14.461: Technological Change, Lecture 8
Modeling Firm Dynamics

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Pareto Distribution

- Many quantities in economics, other social sciences and physical sciences appear to be well approximated by Pareto distribution.
- Pareto distribution or the power law has the following counter-cumulative distribution function:

$$\mathbb{G}(y) \equiv 1 - \Pr[\tilde{y} \leq y] = \Gamma y^{-\lambda},$$

  where $\lambda \geq 1$ is the shape parameter.
- When the literature refers to the Pareto or the power law distribution, this generally means that the distribution has Pareto tails, meaning that it takes this form for y large.

Zipf Distribution

- The Zipf distribution is a special case with $\lambda = 1$ (or sometimes it is used for the case where λ is approximately equal to 1).
- Empirical city size distribution and firm size distributions appear to be well approximated by this distribution, and for city size distributions, this is generally referred to as "Zipf's Law".
- This is also equivalent to a relationship of slope -1 between log rank of the city (according to city size) and log of the population, or:

$$\ln \text{Rank}_j = c - \ln y_j ,$$

where y is size. Thus

$$y_j = \frac{C}{\text{Rank}_j}.$$

- Rewriting this

$$\Pr\left(\tilde{y} > y_j\right) = \frac{C}{y_j},$$

- This may apply exactly or only in the "tails," i.e., y_j large enough.

Firm Size Distribution

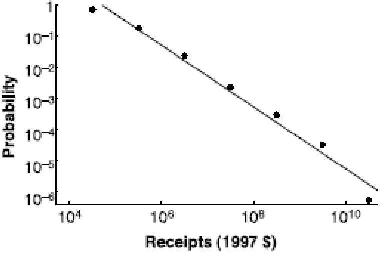
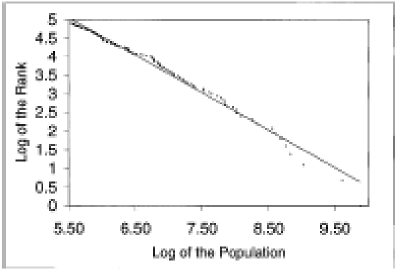


Fig. 2. Tail cumulative distribution function of U.S. firm sizes, by receipts in dollars. Data are for 1997 from the U.S. Census Bureau, tabulated in bins whose width increases in powers of 10. The solid line is the OLS regression line through the data and has slope of 0.994 (SE = 0.064; adjusted R^2 = 0.976).

Axtell 2001

For US Metropolitan Areas in the 1990s



Where Does the Pareto Distribution Come From?

- Many stochastic processes lead to the Pareto distribution. The most famous and most convenient is the so-called **Kesten process**, which takes the form

$$y_{t+1} = \gamma_t y_t + z_t,$$

  where γ_t and z_t are independent random variables, with $\mathbb{E}\gamma_t < \infty$ and $\mathbb{E}z_t < \infty$.
- Suppose that y has a stationary distribution $\mathbb{G}$ (this is not trivial as we will see, so this assumption makes life much more straightforward).

Pareto Distribution (continued)

- $\mathbb{G}(y) \equiv 1 - \Pr[\tilde{y} \leq y] = \Gamma y^{-\lambda}$ can be written as

$$\begin{aligned} \Pr[y_{t+1} \geq y] &= \mathbb{E}\left[\mathbf{1}_{\{y_{t+1} \geq y\}}\right] \\ &= \mathbb{E}\left[\mathbf{1}_{\{\gamma_t y_t + z_t \geq y\}}\right] \\ &= \mathbb{E}\left[\mathbf{1}_{\{y_t \geq (y - z_t)/\gamma_t\}}\right], \end{aligned}$$

  where $\mathbf{1}_{\{\mathcal{P}\}}$ is the indicator function for the event $\mathcal{P}$.
- Then, by the definition of a stationary distribution $\mathbb{G}$, we have

$$\mathbb{G}(y) = \mathbb{E}_{\gamma,z}\left[\mathbb{G}\left(\frac{y-z}{\gamma}\right)\right].$$

- More generally, this is derived from the solution of **Kolmogorov's forward equation** (or the **Fokker-Planck equation**).
- The solution, which exists by assumption, will be Pareto in the tail:

$$\mathbb{G}(y) = \Gamma y^{-\lambda}$$

  for large y, and moreover, λ will be given by $\mathbb{E}_\gamma \gamma^\lambda = 1$.

Pareto Distribution (continued)

- To see this, consider the special case where $z_t = z$. Then

$$\mathbb{G}(y) = \mathbb{E}_\gamma \left[\mathbb{G}\left(\frac{y-z}{\gamma} \right) \right].$$

Suppose $\mathbb{G}(y) = \Gamma y^{-\lambda}$ for large y. Then, again for large y,

$$\begin{aligned} \Gamma y^{-\lambda} &= \mathbb{E}_\gamma \left[\Gamma (y-z)^{-\lambda} \gamma^\lambda \right], \text{ or} \\ \Gamma y^{-\lambda} &= \Gamma (y-z)^{-\lambda} \mathbb{E}_\gamma \gamma^\lambda, \end{aligned}$$

where the term on the left and the first term on the right are approximately equal for large y, giving the desired result.

- When z is random, same reasoning applies for large y.

When Does a Limiting Distribution Exist?

- Take the process

$$y_{t+1} = \gamma_t y_t,$$

  with γ_t independent and mean one. This clearly does not have a limiting distribution, as the empirical distribution (as a function of time) will keep on expanding.
- Essentially for a limiting distribution to exist, the z_t term needs to make sure that there aren't too many small observations (in the exact Pareto case, y_t has to be greater than Γ.
- One way of achieving this is to have a hard or soft lower **reflecting barrier**.

When Does a Limiting Distribution Exist? (continued)

- The following is a well-known theorems from stochastic processes.

Theorem

Suppose that y follows the continuous time reflected geometric Brownian motion process

$$\frac{dy_t}{y_t} = \begin{cases} \gamma dt + \sigma dB_t & \text{if } y_t > y_{\min} \\ \max\{\gamma dt + \sigma dB_t, 0\} & \text{if } y_t \leq y_{\min}, \end{cases}$$

where $\gamma < 0$, then y_t converges in distribution to the Pareto distribution with shape parameter

$$\lambda = \frac{1}{1 - y_{\min}/\bar{y}},$$

where $\bar{y}$ is its stationary distribution average.

- This implies that if the lower barrier $y_{\min}$ is sufficiently small, the exponent is approximately 1.

Innovation Dynamics: Approach

- Simple framework for analysis of growth with innovations by incumbents as well as entry (creative destruction) jointly would firm size distributions.
- Need to go beyond existing models of growth.
  - Schumpeterian models thus far generate growth only by entry (powerful *Arrow's replacement effect*).
  - Models of expanding input or product variety not useful for the study of the set of questions either.

Remainder of This Lecture

- Small modification on textbook Schumpeterian growth models to incorporate innovation by incumbents as well as creative destruction.
- Two types of innovations; incremental and radical (consistent with quantitative and qualitative evidence on innovation).
- A large part of productivity growth generated by continuing establishments.
- Somewhat counterintuitive comparative statics: in contrast to the spirit of Schumpeterian models, entry barriers or taxes on entrants increase growth.
  - because it increases the value of incumbents and encourages their innovation.
- Endogenous firm dynamics (consistent with Gibrat's law, e.g., Sutton, 1997; though see Akcigit 2008; Akcigit and Kerr, 2010).
- Endogenous firm size distribution (consistent with Pareto with exponent$\simeq 1$, e.g. Axtell, 2001).

Model I

- Continuous time, extension of the baseline Schumpeterian model.
- Representative household with the standard CRRA preferences.

$$\int_0^\infty \exp(-\rho t) \frac{C(t)^{1-\theta} - 1}{1-\theta} dt.$$

- Population constant at L and labor is supplied inelastically.
- Resource constraint at time t as usual:

$$C(t) + X(t) + Z(t) \le Y(t), \tag{1}$$

- Production function of the unique final good:

$$Y(t) = \frac{1}{1-\beta} \left[\int_0^1 q(\nu, t)^\beta x(\nu, t \mid q)^{1-\beta} d\nu \right] L^\beta, \tag{2}$$

  where:
  - $x(\nu, t \mid q)$=quantity of the machine of type ν of quality $q(\nu, t)$.

Model (continued)

- Engine of growth: quality improvements, now driven by two types of innovations:
  1. Innovation by incumbents
  2. Creative destruction by entrants.
- "Quality ladder" for each machine type:

$$q(\nu, t) = \lambda^{n(\nu,t)} q(\nu, s) \text{ for all } \nu \text{ and } t,$$

where:

- $\lambda > 1$ and $n(\nu, t)$ is the number of *incremental* innovations on this machine line between $s \leq t$ and t.
- s is the date at which this particular type of technology was first invented, with quality $q(\nu, s)$ at that point.

Model (continued)

- Incumbent has fully enforced patent on machines that it has developed.
- This patent does not prevent entrants leapfrogging the incumbent's machine quality.
- At time $t = 0$, each machine line starts with some quality $q(\nu, 0) > 0$ owned by an incumbent.
- Incremental innovations only by the incumbent producer, i.e. "tinkering" innovations (consistent with case study evidence, e.g., Freemen, 1982, or Scherer, 1984):
  - If spend $z(\nu, t)\, q(\nu, t)$ of the final good for innovation on quality $q(\nu, t)$, then flow rate of innovation $\phi z(\nu, t)$ for $\phi > 0$.
  - More general version $\phi(z)$: see paper.
  - More formally: for any interval $\Delta t > 0$, the probability of one incremental innovation is $\phi z(\nu, t)\, \Delta t$ and the probability of more than one is $o(\Delta t)$ (with $o(\Delta t) / \Delta t \to 0$ as $\Delta t \to 0$).
  - Such innovation results in new machine of quality $\lambda q(\nu, t)$.

Model (continued)

- Alternatively, new firm (entrant) can innovate over existing machines in machine line ν at time t:
    - If current quality is $q(\nu, t)$, spending one unit of the final good gives flow rate of innovation $\eta(\hat{z}(\nu, t)) / q(\nu, t)$.
    - $\eta(\cdot)$ is a strictly decreasing, continuously differentiable function.
    - $\hat{z}(\nu, t)$ is total of R&D by new entrants towards machine line ν at time t.
    - Innovation leads to new machine of quality $\kappa q(\nu, t)$, where $\kappa > \lambda$.
- Note:
    - Innovation by entrants more "radical" than by incumbents, supported from studies of innovation.
    - Incumbents also have access to the technology for radical innovation, but Arrow replacement effect implies they would never use it (entrants will make zero profits from it, so profits of incumbents would be negative).
    - Strictly decreasing function η, captures "external" diminishing returns (new entrants "fishing out of the same pond").

Model (continued)

- Each entrant attempting R&D is potentially small, take $\eta(\hat{z}(\nu, t))$ as given.
- Assume that $z\eta(z)$ is strictly increasing in z: greater aggregate R&D towards a machine line increases the overall probability of discovering a superior machine.
- $\eta(z)$ satisfies Inada-type assumptions:

$$\lim_{z \to \infty} \eta(z) = 0 \text{ and } \lim_{z \to 0} \eta(z) = \infty. \tag{3}$$

- Once a machine of quality $q(\nu, t)$ has been invented, any quantity can be produced at the marginal cost ψ, $\psi \equiv 1 - \beta$.

Model (continued)

- Thus total amount of expenditure on the production of intermediate goods at time t:
$$X(t) = \int_0^1 \psi x(\nu, t)\, d\nu, \tag{4}$$
where $x(\nu, t)$ is the quantity of this machine used in final good production.
- Total expenditure on R&D is sum of R&D by incumbents and entrants ($z(\nu, t)$ and $\hat{z}(\nu, t)$):
$$Z(t) = \int_0^1 [z(\nu, t) + \hat{z}(\nu, t)]\, q(\nu, t)\, d\nu, \tag{5}$$
where $q(\nu, t)$ refers to the highest quality of the machine of type ν at time t (recall: higher-quality machines is proportionately more difficult).

Model (continued)

- Allocation. Time paths of $[C(t), X(t), Z(t)]_{t=0}^{\infty}$, $[z(\nu, t), \hat{z}(\nu, t)]_{\nu\in[0,1],t=0}^{\infty}$, $[p^x(\nu, t \mid q), x(\nu, t), V(\nu, t \mid q)]_{\nu\in[0,1],t=0}^{\infty}$, $[r(t), w(t)]_{t=0}^{\infty}$.
- Equilibrium. Allocation in which R&D decisions by entrants maximize their net present discounted value, pricing, quantity and R&D decisions by incumbents maximize their net present discounted value, consumers choose the path of consumption and allocation of spending across machines and R&D optimally, and the labor market clears.

Model (continued)

- Profit-maximization by the final good sector implies the demand for machines of highest-quality:

$$x(\nu, t \mid q) = p^x (\nu, t \mid q)^{-1/\beta} q(\nu, t) L \quad \text{for all } \nu \in [0,1] \text{ and all } t, \tag{6}$$

- Unconstrained monopoly price is usual formula as a constant markup over marginal cost.
- No limit price assumption:

$$\kappa > \left(\frac{1}{1-\beta}\right)^{\frac{1-\beta}{\beta}}, \tag{7}$$

- By implication, incumbents that make further innovations can also charged the unconstrained monopoly price.

Equilibrium

- Since demand for machines in (6) is iso-elastic and $\psi = 1 - \beta$, profit-maximizing monopoly price:

$$p^x(\nu, t \mid q) = 1. \tag{8}$$

- Combining with (6):

$$x(\nu, t \mid q) = qL. \tag{9}$$

- Flow profits of a firm with monopoly rights on the machine quality q:

$$\pi(\nu, t \mid q) = \beta qL. \tag{10}$$

- Substituting (9) into (2), total output is:

$$Y(t) = \frac{1}{1-\beta} Q(t) L, \tag{11}$$

  with average quality $Q(t) \equiv \int_0^1 q(\nu, t) d\nu$

Equilibrium (continued)

- Aggregate spending on machines:

$$X(t) = (1-\beta) Q(t) L. \tag{12}$$

- Labor market is competitive, wage rate:

$$w(t) = \frac{\beta}{1-\beta} Q(t). \tag{13}$$

- Need to determine R&D effort levels by incumbents and entrants.
- Net present value of a monopolist with the highest quality of machine q at time t in machine line ν satisfies HJB ($V(\nu, t \mid q) = V(q)$, etc.):

$$r(t) V(q) - \dot{V}(q) = \max_{z(\nu,t|q) \geq o} \{\pi(q) - z(q) q \tag{14}$$
$$+\phi z(q) (V(\lambda q) - V(q)) - \eta(\hat{z}(q)) \hat{z}(q) V(q)\},$$

Equilibrium (continued)

- Free entry:

$$\eta\left(\hat{z}(\nu,t\mid q)\right)V\left(\nu,t\mid\kappa q\right) \leq q(\nu,t),\ \text{and} \tag{15}$$
$$\eta\left(\hat{z}(\nu,t\mid q)\right)V\left(\nu,t\mid\kappa q\right) = q(\nu,t)\ \text{if}\ \hat{z}\left(\nu,t\mid q\right)>0,$$

- Incumbent's choice of R&D effort implies similar complementary slackness condition:

$$\phi\left(V\left(\nu,t\mid\lambda q\right)-V\left(\nu,t\mid\kappa q\right)\right) \leq q(\nu,t)\ \text{and} \tag{16}$$
$$\phi\left(V\left(\nu,t\mid\lambda q\right)-V\left(\nu,t\mid\kappa q\right)\right) = q(\nu,t)\ \text{if}\ z\left(\nu,t\mid q\right)>0.$$

- Consumer maximization:

$$\frac{\dot{C}\left(t\right)}{C\left(t\right)}=\frac{1}{\theta}\left(r\left(t\right)-\rho\right), \tag{17}$$

$$\lim_{t\to\infty}\left[\exp\left(-\int_0^t r\left(s\right)ds\right)\int_0^1 V\left(\nu,t\mid q\right)d\nu\right]=0 \tag{18}$$

Equilibrium and Balanced Growth Path

- Equilibrium is thus time paths of
  - $[C(t), X(t), Z(t)]_{t=0}^{\infty}$ that satisfy (1), (5), (12) and (18)
  - $[z(\nu, t), \hat{z}(\nu, t)]_{\nu \in [0,1], t=0}^{\infty}$ that satisfy (15) and (16);
  - $[p^x(\nu, t \mid q), x(\nu, t), V(\nu, t \mid q)]_{\nu \in [0,1], t=0}^{\infty}$ given by (8), (9) and (14);
  - $[w(t), r(t)]_{t=0}^{\infty}$ that satisfy (13) and (17).
- *BGP* (balanced growth path): equilibrium path in which innovation, output and consumption growth a constant rate.
- Note in BGP aggregates grow at the constant rate but there will be firm deaths and births, and the firm size distribution may also change.

Balanced Growth Path (continued)

- From Euler equation, the requirement that consumption grows at a constant rate in the BGP implies

$$r(t) = r^*$$

- In BGP, must also have $z(\nu, t \mid q) = z(q)$ and $\hat{z}(\nu, t \mid q) = \hat{z}(q)$.
- These imply in BGP $\dot{V}(\nu, t \mid q) = 0$ and $V(\nu, t \mid q) = V(q)$.
- Since profits and costs are both proportional to quality q, $z(q) = z$, $\hat{z}(q) = \hat{z}$ and $V(q) = vq$.
- Look for an "interior" BGP equilibrium (will verify below that exists and is unique).

Balanced Growth Path (continued)

- Incumbents undertake research, thus

$$\phi\left(V\left(\nu, t \mid \lambda q\right) - V\left(\nu, t \mid q\right)\right) = q(\nu, t), \tag{19}$$

- Therefore

$$V\left(q\right) = \frac{q}{\phi\left(\lambda - 1\right)}. \tag{20}$$

- The free entry condition then implies $\eta\left(\hat{z}\right) V(\kappa q) = q$ and thus

$$V\left(q\right) = \frac{\beta L q}{r^* + \hat{z}\eta\left(\hat{z}\right)}. \tag{21}$$

- Combining this expression with (19) and (20), we obtain

$$\frac{\phi\left(\lambda - 1\right)}{\kappa\eta\left(\hat{z}\right)} = 1.$$

Balanced Growth Path (continued)

- Hence the BGP R&D level by entrants $\hat{z}^*$ is defined implicitly by:

$$\hat{z}(q) = \hat{z}^* \equiv \eta^{-1}\left(\frac{\phi(\lambda-1)}{\kappa}\right) \text{ for all } q > 0. \tag{22}$$

- Combining with (21):

$$\begin{aligned} r^* &= \kappa\eta(\hat{z}^*)\beta L - \hat{z}^*\eta(\hat{z}^*) \\ &= \phi(\lambda-1)\beta L - \hat{z}^*\eta(\hat{z}^*). \end{aligned} \tag{23}$$

- From Euler equation, growth rate of consumption and output:

$$g^* = \frac{1}{\theta}\left(\phi(\lambda-1)\beta L - \hat{z}^*\eta(\hat{z}^*) - \rho\right). \tag{24}$$

Balanced Growth Path (continued)

- (24) determines relationship between $\hat{z}^*$ and g^*. In contrast to standard Shcumpeterian models:

  Remark There is a negative relationship between $\hat{z}^*$ and g^*.

- From (24), g^* is decreasing in $\hat{z}^* \eta\left(\hat{z}^*\right)$ (which is always strictly increasing in $\hat{z}^*$).
- (24) and (22), determine BGP growth rate of the economy, but not how much of productivity growth is driven by creative destruction (entrants) and how much by incumbents.

Balanced Growth Path (continued)

- To determine this, let us write:

$$Q(t+\Delta t) = (\lambda \phi z(t)\Delta t) Q(t) + (\kappa \hat{z}(t)\eta(\hat{z}(t))\Delta t) Q(t) + ((1-\phi z(t)\Delta t - \hat{z}(t)\eta(\hat{z}(t))\Delta t)) Q(t) + o(\Delta t).$$

- Subtracting $Q(t)$ from both sides, dividing by Δt and taking the limit as $\Delta t \to 0$:

$$g(t) = \frac{\dot{Q}(t)}{Q(t)} = (\lambda - 1) z(t) + (\kappa - 1)\hat{z}(t)\eta(\hat{z}(t)),$$

  which decomposes growth into the component from incumbent firms (first term) and from new entrants (second term).

- In BGP:

$$g^* = (\lambda - 1)\phi z^* + (\kappa - 1)\hat{z}^*\eta(\hat{z}^*). \tag{25}$$

Balanced Growth Path (continued)

- Can also verify that this economy does not have any transitional dynamics; if an equilibrium with growth exists, it will involve growth at g^*.
- To ensure equilibrium exists, verify R&D is profitable both for entrants and incumbents.
- The condition that r^* should be greater than ρ is sufficient for there to be positive aggregate growth.
- In addition, r^* should not be so high that transversality condition of the consumers is violated.
- Finally, need to ensure that there is also innovation by incumbents.

Balanced Growth Path (continued)

- The following condition ensures all three of these requirements:

$$\begin{aligned} & \kappa\eta\left(\hat{z}^*\right)\beta L-\left(\theta\left(\kappa-1\right)+1\right)\hat{z}^*\eta\left(\hat{z}^*\right) \\ > \quad & \rho>\left(1-\theta\right)\left(\kappa\eta\left(\hat{z}^*\right)\beta L-\hat{z}^*\eta\left(\hat{z}^*\right)\right), \end{aligned} \tag{26}$$

  with $\hat{z}^*$ given by (22).

- To obtain how much of productivity growth and innovation are driven by incumbents and how much by new entrants, from:

$$\left(\lambda-1\right)\phi z^* = \frac{1}{\theta}\left(g^*-\rho\right)-\left(\kappa-1\right)\hat{z}^*\eta\left(\hat{z}^*\right), \tag{27}$$

  with g^* given in (24) and $\hat{z}^*$ in (22).

Firm Size Dynamics

- Firm-size dynamics: size of a firm can be measured by its sales:

$$x(\nu, t \mid q) = qL \text{ for all } \nu \text{ and } t.$$

- Quality of an incumbent firm increases that the flow rate ϕz^*, with z^* given by (27), while the firm is replaced at the flow rate $\hat{z}^* \eta(\hat{z}^*)$.
- Thus, for Δt sufficiently small, the stochastic process for the size of a particular firm is:

$$x(\nu, t + \Delta t \mid q) = \begin{cases} \lambda x(\nu, t \mid q) & \text{w. p. } \phi z^* \Delta t + o(\Delta t) \\ 0 & \text{w. p. } \hat{z}^* \eta(\hat{z}^*) \Delta t + o(\Delta t) \\ x(\nu, t \mid q) & \text{w. p. } (1 - \phi z^* - \hat{z}^* \eta(\hat{z}^*)) \Delta t \\ & \quad + o(\Delta t) \end{cases} \tag{28}$$

for all ν and t.

Firm Size Dynamics (continued)

- Thus firms have random growth, and surviving firms expand on average.
- Firms also face a probability of bankruptcy (extinction)
- Let $P(t \mid s, \nu)$ =probability that a particular incumbent firm that started production in machine line ν at time s will be bankrupt by time $t \geq s$:

$$\lim_{t \to \infty} P(t \mid s, \nu) = 1$$

so that each firm will necessarily die eventually.

Summary of Equilibrium

Proposition Consider the above-described economy starting with an initial condition $Q(0) > 0$. Suppose that (3) and (26) are satisfied. Then there exists a unique equilibrium. In this equilibrium growth is always balanced, and technology, $Q(t)$, aggregate output, $Y(t)$, and aggregate consumption, $C(t)$, grow at the rate g^* as in (24) with $\hat{z}^*$ given by (22). Equilibrium growth is driven both by innovation by incumbents and by creative destruction by entrants. Any given firm expands on average as long as it survives, but is eventually replaced by a new entrant with probability one.

The Effects of Policy on Growth

- Since Schumpeterian structure, it may be conjectured that entry barriers (or taxes on potential entrance) will have negative effects on economic growth.
- Tax τ_e on R&D expenditure by entrants and a tax τ_i on R&D expenditure by incumbents (can be negative and interpreted as subsidies).
- τ_e can also be interpreted as a more strict patent policy than in the baseline model, where the entrant did not have to pay the incumbent for partially benefiting from its accumulated knowledge.
- Focus on the case in which tax revenues are collected by the government rather than rebated back to the incumbent as patent fees.

The Effects of Policy on Growth (continued)

- Repeating the analysis above, equilibrium conditions:

$$\eta\left(\hat{z}^*\right)) V(\kappa q) = (1+\tau_e)\, q \text{ or } V(q) = \frac{q\,(1+\tau_e)}{\kappa\eta\left(\hat{z}^*\right)}. \tag{29}$$

- The equation that determines the optimal R&D decisions of incumbents, (19), is also modified:

$$\phi\left(V(\lambda q) - V(q)\right) = (1+\tau_i)\, q. \tag{30}$$

- Combining (29) with (30):

$$\phi\left(\frac{(\lambda-1)(1+\tau_e)}{\kappa\eta\left(\hat{z}^*\right)(1+\tau_i)}\right) = 1.$$

- Consequently, the BGP R&D level by entrants $\hat{z}^*$, when their R&D is taxed at the rate τ_e, is given by

$$\hat{z}^* \equiv \eta^{-1}\left(\frac{\phi\,(\lambda-1)(1+\tau_e)}{\kappa\,(1+\tau_i)}\right). \tag{31}$$

The Effects of Policy on Growth (continued)

- Equation (21) still applies, so that the the BGP interest rate is

$$r^* = (1+\tau_i)^{-1}\phi(\lambda-1)\beta L - \hat{z}^*\eta(\hat{z}^*), \tag{32}$$

- BGP growth rate is

$$g^* = \frac{1}{\theta}\left((1+\tau_i)^{-1}\phi(\lambda-1) - \hat{z}^*\eta(\hat{z}^*) - \rho\right). \tag{33}$$

- From (33), g^* does not directly depend on τ_e, thus

$$\frac{dg^*}{d\tau_e} = \frac{\partial g^*}{\partial \hat{z}^*}\frac{\partial \hat{z}^*}{\partial \tau_e} > 0.$$

- Opposite of standard Schumpeterian results. Intuition?
- Moreover, as expected

$$\frac{dg^*}{d\tau_i} < 0.$$

Proposition The growth rate is decreasing in the tax rate on incumbents and increasing in the tax rate (entry barriers) on entrants.

The Effects of Policy on Growth (continued)

- Surprising result: in Schumpeterian models, making entry more difficult, either with entry barriers or by taxing R&D by entrants, has negative effects on economic growth. But here blocking entry and protecting the incumbents increases equilibrium growth (and welfare)
- *Intuition:*
  - Engine of growth is still quality improvements, but these improvements are undertaken both by incumbents and entrants.
  - Entry barriers, by protecting incumbents, increase their value and greater value by incumbents encourages more R&D investments and faster productivity growth.
  - Taxing entrants makes incumbents more profitable and this encourages further innovation by the incumbents.
- More general result: if incumbents also have concave technology $\phi(z)$, then the above result applies if ϕ is close to linear, but entry taxes reduce growth if ϕ is sufficiently concave.

Equilibrium Firm Size Distribution

- In equilibrium, there is entry, exit and stochastic growth of firms. $\Longrightarrow$ endogenous of firm size distribution.
- Firm growth consistent with *Gibrat's Law*
  - Gibrat's Law states that firm growth is independent of firm size.
  - Good description of actual firm size dynamics in the data (e.g., Sutton, 1997).
  - Though not always for new firms.
- What about firm size distributions?
- Axtell (2001) US firm size distribution very well approximated by the Pareto distribution with an exponent of one.
- Recall that the *Pareto distribution* is $\Pr\left[\tilde{x} \leq y\right] = 1 - \Gamma y^{-\chi}$ for $\Gamma > 0$ and $y \geq \Gamma$.

Equilibrium Firm Size Distribution (continued)

- In the current economy, the size of average firm measured by sales, $x(t)$, grows.
- To look at the firm size distribution, weneed to normalize firm sizes by the average size of firm, given by $X(t)$.
- Let the *normalized firm size* be

$$\tilde{x}(t) \equiv \frac{x(t)}{X(t)}.$$

Equilibrium Firm Size: Pareto Distribution

- Since in equilibrium $\dot{X}(t)/X(t) = g^* > 0$, law of motion for normalized size of leading firm in each industry:

$$\tilde{x}(t+\Delta t) = \begin{cases} \frac{\lambda}{1+g^*\Delta t}\tilde{x}(t) & \text{w. p. } \phi z^* \Delta t \\ \frac{\kappa}{1+g^*\Delta t}\tilde{x}(t) & \text{w. p. } \hat{z}^* \eta(\hat{z}^*) \Delta t + o(\Delta t) \\ \frac{1}{1+g^*\Delta t}\tilde{x}(t) & \text{w. p. } (1 - \phi z^* - \hat{z}^* \eta(\hat{z}^*)) \Delta t + o(\Delta t) \end{cases}.$$

- Notice that this expression does not refer to the growth rate of a single firm, but to the leading firm in a representative industry, and in particular, when there is entry, this leads to an increase in size rather than extinction.
- It can be verified that if a stationary distribution of (normalized) firm sizes were to exist, then it would be a Pareto distribution with exponent equal to 1, i.e., $\Pr[\tilde{x} \leq y] = 1 - \Gamma/y$ with $\Gamma > 0$.

Pareto Distribution of Firm Sizes

- *Suppose that a stationary distribution exists.*
- Consider an arbitrary time interval of $\Delta t > 0$ and write

$$\begin{aligned}
\Pr\left[\tilde{x}\left(t+\Delta t\right) \leq y\right] &= \mathbb{E}\left[\mathbf{1}_{\{\tilde{x}(t+\Delta t)\leq y\}}\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{\tilde{x}(t)\leq y/(1+g^x(t+\Delta t))\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\tilde{x}(t)\leq y/(1+g^x(t+\Delta t))\}} \mid g^x\left(t+\Delta t\right)\right]\right],
\end{aligned}$$

where $\mathbf{1}_{\{\mathcal{P}\}}$ is the indicator function, so the first equation holds by definition. The second equation also holds by definition once $g^x\left(t+\Delta t\right)$ is designated as the the (stochastic) growth rate of x between t and $t+\Delta t$. Finally, the third equation follows from the Law of Iterated Expectations.

Pareto Distribution of Firm Sizes (continued)

- Next, denoting $\mathbb{G}_t(y) \equiv 1 - \Pr[\tilde{x}(t) \le y]$:

$$\begin{aligned}\Pr[\tilde{x}(t+\Delta t) \le y] &= 1 - \mathbb{G}_{t+\Delta t}(y) \\ &= \mathbb{E}\left[1 - \mathbb{G}_t\left(\frac{y}{1+g^x(t+\Delta t)}\right)\right].\end{aligned}$$

- Thus we obtain a simple form of Kolmogorov's forward equation:

$$\begin{aligned}\mathbb{G}_{t+\Delta t}(y) &= \mathbb{E}\left[\mathbb{G}_t\left(\frac{y}{1+g^x(t+\Delta t)}\right)\right] \\ &= \phi z^* \Delta t \mathbb{G}_t\left(\frac{(1+g^*\Delta t)\,y}{\lambda}\right) \\ &\quad + \hat{z}^* \eta(\hat{z}^*) \Delta t \mathbb{G}_t\left(\frac{(1+g^*\Delta t)\,y}{\kappa}\right) \\ &\quad + \left(1 - \phi z^* \Delta t - \hat{z}^* \eta(\hat{z}^*) \Delta t\right) \mathbb{G}_t\left((1+g^*\Delta t)\,y\right) + o(\Delta\end{aligned}$$

Pareto Distribution of Firm Sizes (continued)

- A stationary equilibrium will correspond to a function $\mathbb{G}(y)$ such that $\mathbb{G}_{t+\Delta t}(y) = \mathbb{G}_t(y) = \mathbb{G}(y)$ for all t and Δt and the previous equation holds.
- Let us conjecture that $\mathbb{G}(y) = \Gamma y^{-\chi}$ with $\Gamma > 0$. Then

$$\begin{aligned}\Gamma y^{-\chi} &= \phi z^* \Delta t \Gamma \left(\frac{(1+g^*\Delta t)\, y}{\lambda} \right)^{-\chi} \\ &\quad + \hat{z}^* \eta(\hat{z}^*) \Delta t \Gamma \left(\frac{(1+g^*\Delta t)\, y}{\kappa} \right)^{-\chi} \\ &\quad + (1 - \phi z^* \Delta t - \hat{z}^* \eta(\hat{z}^*) \Delta t) \Gamma ((1+g^*\Delta t)\, y)^{-\chi} + o(\Delta t).\end{aligned}$$

or

$$\begin{aligned} &\phi z^* \Delta t \left(\frac{(1+g^*\Delta t)}{\lambda} \right)^{-\chi} + \hat{z}^* \eta(\hat{z}^*) \Delta t \left(\frac{(1+g^*\Delta t)}{\kappa} \right)^{-\chi} \\ + \quad &(1 - \phi z^* \Delta t - \hat{z}^* \eta(\hat{z}^*) \Delta t)(1+g^*\Delta t)^{-\chi} + o(\Delta t)\, \Gamma^{-1} y^{-\chi} = 1. \end{aligned} \tag{34}$$

Pareto Distribution of Firm Sizes (continued)

- Now subtracting 1 from both sides, dividing by Δt, and taking the limit as $\Delta t \to 0$, we obtain

$$\lim_{\Delta t \to 0} \left\{ \phi z^* \left(\frac{(1+g^*\Delta t)}{\lambda} \right)^{-\chi} + \hat{z}^* \eta\left(\hat{z}^*\right) \left(\frac{(1+g^*\Delta t)}{\kappa} \right)^{-\chi} + \frac{\left(1 - \phi z^* \Delta t - \hat{z}^* \eta\left(\hat{z}^*\right) \Delta t\right) \left(1+g^*\Delta t\right)^{-\chi} - 1}{\Delta t} + \frac{o\left(\Delta t\right)}{\Delta t} \Gamma^{-1} y^{-\chi} \right\}$$

- Therefore the exponent χ must satisfy

$$\phi\left(\lambda^{\chi} - 1\right) z^* + \left(\kappa^{\chi} - 1\right) \hat{z}^* \eta\left(\hat{z}^*\right) - \chi g^* = 0. \qquad (35)$$

- It can be easily verified that (35) has two solutions $\chi = 0$ and $\chi^* = 1$, since, by definition, $g^* = \phi\left(\lambda - 1\right) z^* + \left(\kappa - 1\right) \hat{z}^* \eta\left(\hat{z}^*\right)$.

Pareto Distribution of Firm Sizes (continued)

- To see that there are no other solutions, consider the derivative of this function, which is given by

$$g'(\chi) = \phi z^{*} \lambda^{\chi} \ln \lambda + \eta(\hat{z}^{*}) \hat{z}^{*} \kappa^{\chi} \ln \kappa - g^{*}.$$

  Since $\ln a < a - 1$ for any $a > 1$, $g'(0) < 0$. Moreover, $g''(\chi) > 0$, so that the right-hand side of (35) is convex and as $\chi \to \infty$, it limits to infinity. Thus there is a unique nonzero solution, which as we saw above, is $\chi^{*} = 1$.
- Finally, note that $\chi = 0$ cannot be a solution, since it would imply $\mathbb{G}(y) = \Gamma$ and thus $\mathbb{G}(y) = 0$, which would imply that all firms have normalized size equal to zero, and violate the hypothesis that a stationary firm-size distribution exists.
- It can also be verified that no other function than $\mathbb{G}(y) = \Gamma y^{-\chi}$ with $\Gamma > 0$ can satisfy this functional equation, completing the proof of the proposition.

Equilibrium Firm Size: Pareto Distribution (continued)

- Unfortunately, previous result was stated under the (italicized) hypothesis that the station redistribution exists.
- But: a stationary firm-size distribution does *not* exist.
- **Proof:**
  - A stationary distribution must take the form $\Pr[\tilde{x} \leq y] = 1 - \Gamma/y$ with $\Gamma > 0$ and Γ should be the minimum normalized firm size.
  - However, the law of motion of firm sizes shows that $\tilde{x}(t)$ can tend to zero. Therefore, Γ must be equal to 0, which implies that there does not exist a stationary firm-size distribution.
- Intuitively, given the random growth process (Gibrat's Law), the distribution of firm sizes will continuously expand.
- The "limiting distribution" will involve all firms being arbitrarily small relative to the average $X(t)$ and a vanishingly small fraction of firms becomes arbitrarily large.

Equilibrium Firm Size: Economy with Imitation

- Can we have a stationary firm size distribution?
- Let us introduce a third type of technology of innovation, "*imitation*".
- A new firm can enter in sector $\nu \in [0, 1]$ with a technology of $q^e(\nu, t) = \omega Q(t)$, where $\omega \geq 0$ and $Q(t)$ is average quality of machines.
- The cost of this type of innovation is $\omega Q(t) / (\phi(\lambda - 1))$ (not necessary, but simplifies life a lot).
- This implies that if a firm could enter into a particular sector, become the monopolist and obtain the BGP value (20), it would be happy to enter (or more precisely, it would be indifferent between entering and not entering, and we suppose that in this case it does enter).

Equilibrium Firm Size (continued)

- This structure implies that they will enter into any sector with quality $q \leq \epsilon Q(t)$, where $\epsilon = \epsilon(\omega)$ is determined endogeneously.
- Assume $\mu_e > \underline{\mu}$ so that an imitator does not enter too early, i.e. $\epsilon \leq \omega (1-\beta)^{(1-\beta)/\beta}$ but once entered, it can charge the monopolistic price without potential threat from the incumbent. Moreover $\mu_e < \overline{\mu}$, so that there is positive entry by imitation.

Roadmap for Derivation of Size Distribution

1. Characterize optimal policies in the economy with imitation and show that they are approximately linear.
2. Derive the Kolmogorov forward equation for the law of motion of firm size and derive the stationary distribution.
3. Derive the growth rate.
4. Conjecture an exact Pareto distribution for the tail.
5. Solve out for the Pareto tail and show that it is approximately 1.

Step 1: Value Functions

- We look for an equilibrium in which

$$V(\nu, t|q) = Q(t)\,\widehat{V}\left(\frac{q(\nu, t)}{Q(t)}\right)$$

- We choose as state variable the relative productivity $\widetilde{q} \equiv q/Q$, which gives the HJB equation

$$\begin{aligned}(r-g)\,\widehat{V}_g(\widetilde{q}) - g\widehat{V}_g'(\widetilde{q}) &= \beta L\widetilde{q} \\ &+ \left\{\max_{z(\nu,t)\geq 0}\{\phi(z(\nu,t))\left(\widehat{V}_g(\lambda\widetilde{q}) - \widehat{V}_g(\widetilde{q})\right) - z(\nu,t)\,\widetilde{q}\right\} \\ &-\widehat{z}(\nu,t)\,\eta(\widehat{z}(\nu,t))\,\widehat{V}_g(\widetilde{q}),\end{aligned} \tag{36}$$

- Additional source of capital gains/losses from changes in aggregate quality, and the threat of being replaced by an imitator.
- When ω and ϵ small, value approximately linear as before.

Step 1: Free Entry Conditions

- Free-entry condition for **Innovative Entrants**:

$$\eta\left(\widehat{z}\left(\nu, t\right)\right)\widehat{V}\left(\kappa\widetilde{q}\left(v, t\right)\right) = \widetilde{q}\left(v, t\right).$$

- Free-entry condition for **Imitators**:

$$\widehat{V}\left(\omega\right) = \mu_e \omega.$$

ϵ is such that $\widehat{V}\left(\epsilon\right) = 0$: entrants enter whenever $q\left(\nu, t\right) \leq \epsilon Q\left(t\right)$.

- Thus boundary condition for value function.

Step 1: Optimal Strategy of Incumbents

- From the value function $\widehat{V}(\widetilde{q})$ we obtain the optimal strategy of incumbents

$$z(\widetilde{q}) = \arg\max_{z} \phi(z)\left(\widehat{V}(\lambda\widetilde{q}) - \widehat{V}(\widetilde{q})\right) - z\widetilde{q}$$

and of the entrants

$$\widehat{z}(\widetilde{q}) = \eta^{-1}\left(\widetilde{q}/\widehat{V}(\kappa\widetilde{q})\right).$$

Step 2: Stationary Distribution

- Functional differential equation of the stationary distribution with cdf $F(.)$ (with intuition similar to the constant growth case)

- If $y > \omega$

$$0 = F'(y)\,yg - \int_{\frac{y}{\lambda}}^{y} \phi\left(z\left(\tilde{q}\right)\right) dF\left(\tilde{q}\right) - \int_{\frac{y}{\kappa}}^{y} \hat{z}\left(\tilde{q}\right)\eta\left(\hat{z}\left(\tilde{q}\right)\right) dF\left(\tilde{q}\right). \tag{37}$$

- If $y < \omega$, then

$$0 = F'(y)\,yg - F'(\epsilon)\,\epsilon g - \int_{\frac{y}{\lambda}}^{y} \phi\left(z\left(\tilde{q}\right)\right) dF\left(\tilde{q}\right) - \int_{\frac{y}{\kappa}}^{y} \hat{z}\left(\tilde{q}\right)\eta\left(\hat{z}\left(\tilde{q}\right)\right) dF \tag{38}$$

and

$$F(y) = 0 \text{ for } y \leq \epsilon.$$

Step 3: Growth Rate

- From the investment of incumbents and entry decision entrants and imitators, as well as stationary distribution we obtain

$$T[g] = \frac{(\lambda - 1)\,\mathbb{E}_{F_g}\left[\phi\left(z\left(\widetilde{q}\right)\right)\widetilde{q}\right] + (\kappa - 1)\,\mathbb{E}_{F_g}\left[\widehat{z}_g\left(\widetilde{q}\right)\eta\left(\widehat{z}_g\left(\widetilde{q}\right)\right)\widetilde{q}\right]}{1 - \epsilon F_g'\left(\epsilon\right)\left(\omega - \epsilon\right)}.$$

- Equilibrium growth rate such that

$$g = T[g].$$

- $T[g^*] = +\infty$ and $T[\overline{g}] < \overline{g}$. Therefore there exists $g(\omega) \in (g^*, \overline{g})$ such that $T[g] = g$.

Step 4: Pareto Distribution in the Tail

- For y large, we have
$0 = F'(y)\, yg - \int_{\frac{y}{\lambda}}^{y} \phi\left(z\left(\widetilde{q}\right)\right) dF\left(\widetilde{q}\right) - \int_{\frac{y}{\kappa}}^{y} \widehat{z}\left(\widetilde{q}\right) \eta\left(\widehat{z}\left(\widetilde{q}\right)\right) dF\left(\widetilde{q}\right),$
which implies (again for y large)

$$0 = F'(y)\, yg - \mathbb{E}_{F_g}[\phi]\left[F(y) - F\left(\frac{y}{\lambda}\right)\right] - \mathbb{E}_{F_g}\left[\widehat{z}\eta\left(\widehat{z}\right)\right]\left[F(y) - F\left(\frac{y}{\kappa}\right.\right.$$

- Conjecture: $F(y) = 1 - \Gamma y^{-\chi}$ for large y.
- Then:

$$\mathbb{E}_{F_g}[\phi]\left[\lambda^{\chi} - 1\right] - \mathbb{E}_{F_g}\left[\widehat{z}\eta\left(\widehat{z}\right)\right]\left[\kappa^{\chi} - 1\right] - \chi g = 0$$

- Solution χ gives the tail Pareto shape coefficient.

Step 5: Pareto Shape Parameter Approximately 1

- When $\epsilon \to 0$,

$$T[g] = \frac{(\lambda - 1)\,\mathbb{E}_{F_g}\left[\phi\left(z\left(\widetilde{q}\right)\right)\widetilde{q}\right] + (\kappa - 1)\,\mathbb{E}_{F_g}\left[\widehat{z}_g\left(\widetilde{q}\right)\eta\left(\widehat{z}_g\left(\widetilde{q}\right)\right)\widetilde{q}\right]}{1 - \epsilon F_g'(\epsilon)(\omega - \epsilon)}$$

$$\to \mathbb{E}_{F_g}[\phi][\lambda - 1] - \mathbb{E}_{F_g}\left[\widehat{z}\eta\left(\widehat{z}\right)\right][\kappa - 1]$$

- Thus

$$\chi \to 1.$$

Formal Derivation of Stationary Distribution

- The stationary distribution has an approximate Pareto tail with the Pareto exponent $\chi = \chi(\omega) > 1$ such that: $\forall \xi > 0$ there exist $\overline{A}$,$\underline{A}$ and y_0 such that
$$f(y) < 2\overline{A} y^{-(\chi-1-\xi)}, \forall y \geq y_0$$
and
$$f(y) > \frac{1}{2}\underline{A} y^{-(\chi-1+\xi)}, \forall y \geq y_0.$$
In other words, $f(y) = y^{-\chi-1}\varphi(y)$, where $\varphi(y)$ is a slow-varying function. Moreover
$$\lim_{\omega \longrightarrow 0} \chi(\omega) = 1.$$
- Proof: Value function is approximately linear for large firms. The R&D investment and entry are approximately constant. Therefore we obtain a constant growth. Combined with lowest barriers $\Longrightarrow$Approximate Pareto distribution.

Simulation result

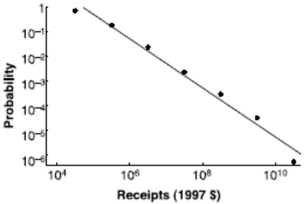
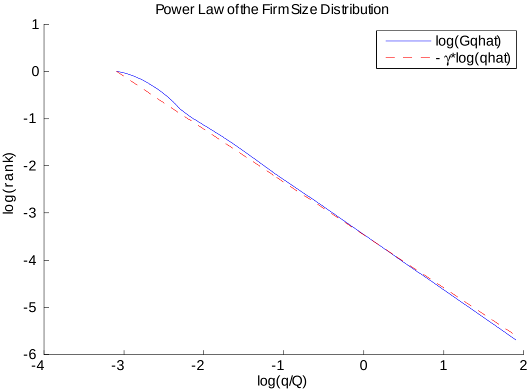


Fig. 2. Tail cumulative distribution function of U.S. firm sizes, by receipts in dollars. Data are for 1997 from the U.S. Census Bureau, tabulated in bins whose width increases in powers of 10. The solid line is the OLS regression line through the data and has slope of 0.994 (SE = 0.064; adjusted R^2 = 0.976).

Axtell 2001



Stationary Distribution of Firm Size

Existence of the Epsilon Economy

Theorem (Existence of the Epsilon Economy)

Suppose that

$$\max_{z>0} \varepsilon_\eta (z) \leq 1 - \frac{1}{\kappa^\theta}.$$

There exists an interval $\left(\underline{\mu}, \overline{\mu}\right)$ *and* $\Delta > 0$, $\overline{\omega} > 0$ *such that given* $\mu_e \in \left(\underline{\mu}, \overline{\mu}\right)$ *and for each* $\omega \in (0, \overline{\omega})$, *there is a BGP with the following properties :*

1. *An imitator pays* $\mu_e \omega Q(t)$ *to buy a product quality* $\omega Q(t)$ *and to enter into a sector* ν *if* $q(\nu, t) \leq \epsilon(\omega) Q(t)$, *where* $0 < \epsilon(\omega) \leq \omega (1-\beta)^{\frac{1-\beta}{\beta}}$

2. *The incumbents solve the net present value maximization problem but they take into account the behavior of the imitators as well as the innovative entrants.*

Existence of the Epsilon Economy (continued)

Theorem (Existence of the Epsilon Economy)

Continued

3. *The equilibrium growth rate of the aggregate product quality is* $g(\omega) \in (g^*, g^* + \Delta)$ *which satisfies*

$$\lim_{\omega \longrightarrow 0} g(\omega) = g^*.$$

4. *The stationary distribution is approximately Pareto with the tail* $\chi(\omega)$

$$\lim_{\omega \longrightarrow 0} \chi(\omega) = 1.$$

Existence of the Epsilon Economy (Sketch of Proof)