\mathcal{S}_{1\cdot C}^{*\emptyset}$. It turns out that $\mathcal{S}^{*V} = \mathcal{S}^{V}$ for all cases because each agent knows v^t, so $\mathbf{E}f_{i|j} = \overline{v}$ makes no difference here. In Case I, for $\mathcal{S}^{*S}$, $\mathcal{S}_{\infty}^{*\emptyset}$, and $\mathcal{S}_{1\cdot C}^{*\emptyset}$, since $\overline{v} \geq c$, any unconnected agent will initiate a link with some j. As a result, there will be no unconnected agent. Since only center-sponsored stars can be the SNE subnetwork structures in these games, the channel of an center-sponsored star would know if that the subnetwork is not connected. If it is not, the channel is strictly better off in expectation by linking with $j \in M_i$. In Case I, since all agents have high-valued goods, the result is that the only SNE are connected, center-sponsored stars. Hence $\mathcal{S}^{*S} = \mathcal{S}_{\infty}^{*\emptyset} = \mathcal{S}_{1\cdot C}^{*\emptyset} = \mathcal{S}^{*VS}$. The same relation holds for Case II where the only element of $\mathcal{S}^{*VS}$ is where the low-valued agent is the channel. The same relation holds for Case III where $\mathcal{S}^{*VS} = \emptyset$. For Case IV, since $\mathcal{S}^{*S} = \mathcal{S}_{\infty}^{*\emptyset} = \mathcal{S}_{1\cdot C}^{*\emptyset} = \mathcal{S}^{*VS}$ already, no further refining of $\mathcal{S}$ can be made.

Now lets consider $\mathcal{S}_{1\cdot A}^{*\emptyset}$, and $\mathcal{S}_0^{*\emptyset}$. Assuming that i believes all $j \in M_i$ to be unconnected, he will have $\mathbf{E}f_{i|j} = \overline{v}$. If i is unconnected, then i will always initiate a link to j unless he knows the true value of j. It follows that for Cases I-III, $\mathcal{S}_0^{*\emptyset}$ and $\mathcal{S}_{1\cdot A}^{*\emptyset}$ contain no networks in which there are unconnected agents. However, this does not mean the network must be connected. If i does not know exactly which agents are in N_i then he might not initiate links if he thinks there is sufficient enough chance to link with someone already in his subnetwork (and receive 0 additional benefits while paying c).

The catch here is that we must know how i forms $\mathbf{E}\left|N_i\right|$. Consider the following procedures that form $\mathbf{E}\left|N_i\right|$. In Procedure A, i assigns positive probability only to those networks that have a number of agents closest to the expected number of agents in the network. For example, suppose i is a stem that receives a flow of 10 from his link in $G_0^{\emptyset}$. If $\overline{v} = 3$, then i assigns positive probability to any structure in which there are exactly 3 agents (besides himself) in his subnetwork. Another possibility, called Procedure B, assigns equal (strictly positive) probability to any network that is possible. If i is the stem receiving flow 10, and $d = 2$ and $D = 4$, then i assigns an equal probability to each structure that has either 3 or 5 agents (besides himself) in his subnetwork.

Since $\mathbf{E}\left|N_i\right|$ is different in each procedure, condition (1) from Section 3.5.1 can hold in Procedure A while not holding in Procedure B for some combination of v^t, s^t, and c. In other combinations of v^t, s^t, and c, we can get (1) to hold in Procedure B but not Procedure A. The result is that there is no simple way to characterize $\mathcal{S}_0^{*\emptyset}$ and $\mathcal{S}_{1\cdot A}^{*\emptyset}$. With this in mind, we will outline some of the general conditions that must be met instead of trying to give a complete characterization of these sets.

For Case I, a network can be a SNE in $\mathcal{S}_0^{*\emptyset}$ if C1 is met and if condition (1) fails for each agent. Condition (1) fails when each agent is connected to a subnetwork that is (i) sufficiently large and (ii) of shape that prevents each agent from believing he can link with an non-subnetwork agent with high probability. This implies that the network must have no unconnected agents, and it also implies that the flow that each agent receives must be high enough to suggest that his subnetwork is of sufficient size to make the agent overly concerned about accidently linking with someone already in his subnetwork. As before, $\mathcal{S}_0^{*\emptyset} \supset \mathcal{S}_{1\cdot A}^{*\emptyset}$, because if two link-initiators are directly connected then the strictness condition is violated. Because non-connected networks are possible in these sets, $\mathcal{S}_0^{*\emptyset} \supset \mathcal{S}_{1\cdot A}^{*\emptyset} \supset \mathcal{S}^{*VS}$. The same relationship holds for Cases II, III, and IV.

4.6 APPENDIX

3.6.1 Proof of PI-8

Proof: It is sufficient to show that the probability that any combination of sums of values equals any other combination of sums of values equals goes to zero as $d \to 0$ or $D \to 0$.

Suppose a game where n draws from the same i.i.d. distribution over $[d, D]$ are made. Denote Y_1 to be the set of draws $\{y_1, y_2, ..., y_n\}$ where $3 \leq n < \infty$. Denote

Y_2 to be the set of sums of combinations of size two, i.e., Y_2 is defined as

$$\begin{aligned}\{y_1 + y_2, y_1 + y_3, ..., y_1 + y_n,&\\ y_2 + y_3, ..., y_2 + y_n,&\\ \ddots \qquad\qquad \vdots&\\ y_{n-1}, y_n\}.&\end{aligned}$$

Similarly denote $Y_3, ..., Y_n$. Further denote Y to be $Y_1 \times Y_2 \times \cdots \times Y_n$. For our proof, it is sufficient to show that, for any two draws without replacement from Y, the probability that those two draws are equal goes to zero as $d \to 0$.

Step 1: The General Expression. Consider drawing x' and z' without replacement from Y. We are interested in the ex ante probability that these two draws will be equal, $\Pr[x' = z']$. Notice that if $x' = (y_k + y_l)$ and $z' = (y_k + y_m + y_n)$ then

$$Pr\left[x' = y'\right] = \Pr\left[y_k + y_l = y_k + y_m + y_n\right] = \Pr\left[y_l = y_m + y_n\right].$$

If one y_i is included in both draws then it can be subtracted, so that in our probabilities we need only concentrate on the y_i that are different in x' and z'. After subtracting common elements, we are left with x and z. Denote the draw with fewer elements (x in the immediate example) to be the draw from Y_i and say the other is from Y_j. (In the example, $i = 1$ and $j = 2$). It follows that $Pr\left[x' = z'\right] = Pr\left[x = z\right]$.

If $i = j = 1$, then $\Pr\left[x = z\right]$ equals the probability of a particular set of draws times the number of combinations that they are equal. Thus, $\Pr\left[x = z\right] = \left(\frac{d}{D}\right)^2$. $\left(\frac{D}{d}\right) = \frac{d}{D}$.

Suppose $i = 1$ and $j = 2$. Then the probability of a particular set of draws is $\left(\frac{d}{D}\right)^3$ since there is one element in x and two elements in z. The number of combinations where they are equal is

$$\sum_{t=1}^{\frac{D}{d}} \sum_{q=1}^{t-1} 1 = \sum_{t=1}^{\frac{D}{d}} (t-1) = 0 + 1 + 2 + ... + \left(\frac{D}{d} - 1\right).$$

There are 0 combinations if $x = d$, 1 combination if $x = 2d$, 2 combinations if $x = 3d$, and so on until there are $\frac{D}{d} - 1$ combinations if $x = D$. It follows that $\Pr[x = y] = \left(\frac{d}{D}\right)^3 \sum_{t=1}^{\frac{D}{d}} \sum_{q=1}^{t-1} 1$. Suppose $i = 2$ and $j = 2$. The probability of a particular set of draws is $\left(\frac{d}{D}\right)^4$, and the number of combinations is

$$\begin{aligned}
\sum_{t_1=1}^{\frac{D}{d}} \sum_{t_2=1}^{\frac{D}{d}} \sum_{q_2=1}^{t_1-t_2-1} 1 \quad = \quad & (0 + 0 + ... + 0)\,|_{t_1=1} + \\
& (0 + 0 + 0 + 0 + 0 + ... + 0)\,|_{t_1=2} + \\
& (1 + 0 + 0 + 0 + 0 + ... + 0)\,|_{t_1=3} + \\
& (1 + 2 + 0 + 0 + 0 + ... + 0)\,|_{t_1=4} + \\
& (1 + 2 + 3 + 0 + 0 + ... + 0)\,|_{t_1=5} + \\
& (1 + 2 + 3 + 4 + 0 + ... + 0)\,|_{t_1=6} + \\
& ... + \\
& \left(1 + 2 + 3 + 4 + 5 + ... + \left(\frac{D}{d} - 1\right) + 0\right)\,|_{t_1=\frac{D}{d}},
\end{aligned}$$

where $\sum_{s=s'}^{s''} 1 \equiv 0$ if $s'' < s'$. So $\Pr[x = y] = \left(\frac{d}{D}\right)^4 \sum_{t_1=1}^{\frac{D}{d}} \sum_{t_2=1}^{\frac{D}{d}} \sum_{q_2=1}^{t_1-t_2-1} 1$.

Suppose $i = 1$ and $j = 3$. Then the probability of a particular set of draws is $\left(\frac{d}{D}\right)^4$, and the number of combinations where they are equal is $\sum_{t=1}^{\frac{D}{d}} \sum_{q_1=1}^{\frac{D}{d}} \sum_{q_2=1}^{t-q-1} 1$. If follows that $\Pr[x = y] = \left(\frac{d}{D}\right)^4 \sum_{t=1}^{\frac{D}{d}} \sum_{q_1=1}^{\frac{D}{d}} \sum_{q_2=1}^{t-q_1-1} 1$.

By continuing to raise i and j, we can obtain the general expression

$$\begin{aligned} & \Pr\left[x = z\right] \\ = & \left(\frac{d}{D}\right)^{i+j} \sum_{t_1=1}^{\frac{D}{d}} \sum_{t_2=1}^{\frac{D}{d}} \cdots \sum_{t_i=1}^{\frac{D}{d}} \sum_{q_1=1}^{\frac{D}{d}} \sum_{q_2=1}^{\frac{D}{d}} \cdots \sum_{q_{j-1}=\max\left\{1, t_1+t_2+\ldots+t_x-(q_1+q_2+\ldots+q_{j-1})-\frac{D}{d}\right\}}^{\min\left\{\frac{D}{d}, t_1+t_2+\ldots+t_x-(q_1+q_2+\ldots+q_{j-1})-1\right\}} 1, \end{aligned}$$

where $i \leq j$ and $\sum_{s=s'}^{s''} 1 \equiv 0$ if $s'' < s'$. The summations part is the number of possible combinations that a draw from Y_i equals a draw from Y_j. The $\left(\frac{d}{D}\right)^{i+j}$ part is the probability of a single combination.

Step 2: Convergence. We can show that $\Pr\left[x = z\right] \to 0$ as $d \to 0$. Since D will be fixed, we also need the number of possible draws $\Delta = \frac{D}{d}$ to be increasing at the rate that d is decreasing.

$$\begin{aligned} & \left(\frac{d}{D}\right)^{x+y} \sum_{t_1=1}^{\frac{D}{d}} \sum_{t_2=1}^{\frac{D}{d}} \cdots \sum_{t_x=1}^{\frac{D}{d}} \sum_{q_1=1}^{\frac{D}{d}} \sum_{q_2=1}^{\frac{D}{d}} \cdots \sum_{q_{y-1}=\max\left\{1, t_1+t_2+\ldots+t_x-(q_1+q_2+\ldots+q_{y-1})-\frac{D}{d}\right\}}^{\min\left\{\frac{D}{d}, t_1+t_2+\ldots+t_x-(q_1+q_2+\ldots+q_{y-1})-1\right\}} 1 \\ \leq & \left(\frac{d}{D}\right)^{x+y} \sum_{t_1=1}^{\frac{D}{d}} \sum_{t_2=1}^{\frac{D}{d}} \cdots \sum_{t_x=1}^{\frac{D}{d}} \sum_{q_1=1}^{\frac{D}{d}} \sum_{q_2=1}^{\frac{D}{d}} \cdots \sum_{q_{y-1}=1}^{\frac{D}{d}} 1 \\ = & \left(\frac{d}{D}\right)^{x+y} \sum_{t_1=1}^{\frac{D}{d}} \sum_{t_2=1}^{\frac{D}{d}} \cdots \sum_{t_x=1}^{\frac{D}{d}} \sum_{q_1=1}^{\frac{D}{d}} \sum_{q_2=1}^{\frac{D}{d}} \cdots \sum_{q_{y-2}=1}^{\frac{D}{d}} \frac{D}{d} \\ = & \left(\frac{d}{D}\right)^{x+y} \left(\frac{D}{d}\right)^{x+y-1} \\ = & \frac{d}{D} = \frac{1}{\Delta} \to 0 \text{ as } d \to 0 \text{ and } \Delta \to \infty. \end{aligned}$$

Since a value greater than the value in question converges as $d \to 0$, the value in question must converge. This holds for all $i \leq j \leq n$.

In G^S (with any level of revelation), an agent is not able to deduce with whom he is connected if more than one sum of agents values equals the flow that he receives.

But since the probability that any two sums are equal converges to 0 as the d goes to 0, the probability that any agent i in the network cannot deduce with whom he is connected also goes to zero. If i knows everyone with whom he is connected, he will initiate a link with some other agent j not connected with him until he receives the network goods of all agents. Only the center-sponsored star can be maintained as an equilibrium network in this instance. Hence, $\Pr\left[\mathcal{S}^S = \mathcal{S}^C\right] \to 1$ as $d \to 0$. The same can be shown as $D \to 0$. ■

3.6.2 Proof that $\mathcal{R}^V = \mathcal{S}^{VS}$ when $v_i^t = v_j^t \geq c$, $\forall i, j$

Here is a summary in words. In order for i to not form a new link, he must believe that the other networks out there are smaller than his own. He must further believe that any $j \notin N_i$ believes that any network besides N_j is also smaller than N_j. Since agents all know the size of their own networks (since v^t and flows are known), j must believe there is some k in an even smaller network, and so on until we finally get some agent that must be unconnected, and being unconnected is never a best-response in Case I when the v^t is commonly known. The only rationalizable structure is thus the connected, center-sponsored star.

Proof: We need to only show this for Case I.

(1) If i is in a connected network that is not a center-sponsored star, then by their flows, the agents know they can switch links and be no worse off. The only connected network that is rationalizable is thus the center-sponsored star.

(2) Being unconnected is not a best response since i knows by v^t known that he can link to some other agent and be strictly better off.

(3) Consider partial networks. If a strict subnetwork has less than four agents, then one of them must know the structure of the subnetwork with certainty. Because we are in Case I and v^t is known, that agent i in that subnetwork is strictly better off by initiating a link with some agent j not in the subnetwork. Hence, it follows that any strict network with less than four agents cannot exist in an SNE in Case I. We now consider a strict subnetwork with at least four agents.

(i) *We show that* $Pr[j \notin N_i] \geq \frac{n-|N_i|}{n-3}$. A subnetwork in which all agents are linked in a line is the subnetwork that provides the least amount of information in signals. If we can show this for the line network then it holds for the others. A channel i in such a subnetwork must have $|L_i| = 2$. Since $v_i^t = v_j^t = v$, i knows $|N_i|$. Then i knows that there are $n - |N_i|$ agents not in N_i. Consider all $j \notin \{L_i \cup \{i\}\}$. If i assigns equal likelihood to any of those j's to not be in N_i then $\Pr[j \notin N_i] = \frac{n-|N_i|}{n-1-|L_i|}$ which will be equal to $\frac{n-|N_i|}{n-3}$ if $|L_i|$ is it's lowest amount.

If i does not assign these probabilities uniformly across all $j \notin \{L_i \cup \{i\}\}$ then there is still some $j \notin \{L_i \cup \{i\}\}$ for which $\Pr[j \notin N_i] \geq \frac{n-|N_i|}{n-3}$. This is true because if some $k \notin \{L_i \cup \{i\}\}$ is assigned less than that amount, then k is more likely relative to our uniform probability to be in N_i which means there is some j that must be more likely to not be in N_i relative to the uniform probability. Hence, $\Pr[j \notin N_i] \geq \frac{n-|N_i|}{n-3}$.

(ii) *We show that if* $|N_j| \geq |N_i|$ *then* i *will initiate a link with* j. Agent i will initiate a link iff $\Pr[j \notin N_i] f_j = \Pr[j \notin N_i] |N_j| v \geq c$ since it would be a best response. This expression becomes $\Pr[j \notin N_i]\,|N_j| \geq \frac{c}{v}$. We want to see if i initiates this link whenever $|N_j| \geq |N_i|$ for all values of $|N_j|$, $|N_i|$, and n, and it will if the LHS's lowest value is greater than $\frac{c}{v}$. Over $|N_j| \geq |N_i|$, LHS is lowest when $|N_j| = |N_i|$, $n = |N_j| + |N_i| = 2|N_i|$, and $\Pr[j \notin N_i] = \frac{n-|N_i|}{n-3}$. These values yield $\Pr[j \notin N_i]\,|N_j| = \frac{n-|N_i|}{n-3}|N_i| = \frac{|N_i||N_i|}{n-3}$ which some algebra reveals to always be greater than 1 for positive values of $|N_i|$. Since $\frac{c}{v}$ is less than 1 by Case I, the LHS must always be higher than $\frac{c}{v}$, which means that i will always initiate a link with j should he believe $|N_j| \geq |N_i|$.

(iii) *It follows that for i to not want to initiate a link, he must assign positive probability to $|N_j| < |N_i|$.*

(iv) *We show that any strict subnetwork is not rationalizable.* If BR_i is no new link then i must believe there is some $|N_j| < |N_i|$ and for this j to not initiate a new link then j must believe $|N_k| > |N_j|$, ..., and so on. Since $|N_i| > |N_j| > |N_k| > ...$ is a strictly decreasing sequence of inequalities involving integers, eventually some y with $|N_y| = 4$ must believe there is some $|N_z| < |N_y|$. But if $z \in N_z$ where $|N_z| < 4$, then z is not playing a best response. Thus the best-response chain is violated, and the conjecture is not rationalizable. Hence, any partial network with v^t known and $v_i^t = v_j^t \geq c$, $\forall i, j$ is not rationalizable. ■

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