Implementation via Information Design in Binary Action Supermodular Games

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Partial Implementation by Information Design

- Fix a game. What outcomes (joint distributions over action profiles and states) can be induced by picking an information structure and an equilibrium?

- Enough to offer players action recommendations that players have an incentive to follow ("obedience constraints")

- The set of obedient outcomes corresponds to an (incomplete information version of) correlated equilibria of the original game; "Bayes correlated equilibria" or BCE)

- This is the many player generalization of "Bayesian persuasion," much explored in recent years

- Will review shortly if you are not familiar with this stuff.

Question 1: Full Implementation

- What outcomes can be induced if you can pick the information structure but not the equilibrium?
- We will give a complete characterization of full implementation by information design for binary action supermodular (BAS) games, closely analogous to that for partial implementation.

Question 2: Smallest Equilibrium Implementation

- What outcomes can be induced if you can pick the information structure and expect the smallest equilibrium to be played?
- Well posed in BAS games
- Easier question to pose and answer than full implementation
- Full implementation result then follows easily
- Smallest equilibrium implementation is more relevant for applications

Main Result

Under a dominant state property, an outcome can be smallest equilibrium implemented if and only if it satisfies not only obedience but also *sequential obedience*

- "Sequential obedience" =
  - designer recommends players to switch to high action according to a randomly chosen sequence
  - a player has a strict incentive to switch when told to do so even if he thinks only players before him in the sequence have switched (without knowing the true state or realized sequence)

- Implementible outcomes characterized by a finite linear program

- Full implementation requires in addition a reverse sequential obedience condition, where designer recommends switches from high action to low action.

Two Applications in Paper

1. (will briefly summarize today) Information Design with Adversarial Equilibrium Selection

   ▶ If the game is a convex potential game (meaning not too much heterogeneity....) and the designer has monotonic and partially convex preferences

     ▶ the optimal outcome satisfies the *perfect coordination property*: the optimal policy always has either all players choosing the high action or all players choosing the low action.
     ▶ the optimal outcome has all choosing the high action on the highest probability event where the high action profile is potential maximizing

2. (see paper) Adding Bonuses

Literature: Higher Order Beliefs Literature and The Main Result

- ▶ Arguments in Rubinstein (1989) and Carlsson and van Damme (1993) show that you can fully implement the risk dominant equilibrium of a two player BAS game by choose of information structure (and a small perturbation of payoffs)

- ▶ Kajii and Morris (1997) showed (among other things) that you cannot fully implement the risk dominated equilibrium.

- ▶ literature on "robustness to incomplete information" follows (in parallel with closely related global games literature)....

- ▶ Oyama and Takahashi (2020) characterize which equilibria you can fully implement in general BAS games.

- ▶ Our main result extends those arguments and results beyond complete information games.

Literature: Applications

1. Information Design with Adversarial Equilibrium Selection
   - Inostroza and Pavan (2020) show you can restrict attention to perfect coordination outcomes in regime change games
   - Problem discussed in Mathevet, Perego and Taneva (2019), who solve a two player two action example
   - Li, Song and Zhou (2020) contemporaneously with us solve regime change game problem
   - Relative to these papers:
     - We provide a result that unifies and generalizes known results, with a simple characterization and intuition for optimal outcome
     - We show perfect coordination property holds even with asymmetric payoffs
     - MPT and LSZ implement with simpler information structures tailored to examples; we provide canonical method that works for all BAS games

2. Adding bonuses
   - Winter (2004), Halac, Lipnowski and Rappoport (last week's seminar!)
   - We illustrate applicability of our results by extending

Comment on Higher Order Beliefs and Rank Beliefs

- ▶ IP, MPT and others have emphasized the importance of higher-order beliefs and expressed results in terms of the universal type space.
- ▶ But they (like us) impose the common prior assumption and the implications of the common prior assumption on higher order beliefs are not very well understood / easy to state.
- ▶ We build on work in the "higher-order beliefs" literature where belief operators (Monderer and Samet (1989)) have been used to characterize relevant properties of higher order beliefs (Kajii and Morris (1997)).
- ▶ Morris and Shin (2007) and Morris, Shin and Yildiz (2017) defined generalized belief operators and highlighted the importance of rank beliefs, and these generalizations play central role in Oyama and Takahashi (2020).
- ▶ In this talk and paper, we do not express results in terms of higher order beliefs (via belief operators or any other way) although we could. It is a pedagogical choice.

Binary-Action Supermodular (BAS) Games

- $I = \{1, \ldots, |I|\}$: the set of players.
- Θ: a finite set of states.
- $\mu \in \Delta(\Theta)$: a common prior.
    - Without loss of generality, we assume $\mu(\theta) > 0$ for any θ.
- $A_i = \{0, 1\}$: the binary-action set for player i.
    - $A = \{0, 1\}^I$.
- $u_i : A \times \Theta \to \mathbb{R}$: player i's payoff, supermodular so that

$$d_i(a_{-i}, \theta) = u_i(1, a_{-i}, \theta) - u_i(0, a_{-i}, \theta)$$

  is increasing in a_{-i}
- Dominant state assumption: there exists $\overline{\theta} \in \Theta$ such that $d_i(\mathbf{0}_{-i}, \overline{\theta}) > 0$ for all i.

Information

- T_i: a countable set of signals for player i.
- $T = \prod_{i \in I} T_i$.
- $\pi \in \Delta(T \times \Theta)$: a common prior.
- Without loss of generality, we assume $\pi(\{t_i\} \times T_{-i}) > 0$ for any t_i.
- Given $\mathcal{T} = (T, \pi)$, the notion of Bayes Nash equilibrium $\sigma = (\sigma_i)_{i \in I}$, $\sigma_i \colon T_i \to \Delta(A_i)$, is defined as usual.
- Information structure $\mathcal{T}$ and strategy profile σ induce an outcome:
$$\nu(a, \theta) = \sum_t \pi(t, \theta) \prod_{i \in I} \sigma_i(t_i)(a_i).$$

Definition. Outcome ν is **partially implementable** if there exists an information structure $\mathcal{T}$ and an equilibrium σ such that $(\mathcal{T}, \sigma)$ induces ν.

Definition. Outcome ν satisfies **consistency** if

$$\nu(A \times \{\theta\}) = \mu(\theta)$$

Definition. Outcome ν satisfies **obedience** if

$$\sum_{a_{-i} \in A_{-i}, \theta \in \Theta} \nu(a_i, a_{-i}, \theta)(u_i(a_i, a_{-i}, \theta) - u_i(a_i', a_{-i}, \theta)) \geq 0$$

for any $i \in I$ and $a_i, a_i' \in A_i$.

Proposition. Outcome ν is partially implementable if and only if it satisfies consistency and obedience.

- ▶ Bergemann and Morris (2016) called the set of such outcomes Bayes correlated equilibria.

Smallest Equilibrium Implementation and Full Implementation

- Outcome ν is **fully implementable** if there exists an information structure $\mathcal{T}$ such that $(\mathcal{T}, \sigma)$ induces ν for all Bayes Nash equilibria of $\mathcal{T}$.
- Because game is BAS, there is a smallest (pure strategy) Bayes Nash equilibrium, $\underline{\sigma}$
- Outcome ν is **smallest equilibrium implementable** if there exists an information structure $\mathcal{T}$ such that $(\mathcal{T}, \underline{\sigma})$ induces ν.
- We will call the set of such outcomes "smallest equilibrium implementable outcomes" *SI*.

Two Player Two State Example

Payoffs

g	Invest	Not	**b**	Invest	Not
Invest	$4, 5$	$1, 0$	Invest	$-5, -4$	$-8, 0$
Not	$0, 2$	$0, 0$	Not	$0, -7$	$0, 0$

- ▶ Supermodular (payoff gain of 3 to investing if the other player invests)
- ▶ Both players have dominant strategy to invest in good state and not invest in bad state
- ▶ Asymmetric: Column player 2 gets higher payoff $(+1)$ from investing relative to row player 1

Partial Implementation

Payoffs

g	Invest	Not	**b**	Invest	Not
Invest	4, 5	1, 0	Invest	−5, −4	−8, 0
Not	0, 2	0, 0	Not	0, −7	0, 0

▶ The following outcome is partially implementable (and is the "best" partially implementable outcome)

g	Invest	Not	**b**	Invest	Not
Invest	$\frac{1}{2}$	0	Invest	$\frac{2}{5}$	0
Not	0	0	Not	$\frac{1}{10}$	0

▶ But there is a strict equilibrium of the direct implementation where both players never invest

▶ Nothing close to this outcome is fully implementable

Full Implementation

Payoffs

g	Invest	Not	**b**	Invest	Not
Invest	$4, 5$	$1, 0$	Invest	$-5, -4$	$-8, 0$
Not	$0, 2$	$0, 0$	Not	$0, -7$	$0, 0$

- The following outcome is fully implementable for every $\eta > 0$ (and the $\eta \to 0$ limit is the supremum)

g	Invest	Not	**b**	Invest	Not
Invest	$\frac{1}{2}$	0	Invest	$\frac{1}{4} - \eta$	0
Not	0	0	Not	0	$\frac{1}{4} + \eta$

- Perfect Coordination Outcome is optimal in asymmetric game

Explanation 1: Risk Dominance

► Complete information game conditional on both being told to invest (and $\eta = 0$)

Complete Information Payoffs

	Invest	Not
Invest	$1, 2$	$-2, 0$
Not	$0, -1$	$0, 0$

► "Invest, Invest" is (just) risk dominant (Harsanyi and Selten (1988)), so we can fully implement it following the logic of Rubinstein (1989) and Carlsson and van Damme (1993).

► Kajii and Morris (1997) show you cannot fully implement a higher probability of investment in this way because the (invest,invest) would not be risk dominant in the induced complete information game

Explanation 2: Sequential Obedience

Consider an "ordered outcome" which is a probability distribution over a state $\theta \in \{\mathbf{g}, \mathbf{b}\}$ and a sequence $\gamma \in \{\{12\}, \{21\}, \varnothing\}$

ν_γ	$\{21\}$	$\{12\}$	$\varnothing$
$\mathbf{g}$	$\frac{1}{3}$	$\frac{1}{6}$	0
$\mathbf{b}$	$\frac{1-\delta}{6}$	$\frac{1-\delta}{12}$ —	$\frac{1+\delta}{4}$

- this ordered outcome establishes sequential obedience:
  - player 1 assigns probability $\frac{2}{3}$ to player 2 having invested and independent probability $> \frac{2}{3}$ to the state being good
  - player 2 assigns probability $\frac{1}{3}$ to player 1 having invested and independent probability $> \frac{2}{3}$ to the state being good

Explanation 3: An Email Game Implementation

- Draw an integer $m \in \{0, 1, 2, ...\}$ almost uniformly, i.e., with probability $\eta (1 - \eta)^m$
- Draw (γ, θ) according to

ν_γ	$\{21\}$	$\{12\}$	$\varnothing$
g	$\frac{1}{3}$	$\frac{1}{6}$	0
b	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{4}$

- Each player observes a signal $t_i \in \{1, 2, 3, ...\} \cup \{\infty\}$; let

$$(t_1, t_2) = \begin{cases} (\infty, \infty), \text{ if } \gamma = \varnothing \\ (m + 1, m + 2), \text{ if } \gamma = \{12\} \\ \{m + 2, m + 1\}, \text{ if } \gamma = \{21\} \end{cases}$$

unless $m = 0$ and $\theta = $**b**, in which case $(t_1, t_2) = (\infty, \infty)$

Explanation 3: An Email Game Implementation

- If a player observes signal $t_i = 1$, he knows the state is good and has a dominant strategy to invest
- Knowing this, players observing $t_i = 2$ will have an incentive to invest (messy calculation, trust me)
- If player 1 observes signal $t_1 \geq 3$, he assigns probability $\frac{2}{3-\eta} > \frac{2}{3}$ to player 2 having a lower signal and independent probability $\frac{2}{3}$ to the state being good
- If player 2 observes signal $t_2 \geq 3$, she assigns probability $\frac{1}{3-2\eta} > \frac{1}{3}$ to player 1 having a lower signal and independent probability $> \frac{2}{3}$ to the state being good
- by induction, invest is only rationalizable action for $t_i \neq \infty$

Explanation 3: An Email Game Implementation

Induced outcome is

g	Invest	Not	**b**	Invest	Not
Invest	$\frac{1}{2}$	0	Invest	$\frac{1}{4} - \frac{1}{4}\eta$	0
Not	0	0	Not	0	$\frac{1}{4} + \frac{1}{4}\eta$

Sequences

- We care about who eventually plays action 1 in the smallest BNE, but the characterization is based on a hypothetical order in which players change actions under an iterative procedure.

- Let Γ be the set of all finite sequences of distinct players; for example, if $I = \{1, 2, 3\}$, then

  $$\Gamma = \{\emptyset, 1, 2, 3, 12, 13, 21, 23, 31, 32, 123, 132, 213, 231, 312, 321\}.$$

- An "ordered outcome" is distribution over sequences and states $\nu_{\Gamma} \in \Delta(\Gamma \times \Theta)$

Sequences

- For $\gamma \in \Gamma$, $\bar{a}(\gamma)$ denotes the action profile where player i plays action 1 iff player i appears in γ;

- Each "ordered outcome" $\nu_\Gamma \in \Delta(\Gamma \times \Theta)$ induces outcome $\nu \in \Delta(A \times \Theta)$ by forgetting the ordering, i.e.,

$$\nu(a, \theta) = \sum_{\gamma \in \Gamma : \bar{a}(\gamma) = a} \nu_\Gamma(\gamma, \theta).$$

- Let $\Gamma_i = \{\gamma \in \Gamma \mid \text{player } i \text{ appears in } \gamma\}$.

- For $\gamma \in \Gamma_i$, $a_{-i}(\gamma)$ denotes the action profile of player i's opponents where player j plays action 1 iff player j appears in γ before player i.

Sequential Obedience

Definition. Ordered outcome ν_Γ satisfies **sequential obedience** if

$$\sum_{\gamma \in \Gamma_i, \theta \in \Theta} \nu_\Gamma(\gamma, \theta) d_i(a_{-i}(\gamma), \theta) > 0$$

for all i such that Γ_i is non-empty. Ordered outcome ν_Γ satisfies **weak sequential obedience** if we replaces inequalities with equalities.

Definition. Outcome $\nu \in \Delta(A \times \Theta)$ satisfies (weak) sequential obedience if there exists ordered outcome $\nu_\Gamma \in \Delta(\Gamma \times \Theta)$ that induces ν and satisfies (weak) sequential obedience.

Definition. Outcome $\nu \in \Delta(A \times \Theta)$ satisfies upper dominance if $\nu\left(\mathbf{1}, \overline{\theta}\right) > 0$

Result

Theorem 1. If an outcome is smallest equilibrium implementable, then it satisfies consistency, obedience and sequential obedience. If an outcome satisfies consistency, obedience, sequential obedience and upper dominance, then ν is smallest equilibrium implementable.

Corollary 1. $\nu \in \overline{SI}$ if and only if it is satisfies consistency, obedience and weak sequential obedience.

Necessity of Sequential Obedience

- Suppose that ν is smallest equilibrium implementable
- Let $\mathcal{T} = (T, \pi)$ be a consistent information structure whose smallest equilibrium induces ν
- Starting from constant 0 strategies, iteratively apply myopic best responses (say, round robin by player)
- This process will converge to the smallest equilibrium
- For each type $t_i \in T_i$, if type t_i changes from action 0 to action 1 in the n-th step, we denote by $n_i(t_i) = n$; if he never changes, then we denote by $n_i(t_i) = \infty$.
- Define

$$\nu_\Gamma(\gamma, \theta) = \sum_{t \colon (n_i(t_i)) \text{ is ordered according to } \gamma} \pi(t, \theta)$$

- Because this process converges to smallest equilibrium, we know that ν_Γ induces ν

Necessity of Sequential Obedience

- To show sequential obedience, note that for each $t_i \in T_i$ with $n_i(t_i) < \infty$, we have

$$\sum_{t_{-i}, \theta} \pi((t_i, t_{-i}), \theta) d_i(a_{-i}(t), \theta) > 0,$$

where $a_{-i}(t)$ is the action profile of player i's opponents in the myopic best response process when i switches; so player j plays action 1 iff $n_j(t_j) < n_i(t_i)$.

- By adding up these inequalities over all such t_i, we have

$$\sum_{\gamma \in \Gamma_i, \theta} \nu_\Gamma(\gamma, \theta) d_i(a_{-i}(\gamma), \theta)$$
$$= \sum_{t_i : \, n_i(t_i) < \infty} \sum_{t_{-i}, \theta} \pi(t, \theta) d_i(a_{-i}(t), \theta)$$
$$> 0$$

for any $i \in I$ such that $\Gamma_i \neq \varnothing$.

Sufficiency of Sequential Obedience and Upper Dominance

- We construct information structure 1 as follows
- Let ν_Γ be an ordered outcome establishing sequential obedience
- Draw γ according to ν_Γ and draw integer m from $\mathbb{Z}_+$ with almost uniform probability $\eta (1 - \eta)^m$
- Let the type of player i be given by

$$t_i = \widetilde{t}_i (m, \gamma) = \begin{cases} m + \text{ranking of } i \text{ in } \gamma & \text{if } \gamma \in \Gamma_i, \\ \infty & \text{otherwise,} \end{cases}$$

Inductive Step

CLAIM: Suppose that for some $k \geq |I|$, we knew that all types $t_i \leq k$ choose action 1. Then types $k + 1$ of all players must choose action 1.

- Consider type $k + 1$ of player i. He will know that all players before him in the realized sequence γ are playing 1. As $\eta \to 0$, his belief over sequences will approximate the belief in the sequential obedience condition and his payoff to action 1 will approach

$$\sum_{\gamma \in \Gamma_i, \theta \in \Theta} \nu_\Gamma(\gamma, \theta) d_i(a_{-i}(\gamma), \theta) > 0$$

- Thus claim holds for sufficiently small η

Perturb Payoffs

- Now construct information structure 2 by re-arranging payoffs in information structure 1 so that types $1, ..., |I|$ have dominant strategy to play action 1
- Possible because we assumed $\nu\left(\mathbf{1}, \overline{\theta}\right) > 0$ and we can choose η sufficiently close to 0 so that $\nu\left(\mathbf{1}, \overline{\theta}\right)$ is much larger than

$$\eta\left(1 + (1 - \eta) + ... + (1 - \eta)^{|I|-1}\right)$$

- Now inductive argument implies all types $t_i < \infty$ choose action 1

Simpler Constructions

- ▶ Mathevet, Perego and Taneva (2019) and Li, Song and Zhao (2019) give alternative simpler constructions in examples
- ▶ In potential games (which we will now discuss) we can give a generic global game construction that works (Frankel, Morris and Pauzner (2003))

Applying Sequential Obedience

- I think the sequential obedience characterization is cute
- It gives rise to a finite linear program that has the flavor of obedience
- But what is it good for?
- A fair amount of extra work is required to get concrete characterizations
- Our paper derives progressively simpler characterizations of sequential obedience: "coalitional obedience" and "grand coalitional obedience" that hold under additional assumptions. These should be useful in many contexts.
- Today, I will (maybe) mention these tools in words and report the main application they imply

Potential Games

Definition. The game is a potential game if, for each θ, there exists $\Phi \colon A \times \Theta \to \mathbb{R}$ such that

$$d_i(a_{-i}, \theta) = \Phi(1, a_{-i}, \theta) - \Phi(0, a_{-i}, \theta).$$

Normalize $\Phi(\mathbf{0}, \theta) = 0$ for all θ.

Investment Game

- Payoff to action 1 (investing) is $\theta + h_n - c_i$ where n is the number of players investing where
  - Assume h_n is increasing in n
  - Without loss of generality, $c_1 \leq c_2 \leq ... \leq c_{|I|}$
- Payoff to action 0 normalized to 0
- This game has potential:

$$\Phi(a, \theta) = n(a) . \theta + \sum_{k=1}^{n(a)} h_k - \sum_{i \in I} a_i c_i$$

where

$$n(a) = \# \{i | a_i = 1\}$$

- Up to normalization, for fixed θ, this is general binary action supermodular game at each state with anonymous interactive component (making it a potential game).

Regime Change Game

- Payoff to action 1 (attacking) is....
    - $1 - c_i$ if the number of players attacking is greater than $|I| - k(\theta)$
    - c_i if the number of players attacking is less than $|I| - k(\theta)$
- This game has potential:

$$
\Phi(a, \theta) = \begin{cases} n(a) - (|I| - k(\theta)) - \displaystyle\sum_{i \in I} a_i c_i, \text{ if } n(a) > |I| - k(\theta) \\ -\displaystyle\sum_{i \in I} a_i c_i, \text{ otherwise} \end{cases}
$$

Simplifying Sequential Obedience I

- An outcome satisfies *coalitional obedience* if there does not a subset of players who could increase the potential by always disobeying recommendations to play action 1
- More constraints than obedience but on outcomes not ordered outcomes

PROPOSITION 3: In a potential game, an outcome satisfies sequential obedience if and only if it satisfies coalitional obedience.

- intuition:
  - existence of potential allows comparison across deviations across players
  - appeals to dual characterization of sequential obedience which involves adding payoff gains to players along sequences

Simplifying Sequential Obedience II

- An outcome satisfies *grand coalitional obedience* if the grand coalition of all players cannot increase the potential by always disobeying recommendations to play action 1
- An outcome ν satisfies *perfect coordination* if

$$\nu(a, \theta) = 0$$

if $a \notin \{\mathbf{0}, \mathbf{1}\}$.

- The potential game is *convex* if

$$\Phi(a, \theta) \leq \frac{n(a)}{|I|} \Phi(\mathbf{1}, \theta)$$

for all θ.

PROPOSITION 4: In a convex potential game, a perfect coordination outcome satisfies sequential obedience if and only if it satisfies grand coalitional obedience.

Convexity

- The potential game is *convex* if

$$\Phi\left(a,\theta\right) \leq \frac{n\left(a\right)}{|I|}\Phi\left(\mathbf{1},\theta\right)$$

  for all θ.

- Because of supermodularity, this is automatically satisfied if Φ is symmetric
- The game is convex if and only if the game is not too asymmetric
- In investment game,
  - convexity if $I \sum_{k=1}^{|I|} \left(h_k - c_k\right) \geq |I| \sum_{k=1}^{I} \left(h_k - c_k\right)$
  - simple sufficient condition: $h_{k+1} - c_{k+1} \geq h_k - c_k$ for all k
- In regime change game, convexity requires $c_1 = c_2 = ... = c_{|I|}$
- Convexity ensures that the benefit of coalitional deviation is bounded below by a constant times the benefit of a grand coalitional deviation

(with adversarial equilibrium selection)

Now suppose an information designer seeks to maximize

$$V : A \times \Theta \to \mathbb{R}.$$

- ▶ Normalize $V(\mathbf{0}, \theta) = 0$ for all θ
- ▶ Two assumptions on V:
    - ▶ Monotonicity: $V(a, \theta)$ is increasing in a.
    - ▶ Restricted convexity:

$$V(a, \theta) \le \frac{n(a)}{|I|} V(\mathbf{1}, \theta)$$

whenever $\Phi(a, \theta) > \Phi(\mathbf{1}, \theta)$.

Restricted Convexity

- Restricted convexity:

$$V(a, \theta) \leq \frac{n(a)}{|I|} V(\mathbf{1}, \theta)$$

  whenever $\Phi(a, \theta) > \Phi(\mathbf{1}, \theta)$.

- Restricted convexity is satisfied when $V(a, \theta) = \left(\frac{n(a)}{|I|}\right)^{\alpha}$ for some $\alpha \geq 1$

  - For $\alpha = 1$, maximize the sum of probabilities that players invest
  - For $\alpha = \infty$, maximize the probability that all players invest

- In regime change game,

$$V(a, \theta) = \begin{cases} 1, & \text{if } n(a) > |I| - k(\theta) \\ 0, & \text{if } n(a) \leq |I| - k(\theta) \end{cases}$$

  satisfies restricted convexity (because $n(a) \leq |I| - k(\theta)$ whenever $\Phi(a, \theta) > \Phi(\mathbf{1}, \theta)$)

Information Design Problem

- Now consider the problem of an information designer choosing an information structure to maximize his expected payoff in the worst possible equilibrium.
- Equivalent to choosing $v \in \overline{SI}$ to maximize

$$\sum_{a,\theta} v(a,\theta) V(a,\theta).$$

Perfect Coordination Solution

- ▶ An outcome ν satisfies *perfect coordination* if

$$\nu(a, \theta) = 0$$

if $a \notin \{\mathbf{0}, \mathbf{1}\}$.

Theorem 2 If the information designer has monotone preferences satisfying restricted convexity, and the game has a convex potential, then the adversarial information design problem has an optimal solution satisfying perfect coordination.

- ▶ Intuition: If $\nu(a', \theta) > 0$ for $a' \notin \{\mathbf{0}, \mathbf{1}\}$, we can replace with probability $\alpha(a', \theta)\nu(a', \theta)$ on $\mathbf{1}$ and probability $(1 - \alpha(a', \theta))\nu(a', \theta)$ on $\mathbf{0}$ such that coalitional obedience is maintained and V is increased.
- ▶ Does not require symmetry (Inostroza and Pavan (2020), Mathevet, Perego and Taneva (2019) and Li, Song and Zhou (2020))
- ▶ Covers known BAS applications including regime change game
- ▶ Failure of sufficient conditions leads to failure of perfect

Easy Characterization of Optimal Solution

- ▶ Given potential and perfect coordination property, optimal solution is easy to characterize
- ▶ Choose $\mathbf{1}$ whenever $\Phi(\mathbf{1}, \theta) > 0$ and include as many states as possible with $\Phi(\mathbf{1}, \theta) < 0$ subject to cost benefit analysis
- ▶ Order states so $\frac{\Phi(\mathbf{1}, \theta)}{V(\mathbf{1}, \theta)}$ is increasing in θ
- ▶ Ignoring integer issues (or assuming continuum of states), find θ^* solving

$$\sum_{\theta \geq \theta^*} \mu(\theta) \Phi(\mathbf{1}, \theta) = \sum_{\theta \geq \theta^*} \mu(\theta) \Phi(\mathbf{0}, \theta) = 0$$

- ▶ Let

$$\nu^*(a, \theta) = \begin{cases} \mu(\theta), & \text{if } a = \mathbf{1} \text{ and } \theta \geq \theta^* \\ \mu(\theta), & \text{if } a = \mathbf{0} \text{ and } \theta < \theta^* \\ 0, & \text{otherwise} \end{cases}$$

- ▶ I.e., maximize the probability that high action profile is "ex ante risk dominant" (ex ante potential maximizing)

Optimal Solution In Investment Game

- potential:

$$\Phi(a, \theta) = n(a) \cdot \theta + \sum_{k=1}^{n(a)} h_k - \sum_{i \in I} a_i c_i$$

- optimal cutoff solves:

$$\mathbb{E}(\theta | \theta \geq \theta^*) = \frac{1}{|I|} \left(\sum_{i \in I} c_i - \sum_{k=1}^{|I|} h_k \right)$$

Full Implementation

- We focussed on "smallest equilibrium implementation" because it is easier to state than full implementation and it is more relevant for applications

- But full implementation characterization turns out to be a mechanical extension of smallest equilibrium implementation result

- Sequential obedience requires tightened obedience conditions when switching from 0 to 1

- Can define reverse sequential obedience condition (that would characterize largest equilibrium implementation)

- Full implementation requires both SO and reverse SO

Summary

1. Sequential obedience characterization of smallest equilibrium implementation and full implementation
   - Explained wh