

# 14.452 Economic Growth: Lecture 13, Stochastic Growth

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# Stochastic Growth Models

- Brock and Mirman (1972): generalization of neoclassical growth and starting point of *Real Business Cycle* models
  - Baseline neoclassical growth: complete markets, households and firms can trade using any Arrow-Debreu commodity.
  - Complete markets: full set of *contingent claims* traded competitively.
  - Implies that individuals can fully insure against idiosyncratic risks.
  - Source of interesting uncertainty thus is aggregate shocks.
- Bewley (1970s and the 1980s): households cannot use contingent claims and can only trade in riskless bonds.
  - Explicitly prevent risk-sharing and thus “incomplete markets”.
  - Stochastic stream of labor income: can only achieve smoothing via “self-insurance”.
  - Does *not* admit a representative household; trading in contingent claims not only sufficient, but also necessary for representative household assumption with uncertainty.
  - Key for study of questions related to risk, income fluctuations and policy.

# The Brock-Mirman Model I

- With competitive and complete markets, the First and Second Welfare Theorems so equilibrium growth path is identical to the optimal growth path.
- But analysis is more involved and introduces new concepts.
- Economy as baseline neoclassical growth model, but production technology now given by

$$Y(t) = F(K(t), L(t), z(t)), \quad (1)$$

- $z(t)$  = stochastic *aggregate* productivity term
- Suppose  $z(t)$  follows a monotone Markov chain (as defined in Assumption 16.6) with values in the set  $\mathcal{Z} \equiv \{z_1, \dots, z_N\}$ .
- Many applications assume aggregate production function takes the form  $Y(t) = F(K(t), z(t) L(t))$ .

## The Brock-Mirman Model II

- Assume that the production function  $F$  satisfies usual assumptions and define

$$\begin{aligned}y(t) &\equiv \frac{Y(t)}{L(t)} \\ &\equiv f(k(t), z(t)),\end{aligned}$$

- Fraction  $\delta$  of the existing capital stock depreciates at each date.
- Suppose  $z_1, \dots, z_N$  are arranged in ascending order and that  $j > j'$  implies  $f(k, z_j) > f(k, z_{j'})$  for all  $k \in \mathbb{R}_+$ .
- Thus higher values of the stochastic shock  $z$  correspond to greater productivity at *all* capital-labor ratios.
- Representative household with instantaneous utility function  $u(c)$  that satisfies the standard assumptions.
- Supplies one unit of labor inelastically, so  $K(t)$  and  $k(t)$  can be used interchangeably (and no reason to distinguish  $C(t)$  from  $c(t)$ ).

# The Brock-Mirman Model III

- Consumption and saving decisions at time  $t$  are made after observing  $z(t)$ .
- Sequence version of the expected utility maximization problem of a social planner:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c(t)) \quad (2)$$

subject to

$$k(t+1) = f(k(t), z(t)) + (1 - \delta)k(t) - c(t) \text{ and } k(t) \geq 0, \quad (3)$$

with given  $k(0) > 0$ .

- To characterize the optimal growth path using the sequence problem: define feasible plans, mappings  $\tilde{k}[z^t]$  and  $\tilde{c}[z^t]$  with  $z^t \equiv (z(0), \dots, z(t))$ .

# The Brock-Mirman Model IV

- Instead look at the recursive version:

$$V(k, z) = \max_{k' \in [0, f(k, z) + (1 - \delta)k]} \{ u(f(k, z) + (1 - \delta)k - k') + \beta \mathbb{E} [V(k', z')] \} \quad (4)$$

**Proposition** In the stochastic optimal growth problem described above, the value function  $V(k, z)$  is uniquely defined, strictly increasing in both of its arguments, strictly concave in  $k$  and differentiable in  $k > 0$ . Moreover, there exists a uniquely defined policy function  $\pi(k, z)$  such that the capital stock at date  $t + 1$  is given by  $k(t + 1) = \pi(k(t), z(t))$ .

- Proof: verifying that Assumptions 16.1-16.6 from the previous chapter are satisfied and apply Theorems.
- To do this, first define  $\bar{k}$  such that  $\bar{k} = f(\bar{k}, z_N) + (1 - \delta)\bar{k}$ , and show that starting with  $k(0) \in (0, \bar{k})$ , the capital-labor ratio will always remain within the compact set  $(0, \bar{k})$ .

# The Brock-Mirman Model V

**Proposition** In the stochastic optimal growth problem described above, the policy function for next period's capital stock,  $\pi(k, z)$ , is strictly increasing in both of its arguments.

- Proof:

- By assumption  $u$  is differentiable and from the Proposition above  $V$  is differentiable in  $k$ .
- By the same argument as before,  $k \in (0, \bar{k})$ ; thus we are in the interior of the domain of the objective function.
- Thus, the value function  $V$  is differentiable in its first argument and

$$u'(f(k, z) + (1 - \delta)k - k') - \beta \mathbb{E}[V'(k', z') | z] = 0,$$

- Proposition above:  $V$  is strictly concave in  $k$ . Thus this can hold when  $k$  or  $z$  increases only if  $k'$  also increases.
- For example, an increase in  $k$  reduces the first-term (because  $u$  is strictly concave), hence an increase in  $k'$  is necessary to increase the first term and to reduce the second term (by the concavity of  $V$ ).
- Argument for increase in  $z$  is similar.

# The Brock-Mirman Model VI

- Define the policy function for consumption as

$$\pi^c(k, z) \equiv f(k, z) + (1 - \delta)k - \pi(k, z),$$

where  $\pi(k, z)$  is the optimal policy function for next date's capital stock determined in Proposition above.

- Using this notation, the stochastic Euler equation can be written as

$$u'(\pi^c(k, z)) = \beta \mathbb{E} [(f'(\pi(k, z), z') + (1 - \delta)) u'(\pi^c(\pi(k, z), z'))] \quad (5)$$

- A different way of expressing this equation makes it both simpler and more intuitive:

$$u'(c(t)) = \beta \mathbb{E}_t [p(t+1) u'(c(t+1))] \quad (6)$$

- $p(t+1)$  is the stochastic marginal product of capital (including undepreciated capital) at date  $t+1$ .



# The Brock-Mirman Model VII

- Also useful for comparison with the competitive equilibrium because  $p(t+1)$  corresponds to the stochastic (date  $t+1$ ) dividends paid out by one unit of capital invested at time  $t$ .
- Proposition above characterizes form of the value function and policy functions, but:
  - 1 Not an analog of the “Turnpike Theorem”: does not characterize the long-run behavior of the neoclassical growth model under uncertainty.
  - 2 Qualitative results about the value and the policy functions, but no comparative static results.
- Stochastic law of motion of the capital-labor ratio:

$$k(t+1) = \pi(k(t), z(t)), \quad (7)$$

# The Brock-Mirman Model VIII

- Defines a general Markov process, since before the realization of  $z(t)$ ,  $k(t+1)$  is a random variable, with its law of motion governed by the last period's value of  $k(t)$  and the realization of  $z(t)$ .
- If  $z(t)$  has a non-degenerate distribution,  $k(t)$  does not typically converge to a single value.
- But may hope that it will converge to an *invariant limiting distribution*.
- Markov process (7): starting with any  $k(0)$ , converges to a unique invariant limiting distribution.
- I.e., when we look at sufficiently faraway horizons, the distribution of  $k$  should be independent of  $k(0)$ .

# The Brock-Mirman Model IX

- Moreover, the average value of  $k(t)$  in invariant limiting distribution will be the same as the time average of  $\{k(t)\}_{t=0}^T$  as  $T \rightarrow \infty$  (stochastic process for the capital stock is “ergodic”).
- A “steady-state” equilibrium now corresponds not to specific values but to invariant limiting distributions.
- If  $z(t)$  takes values within a sufficiently small set, this limiting invariant distribution would hover around some particular values (“quasi-steady-state” values)
- But in general the range of the limiting distribution could be quite wide.

## Example: Brock-Mirman with Closed-form Solution I

- Suppose  $u(c) = \log c$ ,  $F(K, L, z) = zK^\alpha L^{1-\alpha}$ , and  $\delta = 1$ .
- Again  $z$  follows a Markov chain over the set  $\mathcal{Z} \equiv \{z_1, \dots, z_N\}$ , with transition probabilities denoted by  $q_{jj'}$ .
- Let  $k \equiv K/L$ . The stochastic Euler equation (5):

$$\frac{1}{zk^\alpha - \pi(k, z)} = \beta \mathbb{E} \left[ \frac{\alpha z' \pi(k, z)^{\alpha-1}}{z' \pi(k, z)^\alpha - \pi(\pi(k, z), z')} \middle| z \right], \quad (8)$$

- Relatively simple functional equation in a single function  $\pi(\cdot, \cdot)$ .
- Here “guessing and verifying” is handy. Conjecture that

$$\pi(k, z) = B_0 + B_1 z k^\alpha.$$

## Example: Brock-Mirman with Closed-form Solution II

- Substituting this guess into (8):

$$\frac{1}{(1 - B_1) z k^\alpha - B_0} = \beta \mathbb{E} \left[ \frac{\alpha z' (B_0 + B_1 z k^\alpha)^{\alpha-1}}{z' (B_0 + B_1 z k^\alpha)^\alpha - B_0 - B_1 z' (B_0 + B_1 z k^\alpha)^\alpha} \middle| z \right]. \quad (9)$$

- This equation cannot be satisfied for any  $B_0 \neq 0$ .
- Thus imposing  $B_0 = 0$  and writing out the expectation explicitly with  $z = z_{j'}$ , this expression becomes

$$\frac{1}{(1 - B_1) z_{j'} k^\alpha} = \beta \sum_{j=1}^N q_{jj'} \frac{\alpha z_j (B_1 z_{j'} k^\alpha)^{\alpha-1}}{z_j (B_1 z_{j'} k^\alpha)^\alpha - B_1 z_j (B_1 z_{j'} k^\alpha)^\alpha}.$$

- Simplifying each term within the summation:

$$\frac{1}{(1 - B_1) z_{j'} k^\alpha} = \beta \sum_{j=1}^N q_{jj'} \frac{\alpha}{B_1 (1 - B_1) z_{j'} k^\alpha}.$$

## Example: Brock-Mirman with Closed-form Solution III

- Now taking  $z_{j'}$  and  $k$  out of the summation and using the fact that, by definition,  $\sum_{j=1}^N q_{jj'} = 1$ , we can cancel the remaining terms and obtain

$$B_1 = \alpha\beta,$$

- Thus irrespective of the exact Markov chain for  $z$ , the optimal policy rule is

$$\pi(k, z) = \alpha\beta z k^\alpha.$$

- Identical to deterministic case, with  $z$  there corresponding to a non-stochastic productivity term.
- Thus stochastic elements have not changed the form of the optimal policy function.
- Same result applies when  $z$  follows a general Markov process rather than a Markov chain.

## Example: Brock-Mirman with Closed-form Solution IV

- Here can fully analyze the stochastic behavior of the capital-labor ratio and output per capita.
- Stochastic behavior of the capital-labor ratio in this economy is identical to that of the overlapping generations model
- But just one of the few instances of the neoclassical growth model that admit closed-form solutions.
- In particular, if the depreciation rate of the capital stock  $\delta$  is not equal to 1, the neoclassical growth model under uncertainty does not admit an explicit form characterization.

# Equilibrium Growth under Uncertainty I

- Environment identical to that in the previous section,  $z$  an aggregate productivity shock affecting all production units and households.
- Arrow-Debreu commodities defined so that goods indexed by different realizations of the history  $z^t$  correspond to different commodities.
- Thus economy with a countable infinity of commodities.
- Second Welfare Theorem applies and implies that the optimal growth path characterized in the previous section can be decentralized as a competitive equilibrium
- Moreover, since we are focusing on an economy with a representative household, this allocation is a competitive equilibrium without any redistribution of endowments.
- Justifies the frequent focus on social planner's problems in analyses of stochastic growth models in the literature.



# Equilibrium Growth under Uncertainty II

- But explicit characterization of competitive equilibria shows the equivalence, and introduces ideas related to pricing of contingent claims.
- Complete markets: in principle, any commodity, including any contingent claim, can be traded competitively.
- In practice no need to specify or trade all of these commodities; a subset sufficient to provide all necessary trading opportunities.
- Will also show what subsets are typically sufficient.
- Preferences and technology as in previous model: economy admits representative household and production side can be represented by a representative firm.
- Household maximize the objective function given by (2) subject to the lifetime budget constraint (written from the viewpoint of time  $t = 0$ ).

# Equilibrium Growth under Uncertainty III

- No loss of generality in considering the viewpoint of time  $t = 0$  relative to formulating with sequential trading constraints.
- $\mathcal{Z}^t$  = set of all possible histories of the stochastic variable  $z^t$  up to date  $t$ .
- $\mathcal{Z}^\infty$  = set of infinite histories.
- $z^t \in \mathcal{Z}^\infty$  = a possible history of length  $t$ .
- $p_0 [z^t]$  = price of the unique final good at time  $t$  in terms of the final good of date 0 following a history  $z^t$ ,
- $c [z^t]$  and  $w_0 [z^t]$  similarly defined.
- Household's lifetime budget constraint:

$$\sum_{t=0}^{\infty} \sum_{z^t \in \mathcal{Z}^\infty} p_0 [z^t] c [z^t] \leq \sum_{t=0}^{\infty} \sum_{z^t \in \mathcal{Z}^\infty} w_0 [z^t] + k(0). \quad (10)$$

# Equilibrium Growth under Uncertainty IV

- No expectations:
  - Complete markets: all trades at  $t = 0$  at price vector for all Arrow-Debreu commodities.
  - Household buys claims to different “contingent” consumption bundles; i.e. conditioned on  $z^t$ .
- Left-hand=total expenditure taking the prices of all possible claims as given.
- Right-hand side=labor earnings and value of initial capital stock per capita.
- Right-hand side of (10) could also include profits accruing to the individuals, but constant returns and competitive markets implies that equilibrium profits will be equal to 0.
- Objective function at time  $t = 0$ :

$$\sum_{t=0}^{\infty} \beta^t \sum_{z^t \in \mathcal{Z}^{\infty}} q [z^t | z^0] u (c [z^t]), \quad (11)$$

# Equilibrium Growth under Uncertainty V

- $q [z^t | z^0]$  = probability at time 0 that the history  $z^t$  will be realized at time  $t$ .
- Sequence problem of maximizing (11) subject to (10). Assuming interior solution, first-order conditions: is

$$\beta^t q [z^t | z^0] u' (c [z^t]) = \lambda p_0 [z^t] \quad (12)$$

for all  $t$  and all  $z^t$ .

- $\lambda$  is the Lagrange multiplier on (10) corresponding to the marginal utility of income at date  $t = 0$
- Combining two different date  $t$  histories  $z^t$  and  $\hat{z}^t$ :

$$\frac{u' (c [\hat{z}^t])}{u' (c [z^t])} = \frac{p_0 [\hat{z}^t] / q [\hat{z}^t | z^0]}{p_0 [z^t] / q [z^t | z^0]},$$

- Right-hand side = relative price of consumption claims conditional on histories  $z^t$  and  $\hat{z}^t$

# Equilibrium Growth under Uncertainty VI

- Combining for histories  $z^t$  and  $z^{t+1}$  such that  $z^{t+1} = (z^t, z(t+1))$ :

$$\frac{\beta u'(c[z^{t+1}])}{u'(c[z^t])} = \frac{p_0[z^{t+1}] / q[z^{t+1} | z^0]}{p_0[z^t] / q[z^t | z^0]},$$

- Right-hand side=contingent interest rate between date  $t$  and  $t+1$  conditional on  $z^t$  (and contingent on the realization of  $z^{t+1}$ ).
- To characterize equilibrium need prices  $p_0[z^t]$ , from the profit maximization problem of firm.
- $R_0[z^t]$ =price of one unit of capital after the state  $z^t$
- $K^e[z^t]$  and  $L[z^t]$ =capital and labor employment levels of the representative firm after history  $z^t$ .
- Value of the firm:

$$\sum_{t=0}^{\infty} \beta^t \sum_{z^t \in \mathcal{Z}^{\infty}} \left\{ \begin{array}{l} p_0[z^t] (F(K^e[z^t], L[z^t], z(t)) + (1-\delta)K^e[z^t]) \\ - R_0[z^t] K^e[z^t] - w_0[z^t] L[z^t] \end{array} \right\}$$

# Equilibrium Growth under Uncertainty VII

- Profit maximization implies:

$$\begin{aligned} p_0 [z^t] \left( \frac{\partial F (K^e [z^t], L [z^t], z (t))}{\partial K^e} + (1 - \delta) \right) &= R_0 [z^t] \\ p_0 [z^t] \frac{\partial F (K^e [z^t], L [z^t], z (t))}{\partial L} &= w_0 [z^t]. \end{aligned}$$

- Using constant returns to scale:

$$\begin{aligned} p_0 [z^t] (f' (k^e [z^t], z (t)) + (1 - \delta)) &= R_0 [z^t] \\ p_0 [z^t] (f (k^e [z^t], z (t)) - k^e [z^t] f' (k^e [z^t], z (t))) &= w_0 [z^t] \end{aligned} \tag{13}$$

- Relation between prices and marginal productivity of factors.
- But (13) also stating that  $R_0 [z^t]$  is equal to the value of dividends paid out by a unit of capital inclusive of undepreciated capital.

## Equilibrium Growth under Uncertainty VIII

- Alternative, equivalent, way of formulating competitive equilibrium and writing (13) is to assume that capital goods are *rented*.
- Labor market clearing:

$$L [z^t] = 1 \text{ for all } z^t. \quad (14)$$

- Production after history  $z^t$  is  $f(k^e [z^t], z(t)) + (1 - \delta) k^e [z^t]$ , divided between consumption  $c [z^t]$  and savings  $s [z^t]$ .
- Capital used at time  $t + 1$  (after history  $z^{t+1}$ ) must be equal to  $s [z^t]$ .
- Market clearing for capital implies that for any  $z^{t+1} = (z^t, z(t + 1))$ ,

$$k^e [z^{t+1}] = s [z^t], \quad (15)$$

- Capital market clearing condition:

$$c [z^t] + s [z^t] \leq f(s [z^{t-1}], z(t)) + (1 - \delta) s [z^{t-1}] \quad (16)$$

for any  $z^{t+1} = (z^t, z(t + 1))$ .

# Equilibrium Growth under Uncertainty IX

- Capital market clearing condition also implies *no arbitrage* condition linking  $R_0 [z^{t+1}]$  to  $p_0 [z^t]$ .
- Consider the following riskless arbitrage:
  - Buy one unit of the final good after  $z^t$  to be used as capital at time  $t + 1$  and simultaneously sell claims on capital goods for each  $z^{t+1} = (z^t, z(t+1))$ .
  - No risk, since unit of final good bought after history  $z^t$  will cover the obligation to pay capital good after any  $z^{t+1} = (z^t, z(t+1))$ .
- Implies the no arbitrage condition

$$p_0 [z^t] = \sum_{z(t+1) \in \mathcal{Z}} R_0 [(z^t, z(t+1))] . \quad (17)$$



# Equilibrium Growth under Uncertainty X

- Competitive equilibrium:  $\{c [z^t], s [z^t], k^e [z^{t+1}]\}_{z^t \in \mathcal{Z}^t}$ , and  $\{p_0 [z^t], R_0 [z^t], w_0 [z^t]\}_{z^t \in \mathcal{Z}^t}$ , such that households maximize utility (i.e., satisfy (12)), firms maximize profits (i.e., satisfy (13) and (17)), and labor and capital markets clear (i.e., (14), (15), and (16) are satisfied).
- Substitute from (13) and (17) into (12) and rearrange:

$$u' (c [z^t]) = \sum_{z^{(t+1)} \in \mathcal{Z}} \frac{\lambda p_0 [z^{t+1}]}{\beta^t q [z^t | z^0]} (f' (k [z^{t+1}], z (t+1)) + (1 - \delta)) \quad (18)$$

- Next using (12) for  $t + 1$ :

$$\begin{aligned} \beta u' (c [z^{t+1}]) &= \frac{\lambda p_0 [z^{t+1}]}{\beta^t q [z^{t+1} | z^0]} \\ &= \frac{\lambda p_0 [z^{t+1}]}{\beta^t q [z^{t+1} | z^t] q [z^t | z^0]}, \end{aligned}$$

# Equilibrium Growth under Uncertainty XI

- Second line uses the law of iterated expectations,  
 $q [z^{t+1} | z^0] \equiv q [z^{t+1} | z^t] q [z^t | z^0]$ .
- Substituting into (18), we obtain

$$\begin{aligned}
 & u' (c [z^t]) \\
 = & \beta \sum_{z^{(t+1)} \in \mathcal{Z}} q [z^{t+1} | z^t] u' (c [z^{t+1}]) \left( f' (k [z^{t+1}], z (t+1)) \right. \\
 & \left. + (1 - \delta) \right) \\
 = & \beta \mathbb{E} [u' (c [z^{t+1}]) (f' (k [z^{t+1}], z (t+1)) + (1 - \delta)) | z^t],
 \end{aligned}$$

- Identical to (6).

**Proposition** In the above-described economy, optimal and competitive growth path coincide.

## Equilibrium Growth under Uncertainty XII

- Equilibrium problem in its equivalent form with sequential trading rather than all trades taking place at the initial date  $t = 0$ .
- Write the budget constraint of the representative household somewhat differently.
- Normalize the price of the final good at each date to 1.
- $a[z^t]$  = Basic Arrow securities that pay out only in specific states on nature.
- $\{a[z^t]\}_{z^t \in Z^t}$  = set of contingent claims that the household has purchased that will pay  $a[z^t]$  units of the final good at date  $t$  when history  $z^t$  is realized.
- Price of claim to one unit of  $a[z^t]$  at time  $t - 1$  after history  $z^{t-1}$  denoted by  $\bar{p}[z(t) | z^{t-1}]$ , where  $z^t = (z^{t-1}, z(t))$ .
- Amount of these claims purchased by the household is denoted by  $a[(z^{t-1}, z(t))]$ .

# Equilibrium Growth under Uncertainty XIII

- Thus flow budget constraint of the household:

$$c [z^t] + \sum_{z(t+1) \in \mathcal{Z}} \bar{p} [z(t+1) | z^t] a [(z^{t-1}, z(t))] \leq w [z^t] + a [z^t],$$

- $w [z^t]$  = equilibrium wage rate after history  $z^t$  in terms of final goods dated  $t$ .
- Let  $a$  denote the current asset holdings of the household (realization of current assets after some  $z^t$  has been realized).
- Then flow budget constraint of the household can be written as

$$c + \sum_{z' \in \mathcal{Z}} \bar{p} [z' | z] a' [z' | z] \leq w + a,$$

- Function  $\bar{p} [z' | z]$  = prices of contingent claims (for next date's state  $z'$  given current state  $z$ ).
- $a' [z' | z]$  = corresponding asset holdings.

# Equilibrium Growth under Uncertainty XIV

- $V(a, z)$  = value function of the household.
- Choice variables:  $a' [z' | z]$  and consumption today,  $c [a, z]$ .
- $q [z' | z]$  = probability that next period's stochastic variable will be equal to  $z'$  conditional on today's value being  $z$ .
- Then taking the sequence of equilibrium prices  $\bar{p}$  as given, the value function of the representative household:

$$V(a, z) = \sup_{\{a'[z'|z]\}_{z' \in \mathcal{Z}}} \left\{ u(a + w - \sum_{z' \in \mathcal{Z}} \bar{p} [z' | z] a' [z' | z]) + \beta \sum_{z' \in \mathcal{Z}} q [z' | z] V(a' [z' | z], z') \right\}. \quad (19)$$

- All Theorems on the value function can again be applied to this value function.

# Equilibrium Growth under Uncertainty XV

- First-order condition for current consumption:

$$\bar{p} [z' | z] u' (c [a, z]) = \beta q [z' | z] \frac{\partial V (a' [z' | z], z')}{\partial a}$$

for any  $z' \in \mathcal{Z}$ .

- Capital market clearing:

$$a' [z' | z] = a' [z],$$

- Thus in the aggregate the same amount of assets will be present in all states at the next date.
- Thus first-order condition for consumption can be alternatively written as

$$\bar{p} [z' | z] u' (c [a, z]) = \beta q [z' | z] \frac{\partial V (a' [z], z')}{\partial a}. \quad (20)$$

# Equilibrium Growth under Uncertainty XVI

- No arbitrage condition implies

$$\sum_{z' \in Z} \bar{p} [z' | z] R [z' | z] = 1, \quad (21)$$

where  $R [z' | z]$  is the price of capital goods when the current state is  $z'$  and last period's state was  $z$ .

- Intuition:
  - Cost of one unit of the final good now, 1, has to be equal to return of carrying it to the next period and selling it as a capital good then.
  - Summing over all possible states  $z'$  tomorrow must have total return of 1 to ensure no arbitrage
- Combine (20) with the envelope condition

$$\frac{\partial V (a, z)}{\partial a} = u' (c [a, z]),$$

# Equilibrium Growth under Uncertainty XVII

- Multiply both sides of (20) by  $R[z' | z]$  and sum over all  $z' \in \mathcal{Z}$  to obtain the first-order condition of the household as

$$\begin{aligned} u'(c[a, z]) &= \beta \sum_{z' \in \mathcal{Z}} q[z' | z] R[z' | z] u'(c[a', z']) . \\ &= \beta \mathbb{E} [R[z' | z] u'(c[a', z']) | z] . \end{aligned}$$

- Market clearing condition for capital, combined with the fact that the only asset in the economy is capital, implies:

$$a = k.$$

- Therefore first-order condition can be written as

$$u'(c[k, z]) = \beta \mathbb{E} [R[z' | z] u'(c[k', z']) | z]$$

which is identical to (6).



# Equilibrium Growth under Uncertainty XVIII

- Again shows the equivalence between the social planner's problem and the competitive equilibrium path.
- Social planner's problem (the optimal growth problem) is considerably simpler, characterizes the equilibrium path of all the real variables and various different prices are also straightforward to obtain from the Lagrange multiplier.

# Application: Real Business Cycle Models I

- Real Business Cycle (RBC): one of the most active research areas in the 1990s and also one of the most controversial.
- Conceptual simplicity and relative success in matching certain moments of employment, consumption and investment fluctuations vs. the absence of monetary factors and demand shocks.
- But exposition of RBC model useful for two purposes:
  - 1 one of the most important applications of the neoclassical growth model under uncertainty
  - 2 new insights from introduction of labor supply choices into the neoclassical growth model under uncertainty generates.
- Only difference is instantaneous utility function of the representative household now takes the form

$$u(C, L),$$

## Application: Real Business Cycle Models II

- $u$  is jointly concave and continuously differentiable in both of its arguments and strictly increasing in  $C$  and strictly decreasing in  $L$ .
- Also assume that  $L$  has to lie in some convex compact set  $[0, \bar{L}]$ .
- Focus on the optimal growth formulation: maximization of

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(C(t), L(t))$$

subject to the flow resource constraint

$$K(t+1) \leq F(K(t), L(t), z(t)) + (1 - \delta)K(t) - C(t).$$

- $z(t)$  again represents an aggregate productivity shock following a monotone Markov chain.

## Application: Real Business Cycle Models III

- Social planner's problem can be written recursively as

$$V(K, z) = \sup_{\substack{L \in [0, \bar{L}] \\ K' \in [0, F(K, L, z) + (1 - \delta)K]}} \begin{cases} u(F(K, L, z) + (1 - \delta)K - K', L) \\ + \beta \mathbb{E}[V(K', z') | z] \end{cases} \quad (22)$$

**Proposition** The value function  $V(K, z)$  defined in (22) is continuous and strictly concave in  $K$ , strictly increasing in  $K$  and  $z$ , and differentiable in  $K > 0$ . There exist uniquely defined policy functions  $\pi^k(K, z)$  and  $\pi^l(K, z)$  that determine the level of capital stock chosen for next period and the level of labor supply as a function of the current capital stock  $K$  and the stochastic variable  $z$ .

- Assuming an interior solution, relevant prices can be obtained from the appropriate multipliers and the standard first-order conditions characterize the form of the equilibrium.

## Application: Real Business Cycle Models IV

- Define the policy function for consumption:

$$\pi^c(K, z) \equiv F(K, \pi^l(K, z), z) + (1 - \delta)K - \pi^k(K, z),$$

- Key first order conditions (write  $\pi^J$  short for  $\pi^J(K, z)$ ,  $J = c, l, k$ ):

$$u_c(\pi^c, \pi^l) = \beta \mathbb{E} \left[ R(\pi^k, z') u_c(\pi^c(\pi^k, z'), \pi^l(\pi^k, z')) \right]$$

$$w(K, z) u_c(\pi^c, \pi^l) = -u_l(\pi^c, \pi^l).$$

where

$$R(K, z) = F_k(K, z) + (1 - \delta)$$

$$w(K, z) = F_l(K, z)$$

## Application: Real Business Cycle Models V

- First condition in (23) is essentially identical to (5), whereas the second is a static condition determining the level of equilibrium (or optimal) labor supply.
- Second condition does not feature expectations: conditional on the current value  $K$  and the current  $z$ .
- Analysis of macroeconomic fluctuations: period in which  $z$  is low.
  - If no offsetting change in labor supply, “recession”.
  - Under standard assumptions,  $w(K, z)$  and labor supply decline: low employment and output.
  - If Markov process for  $z$  exhibits persistence, *persistent fluctuations*.
  - Provided  $F(K, L, z)$  is such that low output is associated with low marginal product of capital, expectation of future low output will typically reduce savings and thus future levels of capital stock
    - This effect depends also on form of utility function (consumption smoothing and income and substitution effects).

## Application: Real Business Cycle Models VI

- Thus model may generate some of the major qualitative features of macroeconomic fluctuations.
- RBC literature argues it generates the *major* quantitative features such as correlations between output, investment, and employment.
- Debate on whether:
  - 1 the model did indeed match these moments in the data;
  - 2 these were the right empirical objects to look at; and
  - 3 focusing on exogenous changes in aggregate productivity sidestep why there are shocks.
- RBC debate is not as active today as it was in the 1990s, but not a complete agreement.

## Example: RBC model with closed-form solution I

- $u(C, L) = \log C - \gamma L$ ,  $F(K, L, z) = zK^\alpha L^{1-\alpha}$ , and  $\delta = 1$ .
- $z$  follows a monotone Markov chain over the set  $\mathcal{Z} \equiv \{z_1, \dots, z_N\}$ , with transition probabilities denoted by  $q_{jj'}$ .

- Conjecture that

$$\pi^k(K, z) = BzK^\alpha L^{1-\alpha}.$$

- Then with these functional forms, the stochastic Euler equation for consumption (23) implies

$$\frac{1}{(1-B)zK^\alpha L^{1-\alpha}} = \beta \mathbb{E} \left[ \frac{\alpha z' (BzK^\alpha L^{1-\alpha})^{-(1-\alpha)} (L')^{1-\alpha}}{(1-B)z' (BzK^\alpha L^{1-\alpha})^\alpha (L')^{1-\alpha}} \middle| z \right],$$

where  $L'$  denotes next period's labor supply.



## Example: RBC model with closed-form solution II

- Canceling constants within the expectations and taking terms that do not involve  $z'$  out of the expectations:

$$\frac{1}{zK^\alpha L^{1-\alpha}} = \beta \mathbb{E} \left[ \alpha (BzK^\alpha L^{1-\alpha})^{-1} \mid z \right],$$

which yields

$$B = \alpha\beta.$$

- Resulting policy function for the capital stock is therefore

$$\pi^k(K, z) = \alpha\beta zK^\alpha L^{1-\alpha},$$

which is identical to that in Example before.

- Next, considering the first-order condition for labor:

$$\frac{(1-\alpha)zK^\alpha L^{-\alpha}}{(1-B)zK^\alpha L^{1-\alpha}} = \gamma.$$

## Example: RBC model with closed-form solution III

- The resulting policy function for labor as

$$\pi^l(K, z) = \frac{(1 - \alpha)}{\gamma(1 - \alpha\beta)},$$

- Labor supply is constant: with the preferences as specified here, the income and the substitution effects cancel out, increase in wages induced by a change in aggregate productivity has no effect on labor supply.
- Same result obtains whenever the utility function takes the form of  $U(C, L) = \log C + h(L)$  for some decreasing and concave function  $h$ .
- Replicates the covariation in output and investment, but does not generate labor fluctuations.

# Growth with Incomplete Markets: The Bewley Model I

- Economy is populated by a continuum 1 of households and the set of households is denoted by  $\mathcal{H}$ .
- Each household has preferences given by (2) and supplies labor inelastically.
- Suppose also that the second derivative of this utility function,  $u''(\cdot)$ , is increasing.
- Efficiency units that each household supplies vary over time.
- In particular, each household  $h \in \mathcal{H}$  has a labor endowment of  $z^h(t)$  at time  $t$ , where  $z^h(t)$  is an independent draw from the set  $\mathcal{Z} \equiv [z_{\min}, z_{\max}]$ , where  $0 < z_{\min} < z_{\max} < \infty$ .
- Labor endowment of each household is identically and independently distributed with distribution function  $G(z)$  defined over  $[z_{\min}, z_{\max}]$ .
- Production side is the same as in the canonical neoclassical growth model under certainty.

## Growth with Incomplete Markets: The Bewley Model II

- Only difference is  $L(t)$  is now the sum (integral) of the heterogeneous labor endowments of all the agents:

$$L(t) = \int_{h \in \mathcal{H}} z^h(t) dh.$$

- Appealing to a law of large numbers type argument, we assume that  $L(t)$  is constant at each date and we normalize it to 1.
- Thus output per capita in the economy can be expressed as

$$y(t) = f(k(t)),$$

with  $k(t) = K(t)$ .

- No longer any aggregate productivity shock; only uncertainty at the individual level (i.e., it is idiosyncratic).
- Individual households will experience fluctuations in their labor income and consumption, but can imagine a *stationary* equilibrium in which aggregates are constant over time.

# Growth with Incomplete Markets: The Bewley Model III

- Focus on such a stationary equilibrium: wage rate  $w$  and the gross rate of return on capital  $R$  will be constant .
- First take these prices as given and look at the behavior of a typical household  $h \in \mathcal{H}$
- Maximize (2) subject to the flow budget constraint

$$a^h(t+1) \leq Ra^h(t) + wz^h(t) - c^h(t)$$

for all  $t$ , where  $a^h(t)$  is the asset holding of household  $h \in \mathcal{H}$  at time  $t$ .

- Consumption cannot be negative, so  $c^h(t) \geq 0$ .

# Growth with Incomplete Markets: The Bewley Model IV

- Requirement that individual should satisfy its lifetime budget constraint in all histories imposes the endogenous borrowing constraint:

$$\begin{aligned} a^h(t) &\geq -\frac{z_{\min}}{R-1} \\ &\equiv -b, \end{aligned}$$

for all  $t$ .

- Maximization problem of household  $h \in \mathcal{H}$  recursively:

$$V^h(a, z) = \sup_{a' \in [-b, Ra + wz]} \left\{ u(Ra + wz - a') + \beta \mathbb{E} \left[ V^h(a', z') \mid z \right] \right\}. \quad (24)$$

# Growth with Incomplete Markets: The Bewley Model V

**Proposition** The value function  $V^h(a, z)$  defined in (24) is uniquely defined, continuous and strictly concave in  $a$ , strictly increasing in  $a$  and  $z$ , and differentiable in  $a \in (-b, Ra + wz)$ . Moreover, the policy function that determines next period's asset holding  $\pi(a, z)$  is uniquely defined and continuous in  $a$ .

**Proposition** The policy function  $\pi(a, z)$  derived in Proposition ?? is strictly increasing in  $a$  and  $z$ .

- Total amount of capital stock in the economy=asset holdings of all households in the economy, thus in a stationary equilibrium:

$$\begin{aligned} k(t+1) &= \int_{h \in \mathcal{H}} a^h(t) dh \\ &= \int_{h \in \mathcal{H}} \pi(a^h(t), z^h(t)) dh. \end{aligned}$$

# Growth with Incomplete Markets: The Bewley Model VI

- Integrates over all households taking their asset holdings and the realization of their stochastic shock as given.
- Both the average of current asset holdings and also the average of tomorrow's asset holdings must be equal by the definition of a stationary equilibrium.
- Recall policy function  $a' = \pi(a, z)$  defines a general Markov process: under fairly weak it will admit a unique invariant distribution.
- If not economy could have multiple stationary equilibria or even there might be problems of non-existence.
- Ignore this complication and assume the existence of a unique invariant distribution,  $\Gamma(a)$ , so stationary equilibrium capital-labor ratio is:

$$k^* = \int \int \pi(a, z) d\Gamma(a) dG(z),$$

which uses the fact that  $z$  is distributed identically and independently across households and over time.



# Growth with Incomplete Markets: The Bewley Model VII

- Turning to the production side:

$$\begin{aligned} R &= f'(k^*) + (1 - \delta) \\ w &= f(k^*) - k^* f'(k^*). \end{aligned}$$

- Recall neoclassical growth model with complete markets and no uncertainty implies unique steady state in which  $\beta R = 1$ , i.e.,

$$f'(k^{**}) = \beta^{-1} - (1 - \delta), \quad (25)$$

where  $k^{**}$  refers to the capital-labor ratio of the neoclassical growth model under certainty.

- In Bewley economy this is no longer true.
- **Implication:** dynamic inefficiency possible as in Solow and OLG models— but for different reasons.

# Growth with Incomplete Markets: The Bewley Model VIII

**Proposition** In any stationary equilibrium of the Bewley economy, we have that the stationary equilibrium capital-labor ratio  $k^*$  is such that

$$f'(k^*) < \beta^{-1} - (1 - \delta) \quad (26)$$

and

$$k^* > k^{**}, \quad (27)$$

where  $k^{**}$  is the capital-labor ratio of the neoclassical growth model under certainty.

# Growth with Incomplete Markets: The Bewley Model IX

- **Sketch of proof:**

- Suppose  $f'(k^*) \geq \beta^{-1} - (1 - \delta)$ .
- Then each household's expected consumption is strictly increasing.
- This implies that average consumption in the population, which is deterministic, is strictly increasing and would tend to infinity.
- This is not possible since aggregate resources must always be finite.
- This establishes (26).
- Given this result, (27) immediately follows from (25) and from the strict concavity of  $f(\cdot)$ .

# Growth with Incomplete Markets: The Bewley Model X

- Interest rate is “depressed” relative to the neoclassical growth model with certainty because each household has an additional self-insurance (or precautionary) incentive to save.
- These additional savings increase the capital-labor ratio and reduce the equilibrium interest rate.
- Two features, potential shortcomings, are worth noting:
  - ① Inefficiency from overaccumulation of capital unlikely to be important for explaining income per capita differences across countries.
    - model is not interesting because of this but as an illustration of stationary equilibrium in which aggregates are constant while individual households have uncertain and fluctuating consumption and income profiles.
  - ② Incomplete markets assumption in this model may be extreme.