6.207/14.15: Networks
Lectures 17 and 18: Network Effects

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November 16 and 18, 2009

Outline

- Network effects
- Strategic complements
- Equilibria with network effects
- Dynamics of network effects
- Network effects with local interactions
- Network effects in residential choices
- Network effects in the labor market
- Supermodular games
- Contagion in networks and graphical games

Public

- Reading:
- EK, Chapter 17
- Jackson, Chapter 9.6

Network Effects

- We say that there are network effects when the desired behavior of an individual depends on some average of the actions of others.
  - Network effects with local interactions when these effects work through the behavior of "neighbors".
- At this level of generality, several games we have already studied exhibit network effects.
- However, network effects become more interesting in the context of markets, particularly, when we study product, residential or technology choices.
- In what follows, we will first illustrate network effects, then provide several different frameworks for studying them.

Example without Network Effects

- Consider a society consisting of a large number of individuals (for example $i \in \mathcal{I} \equiv [0,1]$, though $i = 1, ..., I$ for I large would also be fine).
- Each individual chooses between two products denoted by $s_i \in \{0,1\}$.
- First suppose that each individual has preferences given by

$$u(s_i, x_i) = [x_i - c]\, s_i,$$

  where $x_i \in \mathbb{R}_+$ could be thought of as the *type* of the individual, representing his utility from taking action $s_i = 1$, and c is the cost of this action.
- Suppose that x_i has a distribution given by G in the population (with continuous density g).

Example without Network Effects (continued)

- It is straightforward that all individuals with

$$x_i > c$$

will take action $s_i = 1$, and those with $x_i < c$ will take action $s_i = 0$.

- Focus on Nash equilibria.
- Then the following is immediate:

Proposition

In the unique equilibrium, a fraction $S = 1 - G(c)$ of the individuals will choose $s_i = 1$.

- So far there are no network effects.

Network Effects in Product Choice

- Now imagine that $s_i = 1$ corresponds to choosing a new product, such as Blu-Ray vs. HD DVD, or signing up to a new website, such as MySpace or Facebook; $\mathcal{I}$ is a set of potential friends ("network").
- The utility of signing up is higher when a greater fraction of one's friends have signed up. This is an example of a *network effect*.
- In the context of the environment described above, we can capture this by modifying each agent's utility function to:

$$u(s_i, x_i, S) = [x_i h(S) - c]\, s_i,$$

where S is the fraction of the population choosing $s_i = 1$ and $h : [0, 1] \to \mathbb{R}$ is an increasing function capturing this network effect.
- This is an **aggregative** game, in the sense that the payoff of each agent depends on some aggregate of the actions of others (here the average action of others).
- Equilibrium concept: again **Nash equilibrium**.
  - In aggregative games, the analysis of Nash equilibria somewhat easier.

Network Effects (continued)

- With a similar argument, taking the action of all other agents and thus S, as given, individual i will choose $s_i = 1$ if
$$x_i > \frac{c}{h(S)}.$$
- With the same argument as before, the fraction of agents choosing $s_i = 1$ must be
$$S = 1 - G\left(\frac{c}{h(S)}\right).$$
- An equilibrium is therefore a fixed point of this equation.

Proposition

An equilibrium in the product choice game exists, but is not necessarily unique.

- Existence follows from standard arguments. We next illustrate the possibility of multiple equilibria.

Network Effects (continued)

- Suppose, for example, that $h(S) = S$ and the distribution G is given by

$$G(x) = \frac{\gamma + \beta}{1 + \beta} \frac{(1 + \beta) e^{-\alpha'/x}}{\gamma + \beta e^{-\alpha'/x}},$$

  where $\alpha' > 0$, $0 < \gamma < \beta$.

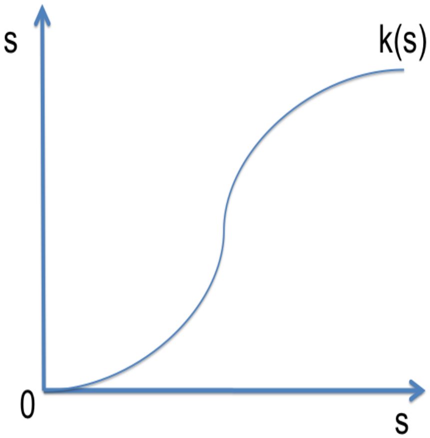
- It can be verified that $G(x)$ is indeed a probability distribution over $\mathbb{R}_+$.
- Then the fixed point equation becomes

$$S = k(S) \equiv \frac{\gamma - \gamma e^{-\alpha S}}{\gamma + \beta e^{-\alpha S}},$$

  where $\alpha \equiv \alpha'/c$.

- Then the right-hand side of this equation has the logistic or the "lazy S" shape.

The Lazy S Shape



Equilibria

- Equilibria are intersections of the $k(S)$ curve with the 45° line. These are just standard Nash equilibria (each agent is best responding to what others are doing, which here is represented by the average fraction of other agents choosing $s_i = 1$).
- The figure shows that there are in general multiple equilibria. The $k(S)$ curve typically has three intersections with the 45° line: one equilibrium at $S = 0$, one in the middle, and one at $S = S^h$.
- Network effects have therefore induced possible multiple equilibria.

Multiple Equilibria

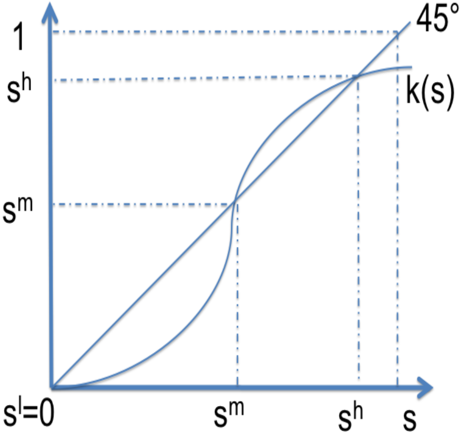
- In particular,

$$k'(S) = \frac{\alpha\gamma e^{-\alpha S}(\gamma+\beta)}{(\gamma+\beta e^{-\alpha S})^2} > 0, \text{and}$$
$$k''(S) = \frac{-\alpha^2\gamma(\gamma+\beta)e^{-\alpha S}[\gamma-\beta e^{-\alpha S}]}{(\gamma+\beta e^{-\alpha S})^3}.$$

- This implies that $k(S)$ is concave for $S > -\log(\gamma/\beta)/\alpha$ and convex otherwise, as shown in the next figure.
- Moreover,

$$k(0) = 0, \text{ and } k(1) < 1.$$

Multiple Equilibria in the Figure



Multiple Equilibria (continued)

- When will there be multiple equilibria? Since $S = 0$ is an equilibrium, $k(S)$ has the lazy S shape, $k(1) < 1$, and $k(S)$ is increasing in α^*, a necessary and sufficient condition is that

$$k(S) - S > 0 \text{ for some } S,$$

which will be true for $\alpha > \alpha^*$ (if $\alpha = \alpha^*$, then tangency rather than intersection between $k(S)$ and the 45° line).

Proposition

In the product choice game, there will be three equilibria if and only if $\alpha >> \alpha^*$.

- The proof follows from the figure and the above derivation.

Externalities and Strategic Complementarities

- At the root of the network effects is the phenomenon of externalities.
- Externalities refer to a situation in which the action of an agent has an effect on the payoff of others. Game theoretic situations are typically situations that generate externalities. In contrast, externalities are assumed to be absent in competitive market economies (except "pecuniary externalities" that do not have first-order effects).
- In this situation, we have **positive externalities**, since higher S (weakly) increases everybody else's utility.
- More important: we also have strategic complements. Each agent is more willing to take the action $s_i = 1$ when others are doing so.
- Multiple equilibria are driven by **strategic complementarities**. Below we will see a more general framework for analyzing strategic complementarities.

Welfare Comparisons

- Multiple equilibria in games with strategic complementarities and positive externalities can generally be **Pareto ranked**.
- We say that a strategy profile (equilibrium) $\tilde{s}$ Pareto dominates another profile $\hat{s}$ if all agents are weakly better off under $\tilde{s}$ than under $\hat{s}$, and at least one agent is strictly better off.
- More formally:

Definition

In a strategic form game $G = \langle \mathcal{I}, (A_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}}$, strategy profile $\tilde{s}$ Pareto dominates profile $\hat{s}$ if

$$u_i(\tilde{s}) \geq u_i(\hat{s}) \text{ for all } i \in \mathcal{I}, \text{ and}$$

$$u_i(\tilde{s}) > u_i(\hat{s}) \text{ for some } i \in \mathcal{I}.$$

Welfare Comparisons (continued)

Proposition

In the product choice game, suppose that $\alpha > \alpha^*$. Then the equilibrium S^h Pareto dominates equilibria S^m and $S^l = 0$ (and S^m Pareto dominates $S^l = 0$).

Proof:

- It is sufficient to prove this proposition for the comparison of S^h and $S^l = 0$.
- First observe that each agent that chooses $s_i = 0$ in the equilibrium S^l will also do so in the equilibrium S^h.
- Conversely, there are no agents choosing $s_i = 1$ in the equilibrium S^l (more generally, in comparing S^h and S^m, we would have that each agent that chooses $s_i = 1$ in S^m will also do so in S^h).

Proof (continued)

- Thus we only have to compare agents who choose $s_i = 1$ in the equilibrium S^h but not in S^l.
- Note that for all of these agents choosing $s_i = 1$ in the equilibrium S^h

$$u\left(s_i = 1, x_i, S^h\right) = \left[x_i h\left(S^h\right) - c\right] s_i \geq 0 = u\left(s_i = 0, x_i, S^l\right).$$

- Since $S^h > 0$, this means that this inequality holds for a positive measure of agents, and since G is a continuous distribution, it must hold as a strict inequality for all but one type that is indifferent.

Comparative Statics

- In game-theoretic situations, we are often interested in comparative statics, which tell us how changes in parameters/environment/network structure affect behavior.
- For example, one could ask how an increase in the value of the product changes the fraction of agents adopting it. Or one could ask how an increase in the value of the product for a subset of the agents affects adoption.
- In games with strategic complementarities, these questions can be answered in a fairly tight manner.

Comparative Statics (continued)

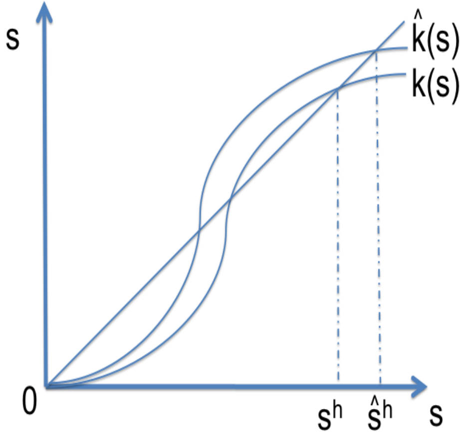
Proposition

In the product choice game, suppose that x_i increases for a fraction $\rho > 0$ of the agents. Then S^h increases or remains the same.

- The idea of the proof:
- First, an increase in x_i, all else equal, will increase s_i or leave it unchanged. This is the *direct effect* of the change.
- Second, there is the indirect effect, since now S^h increases because of the direct affect and thus $\left[x_i h\left(S^h\right) - c\right] s_i$ increases for all agents, other agents will be induced to also increase their actions. (Or if the initial change did not affect any of the agents, then there is no change).
- This captures the notion of strategic complementarity: each agent is more likely to choose the product when others do so in greater proportion.

Comparative Statics in the Figure

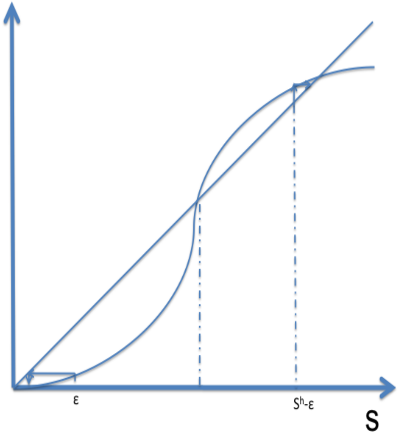
- This proposition can also be seen in the figure.



Myopic Stability and Tipping

- Consider the three equilibria and the myopic dynamic process similar to fictitious play induced by **best response dynamics**.
- Formally, best response dynamics generate a sequence of play, here represented by $\{S_t\}_{t=0}^{\infty}$ such that $S_{t+1} \in BR(S_t)$, where $BR(S)$ is the set of fraction of agents taking action $s_i = 1$ that is a best response to that fraction being S.
- Clearly, the three equilibria satisfy $S \in BR(S)$.
- In the neighborhood of the equilibrium with $S = 0$, if a few agents (a small measure ε of agents) are induced to choose $s = 1$, this will not disturb the equilibrium. The best response to $S = \varepsilon$ is the fraction less than ε as the figure shows, i.e., $BR(\varepsilon) < \varepsilon$.
- Carrying through with the best response dynamics, we will gradually approach $S = 0$.
- The next figure represents these ideas by looking at "myopic dynamics"

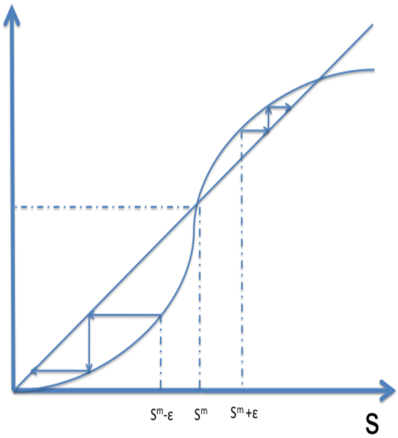
Figure



Myopic Stability and Tipping (continued)

- Also in the neighborhood of the highest equilibrium, S^h, if we induce a small measure ε of agents to change their behavior, this will have a (small) effect in the direction of S^h.
- For example, $BR\left(S^h - \varepsilon\right) > S^h$.
- However, if the same thing is done starting in the intermediate equilibrium, S^m, then in response other agents will start moving in the direction of the small change.
- We therefore have that the middle equilibrium S^m is **asymptotically unstable** under myopic dynamics, while S^l and S^h are **asymptotically stable** (recall the terminology introduced in the context of evolutionary dynamics).
- The next figure shows the tipping phenomenon from the "myopic dynamics"

Figure



Tipping

- We therefore have an example of tipping: a small change ("perturbation") in the neighborhood of the unstable equilibrium S^m will take the equilibrium in one or another direction.
- This term goes back to Thomas Schelling's description of *neighborhood tipping*, where because of slight preferences in one group's preferences about who their neighbors should be, ratio structure of neighborhoods can suddenly tip from all white to all black etc.
- Tipping is a general representation of phenomena in the presence of network effects, whereby some configurations are "unstable" and will change in response to small disturbances taking the system to a different (potentially far away) equilibrium.

Dynamic Version of Network Effects

- The above description of dynamics was based on "myopic dynamics" in the sense that even though we looked at the Nash equilibrium, the dynamics were driven by myopic best responses of individuals to what others are doing rather than their anticipation of what will happen in the future.
- This was for simplicity. We can extend the same ideas to proper dynamic equilibria.
- Imagine that this game is played by a sequence of (generations of) players, $t = 0, 1, ...$, (each generation represented by a set of players $[0, 1]$).
- For simplicity, suppose that preferences of generation $t \geq 1$ are given by

$$u(s_{i,t}, x_{i,t}, S_{t-1}) = [x_{i,t} h(S_{t-1}) - c] \, s_{i,t},$$

so that they care about the actions taken by the previous generation.
- Similar justification for this type of network effects.

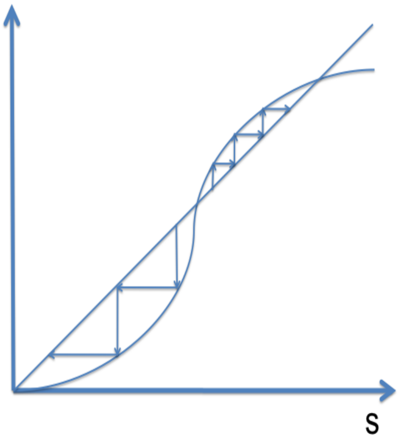
Equilibrium Dynamics

- Now with the same reasoning as before, we have that in generation t, the fraction of agents choosing $s_i = 1$ will be

$$S_t = 1 - G\left(\frac{c}{h(S_{t-1})}\right).$$

- The **dynamic equilibrium path** can now be drawn using the same figure.
- This example also highlights the connections between myopic stability (from best response dynamics) in static games and dynamic equilibrium of related (dynamic) games in which payoff externalities are from past actions.

Figure



Dynamic Tipping

- This figure shows the type of dynamic tipping that will happen in this case. When we start in the neighborhood of $S = 0$, say at $S_0 = \varepsilon$ for some ε small. We will then move towards $S = 0$. Thus $S = 0$ is **asymptotically stable**, now under equilibrium dynamics.
- The same applies when we are in the neighborhood of $S = S^h$, say either at $S_0 = S^h - \varepsilon$ or $S_0 = S^h + \varepsilon$. In both cases, the dynamic equilibrium path will move towards $S_0 = S^h$, so that S^h is also asymptotically stable.
- In contrast, suppose we now start at $S_0 = S^m$. Now we have a **tipping** phenomenon. If we disturb the equilibrium with a small perturbation, we will move in the direction of the perturbation, if we have $S_1 = S^m - \varepsilon$, then $S_t \to 0$, and if we have $S_1 = S^m + \varepsilon$, then $S_t \to S^h$. This is true regardless of how small ε is.

History vs. Expectations in Games with Network Effects

- The above equilibrium path was derived assuming a backward-looking payoff function $u(s_i, x_i, t) = x_i s_i h(S_{t-1}) - c$, so that only **history** mattered.
- In general, tipping like phenomena imply that changes in **expectations** could have a large role. In the static game, which equilibrium will emerge is purely due to "expectations"— or beliefs.
- Now imagine a more general version of the payoff function, whereby

$$u(s_{i,t}, x_{i,t}, S_t, S_{t-1}) = x_{i,t} [\lambda h(S_{t-1}) + (1-\lambda) h(S_t)] s_{i,t} - c,$$

so that both the past and the present matter. Now clearly, in generation t, the fraction of agents choosing $s_i = 1$ will be

$$S_t = 1 - G\left(\frac{c}{\lambda h(S_{t-1}) + (1-\lambda) h(S_t)}\right).$$

History vs. Expectations in Games with Network Effects

- If λ is large, history matters a lot, and the dynamic equilibrium similar to the above case.
- In particular, "no jump from one equilibrium to another".
- If λ is small, then expectations matter a lot, and one could start in the neighborhood of $S = S^h$ and jump to $S = 0$.
  - Question: are such jumps possible along the equilibrium path?

Network Effects with Local Interactions

- The term network effects originates from the fact that what individuals typically care about is not averages, but what their neighbors do.
- This can be incorporated into models of network interactions and leads to richer dynamics, and to phenomena such as **domino effects**.
- More generally, so far we have looked at **aggregative games**, where payoff externalities are from the aggregate of others's action (such as the average).
- More generally, one could look at **graphical games**, where payoff externalities are from the average behavior of "neighbors" in a social network (directed or undirected graph).
- Here we provide a simple example to illustrate the issues.

Domino Effects on the Circle

- As a simple illustration, consider a structure of local interactions represented by I agents located around a circle, so that agent i is to the right of $i-1$ and 1 is to the right of I.
- Suppose that the utility is now given by

$$u(s_i, x_i, s_{i-1}) = [x_i h(s_{i-1}) - c]\, s_i,$$

  where for $i = 1$, $i - 1$ is, by convention, set to I.
- First suppose that $x_i = 1$ for all i, $c = 1$, $h(0) = 2/3$ and $h(1) = 2$. Then it can be verified that there are two pure strategy equilibria here, one in which all agents choose $s_i = 0$ and one in which they all choose $s_i = 1$.
- Can these equilibria be Pareto "ranked"?
- In this example, the equilibrium with $s_i = 1$ once again **Pareto dominates** the equilibrium with $s_i = 0$: all agents have strictly higher utility in equilibrium with $s_i = 1$.

Domino Effects on the Circle (continued)

- To illustrate the possibility of domino effects, now suppose that we start with the equilibrium with $s_i = 0$ for all i. Now one of the agents, say $i = 1$, is hit by a shock that increases x_1 from 1 to $x_1 = 2$.
- Then it can be verified that the unique equilibrium becomes $s_i = 1$.
- The domino effects arise from the fact that once $i = 1$ chooses $s_i = 1$, then $i = 2$ would also like to choose $s_i = 1$, and so on.
  - Alternatively, we can say that the behavior $s_i = 1$ is *contagious*. We will see more general model of contagion below.
- Conversely, starting from the equilibrium with $s_i = 1$, a shock to one of the agents, say again $i = 1$, so that $x_1 = 0$ will start the reverse domino effects/contagion, and all agents will switch to $s_i = 0$.

Dynamic Domino Effects

- With the same trick we used in the model where externalities were from aggregate behavior, we can also turn the domino effects into dynamic effects.
- In particular, suppose that individual i moves at times $\lfloor \frac{t}{I} \rfloor + i$ (thus once every I periods). Also assume that I is sufficiently large and discounting is sufficiently small so that each agent cares about current actions.
- Suppose utility of agent i acting at time t is

$$u(s_{i,t}, x_i, s_{i-1,t-1}) = [x_i h(s_{i-1,t-1}) - c] \, s_{i,t}.$$

- Suppose we start an equilibrium with $s_{i,t} = 0$ for all i.
- Now a shock to player i that makes it switch from $s_{i,t-I} = 0$ to $s_{i,t} = 1$ will create a dynamic domino effect, whereby in each subsequent period (until all agents are reached) one more agent will switch to $s_i = 1$.

Network Effects in Residential Choices

- Schelling's original tipping model was formulated in the context of residential choices by whites and blacks.
- The model he proposed is quite complex (similar to the Ising model in physics), and Schelling illustrated that rich behavior can arise.
- In general, network effects are possible in residential choices not only because of racial concerns, but because of a host of different types of externalities.
- An important class of externalities are related to **peer effects**, the fact that children are affected by their peers.
- Then parents who care about children's education will also care about their children's peers.

Network Effects in Residential Choices (continued)

- We now discuss a model due to Benabou (1992) "The Workings of a City".
- The model will illustrate both network effects in residential choices and also the idea of **symmetry breaking**, that is, the notion that even with symmetric and homogeneous populations, the equilibrium may involve asymmetries and iniquities.
- All agents are assumed to be ex ante homogeneous, and will ultimately end up either low skill or high skill.
- Utility of agent i is assumed to be

$$U^i = w^i - c^i - r^i$$

where w is the wage, c is the cost of education, which is necessary to become both low skill or high skill, and r is rent.

Network Effects in Residential Choices (continued)

- The cost of education is assumed to depend on the fraction of the agents in the neighborhood, denoted by x, who become high skill. In particular, we have $c_H(x)$ and $c_L(x)$ as the costs of becoming high skill and low skill.
- Both costs are decreasing in x, meaning that when there are more individuals acquiring high skill, becoming high skill is cheaper (positive peer group effects).
- In addition,

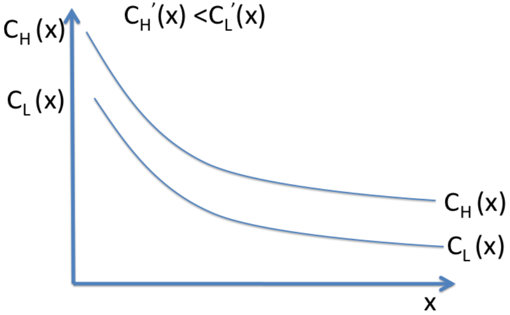
$$c_H(x) > c_L(x)$$

so that becoming high skill is always more expensive.

Network Effects in Residential Choices (continued)

- More importantly, the effect of increase in the fraction of high skill individuals in the neighborhood is bigger on the cost of becoming high skill.

$$c'_H(x) < c'_L(x),$$



Network Effects in Residential Choices (continued)

- Since all agents are ex ante identical, in equilibrium we must have

$$U(L) = U(H)$$

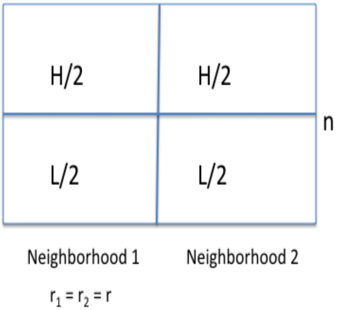
  that is, the utility of becoming high skill and low skill must be the same.
- Assume that the labor market in the economy is global, and takes the constant returns to scale form $F(H, L)$.
- This implies that if the ratio of high to low skill workers, H/L, is high, then w^H/w^L, the wage of a high skill worker relative to a low skill workers, will be low.
- This will guarantee that typically not all workers in the city will be high or low skill. There will be a mixture.

Network Effects in Residential Choices (continued)

- Another important implication is that irrespective of where a worker obtains his or her education, he will receive the same wage as a function of his skill level.
- Also assume that there are two neighborhoods of fixed and equal size, and individuals will compete in the housing market to locate in one neighborhood or the other.
- There are two types of equilibria:
  1. Integrated city equilibrium, where in both neighborhoods there is a fraction $\hat{x}$ of individual obtaining high education.
  2.]d city equilibrium, where one of the neighborhoods is homogeneous. For example, we could have a situation where one neighborhood has $x = 1$ and the other has $\tilde{x} < 1$, or one neighborhood has $x = 0$ and the other has $\bar{x} > 0$.

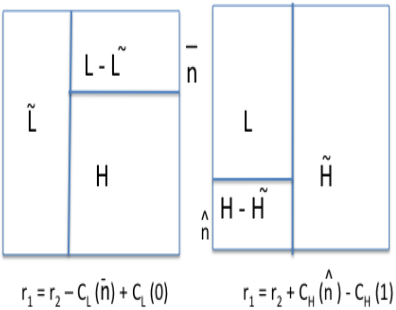
Network Effects in Residential Choices (continued)

- Figure: Integrated City



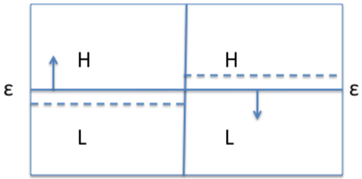
Network Effects in Residential Choices (continued)

- Figure: Segregated Cities



Network Effects in Residential Choices (continued)

- The important observation here is that only segregated city equilibria are **asymptotically stable** (under myopic dynamics).
- To see this consider an integrated city equilibrium, and imagine relocating a fraction ε of the high-skill individuals (that is individuals getting high skills) from neighborhood 1 to neighborhood 2.



Network Effects in Residential Choices (continued)

- This will reduce the cost of education in neighborhood 2, both for high and low skill individuals.
- But by assumption, it reduces it more for high skill individuals, so all high skill individuals now will pay higher rents to be in that city, and they will outbid low-skill individuals, taking the economy toward the segregated city equilibrium.
- This illustrates *symmetry breaking*.

Network Effects in Residential Choices (continued)

- In contrast, the segregated city equilibrium is always **asymptotically stable**.

Proposition

In the residential choice game, there always exists an integrated city and a segregated city equilibrium. The integrated city equilibrium is always asymptotically unstable and segregated city equilibria are asymptotically stable.

Network Effects in Residential Choices (continued)

- Thus segregation arises as the equilibrium (stable equilibrium) outcome, because of "complementarities". However, in this case, we do not have a game of strategic complementarities because the population must consist of a mixture of high and low skill workers.
- In fact, it can be verified that in this case, either one of the three different types of equilibria could Pareto dominate others.
- This is because greater concentration of skilled workers in particular neighborhood reduces the costs of education both for high and low skill workers.

Network Effects in the Labor Market

- An idea going back to Marshall's *Principles of Economics* is that geographic concentration of firms is related to the geographic concentration of certain types of skills. But why would skilled workers be in one place vs. another?
- Here we will discuss a model based on Acemoglu (1997) "Training and Innovation in an Imperfect Labor Market".
- Consider the following two-period model. There is population N of workers and population N of firms (where N is large).
- Initially, each worker is matched with a firm. The firm can produce a baseliner level of output y_0.

Network Effects in the Labor Market (continued)

- In addition, it can adopt a new technology at cost k and also train its employee at cost c. The return to the technology is α if there is a trained worker to operate it (so that the firm produces $y_0 + \alpha$). Without a trained worker, the new technology has no additional return. Training is also not useful without the new technology.
- Also assume that there is no possibility of additional technology adoption or training in the second date and that there is no discounting.
- Suppose that wages are given by a fraction $\beta \in (0, 1)$ of total output.
- Both training and the new technology can be used in the second period as well.
- Finally, with probability $q \in [0, 1)$, the worker and the firm separate (they are no longer a "good match" together), and they find a new partner (since there are the same number of firms and workers to start with, each separate side finds a partner).

Network Effects in the Labor Market (continued)

- First assume that $q = 0$. Then we have the following result:

Proposition

In the technology adoption game, suppose that

$$\beta < 1 - \frac{k + c}{2\alpha}, \qquad (*)$$

then the unique equilibrium involves all firms adopting the new technology and training their employees.

Network Effects in the Labor Market (continued)

- Why is this true?
- Clearly, training by itself or adoption of the new technology by itself cannot be optimal.
- If the firm does not adopt a new technology and train the employee, then it has total profits

$$2(1-\beta)\,y_0.$$

- If it adopts the new technology and trains its employee, then has profits

$$2(1-\beta)(y_0+\alpha)-k-c,$$

which is strictly greater than $2(1-\beta)\,y_0$ under $(*)$

Network Effects in the Labor Market (continued)

- Now suppose that $q > 0$. In this case, suppose that no other firm adopts the technology and trains their employee. Then expected profits from not training are still $2(1-\beta)\,y_0$, while the profits from adoption of technology are

$$(1-\beta)\,[2y_0 + \alpha + (1-q)\,\alpha] - k - c.$$

Proposition

In the technology adoption game, suppose that

$$1 - \frac{k+c}{(2-q)\,\alpha} < \beta < 1 - \frac{k+c}{2\alpha}, \qquad (**)$$

then there exist an equilibrium in which all firms adopt the new technology and train their employees and another equilibrium in which no firm adopts the new technology or trains their employees. The equilibrium with technology adoption Pareto dominates the equilibrium without.

Network Effects in the Labor Market (continued)

- This result is straightforward. Under $(**)$, if no other firm adopts, adopting the new technology and training is not profitable.
- In contrast, if all other firms are expected to adopt the new technology, then $(**)$ implies that adopting new technology and offering training to one's employee is a (strict) best response.
- There are therefore 2 pure strategy (symmetric) equilibria, one in which all firms adopt the new technology and train their workers, and another one in which all of them do not.
- With the same reasoning as above, the equilibrium with adoption is Pareto dominant (in this equilibrium, all firms have the option of not adopting, and strictly prefer to adopt).

Network Effects in the Labor Market (continued)

- Intuitively, the multiplicity of equilibria is because in one case firms expect to be able to fill their vacancies with qualified workers and thus are more willing to adopt new technologies *complementary* to the workers, whereas in the other case they do not expect such qualified workers and adopting the new technology is not sufficiently profitable.
- These expectations are *self-fulfilling* because when all firms adopt the new technology they generate (by investing in their employees) a qualified workforce.
- Now imagine an extension, with two regions, one in which all firms that adopt the new technology and the other one where all existing firms do not. Suppose that a firm that wishes to adopt a new technology will choose its location. Then it will locate together with other adopters, even if residential rents are higher in that area. In contrast, those wishing not to adopt can locate with the non-adopters.
- This captures some of the logic of technology clusters such as Route 128 or the Silicon Valley.

Supermodular Games

- So far we have studied several examples of games with "network effects" or strategic complementarities.
- These are special instances of more general phenomena.
- We can get a sense of the general results in such games by considering the class of games called supermodular games.
- Supermodular games $\equiv$ games with strategic complementarities
- Informally, this means that the marginal return to increasing a player's strategy increases in the other players' strategies.
  - Implication $\Rightarrow$ best response of a player is a nondecreasing function of other players' strategies
- In addition to unifying what we have seen before, the analysis here will show that these games always have pure strategy equilibria (without assuming concavity of payoff functions), lead to similar predictions under different equilibrium concepts, have nice stability properties under myopic or best response dynamics, and lead to general comparative statics results.

Monotonicity of Optimal Solutions

- The machinery needed to study supermodular games is lattice theory and monotonicity results in **lattice programming.**
- We first study the monotonicity properties of optimal solutions of parametric optimization problems. Considered a problem

$$x(t) \in \arg\max_{x \in X} f(x, t),$$

where $f : X \times T \to \mathbb{R}$, $X \subset \mathbb{R}$, and $T \subset \mathbb{R}$ (or more generally some partially ordered set). [Throughout the symbol $\subset$ stands for "is included in and possibly equal to," i.e., equivalent to $\subseteq$].
- We will focus on $T \subset \mathbb{R}^K$ with the usual **vector order**, i.e., for some $x, y \in T$, $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, \ldots, k$.
    - Theory extends to general lattices; lattice $\approx$ a set that has least and greatest elements.
- We are interested in conditions under which we can establish that $x(t)$ is a nondecreasing function of t.

Increasing Differences

- Key property: **Increasing differences**.

Definition

Let $X \subset \mathbb{R}$ and $T \subset \mathbb{R}$. A function $f : X \times T \to \mathbb{R}$ has increasing differences in (x, t) if for all $x' \geq x$ and $t' \geq t$, we have

$$f(x', t') - f(x, t') \geq f(x', t) - f(x, t).$$

- Intuitively: incremental gain to choosing a higher x (i.e., x' rather than x) is greater when t is higher, i.e., $f(x', t) - f(x, t)$ is nondecreasing in t.
- The previous definition gives an abstract characterization. The following result makes checking increasing differences easy in many cases.

Increasing Differences

Lemma

Let $X \subset \mathbb{R}$ and $T \subset \mathbb{R}^k$ for some k. Let $f : X \times T \to \mathbb{R}$ be a twice continuously differentiable function. Then, the following statements are equivalent:

- *The function f has increasing differences in (x, t).*
- *For all $t' \geq t$ and all $x \in X$, we have*

$$\frac{\partial f(x, t')}{\partial x} \geq \frac{\partial f(x, t)}{\partial x}.$$

- *For all $x \in X$, $t \in T$, and all $i = 1, \ldots, k$, we have*

$$\frac{\partial^2 f(x, t)}{\partial x \partial t_i} \geq 0.$$

Increasing Differences

- Increasing differences closely linked to strategic complementarities.
- This lemma also shows that when the relevant functions are differentiable, strategic complementarities are the same as "positive cross partial derivatives".
- In general check strategic complementarity through this condition.

Supermodular Games

Definition

The strategic game $\langle \mathcal{I}, (S_i), (u_i) \rangle$ *is a supermodular game if for all* $i \in \mathcal{I}$*:*

- S_i *is a compact subset of* $\mathbb{R}$ *[or more generally* S_i *is a complete lattice in* $\mathbb{R}^{m_i}$*];*
- u_i *is continuous in* s_i*, continuous in* s_{-i} *[or more generally upper semi-continuity suffices];*
- u_i *has increasing differences in* (s_i, s_{-i}) *[or more generally* u_i *is supermodular in* (s_i, s_{-i})*, which is an extension of the property of increasing differences to games with multi-dimensional strategy spaces].*

Network Effects in Supermodular Games

- Consider a generalization of the product choice game discussed in the previous literature.
- A set $\mathcal{I}$ of users can use one of two products X and Y (e.g., Blu-ray and HD DVD).
- $B_i(J, k)$ denotes payoff to i when a subset J of users use technology k and $i \in J$
- There exists a positive externality if

$$B_i(J, k) \leq B_i(J', k), \qquad \text{when } J \subset J',$$

i.e., player i better off if more users use the same technology as him.

Network Effects in Supermodular Games (continued)

- This leads to a strategic form game with actions $S_i = \{X, Y\}$
- Given $s \in S$, let $X(s) = \{i \in \mathcal{I} \mid s_i = X\}$, $Y(s) = \{i \in \mathcal{I} \mid s_i = Y\}$. We write $X(s) \geq Y(s)$ if the size of the first set is greater than the second.
- Then we have a supermodular game with payoff function

$$u_i(s_i, s_{-i}) = \begin{cases} B_i(X(s), X) & \text{if } s_i = X, \\ B_i(Y(s), Y) & \text{if } s_i = Y \end{cases}$$

- It can be verified that increasing differences are satisfied.

Cournot As a Supermodular Game with Change of Order

- Consider Cournot duopoly model. Two firms choose the quantity they produce $q_i \in [0, \infty)$.
- Let $P(Q)$ with $Q = q_i + q_j$ denote the inverse demand (price) function. Payoff function of each firm is $u_i(q_i, q_j) = q_i P(q_i + q_j) - cq_i$.
- Assume $P'(Q) + q_i P''(Q) \leq 0$ (firm i's marginal revenue decreasing in q_j).
- We can now verify that the payoff functions of the transformed game defined by $s_1 = q_1$, $s_2 = -q_2$ have increasing differences in (s_1, s_2).
- Alternatively, Cournot can be modeled as a game of **strategic substitutes**. In such games, there are "decreasing differences" instead of increasing differences. Important class of games, but more difficult to analyze not as salient in the analysis of "network effects". Hence we will not discuss those in this course.

Monotonicity of Optimal Solutions

- Key theorem about monotonicity of optimal solutions:

Theorem

(Topkis) *Let $X \subset \mathbb{R}$ be a compact set and $T \subset \mathbb{R}$. Assume that the function $f : X \times T \to \mathbb{R}$ is continuous [or upper semicontinuous] in x for all $t \in T$ and has increasing differences in (x, t). Define $x(t) \equiv \arg\max_{x \in X} f(x, t)$. Then, we have:*

1. *For all $t \in T$, $x(t)$ is nonempty and has a greatest and least element, denoted by $\bar{x}(t)$ and $\underline{x}(t)$ respectively.*
2. *For all $t' \geq t$, we have $\bar{x}(t') \geq \bar{x}(t)$ and $\underline{x}(t') \geq \underline{x}(t)$.*

- Summary: if f has increasing differences, the set of optimal solutions $x(t)$ is non-decreasing in the sense that the largest and the smallest selections are non-decreasing.

Existence Problems

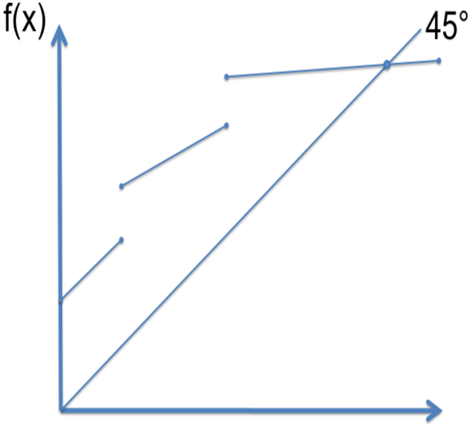
- How do we show existence of Nash equilibria?

Theorem

(Tarski) *Let S be a compact sublattice of $\mathbb{R}^k$ and $f : S \to S$ be an increasing function (i.e., $f(x) \leq f(y)$ if $x \leq y$). Then, the set of fixed points of f, denoted by E, is nonempty.*

- **Line of attack:** Apply Topkis's Monotonicity Theorem to show that best response correspondences are increasing and then apply Tarski's Theorem to best response correspondences to show existence of pure strategy Nash equilibria and their properties.

Tarski's Fixed Point Theorem



Existence of a Pure Nash Equilibrium

- Existence of pure strategy Nash equilibria will then follow from Tarski's fixed point theorem:

Theorem

Assume $\langle \mathcal{I}, (S_i), (u_i) \rangle$ *is a supermodular game. Let*

$$B_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

Then:

1. $B_i(s_{-i})$ *has a greatest and least element, denoted by* $\bar{B}_i(s_{-i})$ *and* $\underline{B}_i(s_{-i})$.
2. *If* $s'_{-i} \geq s_{-i}$, *then* $\bar{B}_i(s'_{-i}) \geq \bar{B}_i(s_{-i})$ *and* $\underline{B}_i(s'_{-i}) \geq \underline{B}_i(s_{-i})$.

- Follows immediately from Topkis's Monotonicity Theorem.

Existence of a Pure Nash Equilibrium

Theorem

(Topkis) *Let* $\langle \mathcal{I}, (S_i), (u_i) \rangle$ *be a supermodular game. Then the set of pure strategy Nash equilibria is nonempty and has greatest and least elements* $\bar{s}$ *and* $\underline{s}$*.*

- **Idea of proof:** Apply Tarski's fixed point theorem to best response correspondences.
- Existence of greatest and least elements common with games with network effects that we have seen above.

Elimination of Strictly Dominated Strategies

Theorem

(Milgrom and Roberts) *Let $\langle \mathcal{I}, (S_i), (u_i) \rangle$ be a supermodular game. Then the set of strategies that survive iterated strict dominance (i.e., iterated elimination of strictly dominated strategies) has greatest and least elements $\bar{s}$ and $\underline{s}$, coinciding with the greatest and the least pure strategy Nash Equilibria.*

- **Proof idea:** Start from the largest or smallest strategy profile and iterate the best-response mapping.
- Same proof idea also implies:

Theorem

(Milgrom and Roberts) *Let $\langle \mathcal{I}, (S_i), (u_i) \rangle$ be a supermodular game. Starting with a strategy profile greater than the greatest equilibrium $\bar{s}$, best response dynamics converge to $\bar{s}$, and starting with a strategy profile less than the least equilibrium, $\underline{s}$, best response dynamics converge to $\underline{s}$.*

Comparative Statics in Supermodular Games

- A different approach to understand the structure of Nash equilibria.

Theorem

(Topkis, Milgrom and Roberts) *Let* $\langle \mathcal{I}, (S_i), (u_i) \rangle$ *be a supermodular game. Consider a change in* u_i *for a subset* $\mathcal{I}' \subset \mathcal{I}$ *such that the marginal return to* s_i *increases for all* $i \in \mathcal{I}'$*. Then the greatest and least equilibria* $\bar{s}$ *and* $\underline{s}$ *both increase (or do not change).*

- Idea: the same as in the games with network effects studied above. The change in the utility function has a direct effect which is to increase the strategy of the affected players. Then the indirect effect working through strategic complementarities reinforces this.

Welfare in Supermodular Games

Theorem

Let $\langle \mathcal{I}, (S_i), (u_i) \rangle$ be a supermodular game and suppose that it also exhibits positive externalities (each u_i is nondecreasing in s_{-i}). Then for any two equilibrium profiles s'', s' such that $s'' \geq s'$ (and $s'' \neq s'$), s'' weakly Pareto dominates s'. If u_i is strictly increasing in s_{-i} for some i, then s'' Pareto dominates s'.

- The proof follows immediately from positive externalities (since in the profile s'' each player could have chosen the strategy under s').

Contagion of Behavior over a Network

- We now discuss a game theoretic model of contagion.
- We will consider a **graphical game**, allowing for rich local interactions based on Morris (2000) "Contagion".
- Consider a society represented by an undirected connected graph $G = (V, E)$. Each agent $i \in V$ chooses $s_i = 0$ or 1.
- Suppose preferences are such that each agent (strictly) prefers to choose $s_i = 0$ if less than a fraction q of her neighbors choose $s = 1$. She prefers $s_i = 1$ if greater than a fraction $q \in (0, 1)$ of her neighbors choose $s_i = 1$. This will result, for example, from a "coordination game".
- Clearly, all agents choosing $s_i = 0$ and all agents choosing $s_i = 1$ are equilibria.
- The question is whether there are other equilibria, and if so, do such equilibria exhibit "contagious" behavior in the sense that change by one agent from $s_i = 0$ to $s_i = 1$ induce a large number of others to also switch?

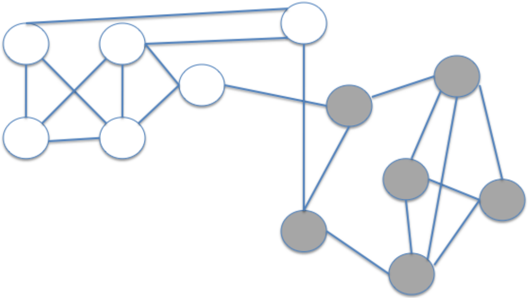
Contagion of Behavior over a Network (continued)

- Let the neighborhood of agent $i \in V$ in G be denoted by $N_i(G)$.
- Let $S = \{i \in V : s_i = 1\}$, that is, the set of agents playing $s_i = 1$ (in this specific equilibrium we have fixed).
- Define S to be r-cohesive with respect to the network G if

$$\max \left\{ \min_{i \in S} \frac{|N_i(G) \cap S|}{|N_i(G)|} \right\} = r.$$

- The denominator $|N_i(G)|$ is the degree of agent (node) i and $|N_i(G) \cap S|$ is the number of edges from i that are also in S (i.e., playing $s_i = 1$).
- This definition implies that r is the maximum real number such that all members of S have at least a fraction r of their neighbors within S.

Figure



Contagion of Behavior over a Network (continued)

Proposition

There exists a pure strategy equilibrium with $\emptyset < S < V$ if and only if S is q- cohesive and $V \backslash S$ is $(1-q)$-cohesive.

Corollary

If there does not exist any S ($\emptyset < S < V$) such that S is q-cohesive and $V \backslash S$ is $(1-q)$-cohesive, then the only pure strategy equilibria are those involving $S = \emptyset$ and $S = V$.

- Both results follow immediately from the description of the game, definition of r-cohesiveness, and the observation that for some $V \backslash S$ to include all players playing $s = 0$, none of them must have a fraction greater than or equal to q playing $s = 1$, and thus $V \backslash S$ must be $(1-q)$-cohesive.

Contagion

- Now start from an equilibrium in which $\emptyset < S < V$, and "infect" some nodes that are not in S so that they also play $s = 1$.
- We ask whether this will start a contagion in the sense that we start with $S' \supset S$ (that is S plus some additional nodes), and then follow best response dynamics, until convergence.
- We say that there is **contagion** if this process converges to all players playing $s = 1$.
- The following result is immediate.

Contagion (continued)

Proposition

Contagion will result from $S' \supset S$ if and only if for any $S'' \supset S'$ we have that $V \backslash S''$ is not $(1-q)$-cohesive.

- **Proof: (Necessity)**
  - If there exists $S'' \supset S'$ such that $V \backslash S''$ is $(1-q)$-cohesive, then every agent in S'' will play $s = 0$ at each iteration of the best resource dynamics, and thus will never switch to $s = 1$.
  - Therefore the stated condition is necessary for contagion.

Proof (continued)

- **Proof: (Sufficiency)**
  - To see that it is sufficient, note that this implies $S'' = S'$ is not $(1-q)$-cohesive.
  - Therefore, in the first iteration, we will move to some $S'_1 \supset S'$.
  - The condition in the proposition implies that this is not $(1-q)$-cohesive, and thus some additional agents will switch to $s = 1$, and we end up with $S'_2 \supset S'_1 \supset S'$. The condition implies that this is not $(1-q)$-cohesive, and thus the process continues. This implies that
  - $\{S'_n\}$ is an increasing sequence of sets, and thus must converge, and can only converge to V. Therefore, in the limit of the best response dynamics, all agents will play $s = 1$, and hence we have contagion, completing the proof.
- However, in practice, difficult to check whether all subsets of V are q- or $(1-q)$-cohesive.