14.773 Political Economy of Institutions and Development.
Lectures 1 and 2: Static Voting Models

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Introduction

- How do democratic societies aggregate preferences?
- How to model elections?
- Do the details of electoral systems matter?

Collective Choices: Introduction

- Abstract economy consisting of a finite set of individuals $\mathcal{H}$, with the number of individuals denoted by H.

- Individual $i \in \mathcal{H}$ has a utility function

$$u\left(x_i, Y\left(x, p\right), p \mid \alpha_i\right).$$

- Here x_i is his action, with a set of feasible actions denoted by X_i;

- p denotes the vector of political choices (for example, institutions, policies or other collective choices), with the menu of policies denoted by $\mathcal{P}$;

- $Y\left(x, p\right)$ is a vector of general equilibrium variables, such as prices or externalities that result from all agents' actions as well as policies, and x is the vector of the x_i's.

Example Preferences

- For example, we could have that given aggregates and policies, individual objective functions are strictly quasi-concave so that each agent has a unique optimal action

$$x_i\left(p, Y\left(x, p\right), \alpha_i\right) = \arg\max_{x \in X_i} u\left(x_i, Y\left(x, p\right), p \mid \alpha_i\right).$$

- Then, *indirect utility function*

$$U\left(p; \alpha_i\right)$$

- The *preferred policy* or the (political) bliss point of individual i

$$p\left(\alpha_i\right) = \arg\max_{p \in \mathcal{P}} U(p; \alpha_i).$$

Preferences More Generally

- Individual individual i weakly prefers p to p',

$$p \succeq_i p'$$

and if he has a strict preference,

$$p \succ_i p'.$$

- Assume: *completeness*, *reflexivity* and *transitivity* (so that $z \succeq_i z'$ and $z' \succeq_i z''$ implies $z \succeq_i z''$).

Collective Preferences?

- Does there exist welfare function $U^S(p)$ that ranks policies for the society.

- Let us simplify the discussion by assuming that the set of feasible policies is, $\mathcal{P} \subset \mathbb{R}^K$

- Let $\Re$ be the set of all weak orders on $\mathcal{P}$, that is, $\Re$ contains information of the form $p_1 \succeq_i p_2 \succeq_i p_3$ and so on, and imposes the requirement of transitivity on these individual preferences.

- An individual ordering R_i is an element of $\Re$, that is, $R_i \in \Re$.

- Since our society consists of H individuals, $\rho = (R_1, ..., R_H) \in \Re^H$ is a *preference profile*.

- Also $\rho_{|\mathcal{P}'} = \left(R_{1|\mathcal{P}'}, ..., R_{H|\mathcal{P}'} \right)$ is the society's preference profile when alternatives are restricted to some subset $\mathcal{P}'$ of $\mathcal{P}$.

Restrictions on Collective Preferences I

- Let $\Im$ be the set of all reflexive and complete binary relations on $\mathcal{P}$ (but notice *not necessarily* transitive).

- A social ordering $R^S \in \Im$ is therefore a reflexive and complete binary relation over all the policy choices in $\mathcal{P}$:

$$\phi : \Re^H \to \Im.$$

- We have already imposed "*unrestricted domain*," since no restriction on preference profiles.

- A social ordering is *weakly Paretian* if

$$\left[p \succ_i p' \text{ for all } i \in \mathcal{H} \right] \implies p \succ^S p'.$$

Restrictions on Collective Preferences II

- Given ρ, a subset $\mathcal{D}$ of $\mathcal{H}$ is *decisive between* $p, p' \in \mathcal{P}$, if

$$\left[p \succeq_i p' \text{ for all } i \in \mathcal{D} \text{ and } p \succ_{i'} p' \text{ for some } i' \in \mathcal{D} \right] \implies p \succ^S p'$$

- If $\mathcal{D}' \subset \mathcal{H}$ is decisive between $p, p' \in \mathcal{P}$ for *all* preference profiles $\rho \in \Re^H$, then it is *dictatorial between* $p, p' \in \mathcal{P}$.

- $\mathcal{D} \subset \mathcal{H}$ is *decisive* if it is decisive between any $p, p' \in \mathcal{P}$

- $\mathcal{D}' \subset \mathcal{H}$ is *dictatorial* if it is dictatorial between any $p, p' \in \mathcal{P}$.

- If $\mathcal{D}' \subset \mathcal{H}$ is dictatorial and a singleton, then its unique element is a *dictator*.

Restrictions on Collective Preferences III

- A social ordering satisfies *independence from irrelevant alternatives*, if for any ρ and $\rho' \in \Re^H$ and any p, $p' \in \mathcal{P}$,

$$\rho_{|\{p,p'\}} = \rho'_{|\{p,p'\}} \implies \phi(\rho)_{|\{p,p'\}} = \phi(\rho')_{|\{p,p'\}}.$$

- This axiom states that if two preference profiles have the same choice over two policy alternatives, the social orderings that derive from these two preference profiles must also have identical choices over these two policy alternatives, regardless of how these two preference profiles differ for "irrelevant" alternatives.

  - While this condition (axiom) at first appears plausible, it is in fact a reasonably strong one. In particular, it rules out any kind of interpersonal "cardinal" comparisons—that is, it excludes information on how strongly an individual prefers one outcome versus another.

Arrow's Impossibility Theorem

Theorem

(**Arrow's (Im)Possibility Theorem**) *If a social ordering, ϕ, is transitive, weakly Paretian and satisfies independence from irrelevant alternatives, then it is dictatorial.*

- An immediate implication of this theorem is that any set of minimal decisive individuals $\mathcal{D}$ within the society $\mathcal{H}$ must either be a singleton, that is, $\mathcal{D} = \{i\}$, so that we have a dictatorial social ordering, or we have to live with intransitivities.

- Also implicitly, political power must matter. If we wish transitivity, political power must be allocated to one individual or a set of individuals with the same preferences.

- How do we proceed? $\rightarrow$ Restrict preferences or restrict institutions.

Proof of Arrow's Impossibility Theorem I

- Suppose to obtain a contradiction that there exists a non-dictatorial and weakly Paretian social ordering, ϕ, satisfying independence from irrelevant alternatives. Contradiction in two steps.

- **Step 1:** Let a set $\mathcal{J} \subset \mathcal{H}$ be *strongly decisive* between $p_1, p_2 \in \mathcal{P}$ if for any preference profile $\rho \in \Re^H$ with $p_1 \succ_i p_2$ for all $i \in \mathcal{J}$ and $p_2 \succ_j p_1$ for all $j \in \mathcal{H} \backslash \mathcal{J}$, $p_1 \succ^S p_2$ ($\mathcal{H}$ itself is strongly decisive since ϕ is weakly Paretian).

- We first prove that if $\mathcal{J}$ is strongly decisive between $p_1, p_2 \in \mathcal{P}$, then $\mathcal{J}$ is dictatorial (and hence decisive for all $p, p' \in \mathcal{P}$ and for all preference profiles $\rho \in \Re^H$).

- To prove this, consider the restriction of an arbitrary preference profile $\rho \in \Re^H$ to $\rho_{|\{p_1, p_2, p_3\}}$ and suppose that we also have $p_1 \succ_i p_3$ for all $i \in \mathcal{J}$.

Proof of Arrow's Impossibility Theorem II

- Next consider an alternative profile $\rho'_{|\{p_1,p_2,p_3\}}$, such that $p_1 \succ'_i p_2 \succ'_i p_3$ for all $i \in \mathcal{J}$ and $p_2 \succ'_i p_1$ and $p_2 \succ'_i p_3$ for all $i \in \mathcal{H} \backslash \mathcal{J}$.

- Since $\mathcal{J}$ is strongly decisive between p_1 and p_2, $p_1 \succ'^S p_2$. Moreover, since ϕ is weakly Paretian, we also have $p_2 \succ'^S p_3$, and thus $p_1 \succ'^S p_2 \succ'^S p_3$.

- Notice that $\rho'_{|\{p_1,p_2,p_3\}}$ did not specify the preferences of individuals $i \in \mathcal{H} \backslash \mathcal{J}$ between p_1 and p_3, but we have established $p_1 \succ'^S p_3$ for $\rho'_{|\{p_1,p_2,p_3\}}$.

- We can then invoke independence from irrelevant alternatives and conclude that the same holds for $\rho_{|\{p_1,p_2,p_3\}}$, i.e., $p_1 \succ^S p_3$.

- But then, since the preference profiles and p_3 are arbitrary, it must be the case that $\mathcal{J}$ is dictatorial between p_1 and p_3.

Proof of Arrow's Impossibility Theorem III

- Next repeat the same argument for $\rho_{|\{p_1,p_2,p_4\}}$ and $\rho'_{|\{p_1,p_2,p_4\}}$, except that now $p_4 \succ_i p_2$ and $p_4 \succ'_i p_1 \succ'_i p_2$ for $i \in \mathcal{J}$, while $p_2 \succ'_j p_1$ and $p_4 \succ'_j p_1$ for all $j \in \mathcal{H} \backslash \mathcal{J}$.

- Then, the same chain of reasoning, using the facts that $\mathcal{J}$ is strongly decisive, $p_1 \succ'^S p_2$, ϕ is weakly Paretian and satisfies independence from irrelevant alternatives, implies that $\mathcal{J}$ is dictatorial between p_4 and p_2 (that is, $p_4 \succ^S p_2$ for any preference profile $\rho \in \Re^H$).

- Now once again using independence from irrelevant alternatives and also transitivity, for any preference profile $\rho \in \Re^H$, $p_4 \succ_i p_3$ for all $i \in \mathcal{J}$.

- Since $p_3, p_4 \in \mathcal{P}$ were arbitrary, this completes the proof that $\mathcal{J}$ is dictatorial (i.e., dictatorial for all $p, p' \in \mathcal{P}$).

Proof of Arrow's Impossibility Theorem IV

- **Step 2:** Given the result in Step 1, if we prove that some individual $h \in \mathcal{H}$ is strongly decisive for some $p_1, p_2 \in \mathcal{P}$, we will have established that it is a dictator and thus ϕ is dictatorial. Let $\mathcal{D}_{ab}$ be the strongly decisive set between p_a and p_b.

- Such a set always exists for any $p_a, p_b \in \mathcal{P}$, since $\mathcal{H}$ itself is a strongly decisive set. Let $\mathcal{D}$ be the minimal strongly decisive set (meaning the strongly decisive set with the fewest members).

- This is also well-defined, since there is only a finite number of individuals in $\mathcal{H}$.

- Moreover, without loss of generality, suppose that $\mathcal{D} = \mathcal{D}_{12}$ (i.e., let the strongly decisive set between p_1 and p_2 be the minimal strongly decisive set).

- If $\mathcal{D}$ a singleton, then Step 1 applies and implies that ϕ is dictatorial, completing the proof.

Proof of Arrow's Impossibility Theorem V

- Thus suppose that $\mathcal{D} \neq \{i\}$. Then, by unrestricted domain, the following preference profile (restricted to $\{p_1, p_2, p_3\}$) is feasible

$$
\begin{array}{ll}
\text{for } i \in \mathcal{D} & p_1 \succ_i p_2 \succ_i p_3 \\
\text{for } j \in \mathcal{D} \setminus \{i\} & p_3 \succeq_j p_1 \succ_j p_2 \\
\text{for } k \notin \mathcal{D} & p_2 \succ_k p_3 \succ_k p_1.
\end{array}
$$

- By hypothesis, $\mathcal{D}$ is strongly decisive between p_1 and p_2 and therefore $p_1 \succ^S p_2$.
- Next if $p_3 \succ^S p_2$, then given the preference profile here, $\mathcal{D} \setminus \{i\}$ would be strongly decisive between p_2 and p_3, and this would contradict that $\mathcal{D}$ is the minimal strongly decisive set.
- Thus $p_2 \succsim^S p_3$. Combined with $p_1 \succ^S p_2$, this implies $p_1 \succ^S p_3$. But given the preference profile here, this implies that $\{i\}$ is strongly decisive, yielding another contradiction.
- Therefore, the minimal strongly decisive set must be a singleton $\{h\}$ for some $h \in \mathcal{H}$. Then, from Step 1, $\{h\}$ is a dictator and ϕ is dictatorial, completing the proof.

Voting

- Could be voting help?
- No because Arrow's Theorem already covers voting.
- But voting may impose additional "institutional structure"
- But in fact *the Condorcet paradox* has many of the same features as Arrow's Theorem.

The Condorcet Paradox

- Imagine a society consisting of three individuals, 1, 2, and 3, three choices and preferences

$$
\begin{array}{ll}
1 & a \succ c \succ b \\
2 & b \succ a \succ c \\
3 & c \succ b \succ a
\end{array}
$$

- Assume "open agenda direct democracy" system.

A1. *Direct democracy.* The citizens themselves make the policy choices via majoritarian voting.

A2. *Sincere voting.* Individuals vote "truthfully" rather than strategically.

A3. *Open agenda.* Citizens vote over pairs of policy alternatives, such that the winning policy in one round is posed against a new alternative in the next round and the set of alternatives includes all feasible policies.

- Implication: *cycling*

The Condorcet Winner

- We can avoid the Condorcet paradox when there is a Condorcet winner.

Definition

A **Condorcet winner** is a policy p^* that beats any other feasible policy in a pairwise vote.

Single-Peaked Preferences

Definition

Consider a finite set of $\mathcal{P} \subset \mathbb{R}$ and let $p(\alpha_i) \in \mathcal{P}$ be individual i's unique bliss point over $\mathcal{P}$. Then, the policy preferences of citizen i are **single peaked** iff:

$$\text{For all } p'', p' \quad \in \quad \mathcal{P}, \text{ such that } p'' < p' \leq p(\alpha_i) \text{ or } p'' > p' \geq p(\alpha_i),$$
$$\text{we have } U(p''; \alpha_i) \quad < \quad U(p'; \alpha_i).$$

- Essentially strict quasi-concavity of U
- *Median voter*: rank all individuals according to their bliss points, the $p(\alpha_i)$'s. Suppose that H odd. Then, the median voter is the individual who has exactly $(H-1)/2$ bliss points to his left and $(H-1)/2$ bliss points to his right.
- Let us denote this individual by α_m, and his bliss point (ideal policy) is denoted by p_m.

Median Voter Theorem

Theorem

(The Median Voter Theorem) *Suppose that H is an odd number, that A1 and A2 hold and that all voters have single-peaked policy preferences over a given ordering of policy alternatives, $\mathcal{P}$. Then, a Condorcet winner always exists and coincides with the median-ranked bliss point, p_m. Moreover, p_m is the unique equilibrium policy (stable point) under the open agenda majoritarian rule, that is, under A1-A3.*

Proof of the Median Voter Theorem

- The proof is by a "separation argument".
- Order the individuals according to their bliss points $p(\alpha_i)$, and label the median-ranked bliss point by p_m.
- By the assumption that H is an odd number, p_m is uniquely defined (though α_m may not be uniquely defined).
- Suppose that there is a vote between p_m and some other policy $p'' < p_m$.
- By definition of single-peaked preferences, for every individual with $p_m < p(\alpha_i)$, we have $U(p_m; \alpha_i) > U(p''; \alpha_i)$.
- By A2, these individuals will vote sincerely and thus, in favor of p_m.
- The coalition voting for supporting p_m thus constitutes a majority.
- The argument for the case where $p'' > p_m$ is identical.

Median Voter Theorem: Discussion

- Odd number of individuals to shorten the statement of the theorem and the proof. It is straightforward to generalize the theorem and its proof to the case in which H is an even number.
- More important: sincere voting
- Alternative: *Strategic voting.*

Strategic Voting

A2′. *Strategic voting.* Define a *vote function* of individual i in a pairwise contest between p' and p'' by $v_i(p', p'') \in \{p', p''\}$. Let a voting (counting) rule in a society with H citizens be $V:\{p', p''\}^H \rightarrow \{p', p''\}$ for any $p', p'' \in \mathcal{P}$.

Let $V(v_i(p', p''), v_{-i}(p', p''))$ be the policy outcome from voting rule V applied to the pairwise contest $\{p', p''\}$, when the remaining individuals cast their votes according to the vector $v_{-i}(p', p'')$, and when individual i votes $v_i(p', p'')$.

Strategic voting means that

$$v_i(p', p'') \in \arg \max_{\tilde{v}_i(p', p'')} U\left(V\left(\tilde{v}_i(p', p''), v_{-i}(p', p'')\right); \alpha_i\right).$$

- A *weakly-dominant* strategy for individual i is a strategy that gives weakly higher payoff to individual i than any of his other strategies regardless of the strategy profile of other players.

Median Voter Theorem with Strategic Voting

Theorem

(The Median Voter Theorem With Strategic Voting) *Suppose that H is an odd number, that A1 and A2′ hold and that all voters have single-peaked policy preferences over a given ordering of policy alternatives, $\mathcal{P}$. Then, sincere voting is a weakly-dominant strategy for each player and there exists a unique weakly-dominant equilibrium, which features the median-ranked bliss point, p_m, as the Condorcet winner.*

- Notice no more "open agenda". Why not?
- Why emphasis on weakly-dominant strategies?

Proof of the Median Voter Theorem with Strategic Voting

- The vote counting rule (the political system) in this case is majoritarian, denoted by V^M.
- Consider two policies $p', p'' \in \mathcal{P}$ and fix an individual $i \in \mathcal{H}$.
- Assume without loss of any generality that $U(p'; \alpha_i) \geq U(p''; \alpha_i)$.
- Suppose first that for any $v_i \in \{p', p''\}$, $V^M(v_i, v_{-i}(p', p'')) = p'$ or $V^M(v_i, v_{-i}(p', p'')) = p''$, that is, individual i is *not* pivotal.
- This implies that $v_i(p', p'') = p'$ is a best response for individual i.
- Suppose next that individual i is pivotal, that is, $V^M(v_i(p', p''), v_{-i}(p', p'')) = p'$ if $v_i(p', p'') = p'$ and $V^M(v_i(p', p''), v_{-i}(p', p'')) = p''$ otherwise. In this case, the action $v_i(p', p'') = p'$ is clearly a best response for i.
- Since this argument applies for each $i \in \mathcal{H}$, it establishes that voting sincerely is a weakly-dominant strategy and the conclusion of the theorem follows.

Strategic Voting in Sequential Elections

- Sincere voting no longer optimal in dynamic situations.

$$
\begin{array}{lll}
1 & a \succ b \succ c \\
2 & b \succ c \succ a \\
3 & c \succ b \succ a
\end{array}
$$

- These preferences are clearly single peaked (e.g., alphabetical order).
- Consider the following dynamic voting set up: first, there is a vote between a and b. Then, the winner goes against c, and the winner of this contest is the social choice.
- Sincere voting: in the first round players 2 and 3 vote for b, and in the second round, 1 and 2 vote for b, which becomes the social choice.
- However, when players 1 and 2 are playing sincerely, in the first round player 3 can deviate and vote for a (even though she prefers b), then a will advance to the second round and would lose to c.
- Consequently, the social choice will coincide with the bliss point of player 3. What happens if all players are voting strategically?

Moving Away from Direct Democracy

- Towards representative democracy, with parties.
- Two parties that can announce and *commit* to policies.
- Rent $Q > 0$ from coming to power and no ideological bias.
- Thus the maximization problem of the two parties are

$$\text{Party } A \quad : \quad \max_{p_A} \ \mathbb{P}(p_A, p_B) Q$$
$$\text{Party } B \quad : \quad \max_{p_B} \ \left(1 - \mathbb{P}(p_A, p_B)\right) Q$$

- $\mathbb{P}(p_A, p_B)$ is the probability that party A comes to power when the two parties' platforms are p_A and p_B respectively.

Party Competition

- Let the bliss point of the median voter be p_m.
- When the median voter theorem applies, we have

$$\mathbb{P}(p_A, p_B = p_m) = 0, \ \mathbb{P}(p_A = p_m, p_B) = 1, \text{ and}$$
$$\mathbb{P}(p_A = p_m, p_B = p_m) \in [0, 1].$$

A4. *Randomization:*

$$\mathbb{P}(p_A = p_m, p_B = p_m) = 1/2.$$

Why?

Downsian Policy Convergence Theorem

Theorem

(Downsian Policy Convergence Theorem) *Suppose that there are two parties that first announce a policy platform and commit to it and a set of voters $\mathcal{H}$ that vote for one of the two parties. Assume that A4 holds and that all voters have single-peaked policy preferences over a given ordering of policy alternatives, and denote the median-ranked bliss point by p_m. Then, both parties will choose p_m as their policy platform.*

Proof of the Downsian Policy Convergence Theorem

- The proof is by contradiction.
- Suppose not, then there is a profitable deviation for one of the parties.
- For example, if $p^A > p^B > p_m$, one of the parties can announce p_m and win the election for sure.
- When $p^A \neq p_m$ and $p^B = p_m$, party A can also announce p_m and increase its chance of winning to $1/2$.

Downsian Policy Convergence Theorem: Discussion

- What happens without Assumption A4?
- Why is this theorem important?
- A natural generalization of this theorem would be to consider three or more parties. What happens with three parties?

Multidimensional Policies?

- Unfortunately, single-peakedness does not work would multidimensional policies.
- But political economy is interesting with multidimensional policies.
- Generalizations, e.g., *intermediate preferences*.
- But not widely applicable.

Single Crossing

- More useful:

Definition

Consider an ordered policy space $\mathcal{P}$ and also order voters according to their α_i's. Then, the preferences of voters satisfy the **single-crossing property** over the policy space $\mathcal{P}$ when the following statement is true:

$$\text{if } p > p' \text{ and } \alpha_{i'} > \alpha_i, \text{ or if } p < p' \text{ and } \alpha_{i'} < \alpha_i, \text{ then}$$
$$U(p; \alpha_i) > U(p'; \alpha_i) \text{ implies that } U(p; \alpha_{i'}) > U(p'; \alpha_{i'}).$$

- Notice that while single peakedness is a property of preferences only, the single-crossing property refers to a set of preferences over a given policy space $\mathcal{P}$. It is therefore a joint property of preferences and choices.

Single Crossing versus Single Peakedness

- Single-crossing property is does not imply single-peaked preferences.

$$
\begin{array}{ccc}
1 & a \succ b \succ c \\
2 & a \succ c \succ b \\
3 & c \succ b \succ a
\end{array}
$$

- These preferences are not single peaked. But they satisfy single crossing

- The natural ordering is $a > b > c$:

$$
\begin{aligned}
\alpha & = 2: c \succ b \implies \alpha = 3: c \succ b \\
\alpha & = 2: \begin{array}{c} a \succ c \\ a \succ b \end{array} \implies \alpha = 1: \begin{array}{c} a \succ c \\ a \succ b \end{array}.
\end{aligned}
$$

Single Crossing versus Single Peakedness (continued)

- The following preferences are single peaked with the natural order $a > b > c > d$:

  $$\begin{array}{ll} 1 & a \succ b \succ c \succ d \\ 2 & b \succ c \succ d \succ a \\ 3 & c \succ b \succ a \succ d \end{array}$$

- For them to satisfy single crossing, we need to adopt the same order over policies (given 1's preferences) and the order $3 > 2 > 1$ over individuals.

- But then the fact that $d \succ_2 a$ should imply that $d \succ_3 a$, which is not the case. (It is easy to verify that if one chooses the order $2 > 3 > 1$ over individuals, one would obtain a similar contradiction as $c \succ_3 b$, but $b \succ_2 c$).

- This shows that single peakedness does not ensure single crossing.

Median Voter Theorem with Single Crossing

Theorem

(Extended Median Voter Theorem) *Suppose that A1 and A2 hold and that the preferences of voters satisfy the single-crossing property. Then, a Condorcet winner always exists and coincides with the bliss point of the median voter (voter α_m).*

Proof

- The proof works with exactly the same separation argument as in the proof of Theorem 4.
- Consider the median voter with α_m, and bliss policy p_m.
- Consider an alternative policy $p' > p_m$. Naturally, $U(p_m; \alpha_m) > U(p'; \alpha_m)$.
- Then, by the single crossing property, for all $\alpha_i > \alpha_m$, $U(p_m; \alpha_i) > U(p'; \alpha_i)$.
- Since α_m is the median, this implies that there is a majority in favor of p_m.
- The same argument for $p' < p_m$ completes the proof

Extended Policy Convergence

Theorem

(Extended Downsian Policy Convergence) *Suppose that there are two parties that first announce a policy platform and commit to it and a set of voters that vote for one of the two parties. Assume that A4 holds and that all voters have preferences that satisfy the single-crossing property and denote the median-ranked bliss point by p_m. Then, both parties will choose p_m as their policy.*

Application: Redistributive Taxation I

- Consider situation with two parties competing to come to power.
- Suppose that agents have the following preferences

$$u^i \left(c^i, x^i \right) = c^i + h(x^i)$$

where c^i and x^i denote individual consumption and leisure, and $h(\cdot)$ is a well-behaved concave utility function.

- There are only two policy instruments, linear tax on earnings τ on lump-sum transfers $T \geq 0$ (and this is important).
- The budget constraint of each agent is

$$c^i \leq (1 - \tau)l^i + T,$$

- The real wage is exogenous and normalized to 1.
- Individual productivity differs, such that the individuals have different amounts of "effective time" available. That is, individuals are subject to the "time constraint"

$$\alpha^i \geq x^i + l^i,$$

Application: Redistributive Taxation II

- Assume that α^i is distributed in the population with mean α and median α^m.

- Since individual preferences are linear in consumption, optimal labor supply satisfies

$$l^i = L(\tau) + (\alpha^i - \alpha),$$

where $L(\tau) \equiv \alpha - (h')^{-1}(1 - \tau)$ is decreasing in τ by the concavity of $h(\cdot)$.

- A higher tax rate on labor income distorts the labor-leisure choice and induces the consumer to work less. This will be the cost of redistributive taxation in this model.

- Let l denote average labor supply. Since the average of α^i is α, we have $l = L(\tau)$. The government budget constraint can therefore be written:

$$T \leq \tau l \equiv \tau L(\tau).$$

Application: Redistributive Taxation III

- Let $U(\tau; \alpha^i)$ be utility for α^i from tax τ with T determined as residual. By straightforward substitution into the individual utility function, we can express the policy preferences of individual i as

$$U(\tau; \alpha^i) \equiv L(\tau) + h(\alpha - L(\tau)) + (1 - \tau)(\alpha^i - \alpha). \tag{1}$$

- Are the preferences represented by (1) single-peaked?
- The answer depends on the shape of the average labor supply function $L(\tau)$. By putting enough structure on dysfunction, we could ensure that $U(\tau; \alpha^i)$ is strictly concave or quasi concave, thus satisfying single-peakedness. However, this function could be sufficiently convex that $U(\tau; \alpha^i)$ could have multiple peaks (multiple local maxima). As a result, preferences may not be single peaked.
- But it is straightforward to verify that (1) satisfies the single-crossing property.

Application: Redistributive Taxation IV

- Therefore, we can apply MVT, and party competition gives

$$\tau^m = \arg\max_\tau U(\tau; \alpha^m)$$

Hence, we have

$$L'(\tau^m) \left[1 - h'(\alpha - L(\tau^m)) \right] - (\alpha^m - \alpha) = 0 \qquad (2)$$

- If the mean is greater than the median, as we should have for a skewed distribution of income, it must be the case that $\alpha^m - \alpha < 0$ (that is median productivity must be less than mean productivity).
- This implies that $\tau^m > 0$—otherwise, (2) would be satisfied for a negative tax rate, and we would be at a corner solution with zero taxes (unless negative tax rates, i.e., subsidies, were allowed).
- Now imagine a change in the distribution of α such that the difference between the mean and the median widens. From the above first-order condition, this'll imply that the equilibrium tax rate τ^m increases.

Application: Redistributive Taxation V

- This is the foundation of the general presumption that greater inequality (which is generally, but not always, associated with a widening gap between the mean and the median) will lead to greater taxation to ensure greater redistribution away from the mean towards the median.

- Notice also that greater inequality in this model leads to greater "inefficiency" of policy.

- Why is this? The reason is only weakly related to the logic of redistribution, but more to the technical assumptions that have been made.

- In order to obtain single-peaked preferences, we had to restrict policy to a single dimensional object, the linear tax rate.

Application: Redistributive Taxation VI

- Moreover, is this "inefficiency" the same as *Pareto suboptimality?*
- Imagine, instead, that different taxes can be applied to different people. Then, redistribution does not necessitate distortionary taxation. But in this case, preferences will clearly be non-single-peaked—agent i particularly dislikes policies that tax him a lot, and likes policies that tax agents j and k a lot, where as agent j likes policies that tax i and k a lot, etc.

Inequality and Redistribution

- One of the key conclusions mentioned above is that greater inequality should lead to greater redistribution.
- Despite these claims in the literature, however, there is no such unambiguous prediction.
- More importantly, there is no empirical evidence that greater inequality leads to more distribution.
- In fact, why many highly unequal societies do not adopt more redistributive policies will be one of the teams we will investigate when we come to understanding the nature of institutions

Inequality and Redistribution in the MVT Models I

- Suppose the economy consists of three groups, upper class, middle class and lower class.
- All agents within a class have the same income level, y_r, y_m and y_l.
- Assume that $\bar{y} > y_m$.
- A middle class agent is the median voter, and decides the linear tax rate on incomes.
- Tax revenues are redistributed lump sum and redistributive taxation at the rate τ has a cost $C(\tau)$ per unit of income.
- Then, the median voter will maximize

$$(1 - \tau) y_m + (\tau - C(\tau)) \bar{y}$$

Inequality and Redistribution in the MVT Models I

- The first-order condition is:

$$\frac{\bar{y} - y_m}{\bar{y}} = c'(\tau)$$

- Now imagine a reduction in y_l and a corresponding increase in y_m such that average income, $\bar{y}$, remains unchanged.

- This increase in the income share of the middle class will *reduce* the desired tax rate of the median voter.

- But in this example, this change in the income distribution corresponds to *greater inequality*.

- So we have a situation in which greater inequality reduces taxes.

Understanding Nonexistence

- Game theoretically, the Condorcet paradox is not about "cycling", but nonexistence of pure strategy equilibria.
- **Example: t**hree (groups of) voters, $i = 1, 2, 3$ of equal size with strictly increasing preferences

$$U(p) = u\left(p^i\right),$$

where $p = \left(p^1, p^2, p^3\right)$, with $\sum_{i=1}^{3} p^i = 1$.
- A policy will be the winner if it gets votes from 2 agents.
- Now take a winning policy (p_1, p_2, p_3) where without any loss of generality suppose that $p_1 > 0$.
- Then the following policy will always beat this winning policy $(p_1 - 2\varepsilon, p_2 + \varepsilon, p_3 + \varepsilon)$, proving that there will always be cycling.
- Therefore, no pure strategy Nash equilibrium.
- Intuition: viewed as a cooperative game, this has an empty core.

Probablilistic Voting: Main Idea

- In the above example, it appears that the discontinuity of best responses in policies is important in nonexistence.
- The main idea of probabilistic voting is to "smooth" best responses in order to get existence.
- Intuitively, there are ideological and non-policy factors, so that a small advantage due to policies will not sway all voters.

Probablilistic Voting: Introduction

- G distinct groups, with a continuum of voters within each group having the same economic characteristics and preferences.
- Electoral competition between two parties, A and B.
- π_P^g:fraction of voters in group g voting for party $P = A, B$, and
- λ^g: share of voters in group g
- Expected vote share of party P is

$$\pi_P = \sum_{g=1}^{G} \lambda^g \pi_P^g.$$

- Suppose that individual i in group g has the following preferences:

$$\tilde{U}_i^g(p, P) = U^g(p) + \tilde{\sigma}_i^g(P) \qquad (3)$$

when party P comes to power, where $p \in \mathcal{P} \subset \mathbb{R}^K$.
- As usual $U^g(p)$ is the indirect utility of agents in group g
- $\tilde{\sigma}_i^g(P)$ captures the non-policy related benefits that the individual will receive if party P comes to power.

Probalilistic Voting I

- Let us normalize $\tilde{\sigma}_i^g(A) = 0$, so that

$$\tilde{U}_i^g(p, A) = U^g(p), \text{ and } \tilde{U}_i^g(p, B) = U^g(p) + \tilde{\sigma}_i^g \qquad (4)$$

- In that case, the voting behavior of individual i can be represented as

$$v_i^g(p_A, p_B) = \begin{cases} 1 & \text{if } U^g(p_A) - U^g(p_B) > \tilde{\sigma}_i^g \\ \frac{1}{2} & \text{if } U^g(p_A) - U^g(p_B) = \tilde{\sigma}_i^g \\ 0 & \text{if } U^g(p_A) - U^g(p_B) < \tilde{\sigma}_i^g \end{cases}, \qquad (5)$$

- Suppose that the distribution of non-policy related benefits $\tilde{\sigma}_i^g$ for individual i in group g is given by a smooth cumulative distribution function H^g defined over $(-\infty, +\infty)$, with the associated probability density function h^g.
- The draws of $\tilde{\sigma}_i^g$ across individuals are independent.
- Consequently, the vote share of party A among members of group g is

$$\pi_A^g = H^g(U^g(p_A) - U^g(p_B)).$$

Probablilistic Voting II

- Supposed to start with that parties maximize their expected vote share.
- In this case, party A sets this policy platform p_A to maximize:

$$\pi_A = \sum_{g=1}^{G} \lambda^g H^g (U^g(p_A) - U^g(p_B)). \qquad (6)$$

- Party B faces a symmetric problem and maximizes π_B, which is defined similarly. Since $\pi_B = 1 - \pi_A$, party B's problem is exactly the same as minimizing π_A.
- Equilibrium policies determined as the Nash equilibrium of a (zero-sum) game where both parties make simultaneous policy announcements to maximize their vote share.
- First-order conditions for party A

$$\sum_{g=1}^{G} \lambda^g h^g (U^g(p_A) - U^g(p_B)) DU^g(p_A) = 0,$$

Probablilistic Voting Equilibrium

- Focus first on pure strategy symmetric equilibria. Clearly in this case, we will have policy convergence with $p_A = p_B = p^*$, and thus $U^g(p_A) = U^g(p_B)$.
- Consequently, symmetric equilibrium policies, announced by both parties, must be given by

$$\sum_{g=1}^{G} \lambda^g h^g(0) D U^g(p^*) = 0. \tag{7}$$

- Therefore, the probability quoting equilibrium is given as the solution to the maximization of the following weighted utilitarian social welfare function:

$$\sum_{g=1}^{G} \chi^g \lambda^g U^g(p), \tag{8}$$

where $\chi^g \equiv h^g(0)$ are the weights that different groups receive in the social welfare function.

Weighted Social Welfare Functions

Theorem

(Probabilistic Voting Theorem) *Consider a set of policy choices $\mathcal{P}$, let $p \in \mathcal{P} \subset \mathbb{R}^K$ be a policy vector and let preferences be given by (4), with the distribution function of $\tilde{\sigma}_i^g$ as H^g. Then, if a pure strategy symmetric equilibrium exists, equilibrium policy is given by p^* that maximizes (8).*

- Most important: probabilistic voting equilibria are *Pareto optimal* (given policy instruments).
- Now in fact, looking back, whenever the Median Voter Theorem applies, the equilibrium is again *Pareto optimal*.
- What does this mean?

Existence of Pure Strategy Equilibria

- However, the probability voting model is not always used properly.
- It is a good model to represent certain political interactions.
- But it is not a good model to ensure pure strategy equilibria.
- In fact, pure strategy existence requires that the matrices

$$
\begin{aligned}
B\left(0, p^{*}\right) \quad \equiv \quad & \sum_{g=1}^{G} \lambda^{g} h^{g}(0) D^{2} U^{g}\left(p^{*}\right) \\
& \pm \sum_{g=1}^{G} \lambda^{g} \frac{\partial h^{g}(0)}{\partial x}\left(D U^{g}\left(p^{*}\right)\right) \cdot\left(D U^{g}\left(p^{*}\right)\right)^{T}
\end{aligned}
$$

is negative semidefinite. (Why?)

Existence of Pure Strategy Equilibria I

- Since this is difficult to check without knowing what p^*, the following "sufficient condition" might be useful:

$$B^g\left(x, p\right) \equiv h^g(x)D^2 U^g\left(p\right) + \left|\frac{\partial h^g\left(x\right)}{\partial x}\right|\left(DU^g\left(p\right)\right) \cdot \left(DU^g\left(p\right)\right)^T$$
(9)

  is negative definite for any x and p, and each g.

Theorem

(**Pure Strategy Existence**) *Suppose that (9) holds. Then in the probabilistic voting game, a pure strategy equilibrium always exists.*

Existence of Pure Strategy Equilibria II

- But (9) is a very restrictive condition. In general satisfied only if all the H^g's uniform.
- Thus we have not solved the existence problem at all.
- To understand (9), consider the first and second order conditions in the one-dimensional policy case with first-order condition

$$\sum_{g=1}^{G} h^g\left(U^g(p_A) - U^g(p_B)\right)\frac{\partial U^g(p_A)}{\partial p} = 0$$

$$\sum_{g=1}^{G} h^g\left(U^g(p_A) - U^g(p_B)\right)\frac{\partial^2 U^g(p_A)}{\partial p^2} +$$

$$\sum_{g=1}^{G} \frac{\partial h^g\left(U^g(p_A) - U^g(p_B)\right)}{\partial x}\left(\frac{\partial U^g(p_A)}{\partial p}\right)^2 < 0$$

Existence of Pure Strategy Equilibria III

- Looking at each group's utility separately, this requires

$$-\frac{\partial^2 U^g(p_A)/\partial p^2}{\left(\partial U^g(p_A)/\partial p\right)^2} > \frac{\partial h^g\left(U^g(p_A) - U^g(p_B)\right)/\partial x}{h^g\left(U^g(p_A) - U^g(p_B)\right)}$$

  for all g.

- At the same time, this point must also be a best response for party B, so by the same arguments,

$$-\frac{\partial^2 U^g(p_B)/\partial p^2}{\left(\partial U^g(p_B)/\partial p\right)^2} > -\frac{\partial h^g\left(U^g(p_A) - U^g(p_B)\right)/\partial x}{h^g\left(U^g(p_A) - U^g(p_B)\right)}.$$

- A sufficient condition for both of these inequalities to be satisfied is

$$\sup_x \frac{\left|\partial h^g(x)/\partial x\right|}{h^g(x)} \leq \inf_p \left|\frac{\partial^2 U^g(p)/\partial p}{\partial U^g(p)/\partial p}\right| \quad \text{for all } g.$$

Existence of Mixed Strategy Equilibria

- Naturally, mixed strategy equilibria are easier to guarantee (for example, they are immediate from Glicksberg's Theorem)

Theorem

(**Mixed Strategy Existence**) *In the probabilistic voting game, a mixed strategy equilibrium always exists.*

Application: the Power of the Middle Class I

- Here is an example showing how with *uniform distribution*, probablilistic voting becomes very tractable and useful.
- Also, assume that now parties care about probability of coming to power not vote share.
- Key concepts: "swing voters"—who are more responsive to policy.
- Three distinct groups, $g = R, M, P$, representing the rich, the middle class, and the poor, with preferences

$$U(p) = u\left(p^i\right),$$

- $u\left(\cdot\right)$ is the strictly monotonic utility function common to all groups. The population share of group g is λ^g, with $\sum_{g=1}^{3} \lambda^g = 1$.
- The relevant policy vector is again a vector of redistributions $p = \left(p^1, p^2, p^3\right)$ with $\sum_{g=1}^{3} \lambda^g p^g = 1$.
- At the time of the elections, voters base their voting decision both on the economic policy announcements and on the two parties' ideologies relative to the realization of their own ideology.

Application: the Power of the Middle Class II

- Voter i in group g prefers party A if

$$U^g(p_A) > U^g(p_B) + \sigma^{ig} + \delta.$$

- Let us assume that this parameter for each group g has group-specific uniform distribution on

$$\left[-\frac{1}{2\phi^g}, \frac{1}{2\phi^g} \right].$$

- The parameter δ measures the average (relative) popularity of candidate B in the population as a whole, and also can be positive or negative. Assume that it has a uniform distribution on

$$\left[-\frac{1}{2\psi}, \frac{1}{2\psi} \right].$$

Application: the Power of the Middle Class III

- The "indifferent" voter in group g will be a voter whose ideological bias, given the candidates' platforms, makes him indifferent between the two parties.

$$\sigma^g = U^g(p_A) - U^g(p_B) - \delta.$$

- All voters in group g with $\sigma^{ig} \leq \sigma^g$ prefer party A. Therefore, party A's actual vote share is

$$\pi_A = \sum_g \lambda^g \phi^g \left(\sigma^g + \frac{1}{2\phi^g} \right).$$

- Notice that σ^g depends on the realized value of δ, and thus the vote share π_A is also a random variable.
- Party A's probability of winning is then

$$\mathbb{P}_A = \operatorname*{Prob}_\delta \left[\pi_A \geq \frac{1}{2} \right] = \frac{1}{2} + \psi \left[\sum_{g=1}^3 \lambda^g \phi^g \left[U^g(p_A) - U^g(p_B) \right] \right],$$

Party B wins with probability $1 - \mathbb{P}_A$.

Application: the Power of the Middle Class III

- Suppose party B has announced the equilibrium policy $p_B = p^*$.
- Then

$$\mathbb{P}_A = \underset{\delta}{\text{Prob}}\left[\pi_A \geq \frac{1}{2}\right] = \frac{1}{2} + \psi \left[\begin{array}{c} \lambda_1 \phi_1 \left(u\left(p_{A,1}\right) - u\left(p_1^*\right)\right) \\ + \lambda_2 \phi_2 \left(u\left(p_{A,2}\right) - u\left(p_2^*\right)\right) \\ + \lambda_3 \phi_3 \left(u\left(p_{A,3}\right) - u\left(p_3^*\right)\right) \end{array} \right],$$

(10)

- Party A will maximize (10) subject to the resource constraint.
- The first-order conditions are

$$\phi_1 u'\left(p_{A,1}\right) = \eta$$
$$\phi_2 u'\left(p_{A,2}\right) = \eta$$
$$\phi_3 u'\left(p_{A,3}\right) = \eta$$

where η is the Lagrangean multiplier on the resource constraint.

- Implication: whichever group has higher ϕ, thus approximating "a swing voter group" will have greater influence on policies.