Informational Robustness and Solution Concepts

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Why Informational Robustness?

- we do not know a lot about the information that economic agents have
- we would like to do economic analysis that is not too sensitive to information that they have
- informational robustness closely related to solution concepts in game theory
- this lecture will discuss this connection
- my next lecture (and also ben's earlier lectures) will use (and did use) this connection in mechanism design

Complete Information

- n players
- A *game* $\mathcal{G}$ specifies for each player i...
  - a finite set of actions A_i
  - a utility function $u_i : A \rightarrow \mathbb{R}$ where $A = A_1 \times ... \times A_n$

Solution Concept: Correlated Equilibrium

A correlated equilibrium is a joint distribution over actions $\sigma \in \Delta(A)$ such that a player knowing only his action recommendation has no incentive to deviate, i.e.,

$$\begin{aligned} & \sum_{a_{-i}} \sigma(a_{-i}|a_i)\, u_i(a_i, a_{-i}) \\ \geq \; & \sum_{a_{-i}} \sigma(a_{-i}|a_i)\, u_i(a_i', a_{-i}) \end{aligned}$$

for each i, a_i and a_i'.

Solution Concept: Rationalizability

- An action is (correlated) rationalizable if it survives iterative deletion of never best responses
- Iterative Construction:
  - Let $R_i^0 = A_i$
  - Let R_i^{k+1} be the set of actions such that there exists $\nu_i \in \Delta\left(R_{-i}^k\right)$ such that

$$a_i \in \arg\max_{a_i'} \sum_{a_{-i}} \nu_i\left(a_{-i}\right) u_i\left(a_i', a_{-i}\right)$$

  - Let $R_i^\infty = \cap_{k \geq 0} R_i^k$
- Action a_i is rationalizable if $a_i \in R_i^\infty$

Adding Correlating Device (Payoff Irrelevant Information)

- An expansion of the game specifies for each player i...
  - a finite set of possible signals S_i
  - a belief $\phi_i \in \Delta(S)$
  - maintained full support assumption: for all s_i,

$$\phi_i(s_i) \equiv \sum_{s_{-i}} \phi_i(s_i, s_{-i}) > 0$$

- The expanded game is a game of incomplete information
- It is a common prior expansion if ϕ_i is the same for all players

Equilibrium of the Expanded Game

- A pure strategy is a mapping $\beta_i : S_i \to A_i$
- A pure strategy profile $\beta = (\beta_i)_{i=1}^n$ is a *Bayes Nash equilibrium* of the expanded game if

$$\begin{aligned} & \sum_{s_{-i}} \phi_i (s_{-i}|s_i)\, u_i \left(\beta_i (s_i), \beta_{-i} (s_{-i})\right) \\ \geq & \sum_{s_{-i}} \phi_i (s_{-i}|s_i)\, u_i \left(a_i, \beta_{-i} (s_{-i})\right) \end{aligned}$$

for each i, s_i and a_i

Informational Robustness Foundation of Correlated Equilibrium

A distribution $\sigma \in \Delta(A)$ is a correlated equilibrium if and only if there is a common prior expansion $((S_i)_{i=1}^{n}, \phi)$ and a Bayes Nash equilibrium β of the expanded game that induces σ, i.e.,

$$\sigma(a) = \sum_{\{s:\beta(s)=a\}} \phi(s)$$

Correlated Equilibrium: Proof

- Suppose that $\sigma \in \Delta(A)$ is a correlated equilibrium. Consider the common prior expansion with $S_i = A_i$ for each i and $\phi = \sigma$. Consider the strategy profile with $\beta_i(s_i) = s_i$. The latter is an equilibrium and induces σ
- Consider an expansion $((S_i)_{i=1}^n, \phi)$ and a Bayes Nash equilibrium β of the expanded game that induces σ.
- We have ex ante statement of equilibrium conditions:

$$\begin{aligned} & \sum_s \phi(s_i)\, \phi(s_{-i}|s_i)\, u_i\left(\beta_i(s_i), \beta_{-i}(s_{-i})\right) \\ \geq\ & \sum_s \phi(s_i)\, \phi(s_{-i}|s_i)\, u_i\left(\beta_i'(s_i), \beta_{-i}(s_{-i})\right) \end{aligned}$$

for each i and β_i'.

Correlated Equilibrium: Proof

But

$$
\begin{aligned}
& \sum_{s} \phi(s_i)\,\phi(s_{-i}|s_i)\,u_i\left(\beta_i(s_i), \beta_{-i}(s_{-i})\right) \\
\geq\ & \sum_{a} u_i(a_i, a_{-i}) \sum_{\{s:\beta(s)=a\}} \phi(s_i)\,\phi(s_{-i}|s_i) \\
=\ & \sum_{a} u_i(a_i, a_{-i})\,\sigma(a_i, a_{-i})
\end{aligned}
$$

where

$$
\sigma(a_i, a_{-i}) = \sum_{\{s:\beta(s)=a\}} \phi(s_i)\,\phi(s_{-i}|s_i)
$$

Correlated Equilibrium: Proof

But now

$$\begin{aligned} & \sum_{a} \sigma\left(a_i\right) \sigma\left(a_{-i} | a_i\right) u_i\left(a_i, a_{-i}\right) \\ \geq \; & \sum_{a} \sigma\left(a_i\right) \sigma\left(a_{-i} | a_i\right) u_i\left(\gamma\left(a_i\right), a_{-i}\right) \end{aligned}$$

for each i and $\gamma : A_i \rightarrow A_i$; so

$$\begin{aligned} & \sum_{a_{-i}} \sigma\left(a_{-i} | a_i\right) u_i\left(a_i, a_{-i}\right) \\ \geq \; & \sum_{a_{-i}} \sigma\left(a_{-i} | a_i\right) u_i\left(a_i', a_{-i}\right) \end{aligned}$$

for each i, a_i and a_i'.

Informational Robustness Foundation of Rationalizability

An action a_i is rationalizable if and only if there is an expansion $\left((S_i, \phi_i)_{i=1}^n\right)$ and a Bayes Nash equilibrium β of the expanded game such that

$$\beta_i(s_i) = a_i$$

for some s_i.

Rationalizability: Proof

- Suppose that action a_i is rationalizable for player i. Consider the expansion with $S_j = R_j^\infty$ for each j. Let $\phi_j(a_{-j}|a_j)$ be any belief that rationalizes action a_j. Consider the strategy profile with $\beta_j(a_j) = a_j$ for all j. The latter is an equilibrium and

$$\beta_i(a_i) = a_i$$

- Consider an expansion $\left(S_j, \phi_j\right)_{j=1}^n$ and a Bayes Nash equilibrium β of the expanded game with $a_i = \beta_i(s_i)$ for some i and s_i.
- Let $\widehat{A}_j$ be the range of β_j, i.e.,

$$\widehat{A}_j = \left\{a_j | \beta_j(s_j) = a_j \text{ for some } s_j \in S_j\right\}$$

- Each $a_j \in \widehat{A}_j$ is rationalized by some belief over $\widehat{A}_{-j}$.
- So $\widehat{A}_j \subseteq R_j^\infty$ for each j and so $a_i \in R_i^\infty$

First Change: Add Uncertainty

- n players and finite states Θ
- A *game* $\mathcal{G}$ specifies for each player i...
  - a finite set of actions A_i
  - a utility function $u_i : A \times \Theta \rightarrow \mathbb{R}$ where $A = A_1 \times ... \times A_n$

Second Change: Informed Players

add information about Θ:

- type space consists for each player i of
  - a finite set of types T_i
  - a belief $\pi_i : T_i \rightarrow \Delta(T_{-i} \times \Theta)$
- common prior: interim beliefs generated from common prior $\pi^* \in \Delta(T \times \Theta)$

Adding Information

- An expansion of the game specifies for each player i...
  - a finite set of possible signals S_i
  - a belief $\phi_i : T \times \Theta \to \Delta(S)$
  - maintained full support condition: each player j assigns positive probability only to signals of player i that player i assigns positive probability to
- The expanded game is a game of incomplete information (with strategies depending on types and signals)

Support Condition

Let

$$\overline{S}_i(t_i) = \left\{ s_i \,\middle|\, \sum_{s_{-i}, t_{-i}, \theta} \phi_i(s|t,\theta)\, \pi_i(t_{-i},\theta|t_i) \right\}$$

Now $\phi_i(s|t,\theta) = 0$ if $s_i \notin \overline{S}_i(t_i)$

Solution Concept: Belief-Free Rationalizability in Words

- Iteratively define kth level rationalizable actions for each type
- An action is $(k+1)$th level rationalizable if it is a best response to a conjecture assigning zero probability to...
  - (action + type) pairs of each other player that have been deleted
  - (state + other players' types) profiles that are assigned probability zero by that type on the original type space
- depends only on support of beliefs in type space
- intuition: signals cannot make a player assign (state + other players' types) profiles that are assigned probability zero on the type space; otherwise, there are no restrictions on how beliefs can be changed

Solution Concept: Belief-Free Rationalizability Formal Definition

- Iterative Construction:
  - Let $BFR_i^0(t_i) = A_i$
  - Let $BFR_i^{k+1}(t_i)$ be the set of actions such that there exists $\nu_i \in \Delta(A_{-i} \times T_{-i} \times \Theta)$ such that

$$(1)\ \nu_i(a_{-i}, t_{-i}, \theta) > 0 \Rightarrow a_j \in BFR_j^k(t_j) \text{ for each } j \neq i$$

$$(2)\ \sum_{a_{-i}} \nu_i(a_{-i}, t_{-i}, \theta) > 0 \Rightarrow \pi_i(t_{-i}, \theta | t_i) > 0$$

$$(3)\ a_i \in \arg\max_{a_i'} \sum_{a_{-i}} \nu_i(a_{-i}, t_{-i}, \theta)\, u_i\left(\left(a_i', a_{-i}\right), \theta\right)$$

  - Let $BFR_i^\infty(t_i) = \cap_{k \geq 0} BFR_i^k(t_i)$

Informational Robustness Foundation of Belief-Free Rationalizability

An action a_i is belief-free rationalizable for t_i if and only if there is an expansion $\left(S_j, \phi_j\right)_{j=1}^{n}$ and a Bayes Nash equilibrium β of the expanded game such that

$$\beta_i\left(t_i, s_i\right) = a_i$$

for some s_i.

Idea of Constructing the Expansion

- If a_i is belief-free rationalizable for t_i, we can find a conjecture rationalizing the choice of a_i from the definition of belief-free rationalizability (property 3: best response)
- we can construct a signal space S_i where each (t_i, s_i) will play a belief-free rationalizable action for t_i (property 1: support on actions)
- because of the support condition, we can construct a signal generating that conjecture by property (2: support on (t_{-i}, θ))

Additional Important Assumptions

1. Impose the common prior assumption
2. Require the expansion to be payoff-irrelevant (i.e., a correlating device)
3. Impose "payoff type environment" on the type space

Solution Concept: Bayes Correlated Equilibrium

- A Bayes correlated equilibrium (of a common prior game) is a decision rule $\sigma : T \times \Theta \to \Delta(A)$ such that a player knowing only his type and recommended action has no incentive to deviate, i.e.,

$$\sum_{a,t,\theta} u_i\left(\left(a_i, a_{-i}\right), \theta\right) \sigma\left(a|t,\theta\right) \pi^*\left(t,\theta\right)$$
$$\geq \sum_{a,t,\theta} u_i\left(\left(\gamma_i\left(a_i\right), a_{-i}\right), \theta\right) \sigma\left(a|t,\theta\right) \pi^*\left(t,\theta\right)$$

  for each i and $\gamma_i : A_i \to A_i$.
- A decision rule $\sigma : T \times \Theta \to \Delta(A)$ is a Bayes correlated equilibrium if and only if there is a common prior expansion $\left(\left(S_i\right)_{i=1}^n, \phi\right)$ and a Bayes Nash equilibrium β of the expanded game that induces σ, i.e.,

$$\sigma\left(a|t,\theta\right) = \sum_{\{s:\beta(t,s)=a\}} \phi\left(s|t,\theta\right)$$

Correlating Devices and Belief Invariant Information

- Expansion is belief-invariant if

$$\sum_{s_{-i}} \phi_i \left((s_i, s_{-i}) \mid (t_i, t_{-i}), \theta \right)$$

  is independent of (t_{-i}, θ);
- from player i's point of view, it is noise; but allows correlation
- incomplete information version of a correlation device

Solution Concept: Interim Correlated Rationalizability

- Interim Correlated Rationalizability: Iteratively delete actions for a type that cannot be rationalized by a conjecture that (1) puts zero probability on already deleted actions and (2) is consistent with that type's beliefs on the type space
- Iterative Construction:
  - Let $ICR_i^0\left(t_i\right) = A_i$
  - Let $ICR_i^{k+1}\left(t_i\right)$ be the set of actions such that there exists $\nu_i \in \Delta\left(T_{-i} \times A_{-i} \times \Theta\right)$ such that

$$(1).\ \nu_i\left(a_{-i}, t_{-i}, \theta\right) > 0 \Rightarrow a_j \in BFR_j^k\left(t_j\right) \text{ for each } j \neq i$$

$$(2)\ \sum_{a_{-i}} \nu_i\left(a_{-i}, t_{-i}, \theta\right) = \pi_i\left(t_{-i}, \theta | t_i\right)$$

$$(3)\ a_i \in \arg\max_{a_i'} \sum_{a_{-i}, t_{-i}, \theta} \nu_i\left(t_{-i}, a_{-i}, \theta\right) u_i\left(\left(a_i', a_{-i}\right), \theta\right)$$

  - Let $ICR_i^{\infty}\left(t_i\right) = \cap_{k \geq 0} ICR_i^k\left(t_i\right)$

Correlating Devices and Belief Invariant Information

- Expansion is belief-invariant if

$$\sum_{s_{-i}} \phi_i \left((s_i, s_{-i}) \mid (t_i, t_{-i}), \theta \right)$$

  is independent of (t_{-i}, θ);
- from player i's point of view, it is noise; but allows correlation
- incomplete information version of a correlation device

Informational Robustness Foundation of Interim Correlated Rationalizability

An action a_i is interim correlated rationalizable at t_i if and only if there is a belief invariant expansion $(S_i, \phi_i)_{i=1}^n$ and a Bayes Nash equilibrium β of the expanded game such that

$$\beta_i(t_i, s_i) = a_i$$

for some (t_i, s_i).

No Learning from Actions

An implication of belief-invariance is that action a_i will add no information to t_i about (t_{-i}, θ), i.e.,

$$\Pr(t_{-i}, \theta | a_i, t_i) = \pi_i (t_{-i}, \theta | t_i)$$

Taxonomy

	BI	not BI
CPA	BI bayes correlated equilibrium	bayes correlated equilibrium
not CPA	interim correlated rationalizability	belief free rationalizability

Binary Action Examples

1. Trade:
   - BCE = BIBCE = no trade
   - no trade under BFE and ICR if and only if there is not common possibility of no gains from trade
2. Coordination: BIBCE = ICR by supermodularity

Payoff Type Space

- Assume $\Theta = \Theta_1 \times .. \times \Theta_n$
- Each player i....
  - knows his "payoff type" $\theta_i \in \Theta_i$
  - knows nothing else
  - has full support on others' payoff types

Solution Concept: Belief-Free Rationalizability

- Belief-free rationalizability: Iteratively delete actions for a payoff type that cannot be rationalized by a conjecture that puts zero probability on already deleted actions for any payoff type of others
- Iterative Construction:
  - Let $BFR_i^0(\theta_i) = A_i$
  - Let $BFR_i^{k+1}(\theta_i)$ be the set of actions such that there exists $\nu_i \in \Delta(A_{-i} \times \Theta_{-i})$ such that

$$
\begin{aligned}
(1)\ \nu_i(a_{-i}, \theta_{-i}) &> 0 \Rightarrow a_j \in BFR_j^k(\theta_j) \text{ for each } j \neq i \\
(2).\ a_i &\in \arg\max_{a_i'} \sum_{a_{-i},\theta} \nu_i(a_{-i}, \theta)\, u_i((a_i', a_{-i}), \theta)
\end{aligned}
$$

  - Let $BFR_i^\infty(\theta_i) = \cap_{k \geq 0} BFR_i^k(\theta_i)$

Linear Best Response Example

- Players have payoff types $[0, 1]$
- Players have actions $[0, 1]$
- Let each player have best response:

$$a_i = \theta_i - \gamma \mathbb{E}\left(\sum_{j=1}^{n} (a_j - \theta_j)\right)$$

(subject to $a_i \in [0, 1]$)

- An example of a game with this best response is a common interest game

$$v(a, \theta) = -\sum_{j=1}^{n} (a_j - \theta_j)^2 - \gamma \sum_{j=1}^{n} (a_j - \theta_j) \sum_{k \neq j} (a_k - \theta_k)$$

Belief-Free Rationalizability in the Linear Best Response Example

- If player i thought that all his opponents were going to choose actions within c of their payoff types, then he would have an incentive to choose an action within $|\gamma|(n-1)c$ of his payoff type
- We have $BFR_i^0(\theta_i) = [0,1]$; by induction we have

$$\begin{aligned} & BFR_i^k(\theta_i) \\ = & \left[\min\left\{0, \theta_i - \left[|\gamma|(n-1)\right]^k\right\}, \max\left\{1, \theta_i + \left[|\gamma|(n-1)\right]^k\right\}\right] \end{aligned}$$

Belief-Free Rationalizability in the Linear Best Response Example

- So unique belief-free rationalizable action to set $a_i = \theta_i$ if $|\gamma| < \frac{1}{n-1}$
- Every action in $[0, 1]$ is belief-free rationalizable for every payoff type in $[0, 1]$ if $|\gamma| \geq \frac{1}{n-1}$.

Bayes Correlated Equilibrium in the Linear Best Response Example

- If $-\frac{1}{n-1} < \gamma < 1$, one can show that the "potential" function is strictly concave and there is a unique Bayes correlated equilibrium,
- If $\gamma \leq 0$, we have strategic complementarities...
  - unique Bayes correlated equilibrium if $|\gamma| < \frac{1}{n-1}$
  - Extremal Bayes correlated equilibria where all players choose 0, and where all players choose 1, if $|\gamma| \geq \frac{1}{n-1}$

Binary Action Examples

1. Trade:
   - BCE = BIBCE = no trade
   - no trade under BFE and ICR if and only if there is not common possibility of no gains from trade
2. Coordination: BIBCE = ICR by supermodularity

Belief-Free Rationalizability: Proof

- Suppose that action a_i is belief-free rationalizable for t_i.
- By definition of belief-free rationalizability, if a_j is belief-free rationalizable for t_j, there exists a conjecture $\nu_j^{a_j,t_j} \in \Delta\left(A_{-i} \times T_{-i} \times \Theta\right)$ such that

$$(1)\ \nu_j^{a_j,t_j}\left(a_{-j}, t_{-j}, \theta\right) > 0 \Rightarrow a_k \in BFR_k\left(t_k\right) \text{ for each } k \neq j$$

$$(2)\ \sum_{a_{-j}} \nu_j\left(a_{-j}, t_{-j}, \theta\right) > 0 \Rightarrow \pi_j\left(t_{-j}, \theta | t_j\right) > 0$$

$$(3)\ a_j \in \arg\max_{a_j'} \sum_{a_{-j}, t_{-j}, \theta} \nu_j\left(a_{-j}, t_{-j}, \theta\right) u_j\left(\left(a_j', a_{-j}\right), \theta\right)$$

Belief-Free Rationalizability: Proof

- Now consider expansion $\left(S_j, \phi_j\right)_{j=1}^{n}$ with $S_j = A$Suppose that action a_i is belief-free rationalizable for t_i.
- By definition of belief-free rationalizability, if a_j is belief-free rationalizable for t_j, there exists a conjecture $\nu_j^{a_j, t_j} \in \Delta\left(A_{-i} \times T_{-i} \times \Theta\right)$ such that

$$(1)\ \nu_j^{a_j, t_j}\left(a_{-j}, t_{-j}, \theta\right) > 0 \Rightarrow a_k \in BFR_k\left(t_k\right) \text{ for each } k \neq j$$

$$(2)\ \sum_{a_{-j}} \nu_j\left(a_{-j}, t_{-j}, \theta\right) > 0 \Rightarrow \pi_j\left(t_{-j}, \theta | t_j\right) > 0$$

$$(3)\ a_j \in \arg\max_{a_j'} \sum_{a_{-j}} \nu_j\left(a_{-j}, t_{-j}, \theta\right) u_j\left(\left(a_j', a_{-j}\right), \theta\right)$$

- Consider the expansion with

$$S_j = \left\{\left(t_j, a_j\right) : a_j \in BFR_j^k\left(t_j\right)\right\}.$$

Let $\nu_i\left(s_{-i}, \theta | a_i\right)$ be a belief under which a_i is optimal , that hthe belief that rationalizes action a_i. Consider the strategy profile with $\beta_i\left(s_i\right) = s_i$. The latter is an equilibrium and