

6.207/14.15: Networks
Lecture 11: Introduction to Game Theory—3

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Outline

- Existence of Nash Equilibrium in Infinite Games
- Extensive Form and Dynamic Games
- Subgame Perfect Nash Equilibrium
- Applications

- **Reading:**
- Osborne, Chapters 5-6.

Existence of Equilibria for Infinite Games

- A similar theorem to Nash's existence theorem applies for pure strategy existence in infinite games.

Theorem

(Debreu, Glicksberg, Fan) Consider an infinite strategic form game $\langle \mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}} \rangle$ such that for each $i \in \mathcal{I}$

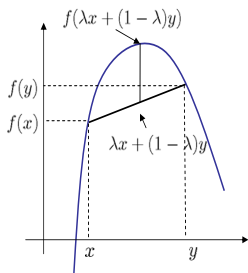
- 1 S_i is compact and convex;
- 2 $u_i(s_i, s_{-i})$ is continuous in s_{-i} ;
- 3 $u_i(s_i, s_{-i})$ is continuous and concave in s_i [in fact quasi-concavity suffices].

Then a pure strategy Nash equilibrium exists.

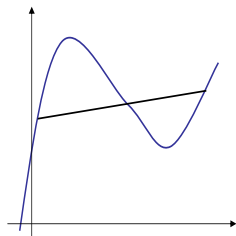
Definitions

- Suppose S is a convex set. Then a function $f : S \rightarrow \mathbb{R}$ is **concave** if for any $x, y \in S$ and any $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y).$$



concave function



not a concave function

Proof

- Now define the best response correspondence for player i ,
 $B_i : S_{-i} \rightrightarrows S_i$,

$$B_i(s_{-i}) = \{s'_i \in S_i \mid u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i}) \text{ for all } s_i \in S_i\}.$$

Thus restriction to pure strategies.

- Define the set of best response correspondences as

$$B(s) = [B_i(s_{-i})]_{i \in \mathcal{I}}.$$

and

$$B : S \rightrightarrows S.$$

Proof (continued)

- We will again apply Kakutani's theorem to the best response correspondence $B : S \rightrightarrows S$ by showing that $B(s)$ satisfies the conditions of Kakutani's theorem.

1 S is compact, convex, and non-empty.

- • By definition

$$S = \prod_{i \in \mathcal{I}} S_i$$

since each S_i is compact [convex, nonempty] and finite product of compact [convex, nonempty] sets is compact [convex, nonempty].

2 $B(s)$ is non-empty.

- • By definition,

$$B_i(s_{-i}) = \arg \max_{s \in S_i} u_i(s, s_{-i})$$

where S_i is non-empty and compact, and u_i is continuous in s by assumption. Then by Weirstrass's theorem $B(s)$ is non-empty.

Proof (continued)

3. $B(s)$ is a convex-valued correspondence.

- This follows from the fact that $u_i(s_i, s_{-i})$ is concave [or quasi-concave] in s_i . Suppose not, then there exists some i and some $s_{-i} \in S_{-i}$ such that $B_i(s_{-i}) \in \arg \max_{s \in S_i} u_i(s, s_{-i})$ is not convex.
- This implies that there exists $s'_i, s''_i \in S_i$ such that $s'_i, s''_i \in B_i(s_{-i})$ and $\lambda s'_i + (1 - \lambda)s''_i \notin B_i(s_{-i})$. In other words,

$$\lambda u_i(s'_i, s_{-i}) + (1 - \lambda)u_i(s''_i, s_{-i}) > u_i(\lambda s'_i + (1 - \lambda)s''_i, s_{-i}).$$

But this violates the concavity of $u_i(s_i, s_{-i})$ in s_i [recall that for a concave function $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$].

- Therefore $B(s)$ is convex-valued.
- ### 4. The proof that $B(s)$ has a closed graph is identical to the previous proof.

Existence of Nash Equilibria

- Can we relax concavity?
- **Example:** Consider the game where two players pick a location $s_1, s_2 \in \mathbb{R}^2$ on the circle. The payoffs are $u_1(s_1, s_2) = -u_2(s_1, s_2) = d(s_1, s_2)$, where $d(s_1, s_2)$ denotes the Euclidean distance between $s_1, s_2 \in \mathbb{R}^2$.
- No pure Nash equilibrium.
- However, it can be shown that the strategy profile where both mix uniformly on the circle is a mixed Nash equilibrium.

A More Powerful Theorem

Theorem

(Glicksberg) Consider an infinite strategic form game $\langle \mathcal{I}, (S_i)_{i \in \mathcal{I}}, (u_i)_{i \in \mathcal{I}} \rangle$ such that for each $i \in \mathcal{I}$

- 1 S_i is nonempty and compact;
- 2 $u_i(s_i, s_{-i})$ is continuous in s_i and s_{-i} .

Then a mixed strategy Nash equilibrium exists.

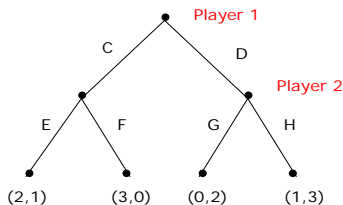
- The proof of this theorem is harder and we will not discuss it.
- In fact, finding mixed strategies in continuous games is more challenging and is beyond the scope of this course.

Extensive Form Games

- Extensive-form games model multi-agent sequential decision making.
- For now, we will focus is on multi-stage games with observed actions.
- Extensive form represented by **game trees**.
- Additional component of the model, **histories** (i.e., sequences of action profiles).
- Extensive form games will be useful when we analyze **dynamic** games, in particular, to understand issues of cooperation and trust in groups.

Histories

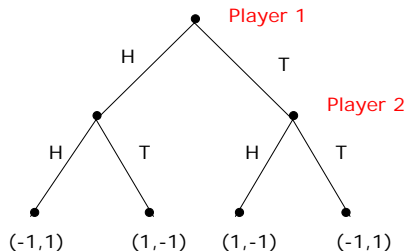
- Let H^k denote the set of all possible stage- k histories
- Strategies are maps from all possible histories into actions:
 $s_i^k : H^k \rightarrow S_i$



- Example:**
 - Player 1's strategies: $s_1 : H^0 = \emptyset \rightarrow S_1$; two possible strategies: C,D
 - Player 2's strategies: $s^2 : H^1 = \{C, D\} \rightarrow S_2$; four possible strategies.

Strategies in Extensive Form Games

- Consider the following two-stage extensive form version of matching pennies.



- How many strategies does player 2 have?

Strategies in Extensive Form Games (continued)

- Recall: strategy should be a *complete contingency plan*.
- Therefore: player 2 has four strategies:
 - 1 heads following heads, heads following tails (HH,HT);
 - 2 heads following heads, tails following tails (HH, TT);
 - 3 tails following heads, tails following tails (TH, TT);
 - 4 tails following heads, heads following tails (TH, HT).

Strategies in Extensive Form Games (continued)

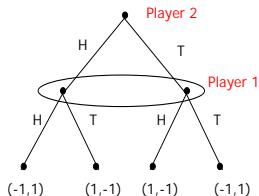
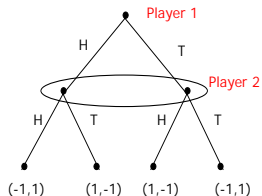
- Therefore, from the extensive form game we can go to the strategic form representation. For example:

Player 1/Player 2	(HH, HT)	(HH, TT)	(TH, TT)	(TH, HT)
heads	$(-1, 1)$	$(-1, 1)$	$(1, -1)$	$(1, -1)$
tails	$(1, -1)$	$(-1, 1)$	$(-1, 1)$	$(1, -1)$

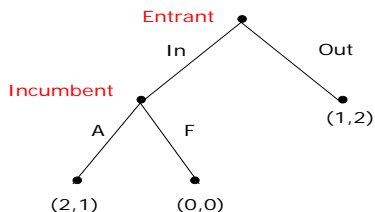
- So what will happen in this game?

Strategies in Extensive Form Games (continued)

- Can we go from strategic form representation to an extensive form representation as well?
- To do this, we need to introduce information sets. If two nodes are in the same information set, then the player making a decision at that point cannot tell them apart. The following two extensive form games are representations of the simultaneous-move matching pennies. These are imperfect information games.
- Note: For consistency, first number is still player 1's payoff.



Entry Deterrence Game



- Equivalent strategic form representation

Entrant \ Incumbent	Accommodate	Fight
In	(2, 1)	(0, 0)
Out	(1, 2)	(1, 2)

- Two pure Nash equilibria: (In,A) and (Out,F).

Are These Equilibria Reasonable?

- The equilibrium (Out,F) is sustained by a **noncredible threat** of the monopolist
- Equilibrium notion for extensive form games: **Subgame Perfect (Nash) Equilibrium**
- It requires each player's strategy to be "optimal" not only at the start of the game, but also after every history
- For finite horizon games, found by *backward induction*
- For infinite horizon games, characterization in terms of **one-stage deviation principle**.

Subgames

- Recall that a game G is represented by a game tree. Denote the set of nodes of G by V_G .
- A game has **perfect information** if all its information sets are singletons (i.e., all nodes are in their own information set).
- Recall that history h^k denotes the play of a game after k stages. In a perfect information game, each node $v \in V_G$ corresponds to a unique history h^k and vice versa. This is not necessarily the case in imperfect or incomplete information games.
- We say that an information set (consisting of a set of nodes) $X \in V_G$ is a successor of node y , or $X \succ y$, if in the game tree we can reach information set X through y .

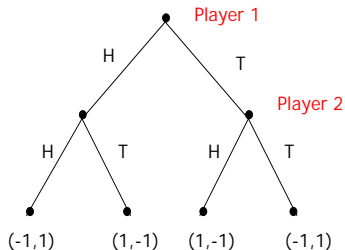
Subgames (continued)

Definition

(Subgames) A subgame G' of game G is given by the set of nodes $V_G^x \subset V_G$ in the game tree of G that are successors of some node $x \in V_G^x$ and are not successors of any node $z \notin V_G^x$; i.e., for all $y \in V_G^x$, there exists an information set (possibly singleton) Y such that $y \in Y$ and $Y \succ x$ and there does not exist $z \notin V_G^x$ such that $Y \succ z$.

- A restriction of a strategy s subgame G' , $s|_{G'}$ is the action profile implied by s in the subgame G' .

Subgames: Examples



- Recall the two-stage extensive-form version of the matching pennies game
- In this game, there are two proper subgames and the game itself which is also a subgame, and thus a total of three subgames.

Subgame Perfect Equilibrium

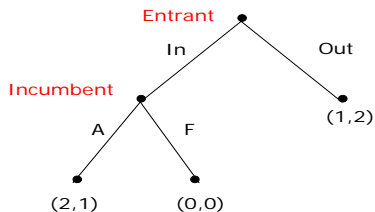
Definition

(Subgame Perfect Equilibrium) A strategy profile s^* is a Subgame Perfect Nash equilibrium (SPE) in game G if for any subgame G' of G , $s^*|_{G'}$ is Nash equilibrium of G' .

- Loosely speaking, subgame perfection will remove noncredible threats, since these will not be Nash equilibria in the appropriate subgames.
- In the entry deterrence game, following entry, F is not a best response, and thus not a Nash equilibrium of the corresponding subgame. Therefore, (Out,F) is not a SPE.
- How to find SPE? One could find all of the Nash equilibria, for example as in the entry deterrence game, then eliminate those that are not subgame perfect.
- But there are more economical ways of doing it.

Backward Induction

- **Backward induction** refers to starting from the last subgames of a finite game, then finding the Nash equilibria or best response strategy profiles in the subgames, then assigning these strategies profiles to be subgames, and moving successively towards the beginning of the game.



Backward Induction (continued)

Theorem

Backward induction gives the entire set of SPE.

Proof: backward induction makes sure that in the restriction of the strategy profile in question to any subgame is a Nash equilibrium.

Existence of Subgame Perfect Equilibria

Theorem

Every finite perfect information extensive form game G has a pure strategy SPE.

Proof: Start from the end by backward induction and at each step one strategy is best response.

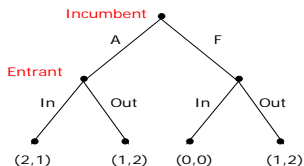
Theorem

Every finite extensive form game G has a SPE.

Proof: Same argument as the previous theorem, except that some subgames need not have perfect information and may have mixed strategy equilibria.

Examples: Value of Commitment

- Consider the entry deterrence game, but with a different timing as shown in the next figure.



- Note: For consistency, first number is still the entrant's payoff.
- This implies that the incumbent can now commit to fighting (how could it do that?).
- It is straightforward to see that the unique SPE now involves the incumbent committing to fighting and the entrant not entering.
- This illustrates the **value of commitment**.

Examples: Stackleberg Model of Competition

- Consider a variant of the Cournot model where player 1 chooses its quantity q_1 first, and player 2 chooses its quantity q_2 after observing q_1 . Here, player 1 is the Stackleberg leader.
- Suppose again that both firms have marginal cost c and the inverse demand function is given by $P(Q) = \alpha - \beta Q$, where $Q = q_1 + q_2$, where $\alpha > c$.
- This is a dynamic game, so we should look for SPE. How to do this?
- **Backward induction**—this is not a finite game, but all we have seen so far applies to infinite games as well.
- Look at a subgame indexed by player 1 quantity choice, q_1 . Then player 2's maximization problem is essentially the same as before

$$\begin{aligned}\max_{q_2 \geq 0} \pi_2(q_1, q_2) &= [P(Q) - c] q_2 \\ &= [\alpha - \beta(q_1 + q_2) - c] q_2.\end{aligned}$$

Stackleberg Competition (continued)

- This gives best response

$$q_2 = \frac{\alpha - c - \beta q_1}{2\beta}.$$

- Now the difference is that player 1 will choose q_1 recognizing that player 2 will respond with the above best response function.
- Player 1 is the Stackleberg leader and player 2 is the follower.

Stackleberg Competition (continued)

- This means player 1's problem is

$$\begin{aligned} \text{maximize}_{q_1 \geq 0} \quad & \pi_1(q_1, q_2) = [P(Q) - c] q_1 \\ \text{subject to} \quad & q_2 = \frac{\alpha - c - \beta q_1}{2\beta}. \end{aligned}$$

- Or

$$\max_{q_1 \geq 0} \left[\alpha - \beta \left(q_1 + \frac{\alpha - c - \beta q_1}{2\beta} \right) - c \right] q_1.$$

Stackleberg Competition (continued)

- The first-order condition is

$$\left[\alpha - \beta \left(q_1 + \frac{\alpha - c - \beta q_1}{2\beta} \right) - c \right] - \frac{\beta}{2} q_1 = 0,$$

which gives

$$q_1^S = \frac{\alpha - c}{2\beta}.$$

- And thus

$$q_2^S = \frac{\alpha - c}{4\beta} < q_1^S$$

- Why lower output for the follower?
- Total output is

$$Q^S = q_1^S + q_2^S = \frac{3(\alpha - c)}{4\beta},$$

which is greater than Cournot output. Why?