Hard Math

for

Middle School:

Solutions for Workbook

Glenn Ellison
Chris Avery

Introduction

Welcome to the solutions manual for the *Hard Math Workbook!*

If you are reading this, we assume that you have just gotten a copy of the *Hard Math for Middle School* textbook and its accompanying workbook creatively titled *Hard Math for Middle School: Workbook.* If not, you should get them. This solution manual is super cheap, but it is of very little use without them.

The *Workbook* supplements the text by presenting a page of problems on each of the 100+ sections of the textbook. Following the *Hard Math* philosophy, the problems in the workbook are intended to be hard because we think it is fun to work on things that are so hard that you can feel good if you get them right, not bad if you get them wrong.

A difficulty with making the problems so hard, of course, is that almost all middle school math teachers also will not be able to do quite a number of them. It can be educational to work on a hard problem even if you don't ever get it. But, working on a problem for a while and then having someone explain what you were missing is even better. We hope this solution manual will make the *Workbook* even more fun and useful. And we hope that it will make teachers and math team coaches more comfortable giving the *Workbook* to their students.

It has taken multiple years to get this solution manual done. In part, this is because we have jobs. In part, it is because we slowed down after our daughters finished middle school. But, it is also because there are lots of problems in the *Workbook* and they really are hard. Writing out solutions took quite a while and made us realize that a number of answers in the first printing of the *Workbook* were wrong. We owe special thanks to all of the people who have helped us out by pointing out mistakes, especially Sam Sun.

Enjoy!

Meet #1

2.1 Basic Definitions

1. An obtuse angle has a measure is greater than 90°. The only such choice is 120°.

2. In the figure, the angle with the smallest degree measure is angle C and the angle with the largest degree measure is angle B. This implies that the measure of angle A is 40° and the measure of angle C is 20°. The positive difference is 20°.

Here, one should be able to recognize which angles are larger by looking at how pointy they are. Another useful fact the largest angle in a triangle is the one opposite the longest side and the smallest angle is the one opposite the smallest side. Here the side opposite angle A (side BC) is clearly longer than the side opposite angle C (side AB), so angle A has a larger measure than angle C.

3. The measure of angle ABD is less than the measure of angle ABC. The largest multiple of 7 less than 40° is 35°.

4. The addition facts are that DAB + CDA = 200° and BCD + CDA = 210°. From the similarity between these expressions (or subtracting the first from the second), we find that BCD = DAB + 10°.

Two of the four angles are obtuse and angle ABC is not obtuse. Hence, exactly two of DAB, BCD, and CDA are obtuse. Because BCD is larger than DAB we there are only two possibilities: (1) CDA and BCD are obtuse and DAB is acute; or (2) BCD and DAB are obtuse and CDA is acute. Consider each case:

Case (1): One of BCD and CDA must have a measure of 100° so the other would have a measure of 110°. The measure of BCD cannot be 110°. If it were, then the measure of DAB would be 100° and there would be three obtuse angles. So in case (1) the only possibility is CDA = 110° and BCD = 100°. We can fill in the other angles and satisfy all conditions from here. DAB = BCD − 10° = 90° is not obtuse, and ABC = 360° − (CDA + BCD + DAB) = 360° − 300° = 60° is acute. The answer is 90°.

If under time pressure you should stop after case (1), but to check your work it could be useful to do case (2) as well. Here, BCD cannot be the 100° angle because that would imply that DAB = 90° is not obtuse. So we would need to have DAB = 100°. However, the first addition fact, DAB + CDA = 200°, then tells us that CDA = 100°. This contradicts that BCD and DAB are the only two obtuse angles. Hence, case (2) does not lead to any valid solutions.

2.2 Adding Up Rules

1. Angles are complements if their measures add up to 90°. 90° – 65° = 25°.

2. Angles BOD′ and D′OH are supplements so the D′OH = 180° – 155° = 25°. Similarly, BOG and GOH are supplements so GOH = 180° – 140° = 40°. The positive difference is 40° – 25° = 15°.

The problem can be done more quickly by realizing that if one angle is 15° larger than another, then its supplement will be 15° smaller than the other's supplement.

3. Let x be the measure of the angle in degrees. The text tells us in words that

$$180 - x = 2(90 - x) + 20$$

Expanding both sides we get 180 – x = 180 – 2x + 20. Cancelling the 180's from the two sides and moving all the x's to one side gives x = 20°.

Students who are not comfortable with algebra could alternately make a table like the one shown below, trying out a couple nice round numbers to start, and then trying larger or smaller numbers to move toward the solution.

Angle	Supplement	Complement	2×Complement+20
40	140	50	120
50	130	40	100
30	150	60	140
20	160	70	160

4. As in many problems like this, a good approach is to just keep looking for angles you can fill in with the facts you have (and have just discovered). First, because BMC and CMB are complements we fill in CMB = 90° – 53° = 37°. Next, we look to use this new fact. Since OMB and CMB are complements, we can fill in OMB = 90° – 37° = 53°. Finally, using that OMG and OMB are supplements we fill in OMG =180 – 53 = 127°.

5. Translating the words to algebra we have

$$180° - (90° - (180° - x)) = x - 7.2°.$$

This simplifies to 270° – x = x – 7.2°, which gives 2x = 277.2° and x = 138.6°.

To translate the words to algebra, some students will find it better to work back from the end step by step instead of trying to do it all at once: the supplement of the angle is 180 – x, the complement of the supplement is 90 – (180 – x) = x – 90, the supplement of this is 180 – (x – 90) = 270 + x.

2.3 Equality Rules

1. DOG and PIG are corresponding angles on parallel lines. Corresponding angles on parallel lines have the same measure so PIG = DOG = 140°.

2. Angles DGF and COF are corresponding angles on parallel lines so COF = 48°. Angles COP and COA are complementary, so COA = 180° – 70° = 110°. Now by subtraction we have FOA = COA – COF = 110° – 48° = 62°.
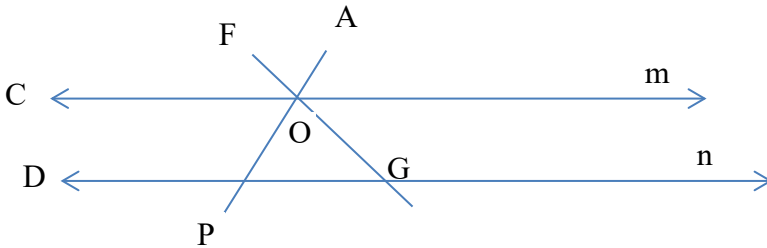
3. In triangle EMB we have EBM = ABM = 42° and BME = 180 – BMD = 90°. The angles in a triangle add up to 180°, so BEM = 180° – (42° + 90°) = 48°. Now, using that BEX and BEM are supplements we have BEX = 180° – 48° = 132°.

Note that in this problem the lines OM and MC were just drawn in to make the diagram more complicated and make it less obvious which angles students should be trying to fill in. Often in angle chasing problems you do want to just blindly start filling in angles one by one, trusting that you'll eventually get to the answer.

But if you're having a hard time getting started or the angle chasing is taking a while it is sometimes better to step back and think about where you're trying to get. Here, I'm trying to find BEX. To find this I'll probably want to first find AEX or BEM and then use supplements. I don't see any way to get AEX so let me think about how I could get BEM. This will have to come from adding up in some triangle, … At which point, you hopefully come up with the first step of using adding up in triangle BEM.

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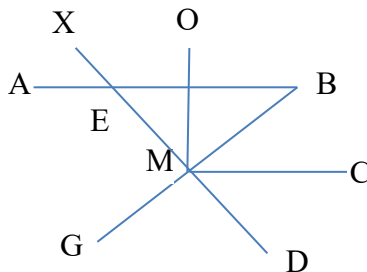
2.4 Angles in Polygons

1. The angles in any quadrilateral (four sided polygon) add up to 360°. Here, the three angles that we are given add up to $130° + 70° + 100° = 300°$. So, the fourth angle we are trying to find is $360° - 300° = 60°$.

2. The exterior angles in an N-gon add up to 360 degrees. So exterior angle in a regular decagon is $360°/10=36°$. The interior angle is the complement if this, $180° - 36° = 144°$.

3. The exterior angle in a regular hexagon is $360°/6 = 60°$. If the exterior angle in a regular N-gon is the complement of this, its measure is $90° - 60° = 30°$. The exterior angles in any N-gon add up to $360°$, so the N-gon must have $360 \div 30 = 12$ sides.

4. To use the fact that lines CB and KS are parallel, extend segment KS so that it intersects ABCDE at F and G as shown below. Angles AFG and ABC are corresponding angles on parallel lines, so they have the same measure. ABC is the interior angle of a regular pentagon so its measure is 108. Hence, the measure of AFG is also 108°.

Now look at triangle AFK. Angle AFK is supplementary to AFG, so its measure is 72°. AKF is an exterior angle of a regular hexagon, so its measure is 60°. (Alternately, you can note that it is supplementary to AKS, which is an interior angle of a regular hexagon.) Using adding up in triangle AFK, the measure of angle FAK is $180° - (72° + 60°) = 48°$.

Now look at all of the angles around point A. EAF is an interior angle of a regular pentagon so it is 108°. We just found that FAK is 48°. KAT is an interior angle of a regular hexagon so it is 120°. The angles around any point add up to 360°, so the measure of EAT must be $360° - (108° + 48° + 120°) = 84°$.

A - 4

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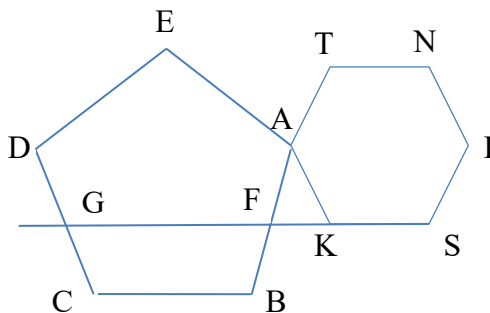
2.5 Problem Solving Strategies

1. This is a classic example of a problem where you find an angle by subtraction. Angle ABK is the difference between angle ABC and angle KBC. Hence, its measure is $108° - 90° = 18°$.

(The measure of ABC is $108°$ because it is the interior angle of a regular pentagon and the measure of KBC is $90°$ because all of the interior angles of a square are right angles.)

2. This is a problem where it pays to be lazy and avoid doing computations until you really need to. Angles ABD and DBC are supplements, so the measure of angle DBC is $180° - 141.5927°$. Angles CBE and DBC are complements, so the measure of angle CBE is $90° - (180° - 141.5927°) = 90° - 180° + 141.5927° = 141.5927° - 90° = 51.5927°$.
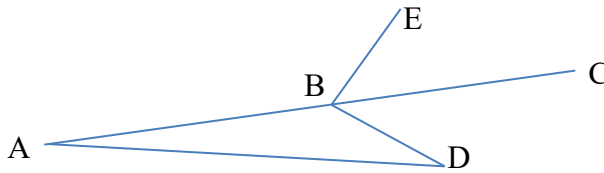

3. This is a complicated problem where one good approach is just to start filling in whatever you can. In triangle ABX we are given that BAX (which is the same as BAD) is $20°$ and ABX (which is the same as ABC) is $115°$, so by adding up the measure of AXB is $180° - (115° + 20°) = 45°$.

Angles AXB and DXC are opposite angles, so the measure of angle DXC is also $45°$. We are given that angles DXC and DFE are complements, so DFE is also $45°$.

Using adding up in triangle DFE, the measure of DEF is $180 - (45 + 20) = 115°$.

We can now immediately find all of the angles in 5-gon BXDEG other than angle AGF. GBX is the supplement of AXB so it is $65°$. BXD is the supplement of AXB, so it is $135°$. XDE is the supplement of EDF, so it is $160°$. Angle DEG is the supplement of DEF, so it is $65°$. The angles in any pentagon add up to $540°$, so the measure of AGF is $540° - (65° + 135° + 160° + 65°) = 540° - 425° = 115°$.

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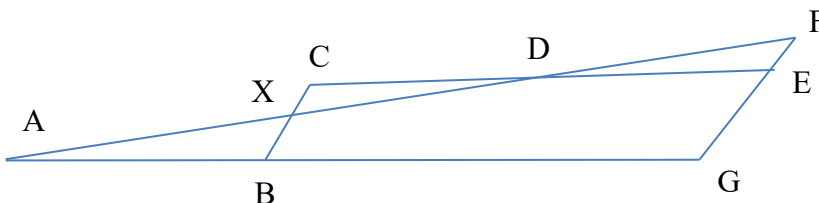
The final calculation involving the pentagon can be avoided if you notice that angles AXB and AFG are corresponding angles on lines CB and FG. The theorem that corresponding angles on parallel lines are equal is an if-and-only-if theorem. When corresponding angles are equal, lines are parallel. This shows that CB and FG are parallel lines. Hence, AGF and ABC are corresponding angles on parallel lines, and the fact that ABC is 115° implies that AGF is also 115°.

While it is always best to try to solve problems, guessing can also be a good strategy if you're short on time and there are other problems you could be doing in the time remaining. In the diagram BC and GF look like they're parallel, so if short on time one could just guess that they are and immediately get 115° from ABC and AGF (hopefully) being corresponding angles.

2.6 Advanced Topics

1. Triangle CDE is an isosceles triangle, so angles DCE and DEC are equal. Angle CDE is 108°, so to make the angles in the triangle add up to 180°, DEC and DCE must add up to 72°. Hence, each is 36°. By identical reasoning, BCA is also 36°.

We can then find angle ECA by subtracting the measures of DCE and BCA from DCB (which again is one of the angles of a regular pentagon). The answer is 108° − (36° + 36°) = 36°.

2. This figure has an obviously missing line segment, AD. One thing you notice immediately upon drawing it in is that ABCD is a square. This implies that ADC is a 90° angle. Because CDE is an equilateral triangle, EDC is a 60° angle.

ADE is an isosceles triangle. Angle ADE is 90° + 60° = 150°. The other two angles, AED and DAE must add up to 30°. They are equal to each other by the isosceles triangle theorem, so each is 15°.
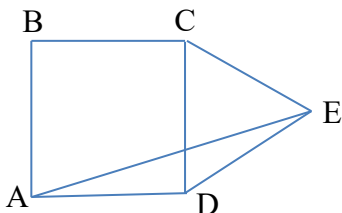
3. Draw in the line segment BD. The diagonals of a rectangle intersect at the midpoints of the diagonals so X is also the midpoint of BD. (You can also prove this by noting that if AC and BD intersect at X', then AX'B and CX'D are similar triangles with AB = CD.)

Because AC = BD (this follows from the Pythagorean theorem or another congruent triangle argument) the segments AB, AX, BX, XC, XD, and CD are all equal so ABX and CDX are both equilateral triangles. This gives that CDX and CXD are both 60°.

Angle ABF is a bisector of a 90° angle so it is 45°. Angle BAF is a right angle, so adding up in triangle BAF gives that BFA = 45°.

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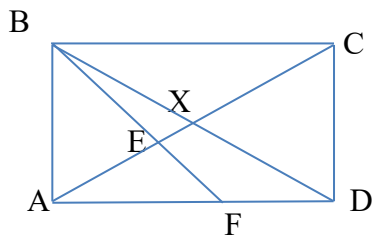
Angle FAE is 30° by subtraction, so adding up in triangle FAE gives that AEF is 105°.

We can now find all of the angles in DFEX as the complements or supplements of angles we found already. DFA = 180° − AFE = 135°. FEX = 180° − AEF = 75°. EXD = 180° − CXD = 120°. XDF = 90° − CDX = 30°.

The median of these four numbers is the average of 75° and 120°, which is 97.5°.

3.1 Prime Numbers

1. For this and other problems it is a good idea to memorize a list of all the primes less than 100. The fact that there's something of a pattern after the first row makes it not so hard to memorize

2	3	5		7
11	13		17	19
	23			29
31			37	
41	43		47	
	53			59
61			67	
71	73			79
	83			89
			97	

The pattern is that there's a row with up to 4 primes, followed by a row in which only the numbers ending in 3 and 9 can be prime, followed by a row in which only the numbers ending in 1 and 7 can be prime. We then start the next repeat with a row with up to 4 primes. It's not an exact pattern, 49 is missing, 77 is missing, and 91 is missing, but it works pretty well.

If you memorize the list you can just write down the answer. Otherwise, use the rule that every non-prime is divisible by a prime less than the square root of the original number. $7^2 = 49$ is greater than any of the numbers from 20 to 30, so we just need to check for divisibility by 2, 3, and 5, the primes less than 7. 2 is a factor of all even numbers: 20, 22, 24, 26, 28, 30. 3 is a factor of 21, 24, 27, 30. 5 is a factor of 20, 25, and 30. This leaves 23 and 29 as the only prime numbers between 20 and 30.

2. Checking the odd numbers in descending order from 40: 39 is divisible by 3, then 37 is prime (not divisible by 2, 3, or 5). So 37 is the largest prime number less than 40. Again, though it's quicker to have memorized the list.

3. Checking the odd numbers between 30 and 40: 31 is prime, 33 is divisible by 3, 35 is divisible by 5, 37 is prime and 39 is divisible by 3. So the primes between 20 and 40 are 23, 29, 31 and 37. The twin primes here are 29 and 31.

4. Once again, it's quickest to have memorized the list. Otherwise, from 31 to 49:

        32, 34, 36, 38, 40, 42, 44, 46 and 48 are divisible by 2;

        33, 39, 45 are divisible by 3;

        35 is divisible by 5;

        49 is divisible by 7.

The remaining numbers 31, 37, 41, 43, and 47 are prime. There are 19 numbers starting at 31 and ending at 49, of which 5 are prime. The remaining 14 numbers listed above are all composite.

The even numbers form an arithmetic sequence above and below 40, so have average 40. Comparing the other numbers to 40: 33 is 7 below, 35 is 5 below, 39

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is 1 below, 45 is 5 above and 49 is 9 above 40. The sum -7 -5 – 1 + 5 + 9 = 1, so the total of these numbers is 1 more than an average of 40. This means that the total is 14 x 40 + 1 = 561 and so the average is 561 ÷ 14 = 40 1/14.

5. At this point we're not going to keep giving answers not using the pattern. The rows that can have four primes are the 10's, 40's, 70's, 100's, 130's, 160's, etc.

41, 43, 47, and 49:	not all prime because 49 = 7 x 7.
71, 73, 77 and 79:	not all prime because 77 = 11 x 7
101, 103, 107 and 109:	are all prime

So x = 10.

6. The early primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.
In this list 5 and 11 differ by 6, and next 7 and 13 differ by 6. Caroline could be reading page 7 x 13 = 91.

3.2 Prime Factorization

1. The prime factorization of 22 is 2×11. This makes 2 and 11 the only prime factors of 22. So the sum of the prime factors is $2 + 11 = 13$.

2. $204 = 2 \times 102 = 2 \times 2 \times 34 = 2 \times 2 \times 2 \times 17$. So its smallest prime factor is 2.

3. $108 = 2 \times 54 = 2^2 \times 27 = 2^2 \times 3 \times 9 = 2^2 \times 3^3$. So $a = 2$, $b = 3$, $c = 0$, $d = 0$ and the sum is 5.

4. $2700 = 27 \times 100 = 3^3 \times 2^2 \times 5^2$. So $a = 2$, $b = 3$, $c = 2$, $d = 0$, $e = 0$. So $a \times b + c \times d = 6$.

5. $555{,}000 = 555 \times 1{,}000 = 5 \times 111 \times 10^3 = 5 \times 3 \times 37 \times 2^3 \times 5^3 = 2^3 \times 3^1 \times 5^4 \times 37^1$. This doesn't look exactly like the form in the question so you have to remember that because $7^0 = 1$ we can always stick this into the factorization and make it $2^3 \times 3^1 \times 5^4 \times 7^0 \times 37^1$. So $a = 3$, $b = 1$, $c = 4$, $d = 0$, $e = 37$, $f = 1$ and the sum is 46.

6. The prime factorization of 105 is $3 \times 5 \times 7$. So the product of these numbers is also 105. (Since 105 is the product of different primes, none of them raised to an exponent greater than 1, the product of its prime factors is still 105.) The sum of these prime factors is $3 + 5 + 7 = 15$. This is 3×5, so its largest prime factor is 5.

7. In a non-leap year, there are 365 days, 24 hours per day, 60 minutes per hour and 60 seconds per minute. So the number of seconds in a non-leap year is S = 365 x 24 x 60 x 60. It's good to be lazy here and not multiply until you need to. The number of minutes in a day is $M = 24 \times 60$. So $S \div M = (365 \times 24 \times 60 \times 60) \div (24 \times 60) = 365 \times 60$. The prime factorization of 365 is 5×73. The prime factorization of 60 is $2^2 \times 3 \times 5$, which means that the prime factorization of their product is $2^2 \times 3 \times 5^2 \times 73$. So $a = 2$, $b = 1$, $c = 2$, $d = 0$, $e = 73$, $f = 1$ and their sum is 79.

8. Since $a + b + c > 3$, this means that x must be at least 2^4 or 16. (This rules out the possibility that $x = 15$, which would otherwise work.) To create a list of numbers from 16 onwards which have only 2's, 3's, and 5's in their prime factorization you just need to list numbers that are not divisible by some larger prime, i.e. numbers that are not a multiple of 7 or 11 or 13 or 17 or 19 or 23, or 29, and so on. Listing numbers like this from 16 on we get 16, 18, 20, 24, 25. So $x = 24$ works. $x = 24$ can be written as $2^3 \times 3^1 \times 5^0$ and $x + 1 = 25$ can be written as $2^3 \times 3^1 \times 5^2$.

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3.3 Counting and Summing Factors

1. The prime factorization of 75 is $3^1 \times 5^2$. The table giving its factors looks like

		1	5	5^2
1	|	1	5	25
3	|	3	15	75

Writing these is order from smallest to largest we get 1, 3, 5, 15, 25, 75.

2. The prime factorization of 36 is $2^2 \times 3^2$. Thinking about the factor table it is easy to see that it will have 3 rows and 3 columns for a total of 9 numbers. You could also do it using the formula in the book: $2^2 \times 3^2$ is of the form $2^a \times 3^b$ for a=2 and b=2 so the formula says the number of factors is $(a + 1)(b + 1) = (2 + 1)(2 + 1) = 9$. If you didn't know about the formula or factor tables you could think that any number that can be written as $2^x\, 3^y$ with $x \le 2$ and $y \le 2$ is a factor of 36. So there are 3 possibilities for x and 3 possibilities for y so $3 \times 3 = 9$ choices for (x, y).

3. $1440 = 10 * 144 = 2 \times 5 \times 12^2 = 2 \times 5 \times (2^2 \times 3)^2 = 2^5 \times 3^2 \times 5$. So the formula for the number of factors gives $(5 + 1)(2 + 1)(1 + 1) = 36$ factors of 1440.

4. The prime factorization of 200 is $2^3 \times 5^2$ so the factor table looks like

		1	5	5^2
1	|	1	5	25
2	|	2	10	50
2^2	|	4	20	100
2^3	|	8	40	200

You could just add all of these numbers up $1 + 2 + 4 + 5 + 8 + 10 + 20 + 25 + 40 + 50 + 100 + 200 = 465$. But it's easier to use the sum-of-factors formula from the book. It says that the sum of the factors is $(1 + 2 + 2^2 + 2^3)\,(1 + 5 + 5^2) = 15 \times 31 = 465$.

5. The prime factorization of 225 is $3^2 \times 5^2$. You could write out its factor table and look for the third biggest number. Or you could think about factors and as coming in factor pairs

$$1 \times 225$$
$$3 \times 75$$
$$\cdots$$

and realize that the third largest factor of 225 will be the factor that's in a pair with the third smallest factor. The third smallest factor is 5, so the third largest factor is $225 \div 5 = 45$.

6. $1935 = 3 \text{ x } 645 = 32 \text{ x } 215 = 3^2 \text{ x } 5 \text{ x } 43$. The only combination of these factors that is a perfect square is $3^2 = 9$, so there is only one factor of 1935 other than 1 that is a perfect square. Counting 1, there are 2 factors of 1935 that are perfect squares.

7. $1080 = 2 \times 540 = 2^2 \times 270 = 2^3 \times 135 = 2^3 \times 3 \times 45 = 2^3 \times 3^2 \times 15 = 2^3 \times 3^3 \times 5$. The odd factors of 1080 are the ones that don't have any powers of 2 in their prime

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factorization which are just the factors of $3^3 \times 5 = 135$. Again, the easiest way to sum these is to use the formula in the book for the sum of the factors of a number. The formula gives $(1 + 3 + 3^2 + 3^3)(1 + 5) = 40 \times 6 = 240$.

8. $726 = 2 \times 363 = 2 \times 3 \times 121 = 2 \times 3 \times 11^2$. There are $2 \times 2 \times 3 = 12$ factors of 726. In a factor table they'd look like

1	1	11	11^2		2	1	11	11^2
1	1	11	121		1	2	22	242
3	3	33	363		3	6	66	726

Squaring all of these and then adding up would take a very long time. Rather than doing this, the best way to do the problem is to think about this problem as being very similar to the calculation we do when we add up all of the factors of a number, but with squares instead. The squares of these twelve factors can be thought of as the numbers in a similar table where all the numbers (including the ones along the top, along the left, and in the corners) are all squared:

1	1	11^2	11^4		2^2	1	11^2	11^4
1	1^2	11^2	121^2		1	2^2	22^2	242^2
3^2	3^2	33^2	333^2		3^2	6^2	66^2	726^2

The same reasoning that gives the sum-of-factors formula tells us that the sum of the twelve numbers in this table will be $(1 + 2^2)(1 + 3^2)(1 + 11^2 + 11^4) = 5 \times 10 \times (1 + 121 + 14641) = 50 \times 14763 = 738{,}150$.

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3.4 Divisibility Rules

1. Every number with last digit 5 or 0 is a multiple of 5, while a number with any other last digit is not a multiple of 5. So A $= 0$ or 5.

2. A number is divisible by 9 if the sum of its digits is divisible by 9. The sum of these digits is equal to $2 + 8 + 4 + B + 2 = 16 + B$. Since B is between 0 and 9, this sum is between 16 and 25, which can only be divisible by 9 if $16 + B = 18$ or B $= 2$.

3. A number is divisible by 12 if it is divisible by 4 and also divisible by 3. This means that the number written 2A is divisible by 4, which means that A $= 0$, 4, or 8. This number is divisible by 3 if the sum of the digits $3 + 1 + A + 2 + A = 6 + 2A$ is divisible by 3, which means that A is a multiple of 3. The only possibility that satisfies both requirements is A $= 0$.

4. The numbers 2, 3, 4, and 6 are factors of 12, so if 35A2A is divisible by 8, 9 and 12, it is also divisible by 2, 3, 4, and 6. Further 12 is a factor of 8 x 9 $= 72$. Therefore, if 35A2A is divisible by 8 and 9, it is also divisible by all of these other numbers. 35A2A is divisibly by 9 if the sum of its digits $10 + 2A$ is divisible by 9. Since $10 + 2A$ is between 10 and 28, this requires $10 + 2A = 18$ or 27, so the only possible solution is A $= 4$. To check if this indeed works we need to check whether 35424 is divisible by 8. If it is, then the answer is A=4. If not, the question would have a mistake, because there would be no value of A that works. One divisibility rule for 8 says that 35424 is a multiple of 8 if 424 is a multiple of 8. Dividing (or factoring out 2's) we realize that $424 = 8 \times 53$ is a multiple of 8, so the answer is A $= 4$.

5. The sum of the digits in AB9BA is $9 + 2A + 2B$. AB9BA is divisible by 72 if it is divisible by both 8 and 9. It is divisible by 9 if $9 + 2A + 2B$ is divisible by 9, which requires $2A + 2B$ to be divisible by 9. This can only happen if A + B is multiple of 9, which means that A + B $= 0$, 9, or 18. If A + B $= 0$, then the number is 900, which is not divisible by 72. If A + B $= 18$, then A $=$ B $= 9$ and the number is 99999, which is odd, so not divisible by 72. So we must have A + B $= 9$. This leaves us with just nine possibilities. The last three digits of these possibilities are 909, 918, 927, 936, 945, 954, 963, 972, 981 and 990. We can rule out the odd numbers, then note that 8 x 110 $= 880$, so 8 x 117 $= 936$ is the only multiple of 8 in this group. This means that A $= 6$, B $= 3$ and the full number 63936. Then A + 2B $= 12$.

A more elegant way to do the final step is to remember that another rule for divisibility by 8 is that AB9BA will be a multiple of 8 if $A + 2B + 4 \times 9 = A + 2 \times (9 - A) + 36 = 54 - A$ is a multiple of 8. The only multiple of 8 that's a little smaller than 54 is 48 which requires A $= 6$.

6. They don't teach this in most middle schools, but there is a good divisibility rule for 37 that should remind you a little of the rules for 9 and 11: add together the 3-

A - 14

digit blocks that make up a number and see if the sum is divisible by 37. In this case, adding the blocks looks like:

```
  524
+ 3A
  55(4+A)
```

By the number of the bottom I mean the three digit number that is 554 if A=0, 555 if A=1, …, but 560 if A=6, 561 if A=7, and so on. Looking for multiples of 37 close to these numbers we realize that $37 \times 15 = 555$, so the answer is A=1.

If you didn't know about this divisibility rule you could start with the observation that 37 divides 111. So 999 is divisible by 37, and so dividing 1,000 by 37 leaves a remainder of 1. That is, the remainder of 30,000 divided by 37 is 30, the remainder of 31,000 divided by 37,000 is 31 and so on, including the fact that dividing 3A,000 by 37 leave s a remainder of $30 + A$. Dividing 524 by 37 leaves a remainder of 6 as $37 \times 14 = 518$. So 3A524 is divisible by 37 if the sum of these two separate remainders, which is $30 + A + 6$ is divisible by 37. This means that $36 + A$ is divisible by 37, so $A = 1$.

7. 6A335B is divisible by 74 if it is even (i.e. B is even) and it is divisible by 37. Again using the divisibility rule for 37 we are looking for values of A and B that make the three digit number that you get by adding 6A3 and 35B a multiple of 37. This number will be at least $603 + 350 = 953$ and at most $693 + 359 = 1052$. And it will be odd because B is even. Recall that 999 is multiple of 37. This gives one immediate answer: we can get 999 by setting A=4 and B=6. It turns out that this is the only answer. The next smaller odd multiple of 37 is $999 - 74 = 924$ which is too small and the next larger odd multiple of 37 is $999 + 74 = 1073$ which is too large.

Here, if you didn't know the divisibility rule you could reason that since the remainder of 1,000 divided by 37 is 1, the remainder when 6A3000 is divided by 37 is equal to $600 + 10A + 3$. (Note that this may include some multiples of 37.) The nearest multiple of 37 to 35B is $37 \times 9 = 333$, so the remainder when 35B is divided by 37 is $17 + B$. So 6A335B is divisible by 74 if B is even and $600 + 10A + 3 + 17 + B = 620 + 10A + B$ is divisible by 37. The closest multiple of 37 below 620 is 592 since $37 \times 16 = 592$, so dividing 620 by 37 leaves a remainder of 28. So we want $28 + 10A + B$ to be divisible by 37 where B is even. $28 + 10A + B$ is between 28 and 127. The multiples of 37 in this range are 37, 74 and 111. Since B is even, $28 + 10A + B$ is even, so it must be that $28 + 10A + B = 74$, which requires $A = 4$ and $B = 6$.

8. Think about how you might converting the base 63 number to base 10. You could compute the powers of 63, multiply each by the appropriate digit, and add them up in a big multicolumn addition problem:

1×1	1×1	1
2×63^1	2×63	126
3×63^2	3×3969	11907

A - 15

Doing this calculation to convert to base 10 would be very tedious, but to figure out if the number is a multiple of 10 we only need to figure out the last digit. For this, we only need to figure out the last digit of each term in the sum, and for this it suffices to keep track of the last digit of each power of 63 and of each number product that comes from multiplying them by the digits. 10 base 63 has last digit 3, 100 base 63 has last digit 9 (since $3^2 = 9$), 1000 base 63 has last digit 7 (since $3^3 = 27$), 10,000 base 3 has last digit 1 (since $3^4 = 81$) and then the pattern repeats. So the last digit of A7894321 is given by the last digit of

$$1 + 2 \times 3 + 3 \times 9 + 4 \times 7 + 9 \times 1 + 8 \times 3 + 7 \times 9 + A \times 7$$
$$= \quad 1 + 6 + 27 + 28 + 9 + 24 + 63 + 7A$$
$$= 158 + 7A.$$

This has last digit 0 if $8 + 7A$ has last digit 0, which occurs if $7A$ has last digit 2. Thus $7A = 42$ and $A = 6$.

You could also do the problem by looking for your own divisibility rule that could be used to see whether base 63 numbers are divisible by 10. Note that $63^2 \equiv -1$ (mod 10), $63^4 \equiv 1$ (mod 10), so a divisibility rule that works in base 63 is to add the numbers in blocks of four and see if the result is multiple of 10. Usually, this wouldn't help much, but here it implies that $A7894321_{(63)}$ is a multiple of 10 if $A789_{(63)} + 4321_{(63)}$ is a multiple of 10. When adding numbers in base 63 no carrying is needed until the numbers in each position add up to at least 63. So in base 63 the answer to this addition problem is a four digit number with first digit equal to A+4 and all other digits equal to the number we call "ten". This makes it obvious that A=6 will work. When you plug in 6 for A the sum is ten times $1111_{(63)}$.

3.5 Problem Solving Tips

1. This relates to my tip of focusing on whatever gives you a small number of choices. Here, focus on the numbers that are squared to give you the odd perfect squares. The number being squared has to be odd. $3^2 = 9$ only has one digit. So the two digit odd perfect squares will be $5^2 = 25$, $7^2 = 49$, and $9^2 = 81$. After this, $11^2 = 121$ is too big, so there are just three.

2. A quick way to do this one is to use the multiple of 5 clue and the perfect square clue. A square which is a multiple of 5 is a multiple of 25. The other two-digit multiples of 25, 50 and 75, are not squares so 25 is the only possible answer. You can verify that it satisfies all of the other requirements.

3. Since n is a factor of 15, it must be 1, 3, 5, or 15. For these numbers $10n + 19$ is 29, 49, 69, or 169, of which 29 is the only prime since $49 = 7 \times 7$, $69 = 3 \times 23$ and $169 = 13 \times 13$. So n can only be equal to 1, which satisfies the other criteria of the problem.

4. There are 99 numbers from 1 to 99. Since $2 \times 49 = 98$, 49 of these numbers are even and are divisible by 2. Since $3 \times 33 = 99$, 33 of the numbers from 1 to 99 are divisible by 33 but 16 of them are even and have already been counted. So there are $49 + 17 = 66$ numbers between 1 and 99 that are divisible by either 2, by 3 or by both 2 and 3.

There are 19 numbers between 1 and 99 that are divisible by 5, but 9 of them are even. Of the remaining ten numbers, 5, 15, 25, 35, 45, 55, 65, 75, 85 and 95, there are three (15, 45, 75) that are divisible by 3. So there are seven numbers (5, 25, 35, 55, 65, 85, 95) between 1 and 99 that are divisible by 5 but not by 2 or 3. This leaves a total of $99 - 49 - 17 - 7 = 26$ numbers between 1 and 99 that are not divisible by 2, 3, or 5.

5. Since the prime factorization of 49 is 7×7, this number must be of form $a^6 b^6$ or a^{48} where a and b are different primes as $a^m b^n$ has $(m + 1) \times (n + 1)$ factors. The smallest possible such number has $a = 2$, $b = 3$ where $2^6 = 64$, $3^6 = 729$ and $64 \times 729 = 46{,}656$. The smallest 48th power, 2^{48}, is much bigger.

6. The perfect cubes up to 1,000 are 1, 8, 27, 64, 125, 216, 343, 512, 729, and 1,000. Since the hundreds digit is greater than the tens digit, this leaves possibilities 216, 512 and 729. None of these numbers is a multiple of 17, so we have to find the ones where $10^n - 1$ is divisible by 7.

The remainder when $10^1 = 10$ is divided by 7 is 3.
The remainder when $10^2 = 100$ is divided by 7 is the remainder of 3×10 divided by 7 or 2.
The remainder when $10^3 = 1{,}000$ is divided by 7 is the remainder of 2×10 divided by 7 or 6.

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The remainder when $10^4 = 10,000$ is divided by 7 is the remainder of 2×10 divided by 7 or 4

The remainder when $10^5 = 100,000$ is divided by 7 is the remainder of 4×10 divided by 7 or 5

The remainder when $10^6 = 1,000,000$ is divided by 7 is the remainder of 5×10 divided by 7 or 1

The remainder when $10^7 = 10,000,000$ is divided by 7 is the remainder of 1×10 divided by 7 or 3

From here the pattern repeats. So $10^6 - 1$ is divisible by 7 and further, $10^n - 1$ is divisible by 7 if n is divisible by 6. $216 = 6^3$ is divisible by 6, but $512 = 8^3 = 2^9$ is not, nor is 729 (it is odd, so is not divisible by 2). So the number must be 216.

4.1: Order of Operations

These problems just require careful use of the standard PEMDAS Order of Operations.

1. The numerator is
$$3 + 4 \times 5 + 2^{1+1} + 1 \quad = 3 + 4 \times 5 + 2^2 + 1 = 3 + 4 \times 5 + 4 + 1$$
$$= 3 + 20 + 4 + 1 = 28$$
The denominator is $2^{3+4\div 2} = 2^{3+2} = 2^5 = 32$. So the fraction is $28 \div 32 = 7/8 = .875$

2. The numerator is
$$3^{2+1} + (-2 \times (5+3)) - 3 \div 3 = 3^3 + (-2 \times 8) - 3 \div 3 = 27 - 16 - 3 \div 3$$
$$= 27 - 16 - 1 = 10$$
The denominator is
$$44 \div 22 \times 3 - 5 \div 5 = 2 \times 3 - 5 \div 5 = 6 - 1 = 5$$
Therefore the result is $10 \div 5 = 2$

3. The numerator is
$$146 \times 6 + (5 - 5 \times 145) - 2^0 = 146 \times 6 + (5 - 725) - 2^0 = 146 \times 6 - 720 - 1$$
$$= 876 - 721 = 155$$
Using the difference of squares formula the denominator is
$$17^2 - 14^2 = (17 - 14)(17 + 14) = 3 \times 31.$$
Therefore the result is $155 \div (3 \times 31) = (5 \times 31)/(3 \times 31) = 5/3$ or about 1.67.

4. In the numerator, the first exponent is $(4 + 3) \div 2 = 7/2$ and the second exponent has numerator $2 \times 3 = 6$ and denominator $4 \times (4+4) \div 4 = 4 \times 8 \div 4 = 32 \div 4 = 8$.
So the numerator is $2^{7/2} - 4^{6/8} = 2^{7/2} - 4^{3/4} = 2^{7/2} - 2^{3/2}$
The exponent in the denominator is $2^{-1} = 1/2$, so the denominator is $2^{1/2}$.
The first term in the numerator divided by the denominator is $2^{7/2} \div 2^{1/2} = 2^3 = 8$.
The second term in the numerator divided by the denominator is $2^{3/2} \div 2^{1/2} = 2^1 = 2$.
So the final result is $8 - 2 = 6$.

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1. The easiest way to compute the average of 397, 810, 596, 1011 and 188 is to write the numbers as sums and differences of nearby multiples of 100:

$(400 - 3)$, $(800 + 10)$, $(600 - 4)$, $(1000 + 11)$, and $(200 - 12)$.

Now sum them and divide by 5 to find the average.

$(400 - 3) + (800 + 10) + (600 - 4) + (1000 + 11) + (200 - 12) =$
$400 + 800 + 600 + 1000 + 200 + (-3 + 10 - 4 + 11 - 12) = 3000 + 2 = 3002.$

So the average is $3002 \div 5 = 600.4$

2. A good way to do this is with the average of averages rule. It says that the average is

$$\frac{12}{21}91 + \frac{9}{21}84 = \frac{4}{7}91 + \frac{3}{7}84 = 4 \times 13 + 3 \times 12 = 52 + 36 = 88.$$

You could also do ahead-behind counting thinking about how much more than the boy's score the average will be. The girls make the total score is $12 \times (91 - 84) = 12 \times 7 = 84$ points ahead of an initial guess that the average is 84. These 84 additional points increase the class average by $84 \div 21 = 4$. So the class average is the boys' average of 84 plus these additional 4 points or 88.

3. There are 14 scores in the plot. The median of an even number of scores is the average between the two middle scores. The symmetric stem-and-leaf plot makes it easy to see that the two scores in the middle are 85 and 87. The average of these two is 86.

One good way to compute the mean is to remember that the mean is the sum of the mean of the stems and the mean of the leaves. The mean of the stems is obviously 80 – there are five stems that are ten points below this and five that are ten above. To average the leaves add $2 + 5 + 9 + 2 + 5 + 7 + 8 + 4 + 6 + 6 + 8 + 8 = 70$ and divide by 14 to get $70 \div 14 = 5$. So the mean is 85

Using ahead and behind counting with a middle score of 85 as a guess also would have worked very well. Relative to 85 the scores the scores can be represented as +5, +5, +7, +10, +14, -3, 0, +2, +3, -11, -9, -9, -7, and -7. These add up to exactly 0 points ahead, which also would have let you find the mean is 85.

Either way, the difference between mean and median is 1 point.

4. Using ahead-behind counting, these scores are 11 points below, 6 points below, exactly equal to, 2 points above and 9 points above 90 for a total of 6 points below an average of 90.

If one score counts twice, then it has to be at least 96 for the final grade to be an A minus, so it must be the highest score of 99.

5. Using ahead-behind counting, Julia's known scores are 6 below, 2 below and 5 above 90 for a total of 3 below 90. Since her average over 5 tests equals 90, her remaining two scores must total 3 above 90. The possible combinations for these scores is then (83, 100), (84, 99), (85, 98), (86, 97), (87, 96), (88, 95), (89, 94), (90,

Meet 1 Solutions

93), (91, 92), so one of these two new scores is above 88 (the median of the first three scores) and the second score might be above 88 as well.

If both new scores are above 88, then the median of all five scores is the lower of the new scores. This can be as small as 89 or as large as 91.

If one new score is above 88 and the other new score is 88 or less, then the median is 88. So the median can be as low as 88 or as high as 91 for a difference of 3.

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5.1: Simplifying Expressions

1. $3(2x - 2) + (x + 5) = 6x - 6 + x + 5 = 7x - 1$

2. $5(2 - x) + 5x + 16(2 - x) = 10 - 5x + 5x + 32 - 16x = -16x + 42$

3. Think of reordering this as $3x - 2 + 3x + 2 + 187(2x - 5) + 13(2x - 5)$
The first terms add up to $6x$ and the last two parts are $187(2x - 5) + 13(2x - 5) = 200(2x - 5) = 400x - 1000$, so the answer is $406x - 1000$

4. $313(2x - 7) + 13(7x - 5) + 313(5x + 7) - 7(13x - 5)$
Here, grouping the terms that have 313 in them simplifies the computation.
$313(2x - 7) + 313(5x + 7) = 313(7x) = 2191x$. The sum of the other two terms is
$13(7x - 5) - 7(13x - 5) = 91x - 65 - (91x - 35) = -30$. The answer is $2191x - 30$

5. $3(3 - 2x) + 27(137x - 3) - 3(2x - 3) - 27(136x - 4) + 5.$
$\quad = 3(3 - 2x - 2x + 3) + 27(137x - 3 - 136x + 4) + 5$
$\quad = 3(6 - 4x) + 27(x + 1) + 5$
$\quad = 18 - 12x + 27x + 27 + 5$
$\quad = 15x + 50$

6. $81(25x + 1) - 5(x - 99) - 45(45x + 11) + 9(x - 9) + (2025x - 243) \div 9^2$
$\quad = 81 \times 25x + 81 - 5x + 495 - 45 \times 45x - 495 + 9x - 81 + (2025x - 243) \div 81$
$\quad = x(81 \times 25 - 5 - 45 \times 45 + 9) + (81 + 495 - 495 - 81) + (25x - 3)$
$\quad = x (2025 - 5 - 2025 + 9 + 25) - 3$
$\quad = 29x - 3$

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5.2: Evaluating Expressions

1. $3(7x - 2) - 20x = 21x - 6 - 20x = x - 6$. If $x = 17$, this is $17 - 6 = 11$.

2. $10 - 5x + 4x - 3 + (2 - 5x) = 10 - 3 + 2 + x(-5 + 4 - 5) = 9 - 6x$. When $x = 1/6$, this is $9 - 6 \times 1/6 = 9 - 1 = 8$

3. $(3x + 17) + 21(x - y) + 5(4y - 3) - 14x = 3x + 17 + 21x - 21y + 20y - 15 - 14x = 10x - y + 2$. If $x = ¼$ and $y = 3.5$, this is $10 / 4 - 3.5 + 2 = 2.5 - 3.5 + 2 = 1$

4. $37(2x - 5) + 7(7x - 5) + 35(5 - 2x) - 6(8x - 6) + (x + 9)$
$$= 74x - 185 + 49x - 35 + 175 - 70x - 48x + 36 + x + 9$$
$$= x(74 + 49 - 70 - 48 + 1) + (-185 - 35 + 175 + 36 + 9)$$
$$= x(74 - 70 + 49 - 48 + 1) + (175 - 185 + 36 - 35 + 9)$$
$$= x(4 + 1 + 1) + (-10 + 1 + 9)$$
$$= 6x$$
If $x = 1/7$, this is $6/7$

5. If $x = 120/17$ and $y = 60/17$, then $x = 2y$ so we can simplify all the analysis by replacing x with 2y before completing any calculations. So $131(x - 5y) + 508(x - y) + 129(x + y) + 500(x - 3y)$
$$= 131 (-3y) + 508(y) + 129(3y) + 500(-y)$$
$$= -393y + 508 y + 387y - 500y$$
$$= (-393 + 387)y + (508 - 500)y = 2y$$
With $y = 60/17$, this is $2 \times 60/17 = 120/17 = 7\frac{1}{17}$

6. Ankur's expression simplifies to $= 137(2x - y) + 79(3x + 4y) = 274x - 137y + 237x + 316y = 511x + 179y$

Mengxi's expression simplifies to $135(2y - x) + 80(4x + 3y) = 270y - 135x + 320x + 240y = 185x + 510y$
So Ankur's answer is $511 \times 22/7 + 179 \times 1/2$ while Mengxi's answer is $185 \times 1/2 + 510 \times 22/7$.
The difference is $(511 - 510) \times 22/7 + (179-185) \times (1/2) = 1 \times 22/7 - 6 \times 1/2 = 22/7 - 3 = 1/7$.

5.3 Solving Equations in One Unknown

1. $5x - 13 = 12$ so $5x = 12 + 13$ or $5x = 25$ with solution $x = 5$.

2. $12 - 3x = 2x + 3$ so $12 - 3 = 2x + 3x$ or $9 = 5x$ with solution $x = 1.8$

3. $3(2 + 5x) - (x + 4) = 3x - 1 + 2(5x + 2)$
  OR $6 + 15x - x - 4 = 3x - 1 + 10x + 4$
  OR $2 + 14x = 13x + 3$
  OR $14x - 13x = 3 - 2$
  OR $x = 1$

4. If a Twix bar costs t, he has $10 - 0.99 - 0.59 - 3t$ left and this is equal to 6.05.
$10 - 0.99 - 0.59 - 3t = 9.01 - 0.59 - 3t = 8.42 - 3t$, so $8.42 - 3t = 6.05$.
Adding $3t - 6.05$ to both sides give $8.42 - 6.05 = 3t$ or $2.37 = 3t$, with solution $t = 0.79$.

5. To simplify calculations, first define $y = 5x + 17$, solve for y, and then replace y $= 5x + 17$ to complete the problem. With this substitution the equation becomes
       $87\,y + 23(16 - y)$ $= 8(2 + 6y)$
 OR $87y + 23 \times 16 - 23\,y$ $= 16 + 48y$
 OR $64y - 48y = 16 - 23 \times 16$
 OR $16y = -22 \times 16$
 OR $y = -22$.
Since $5x + 17 = -22$, it must be that $5x = -22 - 17$ so $5x = -39$ and $x = -7.8$.

6. Denote the ages by G, Y and J. First since Dong Gil is three years younger than Dong Yeop,
     $G + 3 = Y$.
Second, in seven years, Dong Jae will be three years older than Dong Yeop was last year, so $(J + 7) = (Y - 1) + 3$ so $J + 7 = Y + 2$ or
     $J + 5 = Y$.
Third, since the mean of the ages is 14 1/3, $(G + J + Y) / 3 = 14\ 1/3$ or
     $G + J + Y = 43$.
Solving for G and J as functions of Y, we have $G = Y - 3$ and $J = Y - 5$.
Substituting these in the third equation gives $Y - 3 + Y - 5 + Y = 43$ or $3Y = 43 + 3 + 5$ or $3Y = 51$, so $Y = 17$, $G = 14$, $J = 12$. The question asks for Dong Gil's age, which is 14.

It is also possible to solve this as follows without substituting. Dong Gil is three years younger than Dong Yeop and Dong Jae is five years younger than Dong Yeop. Using ahead and behind counting, the sum of their ages is eight less than if all were Dong Yeop's age, so the average is 8/3 less than Dong Yeop's age, meaning that $Y = 14\ 1/3 + 8/3 = 17$.

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5.4 Identities

1. $3(x - 2) = 3x - W$ is equivalent to $3x - 6 = 3x - W$, which requires $W = 6$.

2. $\quad A(x + 3) - 2(x - 5) = x + 19$
OR $\quad Ax + 3A - 2x + 10 = x + 19$
OR $\quad Ax + 3A = 3x + 9$, which requires $A = 3$

Note that in a problem like this you didn't actually need to simplify the full equations. You could have equated the x coefficients, $A - 2 = 1$ or equated the constant terms $3A + 10 = 19$ and not bothered with the other one (assuming that the question can be taken to imply that there is an A that works.)

3. In this problem the K only appears as a coefficient on x so we can ignore the terms that don't involve x. The other terms are identical if
$\quad\quad 18\,(2x) + 6(K - 2)\,x = 25x$
OR $\quad 36x + 6Kx - 12x = 25x$
OR $\quad (24 + 6K)\,x = 25x$
This is an identity if K=1/6.

4. Here the W only appears in the constant term so we will again ignore all terms involving x and just find W to make the constant terms identical. The terms not involving x are
$\quad\quad 3(2W - 5) + 4(2W)/2 = 5(W)$
$\quad\quad 6W - 15 + 4W = 5W$
$\quad\quad 5W = 15$
The answer is W=3.

5. In this problem there are A's in both the x coefficients. Again, assuming that the question implies that some such A exists we are free to either look for the A that makes the coefficients on x equal on both sides or to look for the A that makes the constant terms on both sides equal. The x coefficients have lots of big numbers in front of them, so it seems easier to equate the terms that don't involve x. This gives
$\quad\quad (3A - 2)(-1) + (4 - 5A)(-2) = (6 + A)(1) + 6(A - 2)$
$\quad\quad 7A - 6 = 7A - 6$
Unfortunately, this does not let us find A. It is true for any A. At least by doing this, we learned that an A that equates the x coefficients will make the equation an identity. Looking for this A we want an A that satisfies
$\quad\quad (3A - 2)(237x) + (4 - 5A)(121x) = (6 + A)\,(105x)$
$\quad\quad (711A - 474)x + (484x - 605A)\,x = (630 + 105A)\,x$
$\quad\quad (106\,A + 10)\,x = (630x + 105A)\,x$
This is an identity if $106A + 10 = 105A + 630$ which is true for A=620.

6. The equation will be true whenever x=y if and only if it is of the form $Ax = Ay$ Checking the constant terms we see that they do cancel: we have $2 + 5 \times 3$ on the left side and 17 on the right side. So it just remains to find a B for which the

A - 25

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coefficient on x if we move all x's to the left side matches the coefficient on y if we move all y terms the right side. Collecting x's and y's together and putting them on these sides the equation takes the form

$$(2(3 + B) + 3B + 5)\, x = (2 + 5(4 - B) - 1)\, y$$

The coefficients match if

$$6 + 2B + 3B + 5 = 2 + 20 - 5B - 1$$
$$5B + 11 = 21 - 5B$$
$$10B = 10.$$

The answer is B=1.

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1. $A \triangle B = (A + B) \times A$ so $\frac{1}{2} \triangle \frac{1}{2} = (1/2 + 1/2) \times 1/2 = 1 \times 1/2 = 1/2$.

2. $A \star B = A^2 - B^2$ so $(5 \star 4) \star 8 = (5^2 - 4^2) \star 8 = (25 - 16) \star 8 = 9^2 - 8^2 = 81 - 64 = 17$.

3. $A @ B = (A^2 - B) * B$ and $A \$ B = \sqrt{A - B}$.
    Step 1: $5 \odot 2 = (5^2 - 2) \times 2 = 23 \times 2 = 46$
    Step 2: $(5 \odot 2) \bullet 10 = 46 \bullet 10 = \sqrt{46 - 10} = \sqrt{36} = 6$.
    Step 3: $21 \bullet 5 = \sqrt{21 - 5} = \sqrt{16} = 4$.
So $(5 \odot 2) \bullet 10 - (21 \bullet 5) = 6 - 4 = 2$.

4. Work from the inside of the parentheses at the far right backwards.
    Step 1: $3 \heartsuit (-2) = 3^2 + 3 - 2(-2) = 9 + 3 + 4 = 16$
    Step 2: $0 \heartsuit (3 \heartsuit 2) = 0 \heartsuit 16 = 0^2 - 0 - 2 * 16 = -32$
    Step 3: $0 \heartsuit (0 \heartsuit (3 \heartsuit 2)) = 0 \heartsuit (-32) = 0^2 - 0 - 2 \times (-32) = 64$
    Step 4: $0 \heartsuit (0 \heartsuit (0 \heartsuit (3 \heartsuit 2))) = 0 \heartsuit 64 = 0^2 - 0 - 2 \times (64) = -128$

5. The factors of $11^2 = 121$ are 1, 11 and 121. So the sum of these factors is 133. To find the factors of 133×101, first note that neither is divisible be 2, 3, or 5. Also 101 is not divisible by 7, so it is a prime, while $133 = 7 * 19$.

So the product $133 \times 101 = 7 \times 19 \times 101$. The sum-of-factors formula says that the sum of the factors is $(1 + 7)(1 + 19)(1 + 101) = 8 \times 20 \times 102 = 16320$.

6. If $x \odot y = x^2(x^2 + y)$, then $3 \odot 2 = 9(9+2) = 99$, $2 \odot 3 = 4(4+3) = 28$, $1 \odot 2 = 1(1+2) = 3$, and $2 \odot 1 = 4(4+1) = 20$.

So $(3 @ 2) @ 1 = 99 @ 1 = 99^2 * (99^2 + 1)$ which is going to be the largest number in the list.
$(2 @ 3) @ 1 = 28 @ 1 = 28^2 * (28^2 + 1)$ which is going to be the 2nd largest number in the list.
$(1 @ 2) @ 3 = 3 @ 3 = 3^2 * (3^2 + 3) = 9 * 12 = 108$.
$3 @ (2 @ 1) = 3 @ 20 = 3^2 * (3^2 + 20) = 9 * 29 = 263$.
$(1 @ (2 @ 3) = 1 @ 28 = 1^2 * (1^2 + 28) = 28*29 = 812$.

So the median is the 3rd largest of the 5 numbers, which is 812.

Meet 1 Solutions

Meet #2

2.1 Perimeter

1. In this figure AC = 3cm, BC = 4cm, and using the Pythagorean Theorem (or recognizing the 3-4-5 right triangle) we have $AB = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$ cm. The perimeter is $3 + 4 + 5 = 12$cm.

2. It is easiest to do the problem first pretending that the grid lines are one unit apart and then multiplying the answer you get this way by $\sqrt{5}$ at the end. With a one unit grid we can easily find each of the lengths by the Pythatogorean Theorem (or just counting in the case of the horizontal and vertical sides). $AB = \sqrt{1^2 + 2^2} = \sqrt{5}$. BC is also $\sqrt{1^2 + 2^2} = \sqrt{5}$. $CD = \sqrt{2^2 + 4^2} = \sqrt{20} = \sqrt{2^2 5} = 2\sqrt{5}$. DE is also $\sqrt{2^2 + 4^2} = 2\sqrt{5}$. EF = 1. And FA = 3. So if the grid lines were one unit apart the answer would be $\sqrt{5} + \sqrt{5} + 2\sqrt{5} + 2\sqrt{5} + 1 + 3 = 4 + 6\sqrt{5}$. To get the final answer we multiply this by $\sqrt{5}$ and get $\sqrt{5}(4 + 6\sqrt{5}) = 4\sqrt{5} + 6\sqrt{5}\sqrt{5} = 30 + 4\sqrt{5}$.

3. Three of the sides of ACEOG (AC, CE, and OG) are as long as AC, so the perimeter is 3AC + AG + EO. To find the lengths of these sides draw in lines dividing each hexagon into six equilateral triangles with side length 6. Adding the angles around E we see that FEG is also equilateral with side length 6. AG = AF + FG = 12. I didn't draw in the lines for the right hexagon, but looking at the left one it is also clear that EO is the sum of two triangle sides, and so it is also 12 cm. Finally, note that AC = AX + XC is the sum of two segments that are altitudes of the equilateral triangles. AB = 6 cm and BX = 3 cm, so by the Pythagorean Theorem $AX = \sqrt{6^2 - 3^2} = \sqrt{27} = 3\sqrt{3}$ cm. Hence AC = $6\sqrt{3}$ cm and the perimeter is $24 + 18\sqrt{3}$ cm.

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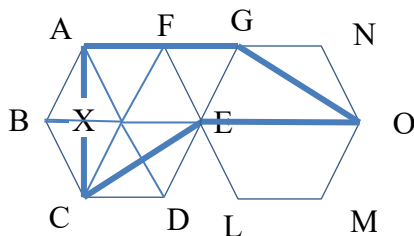
1. The fact that distances are measured in parsecs instead of inches or centimeters doesn't affect how we compute areas. We can still use the standard ½ base × height formula to find the area (in square parsecs). The answer is $\frac{1}{2} \times 36 \times 12\frac{1}{2} = \frac{1}{2} \times 36 \times \frac{25}{2} = 9 \times 25 = 225$ square parsecs.

2. Compute the area of the octagon by computing the areas of each of the parts it is divided into in the figure. The central square has an area of 4 cm². Each of the four rectangles Helen drew next has an area of $2\sqrt{2}$ cm². And each of the four triangles have area $\frac{1}{2}\sqrt{2}\sqrt{2} = 1$ cm². So the total area is $4 + 4 \times 2\sqrt{2} + 4 = 8 + 8\sqrt{2}$ cm².

3. To compute the area of the trapezoid draw altitudes from the upper corners to the lower base. Let h be the height of the trapezoid. Let x and y be the lengths of the segments to the left of the left altitude and to the right of the right altitude. The segment in between the two altitudes has a length of 9 feet because the middle part of the figure is a rectangle. Applying the Pythagorean Theorem to the two triangles we find that $x^2 + h^2 = 5^2$ and $y^2 + h^2 = 5^2$. This can only be true if x = y. Adding up the lengths along the bottom of the trapezoid we have $9 + x + y = 15$. Together these two imply that x = y = 3 feet. Our Pythagorean Theorem equation then tells us that h = 4 feet. And the formula for the area of a trapezoid gives that $A = \frac{1}{2}(9 + 15)\,4 = 48$ square feet.

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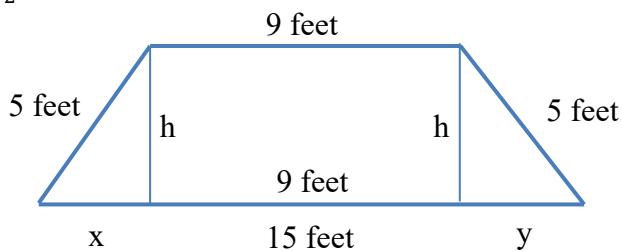
2.3 Perimeters of Rectilinear Figures

1. The figure here is a standard rectilinear figure. It is 4 blocks high and 7 blocks wide. Any walk from A to B that does not involve backtracking will have a length of 11 blocks. (You can count segments of both paths if you want to convince yourself.) The total length that Wei walks is $11 + 11 = 22$ blocks. If each block is 500 feet the total distance is $22 \times 500 = 11,000$ feet.

2. In the first question the rectilinear figure consisted of two paths from its upper left corner to its lower right corner. But this is not necessary for the theorem that the perimeter is equal to be twice the sum of the height plus width to be true. All you need for that is that there is no backtracking, i.e. that the is only one shift from going up to going down and only one shift from going left to going right as you go around the perimeter. This is true in this figure as well. Starting from the lower left corner all steps are up (or horizontal) until you reach the upper horizontal segment after which all future steps are down (or horizontal). And starting at the same point all steps are to the right (or vertical) until you reach the rightmost vertical segment, after which they are all to the left (or vertical). So all we need to do is to find the height and width of the figure.

You can find the total height by adding the segments along the left side: 5 cm + 3 cm = 8 cm. You can find the total width by adding the segments along the bottom side of the figure: 24 cm +2 cm = 26 cm. The perimeter is $2(8 + 26) = 68$ cm.

3. "Hexadecagon" is a fancy word for a sixteen-sided polygon. This one involves backtracking. At the top of a figure there is a 5 cm wide dip where it goes down and then back up. And along the right side of the figure the segment with the curvy arrow pointing to it is going 2 cm back to the left before the figure later continues on to a point that is further to the right. The perimeter of the figure will be 2 (height + width) + 2 (total length of wrong-direction segments). Here, width means the distance in the x dimension from the farthest left to the farthest right point. And height means the y distance from the lowest to the highest point.

The total width is the distance along the bottom of the figure, 38 cm. The total height can be obtained by adding the vertical segments along the right side of the figure: $6 + 4 + 2 + 8 = 20$ cm. The backtracking on the right is 2 cm. The length of the backtracking along the top cannot be calculated just by adding and subtracting segments. We need to use the fact about the area.

Ignoring the backtracking along the top we could divide the figure into rectangles and squares as shown below. The area would be $140 + 320 + 108 + 4 + 18 = 590$ cm^2. For the area to be 20 cm^2 less than this, it must be that the backtracking square along the top has an area of 20 cm. This means that its height is 4cm.

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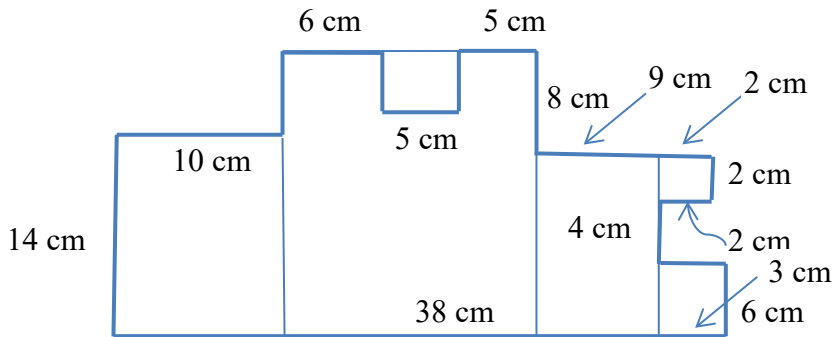
The perimeter is $2(38 + 20) + 2(2 + 4) = 128$ cm.

2.4 Problem Solving Tip: Subtract

1. We compute the area as the difference between the area of ABC and the area of ADC. We can compute both areas using the one-half base times height formula. Base AC has length $AH + HC = 5$ cm $+ 11$ cm $= 16$ cm. Using this we find:

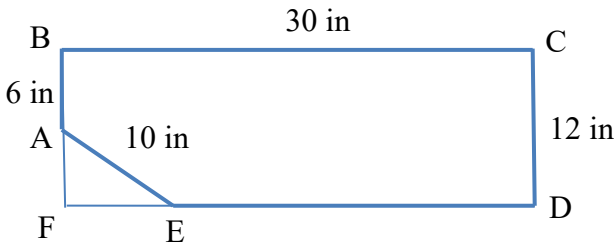Area(ABC) $= \frac{1}{2}(16)(4 + 5) = 72$ cm².

Area(ADC) $= \frac{1}{2}(16)(5) = 40$ cm².

So the area of ABCD is $72 - 40 = 32$ cm².

2. Extend sides BA and DE to meet at F. This makes it clear that pentagon ABCDE is a rectangle with a triangle cut off from the corner, so its area is equal to the area of the rectangle BCDF minus the area of triangle FAE.

Area(BCDF) $= 30 \times 12 = 360$ square inches.

To find the area of triangle FAE we first note that $FA = 6$ inches by subtracting the length of BA from the length of the right side of the rectangle. We can then find that the length of FE is 8 inches using the Pythagorean Theorem: the hypotenuse has length 10 and one leg of a right triangle has length 6 so other leg has length 8 inches. Using this we have

Area(FAE) $= \frac{1}{2}8 \times 6 = 24$ square inches.

By subtraction the area of ABCDE is $360 - 24 = 336$ square inches.

3. Let A be the point at the lower left corner of the grid and K be the point at the upper left corner of the grid. We can then think of triangle ICE as what is left of trapezoid AKIE after we cut triangle ACE off the lower left corner and triangle ICK off the upper left corner.
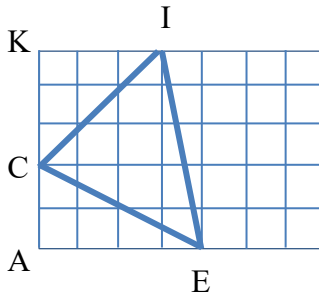
Trapezoid AKIE has an upper base of length 18 inches, a lower base of length 24 inches, and a height of 30 inches, so its area is

Area(AKIE) $= \frac{1}{2}(18 + 24)\,30 = 630$ square inches.

The two cut off triangles are both right triangles. Their areas are:

Area(ACE) $= \frac{1}{2}18 \times 18 = 162$ square inches.

Area(ICK) $= \frac{1}{2}12 \times 24 = 144$ square inches.

Subtracting, the area of ICE is $630 - (162 + 144) = 324$ square inches. One square foot is 144 square inches so the area in square feet is $\frac{324}{144} = \frac{81}{36} = \frac{9}{4} = 2\frac{1}{4}$ sq. feet. One can also do the calculation by keeping the distances in feet which requires working with fractions. Even easier is to find the area in square grid cells and then to convert to square feet at the end using that each grid cell is one fourth of a square foot. But it might be easier to get confused and make a mistake this way.

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2.5 Advanced Topic: More Area Formulas

1. Using Heron's formula after computing s = (13 + 14 + 15)/2 = 21 cm, the area is $\sqrt{21(21-13)(21-14)(21-15)}$. Rather than multiplying things out and needing to find the square root of a big number it is easier to keep everything factored and look for factors that can be grouped together to make squares. The area is

$$\sqrt{(3 \cdot 7) \cdot 8 \cdot 7 \cdot 6} = \sqrt{3^2 \cdot 7^2 \cdot 16} = 3 \cdot 7 \cdot 4 = 84 \text{ square centimeters.}$$

2. To find the inradius r we compute the area in two different ways: using Heron's formula; and using A = rs.
For Heron's formula we have s = (4 + 5 + 7)/2 = 8 so the area is $\sqrt{8(8-4)(8-5)(8-7)} = 4\sqrt{6}$.
Using A = rs then gives $4\sqrt{6} = 8r$ so $r = \frac{4\sqrt{6}}{8} = \frac{\sqrt{6}}{2}$ cm.

3. Connecting the center of the circle to the 8 vertices of the octagon divides the octagon into 8 congruent triangles.
One easy way to find each triangle's area (assuming you know some trig) is to use the formula $A = \frac{1}{2}ab\sin C = \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin(45) = \frac{1}{2} \cdot 1 \cdot 1 \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}$. (The sides connecting the center of the circle to the vertices have length 1 because they are radii of the circle.) The area of the octagon is 8 times this or $2\sqrt{2}$ sq. feet.
It requires more cleverness, but you can also find the area of one of the triangles by the standard one-half base height formula. Think of the octagon as on a grid with the center of the circle at (0, 0) and one vertex of the octagon at (1, 0). The next vertex above this point is equidistant between the x- and y-axes so its location is of the form (x, x) and satisfies $x^2 + x^2 = 1$. This gives $x = \frac{1}{\sqrt{2}}$. Treating the segment connecting (0, 0) and (1, 0) as the base $\frac{1}{\sqrt{2}}$ is the height so the standard formula gives the area we found above.
Note that $2\sqrt{2} \approx 2.83$ is a little less than the area of the circle, which is π. This should give you some confidence that you didn't make a computational error so you can use your time available for checking on other problems.

4. The first step in solving this problem is to drop perpendicular from the upper left and right corners to the base just as we did when solving the final problem on the worksheet for Meet 2 section 2.2. But this time the trapezoid is NOT symmetric so we will need to use some algebra to find the lengths of the segments that the bottom of the trapezoid is divided into and to find the height.
Let x be the length of the leftmost segment into which the base is divided and let y be the length of the rightmost segment. The middle segment has length 145 (the length of the top) so x + y = 70 which I will use to say y = 70 − x.
Applying the Pythagorean Theorem to the triangles on the left and right sides of the figures gives

A - 34

$x^2 + h^2 = 65^2$ and $(70 - x)^2 + h^2 = 75^2$. Normally, getting multiple nonlinear equations is not a sign of going in a productive direction. But here, if you expand $(70 - x)^2 = 4900 - 140x + x^2$, you may notice that the terms with squares are the same in both equations so they'll all cancel out if you subtract one equation from the other. Subtracting the first equation from the second gives $4900 - 140x = 75^2 - 65^2 = (75 - 65)(75 + 65) = 1400$. This simplifies to $140x = 3500$. Dividing both sides by 70 gives $2x = 50$ so $x = 25$.

Once we've found x we can find the height of the trapezoid using either of the Pythagorean Theorem equations above: $25^2 + h^2 = 65^2$ gives $h^2 = 65^2 - 25^2 = (65 - 25)(65 + 25) = 40 \cdot 90 = 3600$. This implies that $h = 60$ inches $= 5$ feet. The area of the trapezoid in square feet $\frac{1}{2}\left(\frac{215+145}{12}\right)5 = \frac{1}{2}30 \cdot 5 = 75$ sq. feet.

145 inches

65 inches   75 inches

h   h

x   215 inches   70-x
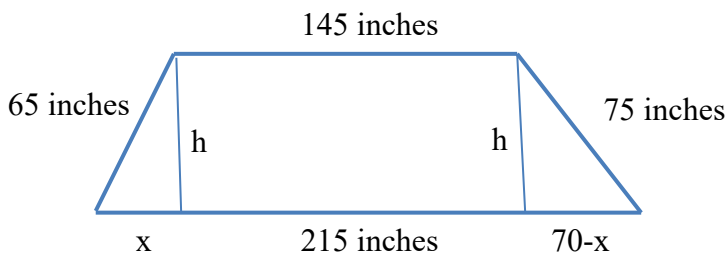
If I got this answer at the end of doing a a long problem I wouldn't feel as confident about skipping checking as I would have after doing problem 3. But I'd still feel pretty confident. It is unlikely that you'd get an answer that was as nice and round as 75 unless someone had designed the problem to come up with a simple answer. Perhaps you could get an answer like this and still get the problem wrong if you accidentally computed twice or half of the answer. But the trapezoid looks roughly as big as a rectangle 15 feet (180 inches) wide and 5 feet high – if you eyeball it the height looks like about one-third of the width – so 75 seems close enough to right that you should be pretty confident that you aren't off by a factor of 2 or 10.

3.1 A Super Quick Review of Prime Factorization

1. The prime factorization of 55 is 5×11 so the sum of the prime factors of 55 is $5 + 11 = 16$.

2. The prime factorization of $169 = 13^2$ so the factors of 169 are $13^0 = 1$, $13^1 = 13$, and $13^2 = 169$.

3. $3500 = 35 \times 100 = 5 \times 7 \times 10 \times 10 = 5 \times 7 \times 2 \times 5 \times 2 \times 5 = 2^2 \times 5^3 \times 7 = 2^2 \times 3^0 \times 5^3 \times 7^1$. So $a = 2$, $b = 0$, $c = 3$, $d = 1$ and $(a + b) \times (c + d) = (2 + 0)(3 + 1) = 8$.

4. $980,000 = 98 \times 10,000 = 2 \times 49 \times 10^4 = 2 \times 7^2 \times (2 \times 5)^4 = 2^5 \times 5^4 \times 7^2$. So $a = 5$, $b = 0$, $c = 4$, $d = 2$ and the maximum of a, b, c, $d = 5$.

5. $1,464,100 = 100 \times 14,641$. 14,641 is not divisible by 2, 3, 5, or 7, but it is divisible by 11 and in fact $14,641 = 11 \times 1331 = 11 \times 11 \times 121 = 11 \times 11 \times 11^2 = 11^4$. So $1,464,100 = 2^2 * 5^2 * 11^4$. Its square root is $2^1 \times 5^1 \times 11^2$ with $p = 2$, $q = 5$, $r = 11$, $a = 1$, $b = 1$, and $c = 2$.

6. If the month and day are not multiples or factors of 3, 5, or 7, each could be 2 (but not 1 since 1 is a factor of 3). She cannot have been born in 2000 since this is a multiple of 5 or in 2001 since this is a multiple of 3. (Note that the sum of digits is $2001 = 3$ so it is divisible by 3.) The number 2002 is not divisible by 3 or 5, but it is divisible by 7 as $7 \times 286 = 2002$. The number 2003 is not divisible by any of 3, 5, or 7, so she could have been born in 2003. In sum, the earliest possible birth date is February 2, 2003.

3.2 Greatest Common Factors

1. The prime factorization of $27 = 3 \times 9 = 3^3$. The prime factorization of $39 = 3 \times 13$. So the/ greatest common factor of 27 and 39 is 3.

2. The prime factorization of $84 = 2 \times 42 = 2^2 \times 21 = 2^2 \times 3 \times 7$. The prime factorization of $119 = 7 \times 17$. So the greatest common factor of 84 and 119 is 7.

3. $34{,}300 = 100 \times 343 = 100 \times 7 \times 49 = 2^2 \times 5^2 \times 7^3$. So the greatest common factor of 34,300 and $2^1\, 3^2\, 5^3\, 7^4\, 11^5$ is $2^1 \times 5^2 \times 7^3$, which is $2 \times 25 \times 343 = 50 \times 343 = 34{,}300 \div 2 = 17{,}150$.

4. 54 yellow cupcakes and 123 chocolate cupcakes fit into separate boxes so the number of cupcakes in each box must divide both 54 and 123 evenly. The prime factorization of 54 is 2×3^3 and the prime factorization of 123 is 3×41. So the only common factor of 54 and 123 (other than 1) is 3. So there must be 3 cupcakes per box.

5. 1 hour is $60 \times 60 = 3600$ seconds, so Mae ran for $3600 + 9 \times 60 + 31 = 3600 + 540 + 31 = 4171$ seconds on Monday. On Tuesday, she ran for $16 \times 60 + 10 = 960 + 10 = 970$ seconds. $970 = 10 \times 97 = 2 \times 5 \times 97$ where 97 is prime since it is not divisible by 2, 3, 5, or 7. 4171 is not divisible by 2 or 5, so it makes sense to check if it is divisible by 97: $97 \times 40 = 3880$, $4171 - 3880 = 291$, and $97 \times 3 = 291$. So $4171 = 43 \times 97$. So, assuming an integer lap time, Mae ran 43 laps of 97 seconds each on Monday and 10 laps of 97 seconds each on Tuesday.

6. In inches, the classroom is $15 \times 12 = 180$ inches wide and $22.5 \times 12 = 270$ inches long. The prime factorizations of these numbers are $180 = 2^2 \times 45 = 2^2 \times 3^2 \times 5$ and $270 = 10 \times 27 = 2 \times 3^3 \times 5$. The greatest common factor of 180 and 270 is then $2 \times 3^2 \times 5 = 18 \times 5 = 90$. So you could cover the floor of the room with 90 inch by 90 inch square tiles. You would need two of these tiles to cover the width of the floor and three of them to cover the length, so this would be $2*3 = 6$ tiles of this size to cover the floor.

7. The easiest way to approach this problem is to work with the smallest of the three numbers, with is 9,282 and find its prime factorization. $9{,}282 = 2 \times 4641$ and 4641 is divisible by 3 since its digits sum to 15, which is divisible by 3. $9{,}282 = 2 \times 3 \times 1547 = 2 \times 3 \times 7 \times 221$, which is equal to $2 \times 3 \times 7 \times 13 \times 17$. So we just need to check if any of 2, 3, 7, 13, and 17 are common factors of the larger numbers 104,101 and 22,303. In fact, none of these numbers are factors of 104,101 (which is actually 19×5479 where 5479 is prime) or 22,303 (which is itself prime).

So GCF(9,282, (GCF(104,101, 22,303)) = 1.

3.3 Least Common Multiples

1. Finding the prime factorizations of these numbers, $30 = 2 \times 3 \times 5$ and $40 = 2^3 \times 5$. So their least common multiples is $2^3 \times 3 \times 5 = 8 \times 3 \times 5 = 24 \times 5 = 120$.

2. The least common multiple of 10 and 22 is $110 = 2 \times 5 \times 11$ since $10 = 2 \times 5$ and $22 = 2 \times 11$.

3. The prime factorizations of these numbers are $420 = 2 \times 210 = 2^2 \times 5 \times 21 = 2^2 \times 3 \times 5 \times 7$ and $1212 = 12 \times 101 = 2^2 \times 3 \times 101$. So the least common multiple is $2^2 \times 3 \times 5 \times 7 \times 101 = 420 \times 101 = 42000 + 420 = 42{,}420$.

4. Sora takes 165 seconds per lap where $165 = 5 \times 33 = 3 \times 5 \times 11$. Ji takes 195 seconds per lap where $195 = 5 \times 39 = 3 \times 5 \times 13$. So Sora arrives back at the starting point at times that are multiples of 165 seconds from the start while Ji arrives back at the starting point at times that are multilples of 195 seconds from the start.
The least common multiple of 165 and 195 is $3 \times 5 \times 11 \times 13 = 195 \times 11 = 1950 + 195 = 2145$. That is, in 2145 seconds, Sora runs 13 laps and Ji runs 11 laps and both of them arrive back at the starting point at the same time. Since Ji runs a lap in 3 minutes and 15 seconds, this is a total of $11 \times 3 + 11 \times 1/4 = 33 + 2\,\frac{3}{4} = 35$ minutes and 45 seconds from the start.
Another way to think about this problem is to recognize that both 2 minutes and 45 seconds and 3 minutes and 15 seconds are multiples of 15 seconds. If Sora takes 11 fifteen-second periods to run a lap and Ji takes 13 fifteen-second periods then they will first finish laps together after $11 \times 13 = 143$ fifteen-second periods.

5. $78 = 2 \times 39 = 2 \times 3 \times 13$ and $778 = 2 \times 389$, where 389 is not divisible by any of 2, 3, or 13. So the least common multiple of 78 and 778 is $2 \times 3 \times 13 \times 389 = 78 \times 389 = 80 \times 389 - 2 \times 389 = 80 \times 400 - 80 \times 11 - 2 * 389 = 32{,}000 - 880 - 778 = 32{,}000 - 1{,}658 = 30{,}342$.

6. Here $221 = 13 \times 17$, $6460 = 20 \times 323 = 2^2 \times 5 \times 17 \times 19$ and $36{,}100 = 100 \times 361 = 2^2 * 5^2 * 19^2$. So LCM(36100, 221) $= 2^2 \times 5^2 \times 13 \times 17 \times 19^2$ and LCM(36100, 6460) $= 2^2 \times 5^2 \times 17 \times 19^2$.

LCM(x, LCM(36100, 6460)) $= 2^2 \times 5^2 \times 17 \times 19^2$ plus any additional factors in the prime factorization of x. So x must be a multiple of 13 that has only 2, 5, 13, 17 and 19 in its prime factorization. (And 2, 5, and 19 cannot be raised to powers above 2 and 13 and 17 must appear at most once.) Since x is a two digit number, it could be 13, 13×2, 13×4 or 13×5, but nothing greater than that since $13 \times 10 = 130$ is already too big. So there are four possible values of x, which are 13, 26, 52 and 65.

1. $14 = 2 \times 7$, $18 = 2 \times 3^2$, $26 = 2 \times 13$, $21 = 3 \times 7$ and $20 = 2^2 \times 5$. These values have no single factor in common so their GCF is 1.

2. Since $x \times y = $ LCM $\times$ GCF, we know LCM $= 2688 \div 8 = 336$.

3. $x \times y = $ LCM $\times$ GCF $= 37 \times 666 = 666 \times 40 - 666 \times 3 = 26640 - 1998 = 26640 - 2000 + 2 = 24640 + 2 = 24642$.

4. We want to find LCM(6, 12, 15, 20), where $6 = 2 \times 3$, $12 = 2^2 \times 3$, $15 = 3 \times 5$, and $20 = 2^2 \times 5$, so LCM $= 2^2 \times 3 \times 5 = 60$. So the next time that all are chasing the same ball will be the 60th ball after the ball that is thrown at 10:27. With one ball thrown per minute, this will be an hour later or 11:27.

5. LCM(n, 60) $= 60$ means that n must be a factor of 60. GCF(n, 6) $= 2$ means that n is even but not a multiple of 3. The prime factorization of 60 is $2^2 \times 3 \times 5$, so if n is not a multiple of 3, it must be an even factor of $2^2 \times 5 = 20$, which means that it must be of form $2^a \, 5^b$ where a is 1 or 2 and b is 0 or 1:
$$2 = 2^1 * 5^0, \quad 4 = 2^2 * 5^0, \quad 10 = 2^1 * 5^1 \quad \text{or } 20 = 2^2 * 51.$$
So there are 4 possible values of n.

6. LCM(a, b) $= 488$ and LCM(c, d) $= 201$, where $488 = 2^3 \times 61$ and $201 = 3 \times 67$. So 488 and 201 have no prime factors in common. So LCM(a, b, c, d) $= 488 \times 201 = 488 \times 2 \times* 100 + 488 = 976 \times 100 + 488 = 97,600 + 488 = 98,088$.

3.5 More on Prime Factorization

1. There are $(3+1)\,(1+1)\,(4+1)\,(1+1)\,(3+1) = 4\times2\times5\times2\times4 = 8\times10\times4 = 320$ factors.

2. $396 = 2\times198 = 2\times2\times99 = 2^2\times3\times33 = 2^2\times3^2\times11$.

3. $84 = 2^2\times21 = 2^2\times3\times7$ is of form $2^a\,3^b\,7^c$ where $a = 2$, $b = 1$, $c = 1$, so it has a total of $(a+1)\,(b+1)\,(c+1) = 3\times2\times2 = 12$ factors.

4. $120 = 2^3\times15 = 2^3\times3\times5$. The sum of the factors formula gives that the sum of the factors is $(1 + 2^1 + 2^2 + 2^3)\,(1 + 3^1)(1 + 5^1) = 15\times4\times6 = 360$.

5. $\text{LCM}(n\,,\,48) = 48$ if and only if n is a factor of 48, so the question is asking for the sum of the factors of 48. The prime factorization of 48 is $2^4\times3$. The sum of the factors is $(1 + 2 + 4 + 8 + 16)\,(1 + 3) = 31\times4 = 124$.

6. First find the prime factorization of $22{,}500 = 100 * 225 = 10^2\times15^2 = 2^2\times3^2\times5^4$. The other number is really big and daunting, but we don't need to find all of its factors, we just need to see whether it is divisible by 2 or 2^2, whether it is divisible by 3 or 3^2 and whether it is divisible by some power of 5 up to the 4th. It is obviously a multiple of $1000 = 2^3\times5^3$. The number that remains after dividing by 1000 ends with an 8, so it is not a multiple of 5 which means that there is not a fourth five in the prime factorization of the big number. Also, the digits add to 39, so the big number is a multiple of 3, but not a multiple of 9. Together, these imply that the greatest common factor of the two numbers is $2^2\times3\times5^3 = 10^2\times3\times15 = 1500$.

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4.1 Fractions and Percents

1. $0.4 \times 85 = 85 \times 4 \div 10 = 340 \div 10 = 34$.

2. Two-thirds of 78 is $78 \times 2 \div 3$. A 25% increase on that number means multiplying by $1.25 = 5/4$. So the answer is $(78 * 2 \div 3) \times 5 \div 4 = (156 \div 3) \times 5 \div 4 = 52 \times 5 \div 4 = 260 \div 4 = 65$. A tip for doing calculations like this it's easier to do the divisions before the multiplications if they divisions produce whole numbers. For example, $(78 \div 3) \times 2$ is quicker than $(78 \times 2) \div 3$ and $(52 \div 4) \times 5$ is quicker than of $(52 \times 5) \div 4$.

3. Mr. Smith's class solves 7/8 of the problems. Mrs. Smith's class solves $85/100 = 17/20$ of the problems. If Mrs. Smith's class attempted 80 problems, then they solved $17 \times 4 = 68$ of them. If Mr. Smith's class attempted 10% more problems, they attempted $80 \times 1.1 = 88$ problems and solved $88 \times 7/8 = 11 * 7 = 77$ problems. So Mr. Smith's class solved $77 - 68$ or 9 more problems than Mrs. Smith's class solved.

4. We want to find $408 \times (55/100) \times (3/11) \times (65 / 100) = 408 \times (11/20) \times (3/11) \times (13/20)$. The product of the fractions is $\frac{11 \times 3 \times 13}{20 \times 11 \times 20} = \frac{3 \times 13}{20 \times 20}$. The original number 408 has prime factorization $8 \times 51 = 3 \times 8 \times 17$. So the answer is $\frac{3 \times 13 \times 3 \times 8 \times 17}{20 \times 20} = \frac{3 \times 13 \times 3 \times 17 \times 2}{100} = 39 \times \frac{102}{100} = 39.78$.

5. Denote the weights by T and P. When you have a really complicated sentence like the first one in this question it help to try to break it up into parts. For example, in this question I might think of it as saying that $T \times 7/9 = X \times 7/8$ where X is the alternate weight of the Prius, i.e. $X = 1.1 \times 4/5 \times P$. This makes it easier to see that the equation is $T \times 7/9 = 7/8 \times 1.1 \times 4/5$ P

      OR $T = (9/7) (7/8) (11/10)(4/5)$ P
      OR $T = P (7 \times 11 \times 9 \times 4) / (8 \times 10 \times 5 \times 7)$
      OR $T = P (11 \times 9) / (2 \times 10 \times 5)$
      OR $T = 99$ P $/ 100$.

So the weight of the Prius is $(100/99)$ times the weight of the Tesla. $2475 = 99 \times 25$, so this is just 2500 pounds.

6. There are nine classrooms so fewer than $9 \times 27 = 243$ students. Also there must be at least $8 \times 27 = 216$ students, as otherwise, there would be only eight classrooms.

The unusual element of the percentages listed in the problem is that $.51 \times .33 = .1683$, which would ordinarily be rounded to .17. This suggests that both the 51% and the 33% must be rounded up. (Checking that $0.505 \times .33$ and 0.51×0.325 are greater than 0.165 shows that this is indeed true.) With standard rounding rules, at least proportion .505 of the students are girls and at least proportion .325 of girls

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wear makeup. Since .325 = 325/1000 = 65/200 = 13/40, it is natural to look for solutions where the number of girls is a multiple of 40.

If there are 120 girls with 39 wearing makeup, then the number of students overall must be somewhere between 120 / .515 and 120 / .505, which means that it is between 233 and 237. For the 39 girls wearing makeup to be 16% of the class, the number of students overall must be between 39 / .155 and .39 / .165 or 237 and 251 so the only value that satisfies both conditions is n = 237. The answer is indeed that there are 237 students, with 120 girls and 39 girls wearing makeup.

4.2 Terminating and Repeating Decimals (Part 1)

1. The repeating fraction $.\overline{45} = 45/99 = 15/33 = 5/11$.

2. Breaking the second fraction into two parts (the part before the repeating decimal and the repeating part) we have

$0.1\overline{6} = 0.1 + 0.0\overline{6} = 0.1 + (0.\overline{6})/10 = 1/10 + (6/9)/10 = 1/10 + 6/90 = 15/90 = 1/6$.

Also, $.\overline{3} = 3/9 = 1/3$. So $0.\overline{3} - 0.1\overline{6} = 1/3 - 1/6 = 1/6$.

3.
$0.1\overline{2} = 0.1 + 0.0\overline{2} = 1/10 + (1/10)(0.\overline{2}) = (1/10) + (1/10)(1/9) = 9/90 + 2/90 = 11/90$.

4.
$0.1\overline{27} = 0.1 + 0.0\overline{27} = 1/10 + (1/10)(0.\overline{27}) = (1/10) + (1/10)(27/99) = 1/10 + (1/10)(3/11) = 14/110$

$0.\overline{27} = 27/99 = 27/99$.

So $\dfrac{0.1\overline{27}}{0.\overline{27}} = (14 \times 99)/(27 \times 110) = (14 \times 9)/(27 \times 10) = 14/30 = 7/15$.

One could also do this problem as $0.1\overline{27} / 0.\overline{27} = .1/\overline{27} + 0.0\overline{27} / 0.\overline{27} = (1/10)/(3/11) + (1/10) = (1/10)(14/3) = 14/30 = 7/15$.

5. $0.10\overline{101} = \frac{1}{10} + \frac{1}{100}\frac{101}{999} = \frac{9990+101}{99900} = \frac{10091}{99900}$. $\overline{10} = \frac{10}{99}$. Subtracting these two fractions turns out to be easier than subtracting fractions with large denominators often is:

$\frac{10091}{99900} - \frac{10}{99} = \frac{999009}{99900 \times 99} - \frac{999000}{99900 \times 99} = \frac{9}{99900 \times 99} = \frac{1}{99900 \times 11} = \frac{1}{1098900}$.

6. In the second decimal, every odd digit after the decimal place is a 1. In the first decimal, after digit 1, digits 3, 5, 6, and 8 are 1's and then the pattern repeats every 6 digits.

Looking just at the six digits starting from the 3rd after the decimal place, we have
        .101101 - .101010 = .000091.
So the difference of these two fractions is $.\overline{000091}$ from the third decimal place on. In decimal form, this is $0.00\overline{000091}$.

You also could have done problem 5 more elegantly by doing problem 8 first. As a fraction, $.\overline{000091}$ is $91/(1,000,000 - 1) = 91/999,999$, so $0.00\overline{000091}$ is $91/99,999,900$. The only thing that keeps us from being done immediately is that this may not yet be in lowest terms. One could try dividing the denominator by 7 and 13 in turn to see if either is a common factor. Indeed, it turns out that the both

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are. The easiest way to see this is to realize that the prime factorization of 999,999 is $999,999 = 3^2 * 111,111 = 3^2 \times 111 \times 1001$ and 1001 is well known to be $7 \times 11 \times 13$ which is a multiple of both 7 and 13. So we can just use long division once dividing the denominator by 91, or cancel factors and multiply to find $91/999,999,900 = 1 / (100 \times 9 \times 111 \times 11) = 1/(100 \times (1000 - 1) \times 11) = 1/((11,000 - 11) \times 100) = 1 / 1,098,900.$

4.3 Terminating and Repeating Decimals (Part 2)

1. $1/30 = 1/10 \times 1/3 = 1/10 \times 0.\overline{3} = 0.0\overline{3}$

2. $5/120 = 1/24 = 1/3 \times 1/8 = (125/1000) \times (1/3) = (1/1000) \times (125/3) = (1/1000) \times 41\tfrac{2}{3} = 0.041\overline{6}$. Every digit from the fourth after the decimal point onwards is a 6, so the 20th digit to the right of the decimal point is a 6.

3. It is easiest to convert this to a repeating decimal by first multiplying the numerator and denominator by 99 so that the denominator is 1 less than a power of 10.

$17/101 = (17 * 99) / (101 * 99) = (1700 - 17) / (9999) = 1683/9999 = .\overline{1683}$. This pattern repeats every four digits. Since 52 is a multiple of 4, the 53rd digit after the decimal point is a 1, the 54th digit after the decimal point is a 6 and the 55th digit after the decimal point is an 8.

4. One way to do this is to figure out that 99,999 is divisible by 41. More likely, however, is to proceed by long division until you see the repeat and find that $3/41 = .\overline{07317}$. Most middle school kids have never seen a repeating decimal with a repeat length of five because $99999 = 9 \times 41 \times 271$ so the only fractions with two digit denominators with such a repeat length are the ones with 41 or 82 in the denominator.

5. The natural starting point here is that we know that $1/99 = .\overline{01}$, so rather than combine the fractions, it may be easiest to simply compute 1/35 as a repeating decimal. The fraction 1/7 has a repeating cycle of 6 digits and since $5 \times 7 = 35$, we should expect 1/35 to have a repeating cycle of six digits as well. Specifically, $1/35 = (1/5) \times (1/7) = (2/10) \times (1/7) = (1/10) \times (2/7) = .0\overline{285714}$. Using the good old fashioned addition with carrying algorithm we see that $1/99 + 1/35 = .0\overline{386724}$, because the addition of 1/99 causes us to add 1 to every other digit in the repeating decimal form of 1/35. This fraction has a repeating cycle of length 6 starting with the 2nd digit to the right of the decimal point. So it begins at digits 2, 8, 14, and 20, indicating that the 20th digit to the right of the decimal point is a 3.

6. By long division, $3/13 = .\overline{230769}$, a repeating decimal of length 6. Since 12 is divisible by 6, the 13th digit after the decimal point is a 2, the 14th digit after the decimal point is a 3 and the 15th digit after the decimal point is a 0.

7. The number we are looking for must have a repeat length of at least 10 (or have several patternless digits and then a repeat length of something close to 10.) We have already seen that the repeat length of 1/3 is 1, the repeat length of 1/7 is 6, the repeat length of 1/11 is 2, and the repeat length of /13 is 6. The repeat length of a product of primes is the LCM of the repeat lengths of the individual primes. And

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multiplying a prime by some number powers of 2 and 5 only adds at most that number of patternless digits to the beginning of the decimal. So the first candidate for having all different digits in its decimal is 1/17. Using long division we find that 1/17 = 0.0588235294117… One could keep dividing to find the full repeating pattern – it has a repeat length of 16 – but it is not necessary once you have gotten to this point because we already have all 10 digits.

5.1 Sums of Arithmetic Sequences

1. This is an arithmetic sequence with 9 terms and difference $d = 10$. Pairing the terms gives (15, 95), (25, 85), (35, 75), (45, 65) with 55 left over. One could add this as four pairs each adding up to 110 with 55 left over so $4 \times 110 + 55 = 495$. Or one can think of it as nine terms with average value of 55. This way the calculation is $9 \times 55 = 495$.

2. Their ages give an arithmetic sequence with $N = 3$ and $d = 5$. Since N is odd, the average is the value in the middle, meaning that the age of the middle brother is $48 \div 3 = 16$. So Artur's age is $16 - 5 = 11$ and the three ages are 11, 16, 21.

3. The number of views on each day for the week gives an arithmetic sequence with $N = 7$ and $d = 101$. The average number of views is $3,619 \div 7 = 517$. Since N is odd, the average must be equal to the middle term so there were 517 views on the 4th day. Then the number of views on the first day is $517 - 3 \times 101 = 214$.

4. There are between 28 and 31 days in a month, so either four or five Sundays in a given month. The 1st Sunday is on a date from 1 to 7 in the month, the 2nd Sunday is on a date from 8 to 14 in the month, the 3rd Sunday is on a date from 15 to 21, the 4th Sunday is on a date from 22 to 28 and the 5th Sunday (if there is one) is on a date from 29 to 31.

If the sum of four dates is 75, then their average is 16.75, but this is not possible as every four term arithmetic sequence has an integer average or an average with decimal 0.5. (This is because it is the average of the first and last numbers.)

If the sum of five dates is 75, then their average is $75 \div 5 = 15$. This is possible if the third Sunday is on the 15th, so the series of Sunday dates is 1, 8, 15, 22, 29. So Sunday is the 15th and the 17th is two days later on a Tuesday.

An alternate way to do this would be to think about whether the month will have four or five Sundays by thinking of how big the sum of the dates would be in each case. If there are four Sundays in the month, then the sum of the dates of those Sundays is an arithmetic sequence with $N = 4$, $d = 7$, and a first term between 1 and 7. The smallest possible sum would be $(1 + 22) \times 2 = 46$ if the first Sunday falls on the 1st and the largest possible sum would be $(7 + 28) \times 2 = 70$ if the first Sunday falls on the 7th. So even the maximum possible sum is too small.

If there are five Sundays in the month, then the middle Sunday could be on the 15th, 16th or 17th. The sum of the dates of the Sundays in these cases are $5 \times 15 = 75$, $5 \times 16 = 80$ and $5 \times 17 = 85$. So Sunday must be on the 15th which means that the 17th is a Tuesday.

5. The 13th term is the 1st term plus 12×21 or 1st term $+ 252$. If the first term is A, then the last term is $A + 252$ so the average of the sequence is the average of these

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two values, which is A + 126. There are 13 terms in the sequence with average A + 126 so the total is

$$13 \times (A + 126) = 13A + 1{,}638.$$

For this sum to end in a zero, 13A must have ones digit equal to 2.
Since the first term A is a multiple of 21, the possibilities for 13 A are

> 13 × 21 which ends in a 3
> 13 × 42 which ends in a 6
> 13 × 63 which ends in a 9
> 13 × 84 which ends in a 2.

So the smallest possible value for the first term of the sequence is 84.

Another way to think about this is to think that the sum will by 13 times the middle term, which is 13× (A + 126). This ends with a zero if and only if it is a multiple of 10, which requires that A be for more than a multiple of 10. The smallest multiple of 21, which is four more than a multiple of 10 is 21 × 4 = 84.

6. Rearrange the terms to identify two arithmetic sequences on each side of the equation:

$$(x + 2x + 3x) - (8{,}092 + 8{,}192 + 8{,}292) =$$
$$(4x + 9x + 14x + 19x + 24x) + (8{,}142 + 8{,}167 + 8{,}192 + 8{,}217 + 8{,}242)$$

The sequences on the left have three terms and averages 2x and 8,192 respectively.
The sequences on the right have five terms and averages 14x and 8,192 respectively
So we can simplify the equation to

	$3 \times 2x - 5 \times 8{,}192$	=	$5 \times 14x + 3 \times 8{,}192$
OR	$6x - 70x$	=	$8 \times 8{,}192$
OR	$-64x$	=	$8 \times 8{,}192$
OR	$-8x$	=	$8{,}192$
OR	x	=	$-1{,}024$

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1. $2x + 5 = 9$ so $2x = 9 - 5 = 4$ and $x = 2$.

2. $2(x + 5) = 10 - 1$ so $2x + 10 = 9$ so $2x = 9 - 10 = -1$ so $x = -1/2$

3. $13(8 - x) = 1 + \frac{1}{4}(5(x + 25\frac{2}{5}))$ so $104 - 13x = 1 + \frac{1}{4}(5x + 5 \times 25 + 5 \times \frac{2}{5}))$
So $104 - 13x = 1 + \frac{1}{4}(5x + 127)$. Multiplying both sides by 4 this is equivalent to $416 - 52x = 4 + 5x + 127$, so $416 - 4 - 127 = 5x + 52x$ so $285 = 57x$ and the solution is $x = 5$.

4. The number sentence is $(x + 7.5) \times 3 = (3x + 7.5) \times 1\frac{2}{3}$. Multiplying by 3 gives $9(x + 7.5) = 5(3x + 7.5) \cdot 5$ so $9x + 67.5 = 15x + 37.5$ so $30 = 6x$ and the solution is $x = 5$.

5. The number sentence is $\frac{3}{7}(1.375)((4\frac{4}{3} - x) + 1) = \frac{1}{4}(0.6x + \frac{1}{7})$. The decimals and percents are all simple fractions, so a good approach is to first write each decimal as a fraction:
$$\frac{3}{7} \times \frac{11}{8}(\frac{19}{3} - x) = \frac{1}{4}(\frac{3}{5}x + \frac{1}{7}).$$
Then, writing each side as a reduced fraction but being lazy about multiplying this is equivalent to
$$\frac{11(19 - 3x)}{7 \times 8} = \frac{21x + 5}{4 \times 5 \times 7}.$$
Mutiplying both sides by 7×8×5 to eliminate the denominators gives
$$55(19 - 3x) = 42x + 10.$$
Now we do have to start actually multiplying.
$$1045 - 165x = 42x + 10 \text{ so } 1035 = 207x \text{ which gives } x=5.$$
In a problem like this any time you get a nice number like 5 at the end you can be pretty sure that you did the problem right even without checking your answer. (Checking, however, is also not such a hard calculation so you should do it unless you're very pressed for time.)

6. The fact that we have to repeat the long set of operations three times suggests that we might look to see if that set of operations does something simpler.
It does simplify: $\frac{3}{4}(\frac{2x}{3} + 2) - 2 = \frac{x}{2} + \frac{6}{4} - 2 = \frac{x-1}{2}$. Applying this operation a second time gives $\frac{1}{2}(\frac{x-1}{2}) - \frac{1}{2} = \frac{x}{4} - \frac{3}{4}$. Applying this operation a third time gives $\frac{1}{2}(\frac{x-3}{4}) - \frac{1}{2} = \frac{x}{8} - \frac{7}{8}$. The other number sentence is easier to understand if you read it backwards. Doing so, we realize that the full equation for x is $\frac{x-7}{8} = 7 + 6 - 5 + 4 - 3 + 2 - 1$ or $\frac{x-7}{8} = 10$ with solution $x = 87$.

A - 49

5.3 Working with Formulas

1. $8^2 + 15^2 = 64 + 225 = 289$. Recognizing that $289 = 17^2$ the length of the hypotenuse is 17.

2. We are given $180\pi = 1/3 \, \pi \, (6^2)$ h or $180\pi = 12 \, \pi$ h. Dividing both sides of the equation by $12 \, \pi$ gives h = 15.

3. $4\pi \, (15^2) = 4\pi \times 225 = 900\pi$. Since π is approximately 3.1415, the surface area is approximately $9 \times 314.15 = 2826 + 9 \times .15$. (Note that further digits of π only affect the second decimal place in the product and so we can ignore them unless we get an answer that ends with something like .45 and makes you worry that the extra terms would make a difference in whether you end up rounding up or down.) In this case the last term is 1.35, so the approximation error which can't be bigger than 0.09 is not a worry and the closest integer is 2,827 square centimeters.

4. Here s = $(5 + 5 + 6) \div 2 = 8$. So we can apply the formula with s = 8, a = 5, b = 5, c = 6 to get the result
$$\sqrt{8(8-5)(8-5)(8-6)} = \sqrt{8 \times 3 \times 3 \times 2} = \sqrt{16 \times 9} = \sqrt{16}\sqrt{9} = 4 \times 3 = 12.$$

5. At 0 degrees Celsius, the temperature is 32 degrees in Fahrenheit. Then each addition of 1 degree in Celsius increases the temperature in Fahrenheit by 1.8 degrees. So the first temperature that is a two digit positive value in each scale is 10 degrees Celsius, which is $1.8 \times 10 + 32 = 50$ degrees Fahrenheit. Then since 5 × 1.8 = 9, each increase of 5 degrees in Celsius also increases the temperature in Fahrenheit by an integer number of degrees (9).
This means that (10C, 50F), (15C, 59F), (20C, 68F), (25C, 77F), (30C, 86F), (35C, 95F) are all of the temperatures that are two digit whole numbers in both Celsius and Fahrenheit. The answer is 6.
If there were going to be more solutions, e.g. if the problem had asked you to how many three-digit Fahrenheit temperatures were also a three-digit Celsius temperature, then you would have wanted to use algebra to find the smallest and largest solutions. For example, you can do the three-digit problem by first saying that 100C = 212F would be the smallest solution and then looking for the largest solution by saying it would the one with the Fahrenheit temperature as close to 1000 with the Celsius temperature still being a multiple of 5. Converting 1000F to Celsius using the formula gives C = $\frac{5}{9}(F - 32) = \frac{5}{9} \times 968 = \frac{5860}{9}$ which is between $650 = \frac{5850}{9}$ and $655 = \frac{5895}{9}$, so the solution would have been to count all multiples of 5 from 100 up to $650 = 100 + 550 = 100 + 110 \times 5$. The answer would have been 111.

6. First we use the formula A = $\frac{abc}{4R}$ to find R. Plugging in we have $12 = \frac{5 \times 5 \times 8}{4R}$ so R = $\frac{5 \times 5 \times 8}{4 \times 12} = \frac{25}{6}$. Next, we use the formula A = rs to find r. The semiperimeter s is

A - 50

$\frac{1}{2}(5 + 5 + 8) = 9$, so $12 = 9r$ gives r $= \frac{4}{3}$. Eulers formula then tells us that $d^2 = R^2 - 2Rr = \frac{25^3}{6^2} - 2\frac{25}{6}\frac{4}{3} = \frac{625}{36} - \frac{400}{36} = \frac{225}{36}$, so d $= \frac{15}{6} = \frac{5}{2}$.

5.4 Word Problems with One Unknown

1. If she watches x videos on Monday, then she watches $2x - 1$ videos on Tuesday for a total of $3x - 1$ videos. So $3x - 1 = 50$ or $3x = 51$ meaning $x = 17$.

2. $A = N + 5$. If $A + N = 13$, then $(N + 5) + N = 13$ or $2N = 8$ so $N = 4$. This means that Nala ate 4 cookies and Alan ate 9.

3. The total of the four ages is $13.5 \times 4 = 54$. Let S be Shyam's age. The sister is $S - 2$ and the twin brothers are both $S + 4$. So $S + (S - 2) + 2(S + 4) = 54$. This gives $4S + 6 = 54$ so $4S = 48$ and $S = 12$. Shyam is 12 years old.

4. In problems like this, you want to look for some order in which you can solve the equations one by one. The third equation shows that $\blacklozenge = 3$. Then the second equation indicates that $\smiley = 2$. And finally $\heartsuit = 11$. You can't always solve systems of equations this way, but in IMLEM meet 2 you usually can.

5. Since class A is the largest and class C is the smallest, class B is the median and has 22 students (the problem states that the median class size is 22). Define x = the number of A's in the A block class. The problem says that there were $x - 7$ A's in the B block class for a total of $2x - 7$ total A's in the three classes. Everyone in the B block class got an A or a B so this leaves $22 - (x - 7) = 29 - x$ grades of B in the B block class. Now using the fact that the number of B's in the B block class was seven more than the number of C's in the C block class says that there were $(29 - x) - 7 = 22 - x$ C's in the C block. Now we use the final fact we're told, that the number of A's was one more than seven times the number of C's she gave out. This gives $2x - 7 = 7(22 - x) - 1$ or $2x - 7 = 154 - 7x - 1$ or $9x = 162$, so $x = 18$. This means that there were $x = 18$ A's in the A-block class, 11 A's in the B-block class for 29 total A's.

6. Sometimes it is less confusing to do problems like this by using letters instead of the funny symbols. Writing $\heartsuit = a$, $\smiley = b$, $\sun = c$ and $\blacklozenge = d$, this problem is
   - (1) $a + b + c = 13$
   - (2) $bc + b = 0$
   - (3) $cd^2 = 9$
   - (4) $b + d = 1$

The second equation is very informative. Since $bc + b = b(c + 1) = 0$, either $b = 0$ or $c = -1$ (or both).
If $b = 0$, then equation (4) tells us that $d = 1$, and then equation (3) tells us that $c = 9$, and then equation (1) gives $a = 4$.
If $c = -1$ then equation (3) says $-d^2 = 9$ which is impossible.

So the only possible solution is the first one we found: $a = 4$, $b = 0$, $c = 9$, $d = 1$. Translating back to the symbols we have $\heartsuit = 4$.

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Meet #3

2.1 Review of Meet 1: Angles in Polygons

1. This problem is easiest if you use that the exterior angles in any polygon add up to 360°. If each interior angle in an N-gon is 144°, then each exterior angle is 180° – 144° = 36°. This gives that 36 × N = 360 so N = 10.

2. The measure of CBF is the sum of the measure of CBA and the measure of ABF. The interior angle in a regular pentagon is 108°. The interior angle in an equilateral triangle is 60°. So CBF = 108° + 60° = 168°.

3. If the interior angle of a regular N-gon and the interior angle of a regular M-gon add up to 180°, then the smaller of the two angles is less than or equal to 90° and the larger of the two angles is at least 90°. This leaves very few choices for the regular polygon with the smaller angle: it must be an equilateral triangle (with a 60° interior angle) or a square (with a 90° interior angle). If we make it a triangle, then the other polygon would need to have a 120° interior angle. This is the interior angle of a regular hexagon so N = 3 and M = 6 is one possibility. If we make it a square, then the other polygon would need to have a 90° interior angle. This also works because it is the interior angle of a square, so N = 4 and M = 4 is another possibility. The sum is larger if we choose the triangle and the hexagon: 3 + 6 = 9.

4. Because ABF is an isosceles triangle, angles ABF and BAF are equal. Adding up implies that they are each 65°. By supplements CBA and BAE are each 115°. Once we know that pentagon ABCDE has three 115° angles we know that the median of its angles measures is 115°. The mean of angle measures in any pentagon is 108°. The difference is 7°.

Students were intended to assume that BF and AF are the equal sides because they look longer than AB in the figure. The problem is well posed because if you assume any other pair of sides in ABF are equal, you also get 7°.

5. The larger is M, the smaller will be its exterior angle, which makes the interior angle of the N-gon smaller. One way to get the answer quickly is to start trying the values of N from the smallest to see if they work. The smallest possible N is N = 3. The interior angle of an equilateral triangle is 60°. The exterior angle of an M-gon is 6° if we choose M = 60. So this is the largest possible value for M.

6. The best way to solve this uses a classic factoring trick. We are given that 10 $\frac{360}{M} = 180 - \frac{360}{N}$. Multiplying through by MN, putting everything on the left side, and dividing by 180 gives $MN - 20N - 2M = 0$. A standard trick for working with equations like this is to recognize that the left side has a lot in common with $(M - 20)(N - 2)$. Specifically, it is $(M - 20)(N - 2) - 40$. So the equation is equivalent to $(M - 20)(N - 2) = 40$. The number of positive integer solutions to

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this with N, M > 2 will be the same as the number of factor pairs for 40. Using 40 = $2^3\, 5^1$ there are $(3 + 1)(1 + 1) = 8$ possible factor pairs. For example, the factor pair 40 × 1 gives M = 60 and N = 3, the factor pair 20 × 2 gives M = 40 and N = 4, and so on until 1 × 40 gives M = 21 and N = 42.

One could have tried to find the solution by just starting with N = 3, then trying N = 4 and so on, but most students who took this approach would probably miss some solutions including the last one. The exterior angle of a regular 21-gon is $\frac{360}{21} = \frac{120}{7} = 17\frac{1}{7}°$. The interior angle of regular 42-gon is $180 - \frac{360}{42} = 180 - \frac{60}{7} = 180 - 8\frac{4}{7} = 171\frac{3}{7}°$ This is 10 times as large as $17\frac{1}{7}°$, but you can miss this even after computing both angles.

# 2.2 Review of Meet 2: Areas of Polygons

1. The information we are given implies that ABCD is a trapezoid. The parallel sides, AB and CD have lengths 10 cm and 18 cm. The height is 6 cm because side BC is perpendicular to both bases. The area is $\frac{1}{2}(10 + 18) \cdot 6 = 84$ cm².

2. Combining BD:DC = 3:5 and BC = 8 we see that BD = 3 cm and DC = 5 cm. The fact that AD is an altitude implies that AD is perpendicular to BC so ADC is a right triangle. The Pythagorean Theorem (or preferably just knowing that 5-12-13 is a Pythagorean triple) gives that AD = 12 cm. Using one-half base times height with BC as the base, the area is $\frac{1}{2}8 \cdot 12 = 48$ cm².

3. First, using the Pythagorean Theorem in triangle ABC we find that $AC = \sqrt{20^2 + 21^2} = \sqrt{400 + 441} = \sqrt{841} = 29$ cm. Next, notice that we can compute the area of ABC in two ways. Viewing BC as the base it is $\frac{1}{2}20 \cdot 21 = 210$ cm². Viewing AC as the base it is $\frac{1}{2}29 \cdot BD$. These two must be equal, so $BD = \frac{420}{29}$ cm.

4. We are given two sides of right triangle ABC. Applying the Pythagorean Theorem to find the third side (or just recognizing it is a 12-16-20 triangle) we have AB = 12 cm, BC = 16 cm, and AC = 20 cm. The area of ADC is 42 cm². Viewing this triangle as having base DC and height AB gives $\frac{1}{2}DC \cdot 12 = 42$ so DC = 7 cm. By subtraction BD = 9 cm. Applying the Pythagorean Theorem to ABD gives that AD = 15 cm. The perimeter of ADC is 15 + 7 + 20 = 42 cm. Solving this problem is a lot like solving the typical angle chasing problem. You don't really need to have a plan. You can just keep trying to use facts you have to fill in lengths until everything is filled in.

5. Let AE be the altitude to side BC from A and similarly let DF be the altitude from D to BC. Let x be the length of BE. Let h be the length of both DF and AE. Triangle ABE and DCF are congruent (by side-angle-side), so CF also has length x and EC = 7 − x and BF = 7 + x. Applying the Pythagorean Theorem to AEC gives $(7 - x)^2 + h^2 = 13^2$. In triangle BDF it gives $(7 + x)^2 + h^2 = 15^2$.
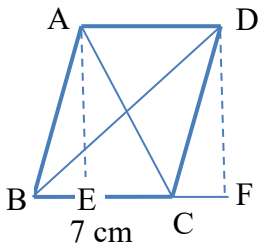
Expanding the first term of each equation the two equations become:
$$49 + 14x + x^2 + h^2 = 225.$$
$$49 - 14x + x^2 + h^2 = 169.$$

A - 55

Subtracting gives 28x = 56 so x = 2. Seeing such a nice round number for x should make you think you are on the right track. Plugging this back into either equation we find h = 12. Using the base times height formula for the area of a parallelogram the area of ABCD is $7 \times 12 = 84$ cm^2.

6. Many level 4 problems require some cleverness in how you approach them, but then work out elegantly in a way that gives you the sense you have done them correctly. This problem is kind of the opposite. The best approach to solving it is fairly straightforward, but after a promising start the calculation turns out to be sufficiently complicated and tedious as to make you think you did something wrong and should start over.

First, note that by applying the Pythagorean Theorem to KEY we have $KY^2 = 162 + 63^2 = 256 + 3969 = 4225 = 65^2$, so KY = 65. The next step is to use similar triangles as we often do in diagrams like this. KAE is similar to KEY so KA/16 = 16/65 which gives KA = 16^2/65. Note that this means that KA/KY = $16^2/65^2$ and AY/KY = $1 - 16^2/65^2 = (65^2 - 16^2)/65^2 = 63^2/65^2$. Using that ATY and KEY are similar we can find other lengths of interest: ET = $(16^2/65^2) \cdot 63$ and AT = $(63^2/65^2) \cdot 16$. The area of KATE is ½ (KE + AT) · ET.

After seeing all of the nice formulas so far it is natural to expect that lots of terms will cancel when you write out these terms. But they don't. The area is

$$\frac{1}{2}\left(16 + \frac{63^2}{65^2}16\right)\frac{16^2 \cdot 63}{65^2} = \frac{16^3(65^2 + 63^2) \cdot 63}{2 \cdot 65^4}$$

You can do a lot of things to simplify the calculation somewhat. If you're familiar with powers of 2 could can note that $16^3/2 = 2^{11} = 2048$. The second term in the numerator can also be calculated more simply by thinking about the binomial expansions: $65^2 + 63^2 = (64 + 1)^2 + (64 - 1)^2 = 2 \cdot 64^2 + 2 = 2^{13} + 2 = 8194$. But at this point, there's no big trick. You can try computing the numerator as $2^{24} + 2^{12}$. You can compute the denominator by squaring 65 using the formula for squaring numbers ending in 5 and then squaring again. But, nothing in the numerator cancels with anything else in the denominator and you just have to multiply out

$$\frac{2048 \times 8194 \times 63}{65^4} = \frac{1,057,222,656}{17,850,625}.$$

Sorry to have put you though such an ugly calculation, but I did think it was a good way to help you remember that sometimes in math there is no trick and you simply need to keep working until you get an answer that isn't elegant.

Meet 3 Solutions

2.3 Diagonals in Polygons

1. The formula for the number of diagonals in a convex N-gon is N(N-3)/2. In a hexagon this gives $6 \times 3 / 2 = 9$. You could also count them by hand. There are two sets of 3 that each form an equilateral triangle, and 3 that go through the center.

2. Here you clearly want to use the formula. It gives $100 \times 97 / 2 = 9700 / 2 = 4850$.

3. Avishek's polygon has seven sides. It has a total of $7 \times 4 / 2 = 14$ diagonals. He has already drawn in two of them, so he needs to draw 12 more.

4. This will involve some guess-and-check. We are looking for a whole number N with $N (N - 3) / 2$ close to 100, which means that $N (N - 3)$ should be close to 200. $15 \times 12 = 180$ and $16 \times 13 = 208$. The latter is closer to 200 so the answer is 16. A 16-gon has 208/2 = 104 diagonals.

5. We are looking for values of N for which $N (N - 3) / 2$ is prime. Consider separately the cases where N is even and N is odd. First, suppose N is even and write $N = 2k$. The number of diagonals is then $2k (2k - 3) / 2 = k (2k - 3)$. This cannot be prime unless k or $2k - 3$ is equal to 1. The first, $k = 1$, does not give a valid solution because $N = 2$ which is not a valid polygon. The second, $2k - 3 = 1$, has $k = 2$ which corresponds to $N = 4$. This is a valid solution: a quadrilateral has two diagonals and two is prime. Next, suppose N is odd and write $N = 2k + 1$. The number of diagonals is $(2k + 1) (2k - 2) / 2 = (2k + 1)(k - 1)$. Again, this cannot be prime unless one of the two numbers that are being multiplied is equal to 1. Only the smaller of the two numbers could be equal to 1, so the only possibility is $k - 1 = 1$ which gives $k = 2$ and $N = 5$. This also works: a pentagon has 5 diagonals and 5 is prime. The question asks for the sum of the values of N that work. The two solutions are $N = 4$ and $N = 5$, so the answer is 9.

6. The fastest way to do this problem is probably just to make a table giving the number of diagonals in each N-gon and then add them

N	5	6	7	8	9	10	11	12
N(N − 3)/2	5	9	14	20	27	35	44	54

| N	13	14	15	16	17	18	19	20
N(N − 3)/2	65	77	90	104	119	135	152	180

What makes this quicker than you might think is that after the first few you notice a pattern that lets you write most of them down without doing the multiplications: the differences between them are 4, then 5, then 6, and so on. They add up to 1120.

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One could also do this problem using formulas. The number of diagonals in an N-gon is $N(N-3)/2 = \frac{1}{2}N - \frac{3}{2}N^2$ so the sum is $\frac{1}{2}(5 + 6 + \cdots 20) - \frac{3}{2}(5^2 + 6^2 + \cdots + 20^2)$. The first of these is an arithmetic sequence. The second can be computed by noting that a formula for the sum of the first N squares $1^2 + 2^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6}$.

7. The M-gon has 7 more diagonals than the N-gon. Extending the table we made in the previous table down to $N = 4$ we see that there are two possibilities: $N = 4$ and $M = 6$; or $N = 8$ and $M = 9$. (Again, notice that the difference between the number of diagonals of an N-gon and an N+1-gon is $N - 1$ so there won't be any solutions with N greater than 8.)

We could compute the area of both the circle circumscribed around a square and around an octagon each with side-length 1 cm to see which is larger. But, this is not really necessary – the area of a circle circumscribed around an N-gon with side length of 1 cm is smaller when N is smaller. (To see this, connect the center of the circle both to one vertex of the N-gon and to the center of one of the sides at that vertex. The radius of the circle is the hypotenuse of a triangle with one side of length ½ cm and the adjacent angle measuring 90 – (180/N) degrees. The hypotenuse is longer when the angle is larger, which occurs when N is larger. So the area is increasing in N.)

So the answer will be the area of a circle circumscribed around a square with side length of 1 cm. Connecting two adjacent vertices to the center of the circle we see that they form an isosceles right triangle. The radius of the circle is the length of the legs of this triangle. By the Pythagorean theorem they satisfy $r^2 + r^2 = 1$ which gives $r^2 = \frac{1}{2}$. The area of the circle is $\pi r^2 = \pi/2$ sq. cm.
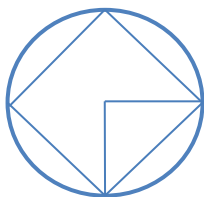
2.4 Pythagorean Theorem (part 1)

1. From the Pythagorean Theorem we know that the length c of the hypotenuse satisfies $c^2 = 6^2 + 8^2 = 36 + 64 = 100$, so c = 10 cm.

2. The problem does not tell us if 60 and 61 are the two legs of the triangle or if one is a leg and the other is the hypotenuse, so we have to try it both ways and see which gives an integer answer.
If we use 60 as one leg and 61 as the hypotenuse, then the length of the other leg a satisfies $a^2 + 60^2 = 61^2$. This gives $a^2 = 61^2 - 60^2 = (61 - 60)(61 + 60) = 121$ so $a = 11$ cm is the desired solution.
If you had first tried 60 and 61 as the two legs you would have gotten $a^2 = 61^2 + 60^2 = 3721 + 3600 = 7321$. This is not a perfect square ($85^2 = 7225$ and $= 86^2 = 7396$) so it does not give a valid solution.

3. An interesting fact about 85 is that it can be written as a sum of squares in two different ways: $6^2 + 7^2 = 36 + 49 = 85$ and $9^2 + 2^2 = 81 + 4 = 85$. The Pythagorean Theorem is an if and only if theorem. If a triangle is a right triangle then $a^2 + b^2 = c^2$. And if $a^2 + b^2 = c^2$, then the triangle is right triangle. In this problem, the fact that AC $= \sqrt{85}$ cm means that the Pythagorean Theorem implies that both ABC and CDA are right triangles. We can then compute the area of the quadrilateral by adding the areas of the two right triangles: $\frac{1}{2} \cdot 6 \cdot 7 + \frac{1}{2} \cdot 9 \cdot 2 = 21 + 9 = 30$ cm².

4. One way to do this problem would be to do it by subtraction. The area of the square is 16 cm². The area of ADF is $\frac{1}{2} \cdot 4 \cdot 2 = 4$ cm². The area of FCE is $\frac{1}{2} \cdot 2 \cdot 1 = 1$ cm². The area of ABE is $\frac{1}{2} \cdot 4 \cdot 3 = 6$ cm². So by subtraction the area of AFE is $16 - (4 + 1 + 6) = 5$ cm².
The problem is in this section because it can also be done with the Pythagorean Theorem. Side AF has length $\sqrt{4^2+2^2} = \sqrt{20}$ cm. Side FE has length $\sqrt{2^2+1^2} = \sqrt{5}$ cm. And side AE has length $\sqrt{4^2+3^2} = 5$ cm. These sides satisfy $AF^2 + FE^2 = AE^2$ so AFE is a right angle and we can compute the area of AFE as $\frac{1}{2} \cdot \sqrt{20} \cdot \sqrt{5} = \frac{1}{2}\sqrt{100} = 5$ cm².

5. The points (x, y) with integer coordinates on a circle with radius 85 centered at the origin are exactly the set of ordered pairs (x, y) of integers that satisfy $x^2 + y^2 = 85^2$.
 Four solutions are the points where the circle hits the x- and y-axes: (85, 0), (−85, 0), (0, 85), and (0, −85).
We will also get eight solutions from every Pythagorean triple (x, y, 85) that has 85 as is largest number: (x, y), (y, x), (−x, y), (−y, x), (x, −y), (y, −x), (−x, −y), and (−y, −x).

A - 59

The problem tells us that there are two primitive Pythagorean triples with 85 as the largest number, (13, 84, 85) and (36, 77, 85). We also need to look for nonprimitive Pythagorean triples that have all three numbers being multiples of some common factor greater than 1. Since $85 = 5 \times 17$ has only two prime factors, the only possible common factors are 5 and 17. To use 5 as the common factor we have to take a Pythagorean triple ending with 17 and multiply by 5. There is exactly one Pythagorean triple ending in 17, (8, 15, 17), so there is one Pythagorean triple ending in 85 with all numbers being multiples of 5, (40, 75, 85). Similarly, if we use 17 as the common factor then we have to take a Pythagorean triple ending with 5 and multiply by 17. There is again exactly one Pythagorean triple ending in 5, (3, 4, 5), so there is one Pythagorean triple ending in 85 with all numbers being multiples of 17, (51, 68, 85).

In total we have shown that there are four relevant Pythagorean triples: (13, 84, 85), (36, 77, 85), (40, 75, 85), and (51, 68, 85). Each gives 8 points on the circle. So we have $4 \times 8 + 4 = 36$ points on the circle with integer coefficients.

6. The distance is $\sqrt{(7-4)^2 + (3-(-1)^2}=\sqrt{3^2 + 4^2} = \sqrt{25}=5$.

7. To solve this problem we first find AC, then AD, and then AE. Don't worry about simplifying any of the square roots that come up in the middle of the calculation because we'll just be squaring the number anyway.
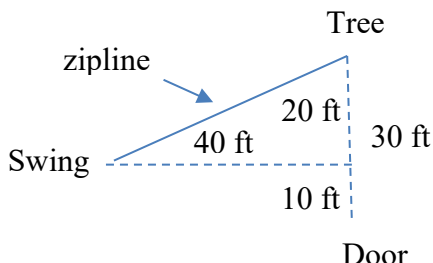
$AC = \sqrt{5^2 + 4^2} = \sqrt{41}$.

$AD = \sqrt{\sqrt{41}^2 + 2^2} = \sqrt{45}$.

$AE = \sqrt{\sqrt{45}^2 + 2^2} = \sqrt{49} = 7$ cm.

You could also notice the pattern of how this is working and at some point realize that the answer will be $AE = \sqrt{5^2 + 4^2 + 2^2 + 2^2}$.

8. The diagram below shows the arrangement of Helen's backyard as seen from above. The three-dimensional distance formula says that the length of the zipline is $\sqrt{40^2 + 20^2 + 5^2} = \sqrt{1600 + 400 + 25} = \sqrt{2025} = 45$ ft.

9. Let X be the point where AD and BE intersect. We then start filling in lengths.
BX = XE = 12 because AD bisects BE.
AX = 5 using the Pythagorean Theorem in triangle AXE.
AB = 12 using the Pythagorean Theorem in triangle ABX.
XD = 16 by subtraction using AD = 21 and AX = 5.
ED = 20 using the Pythagorean Theorem in triangle EXD.
DC = 19 by subtraction using AC = 40 and AD = 21.
XC = 35 by adding XD = 16 and DC = 19.
BC = 37 by the Pythagorean Theorem in BXC. (12-35-37 is a Pythagorean triple.)
 The perimeter is $13 + 37 + 19 + 20 + 13 = 102$ cm.

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10. Draw altitudes AX and BY from A and B to DC. (These segments are perpendicular to DC and meet DC at X and Y, respectively.) By symmetry we know that $DX = YC = 3/2$. Symmetry plus the fact that the perimeter is 16 implies that $AD = BC = 9/2$. The Pythagorean Theorem in triangle BYC gives that $BY^2 + (3/2)^2 = (9/2)^2$ which implies that $BY = \sqrt{\frac{81}{4} - \frac{9}{4}} = \sqrt{18} = 3\sqrt{2}$. We can then find BD by applying the Pythagorean Theorem to right triangle BYD. $YD = DX + XY = 3/2 + 2 = 7/2$ so $BD^2 = (7/2)^2 + \sqrt{18}^2 = \frac{49}{4} + \frac{72}{4} = \frac{121}{4}$.Taking square roots we have BD = 11/2= 5.5 inches.
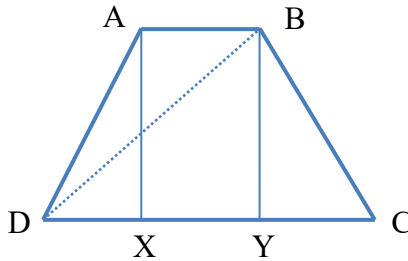

11. The Starbucks is 24 blocks north and 2 blocks west of Gee Young's apartment. Converting both to miles this is 6/5 miles north and ½ miles west. By the Pythagorean Theorem the distance is $\sqrt{\frac{36}{25} + \frac{1}{4}} = \sqrt{\frac{144}{100} + \frac{25}{100}} = \sqrt{\frac{169}{100}} = \frac{13}{10} = 1.3$ miles.

12. As illustrated in the diagram below, altitudes from the endpoints of the base of length 29 cm will divide the base of length 37 cm into segments of length x, 29, and $8 - x$ for some x. In the triangle on the right the Pythagorean Theorem gives $81 = h^2 + (8 - x)^2 = h^2 + 64 - 16x + x^2$.In the triangle on the left it gives $49 = h^2 + x^2$. Subtracting the two equations we find $32 = 64 - 16x$ with solution x = 2. The Pythagorean Theorem then gives $h = \sqrt{49 - 4} = \sqrt{45} = 3\sqrt{5}$. The area is $\frac{1}{2}(29 + 37) \cdot 3\sqrt{5} = 99\sqrt{5}$ cm².
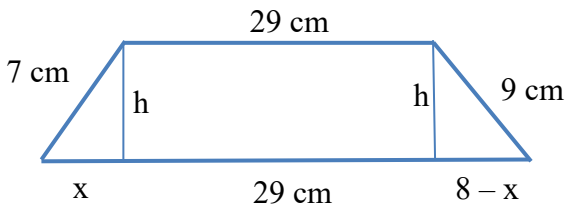
13. Let M be the midpoint of DE. Applying the Pythagorean Theorem to both the left and right halves of triangle ADE we have $AD^2 = DM^2 + AM^2 = ME^2 + AM^2 = AE^2$ so AD = AE = 53 – 2 = 51 cm. Draw in the altitude from A meeting BC at N. Because ABC is an isosceles triangle, N is the midpoint of BC, which

A - 62

gives BN = 28 cm. Applying the Pythagorean Theorem to ABN we find that the height of ABC is $AN = \sqrt{53^2 - 28^2} = \sqrt{(53-28)(53+28)} = \sqrt{25 \cdot 81} = 45$ cm. Next, applying the Pythagorean Theorem in triangle AND gives $ND = \sqrt{51^2 - 45^2} = \sqrt{(51-45)(51+45)} = \sqrt{6 \cdot (6 \cdot 16)} = 24$ cm. The answer is that BD = BN + ND = 28 + 24 = 52 cm. Note that the figure below isn't quite to scale We made the triangles along the right a little bigger than they really are to make things more visible in the diagram.
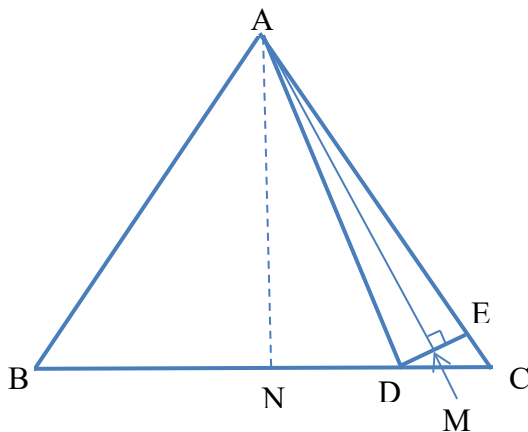
This problem is an example of where knowing your category can be a big advantage. If you just look at the diagram given with so few lines drawn in you would have a hard time knowing where to start. But if you remember that the category is the Pythagorean Theorem, it is more natural to start drawing in altitudes and thinking about what the Pythagorean Theorem implies.

The problem is also an example of the fact that knowing Pythagorean triples can come in handy. Knowing that 28-45-53 and 24-45-51 are both Pythagorean triples would have made some of the calculations quicker. And it could also have helped you figure out how to approach the problem. Seeing the numbers 53 and 51 in the problem, especially if you also noticed that the length of BC was 2 × 28, might have suggested to you that this was a Pythagorean Theorem problem that made use of the multiple Pythagorean triples that involve 45 even if it was not on a worksheet or contest that had problems arranged by category.

Number Theory: 3.1 Scientific Notation

1. $54 = 5.4 \times 10^1$ $\qquad\qquad$ $328 = 3.28 \times 10^2$

$163{,}000 = 1.63 \times 10^5$ $\qquad$ $0.04 = 4 \times 10^{-2}$

$0.0025 = 2.5 \times 10^{-3}$ $\qquad$ $0.000123 = 1.23 \times 10^{-4}$

2. $12.3 \times 10^6 = 1.23 \times 10^7$ $\qquad$ $243 \times 10^{-5} = 2.43 \times 10^{-3}$

$800 \times 10^{-2} = 8 \times 10^0$ $\qquad$ $0.3 \times 10^6 = 3 \times 10^5$

$0.0024 \times 10^{-5} = 2.4 \times 10^{-8}$ $\qquad$ $0.000008 \times 10^1 = 8 \times 10^{-5}$

3. $(8 \times 10^{-2}) \times (3 \times 10^6) = (8 \times 3) \times (10^{-2} \times 10^6) = 24 \times 10^4 = 2.4 \times 10^5$

$(1.6 \times 10^4) \times (3 \times 10^6) = (1.6 \times 3) \times (10^4 \times 10^6) = 4.8 \times 10^1$

$(1.4 \times 10^{-5}) \times (8.2 \times 10^3) = (14 \times 10^{-6}) \times (82 \times 10^2) = (14 \times 82) \times (10^{-6} \times 10^2) =$
$1148 \times 10^{-4} = 1.148 \times 10^{-1}$

4. $(8.2 \times 10^{-3}) \times (2 \times 10^4) = (8.2 \times 2) \times (10^{-3} \times 10^4) = 16.4 \times 10^1 = 164$

$(2.5 \times 10^4) \times (2.5 \times 10^{-6}) = (25 \times 10^3) \times (25 \times 10^{-7}) = (25 \times 25) \times (10^3 \times 10^{-7}) =$
$625 \times 10^{-4} = 0.0625$

5. $(8 \times 10^5) \div (2 \times 10^3) = (8 \div 2) \times (10^5 \div 10^3) = 4 \times 10^2$

$(2 \times 10^4) \div (5 \times 10^6) = (2 \div 5) \times (10^4 \div 10^6) = 0.4 \times 10^{-2} = 4 \times 10^{-3}$

$(3.6 \times 10^{-5}) \div (6 \times 10^3) = (3.6 \div 6) \times (10^{-5} \div 10^3) = 0.6 \times 10^{-8} = 6 \times 10^{-9}$

1. $\frac{(8\times10^5)\times(6\times10^{-2})}{(2\times10^3)\times(3\times10^{-2})}$. We can cancel each of the terms in the denominator from the term directly above it. $8 \div 2 = 4$ and $6 \div 3 = 3$. So the fraction simplifies to $(4 \times 2) \times (10^3 \div 10^1) = 8 \times 10^2$.

2. $\frac{(8.1\times10^5)\times(1.6\times10^{-4})}{(4\times10^4)\times(9\times10^1)} = \frac{(81\times10^4)\times(16\times10^{-5})}{(4\times10^4)\times(9\times10^1)}$. We can cancel a 9 from both 81 and 9 and cancel a 4 from both 16 and 4. This simplifies the ratio to $9 \times 4 \times (10^{-1} \div 10^3) = 36 \times 10^{-4} = 3.6 \times 10^{-5}$.

3. $\frac{(4.2\times10^{-5})\times(1.08\times10^{-2})}{(2.7\times10^3)\times(1.2\times10^{-2})\times(1.4\times10^{-11})} = \frac{(42\times10^{-6})\times(108\times10^{-4})}{(27\times10^2)\times(12\times10^{-3})\times(14\times10^{-12})}$
We can divide both 42 and 14 by 14 leaving just a 3 where the 42 was. $108 = 9 \times 12$ so we can divide both the 108 and 12 by 12 leaving just a 9 where the 108 was. The numerator then has 3×9 and the denominator has 27, which cancel. We are left with just $1 \times (10^{-10} \div 10^{-13}) = 1 \times 10^3$.

4. We want $1 / (6.022 \times 10^{23})$. Rewrite this as $\frac{100\times10^{-2}}{6.022\times10^{23}}$. $100 \div 6.022$ is about 16.6, so this is about 16.6×10^{-25}, which is 1.7×10^{-24} in standard form after rounding to one digit after the decimal point.

5. Light travels 1.86×10^5 miles per second or $1.86 \times 10^5 \times 60$ miles per minute or $1.86 \times 10^5 \times 60 \times 60$ miles per hour. Since it is 2500 miles from Boston to Seattle, the round trip is 5,000 miles and so light makes a total of $1.86 \times 60 \times 60 \times 10^5 \div$ 5,000 round trips in an hour. Write this as $\frac{186\times60\times6\times10^4}{5\times10^3}$. Cancelling the 5 from the 60 this is $186 \times 12 \times 6 \times 10^1 = 72 \times 186 \times 10^1 = (200 - 14) \times 72 \times 10^1 = (14,400 - 1,008) \times 10^1 = 13,392 \times 10^1$ or approximately 1.34×10^5.

6. $\frac{(1.44\times10^4)\times(2.89\times10^{-2})}{(1.7\times10^9)\times(3.6\times10^{-2})\div(1.25\times10^3)} = \frac{(144\times10^2)\times(289\times10^{-4})\times(125\times10^1)}{(17\times10^8)\times(36\times10^{-3})}$. We can cancel a 17 from both 289 $(= 17^2)$ and 17. 144 is 4×36 so we can also cancel 36 from the first term on the top and the last on the bottom. We are left with $4 \times 17 \times 125 \times (10^{-1} \div 10^5) = 500 \times 17 \times 10^{-6} = 8500 \times 10^{-6} = 8.5 \times 10^{-3}$.

Meet 3 Solutions

3.3 Basics of Bases

1. 3043 base 5 in base 10 is
$$= (3 \times 5^3) + (4 \times 5) + 3 = (3 \times 125) + 20 + 3 = 375 + 20 + 3 = 398.$$

2. 55 base 10 is less than $8^2 = 64$ so will be a two-digit number in base 8. Since 55 $= 8 \times 6 + 7$, then base 8 representation of this number is 67.

3. First convert to base 10 then to base 3. $10110_{(2)} = 2^4 + 2^2 + 2^1 = 16 + 4 + 2 = 22$ in base 10. The largest power of 3 that is less than 22 is $3^2 = 9$, so we start converting by dividing by 9. $22 = (\mathbf{2} \times 9) + 4$. Dividing 4 by 3 we get $4 = (\mathbf{1} \times 3)$ + 1. So the base 3 representation of this number is $211_{(3)}$.

4. $BB8_{(16)}$ is equal to $(11 \times 256) + (11 \times 16) + 8 = (2560 + 256) + (160 + 16) + 8 = 3000.$

5. $3333_{(6)} - 3333_{(4)} = 3(6^3 - 4^3) + 3(6^2 - 4^2) + 3(6^1 - 4^1) + 3(1 - 1)$
$$= 3((216 - 64) + (36 - 16) + 2)$$
$$= 3 (152 + 20 + 2) = 3 \cdot 174 = 522.$$

6. The smallest two-digit number in base 5 is $10_{(5)}$, which is 5 in base 10. The largest two-digit number in base 5 is $44_{(5)} = (4 \times 5) + 4 = 24$ in base 10. (You could also realize that the largest two-digit base five number must be one less than $100_{(5)}$, which is 25.) The primes in the set from 5 to 24 inclusive are 5, 7, 11, 13, 17, 19, 23. So seven two-digit base 5 numbers are prime.

7. A positive whole number less than 64 in base 10 is either a one-digit or a two-digit number in base 8. The one-digit numbers 1 through 7 are identical in base 8 or base 10, so have the same sum of their digits. Trying numbers that are two digits and start with 1 in base 8, then numbers that are two digits and start with a two in base 8, and so on, you can see that no other numbers work. The answer is 7.

To do this using algebra instead of a long calculation, suppose there was a two-digit base 8 number that worked. We would have $ab_{(8)} = cd_{(10)}$ and $a + b = c + d$ for some digits a, b, c, d with a $\in \{1, 2, \ldots, 7\}$. The first equation is $8a + b = 10c + d$. Subtracting the two equations gives $7a = 9c$. The left side is not a multiple of 9 for any a from 1 to 7, so there are no solutions involving a number with two digits in base 8. Doing this would also let you have seen that all combinations with a $= c =$ 0 and b $=$ d will work. These are the 7 one-digit base 8 numbers that work.

8. One could just convert to base 10 (it is 5208) and then factor. But it is easier to recognize that multiplication works identically in base 5 and base 10 when there is no carrying so $131313_{(5)} = 13_{(5)} \times 10101_{(5)} = 8 \times (625 + 25 + 1) = 2^3 \; 651 = 2^3 \; 3^1$ $217 = 2^3 \; 3^1 \; 7^1 \; 31^1$. This has $(3+1)(1+1)(1+1)(1+1) = 32$ factors.

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3.4 Converting from Base A from Base B

1. $63_{(8)} = (6 \times 8) + 3 = 51$ in base 10. The largest power of 5 less than 51 is 25. Dividing by 25 we get $51 = (2 \times 25) + 1$. So this is $201_{(5)}$ in base 5.

2. To convert from base 4 into base 2, remember that every base 4 digit is a two-digit number in base 2: $0_{(4)} = 00_{(2)}$; $1_{(4)} = 01_{(2)}$; $2_{(4)} = 10_{(2)}$; $3_{(4)} = 11_{(2)}$. To write a base 4 number in base 2, you just write the base 2 versions of each of these numbers next to each other. I sometimes include small spaces when I'm writing it out to avoid getting confused about where I am. So $312_{(4)} = 11\ 01\ 10_{(2)}$. Without the extra spaces this is $110110_{(2)}$.

3. Every three digits in base 2 corresponds to 1 digit in base 8 with $000_{(2)} = 0$, $001_{(2)} = 1$, $010_{(2)} = 2$, $011_{(2)} = 3$, $100_{(2)} = 4$, $101_{(2)} = 5$, $110_{(2)} = 6$, and $111_{(2)} = 7$. So $101\ 100\ 101_{(2)} = 545_{(8)}$.

4. Every digit in base 9 corresponds to 2 digits in base 3 with $6_{(9)} = 20_{(3)}$, $4_{(9)} = 11_{(3)}$, $2_{(9)} = 02_{(3)}$, and $1_{(9)} = 01_{(3)}$. So $6241_{(9)} = 20\ 02\ 11\ 01_{(3)} = 20021101_{(3)}$.

5. One good way to do this is to first convert to base 2, and then to base 8. Each digit in base 16 corresponds to 4 digits in base 2 with $A_{(16)} = 1010_{(2)}$, $3_{(16)} = 0011_{(2)}$, $5_{(16)} = 0101_{(2)}$, $8_{(16)} = 1000_{(2)}$. Hence, $A358_{(16)} = 1010\ 0011\ 0101\ 1000_{(2)}$. Pushing these digits together and then grouping them into groups of 3 digits from the right side this is $1\ 010\ 001\ 101\ 011\ 000_{(2)}$. Then converting each set of three digits in base 2 to a single digit in base 8, this is $121530_{(8)}$.

Another approach would be to exploit relationships between the powers of 16 and the powers of 8. The powers of 16 up to 16^3 are 1, 16, 256, 4096. The powers of 8 up to 8^4 are 1, 8, 64, 512, 4096. Write each power of 16 in terms of the next lower power of 8 to help with the conversion from base 16 to base 8: $16 = 2 \times 8^1$, $256 = 4 \times 8^2$, $4096 = 8^4$. So,

$$A358_{(16)} = (10 \times 4096) + (3 \times 256) + (5 \times 16) + (1 \times 8)$$
$$= (10 \times 8^4) + (3 \times 4 \times 8^2) + (5 \times 2 \times 8^1) + 8^1$$
$$= (10 \times 8^4) + (12 \times 8^2) + (11 \times 8^1)$$

To make this a base 8 number, we need to adjust this sum so that each coefficient is less than 8.

$$A358_{(16)} = (8 + 2) \times 8^4 + (8 + 4) \times 8^2 + (8 + 3) \times 8^1$$
$$= 8^5 + (2 \times 8^4) + 8^3 + (4 \times 8^2) + 8^2 + (3 \times 8^1)$$
$$= (1 \times 8^5) + (2 \times 8^4) + (1 \times 8^3) + (5 \times 8^2) + (3 \times 8^1) + (0 \times 8^0).$$

So in base 8 this number is $121530_{(8)}$.

6. A good way to do this is to convert $463_{(8)}$ to base 4, and then to add the numbers in base 4. One good way to do the conversion is to go through base 2. Using $4_{(8)} = 100_{(2)}$, $6_{(8)} = 110_{(2)}$, and $3_{(8)} = 011_{(2)}$ we have $463_{(8)} = 100\ 110\ 011_{(2)}$. Grouping the digits two at a time this is $1\ 00\ 11\ 00\ 11_{(2)}$ which converts to $10303_{(4)}$.

You could also get by going through base 10 or by a method similar to the second method we used in the previous section $463_{(8)} = (4 \times 8^2) + (6 \times 8) + 3 = 4 \times 4^3 + (6 \times 2 \times 4^1) + 3 = 4^4 + (12 \times 4^1) + 3 = 4^4 + (3 \times 4^2) + 3 = 10303_{(4)}$.

We can then add using multi-column addition just as we would do in base 10. (This one turns out to be easier than it might be because there is no carrying.)

$$
\begin{array}{r}
1\,0\,3\,0\,3_{(4)} \\
+\quad 3\,0\,2\,0_{(4)} \\
\hline
1\,3\,3\,2\,3_{(4)}
\end{array}
$$

The answer is $13323_{(4)}$.

3.5 Arithmetic in Other Bases

1. Think about this as a multicolumn addition problem. As in base 10 you start from the right and carry when you need to. In the ones column we just have $4 + 0 = 0$. In the next column (the fives column) we have $3 + 1 = 4$. In the third column (the 25's column) we have $2 + 3 = 5$. There is no digit 5 in base 5, so carry a 1 to the next column and write $5 - (1 \times 5) = 0$ in this column. Finally in the leftmost column we just have 1 plus the one that was carried, which gives us 2. The answer is $2044_{(5)}$.

$$
\begin{array}{r}
1 \\
1\ 2\ 3\ 4_{(5)} \\
+\quad 3\ 1\ 0_{(5)} \\
\hline
1\ 0\ 4\ 4_{(5)}
\end{array}
$$

2. Multiplying by $10_{(8)}$ (also called eight) in base 8 is just like multiplying by 10 in base 10. You just add a 0 to the end of the number. So the answer is $1750_{(8)}$.

3. We can again think of this as a multicolumn addition problem. In the ones column we have $1 + 2 + 3 + 4 = 10$ in base 10. But there is no digit 10 in base 9 (just like there isn't in base 10), so we carry a 1 to the next column and put 1 ($= 10 - 9$) below this column. In the second column we now have $1 + 1 + 2 + 3 + 4 = 11$. So we write down a 2 ($=11 - 9$) at the bottom and carry a 1. The same things happens in the third column. And finally we have just a carried one in the fourth column. The answer is $1221_{(9)}$.

$$
\begin{array}{r}
1\ 1\ 1 \\
1\ 1\ 1_{(9)} \\
2\ 2\ 2_{(9)} \\
3\ 3\ 3_{(9)} \\
+\quad 4\ 4\ 4_{(9)} \\
\hline
1\ 2\ 2\ 1_{(9)}
\end{array}
$$

4. In base 8 the number $40_{(8)}$ is one more than $37_{(8)}$. If the subtraction problem were $2237_{(8)} - 137_{(8)}$ the answer would obviously be $2100_{(8)}$ using multicolumn subtraction. But we are subtracting one more than this so the answer is one less. The number $2100_{(8)}$ is $(2 \times 512) + (1 \times 64) = 1088$ in base 10. So the answer is one less than this or 1087.

5. This one is confusing enough that you are probably best off just converting everything to base 10, doing the multiplication in base 10, and then converting back to base 10 at the end. $14_{(6)} = (1 \times 6) + 4 = 10$ in base 10. $34_{(6)} = (3 \times 6) + 4 = 22$ base 10. In base 10 it's easy to compute $10 \times 22 = 220$. Since $6^3 = 216$, this $220 = (1 \times 6^3) + 4$, which is $1004_{(6)}$ in base 6.

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6. You could convert everything to base 10 and back, but here the numbers make it easier to just stay in base 8. $1001_{(2)}$ is $11_{(8)}$ in base 8. You can then do the problem via the same long multiplication algorithm you would use in base 10. It is easy because you are only ever multiplying by 1 and there is no carrying. The answer is $2453_{(8)}$.

$$
\begin{array}{r}
2\ 2\ 3_{(8)} \\
\times\quad\ 1\ 1_{(8)} \\
\hline
2\ 2\ 3_{(8)} \\
+\ 2\ 2\ 3\quad_{(8)} \\
\hline
2\ 4\ 5\ 3_{(8)}
\end{array}
$$

7. Think about how this could work out in a multicolumn addition problem. In the ones column we need for $3 + 2$ to be 5 after carrying. This will only happen if there is no carrying, so we know that b is at least 6. In the second column, we then need for $6 + 3$ to be 2 after carrying. This can only happen if you carry a 1 in base 7. You can check that everything else works out in base 7, so the answer is b=7.

$$
\begin{array}{r}
1 \\
3\ 6\ 3_{(b)} \\
+\ 1\ 0\ 3\ 2_{(b)} \\
\hline
1\ 4\ 2\ 5_{(b)}
\end{array}
$$

8. Again this is probably easiest to solve by converting back and forth to base 10. $1520_{(8)} = 8^3 + (5 \times 8^2) + (2 \times 8) = 512 + 320 + 16 = 848$ in base 10. To solve this problem, it is helpful to think about the prime factorization of 848: $848 = 8 \times 106 = 2^4 \times 53$. (I could have started by taking out the 8 and just converting $152_{(8)}$ to base 10.)

The sum of an arithmetic sequence is equal the product of the number of terms and the average of the first and last terms. In this problem, the average of the first and last terms will be a whole number because all terms are odd. So this means that the number of terms must be a factor of $2^4 \times 53$. The number of terms clearly cannot be as big as 53 (even just the first term is 23 in base 10 and 53×23 is over 1000), so it must be 1 or 2 or 4 or 8 or 16. From the size of the numbers 16 is the obvious guess for the number of terms. If there are 16 terms the first and last term would have to average to 53 (in base 10) so the last term would have to be 30 more than 53 which is 83 (because the first term is 30 less than 53 in base 10). This does work. If the first term is 23, then the 16th term is $23 + (15 \times 4) = 83$.

The question asks N as a base 8 number. $83 = (1 \times 64) + (2 \times 8) + 3$, so the answer is $123_{(8)}$.

Note that some early printings of the workbook had a version of this problem where there is no solution. (The sequence started with $33_{(8)}$ and the sum was supposed to be $1260_{(8)}$.)

3.6 Word Problems Related to Bases

1. Each cube is $5^3 = 125$ minicubes. Each square is 25 minicubes. Each rod is 5 minicubes.

The sandwich is equal to $(1 \times 125) + (3 \times 25) + (1 \times 5) + 2 = 125 + 75 + 5 + 2 = 207$ minicubes. Since each minicube is 1 gram, the total weight is 207 grams.

2. She bought $2 \times 12 \times 12$ or 24×12 bottles of water and had $2 \times 12 + 2$ left over. So $(24 \times 12) - (2 \times 12 + 2) \qquad = (22 \times 12) - 2 = 264 - 2 = 262$ were given out.

3. The cube holds $9^3 = 729$ cubic inches of flour. The pan holds $9^2 = 81$ cubic inches of flour. The tray holds 12 cubic inches of flour.

The total amount is $(1 \times 729) + (2 \times 81) + (3 \times 12) + 4 = 729 + 162 + 36 + 4 = 931$ cubic inches of flour in the 20 pound bag.

4. 3705 in base b $= (3 \times b^3) + (7 \times b^2) + 5$ in base 10. Taylor Swift was presumably born some time between 1980 and the present. Since there is a 7 in the base b representation, b must be at least equal to 8.

If b = 8, then 3705 base 8 $= (3 \times 8^3) + (7 \times 8^2) + 5 = (3 \times 512) + (7 \times 64) + 5 = 1536 + 448 + 5 = 1989$. This must be the answer since 3705 in base 9 will be a much larger number in base 10. So b = 8.

3.7 Adding and Subtracting in Scientific Notation

1. $(4.3 \times 10^6) + (3.1 \times 10^6) = (4.3 + 3.1) \times 10^6 = 7.4 \times 10^6$
   $(8.3 \times 10^3) + (3.8 \times 10^3) = (8.3 + 3.8) \times 10^3 = 1.21 \times 10^4$
   $(2.3 \times 10^4) + (1.1 \times 10^5) = (0.23 \times 10^5) + (1.1 \times 10^5) = (0.23 + 1.1) \times 10^5$
   $= 1.33 \times 10^5$

2. $(4.44 \times 10^4) + (4.4 \times 10^4) = (4.44 - 4.4) \times 10^4 = 0.04 \times 10^4 = 4 \times 10^2.$
   $(8.34 \times 10^3) - (3.8 \times 10^2) = (8.34 \times 10^3) - (0.38 \times 10^3) = 7.96 \times 10^3.$
   $(4.3 \times 10^{-5}) + (7.1 \times 10^{-6}) = (4.3 \times 10^{-5}) + (0.71 \times 10^{-5}) = 5.01 \times 10^{-5}.$

3. Convert all terms into scientific notation at the highest power of 10 in the equation, which is –2.
   $(4.44 \times 10^{-3}) - (1.21 \times 10^{-2}) + (8.17 \times 10^{-3}) + (1.01 \times 10^{-2})$
   $= (0.444 - 1.21 + 0.817 + 1.01) \times 10^{-2}$
   $= (.444 + .817 - 1.21 + 1.01) \times 10^{-2}$
   $= (1.261 - 1.21 + 1.01) \times 10^{-2}$
   $= 1.061 \times 10^{-2}$

4. Step 1: $(1.21 \times 10^{-5}) - (80 \times 10^{-7}) = (1.21 \times 10^{-5}) - (0.80 \times 10^{-5}) = 0.41 \times 10^{-5} = 4.1 \times 10^{-6}$
   Step 2: $(2.6 \times 10^3) \times (4.1 \times 10^{-6}) = (26 \times 10^2) \times (41 \times 10^{-7}) = (26 \times 41) \times (10^2 \times 10^{-7}) = 1066 \times 10^{-5} = 1.066 \times 10^{-2}$

So the answer is 1.066×10^{-2} or .01066 as a decimal.

5. There are 1000 meters in a kilometer, so 1.496×10^{11} meters $= 1.496 \times 10^8$ kilometers.

The distance from the earth to the moon is the distance from the earth to the sun minus the distance from the moon to the sun or $(1.496 - 1.49216) \times 10^8 = 0.00384 \times 10^8 = 3.84 \times 10^5$ km.

The car drives 2,400 km per day or in scientific notation 2.4×10^3 km per day.

At this speed, the number of days to go 3.84×10^5 km is
   $(3.84 \times 10^5) \div (2.4 \times 10^3) = (384 \times 10^3) \div (24 \times 10^2) = (384 \div 24) \times (10^3 \div 10^2)$
   $= 16 \times 10^1$ or 160 days.

4.1 Basics of Exponents

1. $4^3 - 3^4 = 64 - 81 = -17$.

2. $3^4 - 2 \times 3^2 + 3^0 = 81 - 2 \times 9 + 1 = 81 - 18 + 1 = 64$

3. $1.5^{-2} = \left(\frac{3}{2}\right)^{-2} = \left(\frac{2}{3}\right)^2 = 4/9 = 0.\overline{4}$ in repeating decimal form.

4. $4^{-1} - (1/2)^2 = 1/4 - 1/4 = 0$.

5. First simplify the right-hand side of the inequality:
$\qquad (-2)^2 \times 40^{-1} = 4/40 = 1/10$.
Next look at the left-hand side of the inequality. $(3/2)^{-n} = (2/3)^n$
The powers of $(2/3)$ are as follows:

$(2/3)^1$	$=$	2/3
$(2/3)^2$	$=$	4/9
$(2/3)^3$	$=$	8/27
$(2/3)^4$	$=$	16/81
$(2/3)^5$	$=$	32/243
$(2/3)^6$	$=$	64/729

Comparing the numerators and denominators, 32 is more than 1/10 of 243, but 64 is less than 1/10 of 729, so $(2/3)^5 > 1/10$ and $(2/3)^6 < 1/10$, so the inequality holds if $n \leq 5$, which means that the largest integer for which it is true is $n = 5$.

6. This is $5 \times (1 - 1/4 + 1/16 - 1/64 + 1/256 - 1/1024)$
$\qquad = 5 - 5/4 + 5/16 - 5/64 + 5/256 - 5/1024$

The sum of the first three terms is $5 - 15/16 = 4.0625$. The remaining terms in the sum are approximately .08, 02 and .005, so the full sum will be very close to 4. After rounding, the nearest whole number to this sum is 7.

You can quickly guess the answer to this question if you know the formula for the sum of a geometric sequence. (See Meet #4 Number Theory Section 3.3.) The infinite geometric sequence $1 - 1/4 + 1/16 - 1/64 + 1/256 + \ldots$ adds up to $\frac{1}{1-(-\frac{1}{4})} = $ 4/5, so the expression in the problem will be very close to $5 \times \frac{4}{5} = 4$.

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1. $2^5 \times 5^{-1} \times 2^2 \times 5^3 = 2^{5+2} \times 5^{3-1} = 2^7 \times 5^2 = 2^5 \, (2^2 \times 5^2) = 32 \times 100 = 3200.$

2. $(2/3)^{-2} + (3/5)^2 = 9/4 + 9/25 = \dfrac{9 \times 25}{4 \times 25} + \dfrac{9 \times 4}{25 \times 4} = \dfrac{225 + 36}{100} = 2\dfrac{61}{100}.$

3. $(5^5)^2 \times 5^{-5} \times (1/5)^3 = 5^{10} \times 5^{-5} \times 5^{-3} = 5^{10-5-3} = 5^2 = 25.$

4. $\dfrac{5^{-3}(5^5 + 5^3)}{3^{-4}(3^6 + 3^4)} = \dfrac{5^{5-3} + 5^{3-3}}{3^{6-4} + 3^{4-4}} = \dfrac{25+1}{9+1} = 2.6$

5. Combining the first two terms this is $\dfrac{(2/3 \times 4/5 \times 6/7)^2}{(4/3 \times 6/5 \times 8/7)^2} \times (1.5^2 + 2^2).$

Multiplying both the top and the bottom of the initial fraction by $3^2 \, 5^2 \, 7^2$ it simplifies to $\dfrac{2^2 4^2 6^2}{4^2 6^2 8^2} = \dfrac{2^2}{8^2} = \dfrac{1}{16}.$ So the result is $\dfrac{1}{16}\left(\left(\dfrac{3}{2}\right)^2 + 2^2\right) = \dfrac{1}{16}\left(\dfrac{9}{4} + \dfrac{16}{4}\right) = \dfrac{25}{64}.$

6. To make this problem manageable, it is helpful to find prime factorizations of the numbers:

    37 is prime

$429 = 3 \times 143$	$= 3 \times 11 \times 13$
$777 = 7 \times 111$	$= 3 \times 7 \times 37$
$1001 = 7 \times 143$	$= 7 \times 11 \times 13$
$1080 = 2 \times 540 = 2^2 \times 10 \times 27$	$= 2^3 \times 3^3 \times 5$
$5005 = 5 \times 1001$	$= 5 \times 7 \times 11 \times 13$
$8181 = 9 \times 909 = 9 \times 9 \times 101$	$= 3^4 \times 101.$
1030301	$= 101^3$

The only way to know that $1030301 = 101^3$ is to recognize the 1331 pattern and perhaps then to remember that $11^3 = 1331.$

After accounting for all the exponents (especially the negative ones), the numerator is then

$1080^2 \times 777^3 \times 429^3 \times 1030301$

$= (2^3 \times 3^3 \times 5)^2 \times (3 \times 7 \times 37)^3 \times (3 \times 11 \times 13)^3 \times 101^3$

$= 2^{3 \times 2} \times 3^{3 \times 2 + 3 + 3} \times 5^2 \times 7^3 \times 11^3 \times 13^3 \times 101^3$

$= 2^6 \times 3^{12} \times 5^2 \times 7^3 \times 11^3 \times 13^3 \times 101^3$

Meanwhile the denominator is

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$5005^2 \times 8181^3 \times 37^3 \times 1001$

$\quad = (5 \times 7 \times 11 \times 13)^2 \, (3^4 \times 101)^3 \, 37^3 \times 7 \times 11 \times 13$

$\quad = 3^{12} \times 5^2 \times 7^3 \times 11^3 \times 13^3 \times 101^3$

The factors in numerator and denominator cancel except for 2^6, so the answer is 2^6 = 64.

Some of the factorizations here including $111 = 3 \times 37$ and $1001 = 7 \times 11 \times 13$ are things that come in handy quite often on math contests. Rather than doing all of the factoring at the start, you also could have just started by flipping over fractions with negative exponents and putting everything as one big fraction:

$$\frac{1080^2 \, 777^3 \, 429^3 \, 1030301^1}{5005^2 \, 8181^3 \, 37^3 \, 1001^1}$$

A few terms here have obvious initial steps toward factoring. For example $777 = 7 \times 111$, $5005 = 5 \times 1001$, and $8181 = 81 \times 101$. Seeing these could jog your memory about 111 and 1001 being numbers with well known factorizations. And if you had gotten to the point of having 1030301 on the top and nothing left on the bottom but 101^3 it might have prompted you to try dividing 1030301 by 101 to see if it was a multiple if you had not recognized 1030301 as 101^3.

1. $\sqrt{9} - \sqrt[3]{8} = 3 - 2 = 1$

2. $\sqrt{144} \div 16^{1/4} = 12 \div 2 = 6$

3. $\sqrt[3]{7^6} = 7^{6/3} = 7^2 = 49$

4. $(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2}) = \sqrt{3}^2 - \sqrt{2}^2 = 3 - 2 = 1$

5. $\sqrt[4]{50^2} = 50^{2/4} = \sqrt{2 \cdot 5^2} = 5\sqrt{2}$

6. Proceed in a series of steps.

Step 1: $625 = 5^4$ so $\sqrt{625} = 5^2 = 25$ and $4\sqrt{625} = 4*25 = 100$.

Step 2: $\sqrt[3]{4\sqrt{625}} = \sqrt[3]{100} = 100^{1/3} = 10^{2/3} = 2^{2/3} 5^{2/3}$

Step 3: $\sqrt[6]{(2/125)^2} = \dfrac{2^{2/6}}{125^{2/6}} = \dfrac{2^{1/3}}{125^{1/3}} = \dfrac{2^{1/3}}{5}$.

So the ratio is $\dfrac{2^{2/3} 5^{2/3}}{2^{1/3}/5} = 2^{1/3} 5^{5/3} = \sqrt[3]{2 \times 5^5} = 5\sqrt[3]{50}$

7. $(\sqrt{8 + 2\sqrt{15}})(\sqrt{8 - 2\sqrt{15}}) = \sqrt{8^2 - (2\sqrt{15})^2} = \sqrt{64 - 4 \times 15} = \sqrt{4} = 2$

8. $10^3 = 1000$, $11^3 = 1331$, and $12^3 = 1768$. This implies that the cube root of 1700 is between 11 and 12, and with 1768 being so much closer to 1700 than is 1331 it is clear that the cube root of 1700 is closer to 12. (If you want to check this cube 10.5 and see that the result is less than 1700.)

$3^2 = 9$; $3.1^2 = 9.61$; $3.2^2 = 10.24$; $3.3^2 = 10.89$ and $3.4^2 = 11.76$ and $3.5^2 = 12.25$. So the square root of 12 is a little less than 3.5. Therefore 3 is the whole number closest to that square root.

It is also generally useful to have memorized at least the first three digits of the square roots of a few small numbers. Some that come up most often are $\sqrt{2} \approx 1.41421356$, $\sqrt{3} \approx 1.732$, $\sqrt{5} \approx 2.236$, and $\sqrt{10} \approx 3.162$. If you knew this you could have approximated $\sqrt{12} = \sqrt{2^2 3} = 2\sqrt{3} \approx 3.464$.

Meet 3 Solutions

1. $\left(\frac{2}{3}\right)^3 \left(3\frac{1}{3}\right)^{-2} 15^3 = \left(\frac{2}{3}\right)^3 \left(\frac{10}{3}\right)^{-2} 15^3 = \dfrac{2^3 3^2 15^3}{3^3 10^2} = \dfrac{2^3 3^5 5^3}{2^2 3^3 5^2} = 2 \cdot 3^2 \cdot 5 = 90$

2. $2^3 = 8$ and $3^3 = 27$ so $10^{1/3}$ is between 2 and 3. $103^{1/2}$ is a bit more than 10. The whole numbers 3, 4, 5, 6, 7, 8, 9, 10 are between the values, so there are 8 whole numbers between them.

3. Term 1: $8^3\, 3^2\, 2^{-5} = 3^2\, 2^{3\times3}\, 2^{-5} = 3^2\, 2^4$, so squaring this gives $3^4\, 2^8$.
Term 2: $7^0 + 2^3\, 3^1 = 1 + 24 = 25$ so the square root of this number is 5.
Term 3: Cancelling common factors, this is $(1/12)^{-4} = 12^4 = (2^2\, 3)^4 = 2^8\, 3^4$
Step 4: Multiplying $3^4\, 2^8$ by 5 and dividing by $2^8\, 3^4$ gives 5.

4. $\sqrt[5]{3\sqrt{9^3 \times 3^5} \div (3^{-2} \times \sqrt[4]{3^{14}})} =$
Step 1: $9^3 \times 3^5 = 3^6 \times 3^5 = 3^{11}$, so its square root is $3^{11/2}$.
Step 2: 3 times this square root is $3^{13/2}$.
Step 3: The 4th root of 3^{14} is $3^{14/4} = 3^{7/2}$. Multiplying this by 3^{-2} gives $3^{3/2}$.
Step 4: Dividing $3^{13/2}$ by $3^{3/2}$ gives $3^{10/2} = 3^5$. Taking the 5th root of that gives the answer is 3.

5. $7^0 + 7^2 = 1 + 49 = 50$, so $\sqrt{7^0 + 7^2}$ is just slightly more than 7.
$\sqrt{72} = \sqrt{2^3 3^2} = 6\sqrt{2} \approx 6 \times 1.414 = 8.484 \approx 8.5$. (You could also get this by noting that $8^2 = 64$ and $9^2 = 81$, or by noting that $8.5^2 = 72.25$.)
So any integer greater than (about) $7 - 8.5$ or less than $7 + 8.5$ is in the relevant range. These are the integers from -1 to 15, including 0, so there are 17 of them.

6. This is a complicated expression so we proceed in a series of steps working from the inside out.
Step 1: $729 = 9^3$ and $64 = 4^3$ so $(729 / 64)^{1/3} = 9/4$.
Step 2: $222 = 2 \times 111 = 2 \times 3 \times 37$, so $222^2 = 2^2 \times 3^2 \times 37^2$ and $222^2 \div (729/64)^{1/3} = 2^2 \times 3^2 \times 37^2 / (9/4) = 2^4 \times 37^2$. The square root of this is $2^2 \times 37$.
Step 3: $3^5 + 11^2 = 243 + 121 = 364 = 4 \times 91 = 2^2 \times 91$ so $3^5 + 11^2 +$ Step 2 is $2^2 (91 + 37) = 2^2 \times 128 = 2^2 \times 2^7 = 2^9$. The cube root of this is $= 2^3$. Squaring this gives that the value of the first term is $2^6 = 64$.
Step 4: Using the difference of squares formula
$(5\sqrt{13} + 10)(5\sqrt{13} - 10) = (5\sqrt{13})^2 - 10^2 = 325 - 100 = 225$.
Step 5: The whole expression is $\sqrt{64 + 225} = \sqrt{289} = 17$.

7. First multiply numerator and denominator by $(\sqrt{115} + \sqrt{117})$.

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This gives

$$\frac{1}{\sqrt{117}-\sqrt{115}} = \frac{\sqrt{117}+\sqrt{115}}{(\sqrt{117}-\sqrt{115})(\sqrt{117}+\sqrt{115})} = \frac{\sqrt{117}+\sqrt{115}}{117-115} = \frac{\sqrt{117}+\sqrt{115}}{2}$$

Since $10^2 = 100$ and $11^2 = 121$, both square roots $\sqrt{115}$ and $\sqrt{117}$ are between 10 and 11. It is helpful to compute 10.5^2, to see if both would be rounded to the same integer. Since $10.5^2 = 105 + 5.25 = 110.25$, both square roots are between 10.5 and 11. Therefore, the average of $\sqrt{115}$ and $\sqrt{117}$ is also between 10.5 and 11.

So the closest integer to $\dfrac{\sqrt{117}+\sqrt{115}}{2}$ is 11.

Another way to do this problem is to approximate the diameter. A clever way to do this is to say that the denominator $d \equiv \sqrt{117} - \sqrt{115}$ satisfies $\sqrt{115} + d = \sqrt{117}$, so squaring both sides we have $115 + 2d\sqrt{115} + d^2 = 117$. Subtracting $115 + d^2$ from both sides and dividing by $2\sqrt{115}$ gives $d = \frac{2}{2\sqrt{115}} - \frac{d^2}{2\sqrt{115}} = \frac{1}{\sqrt{115}}(1 - \frac{d^2}{2})$. We know that d is not very far from 0.1 ($\sqrt{121} - \sqrt{100} = 1$ so the square roots of numbers in this range that are 21 apart differ by 1 suggesting that the square roots of numbers that are 2 apart differ by about 0.1). So the second term is around 0.995 and can probably be ignored. If we write $d \approx \frac{1}{\sqrt{115}}$, then $\frac{1}{\sqrt{117}-\sqrt{115}} = \frac{1}{d} \approx \sqrt{115} \approx 10.7$. This is far enough from 10.5 and 11.5 so that the fact that our approximation has an error of around one-half of one percent cannot change the implication that 11 is the closest integer.

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5.1 Linear Equations with Absolute Values

1. $|\,5 - |\,7 - 23| + 3\,| = |\,5 - 16 + 3\,| = 8$.

2. If $|\,x - 6\,| = 2$, then either $x - 6 = 2$ or $x - 6 = -2$, so the possible solutions are x = 4, 8.

3. If $|2x - 7| = 3$, then either $2x - 7 = 3$ or $2x - 7 = -3$ so $2x = 10$ or $2x = 4$ with solutions x = 2, 5.

4. If $|3x + 5| = 8$, then either $3x + 5 = 8$ or $3x + 5 = -8$ so $3x = 3$ or $3x = -13$ with solutions x = –13/3 or x = 1. The difference between these solutions is $1 - (-13/3)$ = 16/3.

5. $\frac{1}{2}|\,x - 7\,| = 3$. Multiplying both sides of the equation by 2 gives $|\,x - 7\,| = 6$. So either $x - 7 = 6$ or $x - 7 = -6$. The second gives the smaller solution of x = 1.

6. $|7 - 2x| = x - 2$ so either (1) $7 - 2x = x - 2$ and $7 - 2x \geq 0$; or (2) $2x - 7 = x - 2$ and $7 - 2x \leq 0$. Considering these cases separately:
In case (1) $7 - 2x = x - 2$ implies $9 = 3x$ with solution x = 3. This satisfies the additional condition $7 - 2x \geq 0$, so it is a solution.
In case (2) $2x - 7 = x - 2$ implies x = 5. This satisfies the additional condition $7 - 2x \leq 0$, so it is also a solution.
So the two solutions are x = 3 and x = 5.

7. If $9 + 4x > 0$, then the equation is $9 + 4x = 5x + 18$ or x = –9. This does not satisfy $9 + 4x > 0$, so it is not a valid solution.
If $9 + 4x < 0$, then the equation is $-9 - 4x = 5x + 18$ or $-27 = 9x$, with solution x = 3. This does satisfy $9 + 4x < 0$ so it is a valid solution. The answer is x = 3.

Note: In early printings the numbers in this question were different.

8. $|2 - |2 - x|\,| = |x - 3|$. A good approach here is to first think about the values of x for which each side takes on each of its possible values. This will let us know how to divide the possible x's into cases.

A1. (Right Side) If $x \geq 3$, the right-hand side is x – 3.
A2. (Right Side) If $x \leq 3$, the right-hand side is 3 – x.

B1. (Left Side) If $x \geq 2$, $|2 - x| = x - 2$ and so $|2 - |2 - x|| = |4 - x|$.
    B1(a). So if $x \geq 4$, then the left side is x – 4.
    B1(b). And if x is between 2 and 4, then then left-hand side is 4 – x.
B2. (Left Side) If $x < 2$, then $2 - x$ is positive, $|2 - x| = 2 - x$ and $|2 - |2 - x|\,| = |x|$.
    B2(a). If x is between 0 and 2, then the left-hand side is x.
    B2(b). If x is negative, then the left-hand side is –x.
Breaking these findings into cases:

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Case 1: $x \geq 4$. Combining A1 and B1(a), we have $x - 4 = x - 3$, which is impossible.

Case 2: $3 \leq x \leq 4$. Combining A1 and B1(b), we have $4 - x = x - 3$ with solution $x = 3.5$

Case 3: $2 \leq x \leq 3$. Combining A2 and B1(b), we have $4 - x = 3 - x$, which is impossible.

Case 4: $0 \leq x \leq 2$. Combining A2 and B2(a), we have $x = 3 - x$ with solution $x = 1.5$

Case 5: x is negative. Combining A2 and B2(b), we have $-x = 3 - x$, which is impossible.

There are two solutions: $x = 1.5$ and $x = 3.5$

5.2 Working with Inequalities

1. The graph shows x > 3, so A = 0.

2. $3(x + 1) + 2 < x - 1 \iff 3x + 3 + 2 < x - 1 \iff 2x < -6$ with solution $x < -3$.

3. The graph shows x > 4. The equation is $3x - 3 - Cx > 3$ which is equivalent to $(3 - c)\, x > 6$ or $x > \frac{6}{3-c}$. So we want $6 / (3 - c) = 4$, which means $6 = 4(3 - c)$ or $6 = 12 - 4c$ with solution $c = 1.5$.

4. $11 + 2/x < 17$ is equivalent to $2/x < 6$. If x is positive, this means $2 < 6x$ or $x > 3$. If x is negative, this means $2 > 6x$, which is always true because the left side is positive and the right side is negative. So the equation holds if $x < 0$ or $x > 1/3$.

5. The left side is $5(A - x) + 3(x + A - 1) = 5A - 5x + 3x + 3A - 3 = 8A - 2x - 3$.
So the inequality is $8A - 2x - 3 \geq A \iff 2x \leq 7A - 3 \iff x \leq 3.5A - 1.5$.
This is equivalent to $x \leq 17$ when $3.5\,A - 1.5 = 17$ or $3.5\,A = 18.5$ or $A = 185/35 = 37/7$. Sometimes non-round numbers like this are correct answers.

6. $337(x - A) + 7A(x + 1) - A(x - 335) = 337x - 337A + 7Ax + 7A - Ax + 335A$ which is $337x + 5A + 6Ax$. We want $337x + 5A + 6Ax \leq -7$ to be equivalent to $x \leq 8$.
Rewriting the inequality as $(337 + 6A)x \leq -7 - 5A$, we want $(-7 - 5A) / (337 + 6A) = 8$ and $337 + 6A \geq 0$ so the inequality does not change signs when we divide both sides by $337 + 6A$.
This means $-7 - 5A = 337 \times 8 + 6 \times 8A$ or $-53A = 2703$ or $A = -51$. Here, you should be good when you divide and realize 2703 is a multiple of 53.

7. $x(A - x) + A(x + A) > 2A(x+4)$ is equivalent to $Ax - x^2 + Ax + A^2 > 2Ax + 8A$ or $A^2 - 8A > x^2$.
We want this to be true for x between -3 and +3, which will be true if and only if $A^2 - 8A = 9$. To find such A, we write this as a quadratic in A, $A^2 - 8A - 9 = 0$ which we can factor to give $(A - 9)(A + 1) = 0$. This has two solutions $A = 9$ and $A = -1$.

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5.3 Absolute Values and Inequalities

1. Think of the inequality as asking how many integers are less than 7 away from 7. This is obviously true for all the integers from 1 to 13 but not for integers less than or equal to 0 or greater than 14. So there are 13 such numbers.

2. Note that n cannot be zero. For all other n we can multiply both sides of the equation by $|n|$ to get $3|n| \leq 15$ or $|n| \leq 5$. (Note that these operations do not change the sign of the inequality since $|n|$ cannot be negative.) There are 11 integers satisfying $|n| \leq 5$ – namely all of the integers from –5 to 5. The sum of these integers is 0 because for each negative integer, there is a corresponding positive integer.

3. For $n \neq 1$, $11 / |n - 1| < 4$ is the same as $|n - 1| > 11/4$ or $|n - 1| > 2.75$. This inequality holds for all integers n that are at least 2.75 units away from 1. The only integers that are not this far away from 1 are –1, 0, 1, 2, and 3. Note that the problematic n=1 we noted earlier is in this set already. So there are a total of 5 integers n that are not solutions.

4. Dividing both sides by 7, $|7x - 120| < 7$ is equivalent to $\left| x - \frac{120}{7} \right| < 1$. Written as a mixed number the fraction 120/7 is $17\frac{1}{7}$, so the set of integer solutions is the set of all integers which are within distance 1 of $17\frac{1}{7}$. This is obviously just 17 and 18. The median of this two-element set is 17.5.

5. Again, this of this in terms of distances. The question is asking for the set of values of x that are closer to 3 than they are to 10. This will obviously be true if and only if x is to the left of the point that is half way between 3 and 10, which is 6.5. So the answer is the set of all x with x < 6.5. This is best written as $\{x | x < 6.5\}$, but on middle school contests writing x < 6.5 is usually fine.

6. We look first for solutions to the inequality with $3x + 4 \geq 0$. Note that this requires that $x \geq -4/3$. In this case the equation becomes $15 > 2(3x + 4) \Longleftrightarrow 15 > 6x + 8 \Longleftrightarrow 6x < 7 \Longleftrightarrow x < 7/6$. So this holds for $-4/3 \leq x \leq 7/6$. This holds for integers –1, 0, and 1.
If $3x + 4 < 0$ (which means $x < -4/3$), then this equation becomes $15 > 2(-3x - 4)$ or $15 > -6x - 8$ or $23 > -6x$. Dividing by –6 changes the inequality sign to produce the result $-23/6 < x$. So this holds for $-23/6 < x < -4/3$. So this identifies the additional integer solutions –3 and –2. The largest possible absolute value is 3.

7. Consider three cases:
CASE 1: x < –7. Then both 2x – 7 and x + 7 are negative so the inequality is 7 – 2x < –7 – x or 14 < x. It cannot be that both x < -7 and 14 < x, so there is no solution in this range.

CASE 2: $-7 \leq x \leq 3.5$. Then $2x - 7$ is negative and $x + 7$ is positive so the inequality is $7 - 2x < x + 7$ or $3x > 0$ with solution $x > 0$. So this holds for integers 0, 1, 2, 3 in this range.

CASE 3: $x \geq 3.5$. Then both terms are positive so the inequality is $2x - 7 < x + 7$ or $x < 14$. The solutions that belong in case 3 and the whole numbers from 4 to 13. (Note that $x = 14$ is not a solution because if $x = 14$, the two terms $2x - 7$ and $x + 7$ each equal 21 and so the relation holds with equality, not a strict inequality.)

So all integers between 0 and 13 are solutions. To find the sum of integers 0 to 13, note that there are 14 integers and that we can pair them from the outside in:

(0, 13), (1, 12), (2, 11), (3, 10), (4, 9), (5, 8), (6, 7)

This is 7 pairs, each with sum 13, for a total of $7 \times 13 = 91$.

8. Again, we will use three cases to capture where the numbers in the absolute value signs are positive and negative.

CASE 1: $n > 1.5$. Then both $2n - 3$ and $n + \frac{1}{2}$ are positive so the equation is $8 / (2n - 3) < 2 / (n + 1/2)$ which is equivalent to $8(n + \frac{1}{2}) < 2(2n - 3)$ or $8n + 4 < 4n - 6$ with solution $n < -2.5$. The conditions $n < -2.5$ and $n > 1.5$ cannot hold at the same time so there are no solutions in this range.

CASE 2: n is between -0.5 and 1.5. Then $2n - 3$ is negative and $n + \frac{1}{2}$ is positive, so the equation is $8 / (3 - 2n) < 2 / (n + \frac{1}{2})$. Both denominators are positive so cross-multiplying gives $8n + 4 < 6 - 4n$ with solution $n < \frac{1}{2}$. So both conditions hold if n is between -0.5 and 0.5

CASE 3: n is less than -0.5. Then both $2n - 3$ and $n + \frac{1}{2}$ is negative, so the equation becomes $8/(3–2n) < 2(–1/2–n)$. Again, we have made sure that both denominators are positive so we can cross-multiply to get $-4 - 8n < 6 - 4n$ or $4n > -10$ with solution $n > -2.5$ So both conditions hold if n is between -2.5 and -0.5

Combining cases 2 and 3, the original inequality holds for integers $n = -2, -1$, and 0 for a total of 3 integer solutions.

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Meet #4

2.1 Areas and Perimeters of Circles

1. If the circumference is 8π cm, the radius is 4 cm. The area is then 16π cm^2.

2. Use proportional reasoning here rather than computing circumferences that will have π's in them. Aimee's tires have twice as large a radius. Therefore the circumference will be twice as large, and her tires will only need to make half as many rotations. Aimee's tires make 200 complete rotations.

3. Let r be the radius of the circles. The rectangle has a base of length 4r and a height of 2r. The fact about the area gives $8r^2 = 32$, which implies that r $= 2$ cm. Each circle has an area of 4π cm^2, so the area in between is $32 - 8\pi$ cm^2.

4. The area of the square is x^2 cm^2. The circle has a radius of x/2 cm. This is 5x mm. So the perimeter of the circle is $10x\pi$ mm. If these two are numerically equal we have $x^2 = 10x\pi$ which gives x $= 10\pi$ cm $= 100\pi$ mm. The answer is 314 mm.

5. The square has a side length of 7 cm. The radius of the big circle is 7/2 cm. Thinking about the width of the square we see that twice the radius of the smaller circle plus 7/2 $= 7$. This gives that the radius of the smaller circle is 7/4 cm. The area of the smaller circle is $\frac{49}{16}\pi$ cm^2. The area of the half of the larger circle that is inside the rectangle is $\frac{1}{2}\frac{49}{4}\pi = \frac{49}{8}\pi = \frac{98}{16}\pi$ cm^2. So the area of the blue shaded region is $49 - \frac{147}{16}\pi$ cm^2. Using a calculator this is approximately 20.1 cm^2.

2.2 Arcs and Angles

1. The measure a central angle is equal to the measure of the arc between its endpoints. Here, the measure of central angle AOB is equal to the measure of minor arc AB, which is 60 degrees.

2. The area of the sector with a 45 degree central angle is $\frac{45}{360}\pi r^2 = \frac{1}{8}\pi r^2$. Here the radius is 8 cm, so the area is 8π cm^2.

3. Let X be the point where AC and OB intersect. In triangle OAX, AOB = 60° (because ABC is equilateral), and we are given that OAC = 20°, so by adding-up-to-180°, the measure of OXA is 100°. In triangle CXB we have CXB = 100° (because it and OXA are vertical angles) and ACB = 30° (because ACB is inscribed in a 60° arc and hence its measure is one-half of the angle of the arc), so by adding up in triangle CBX the measure of CBO is 50°.

4. If segments AB and BC are the same length, then minor arcs AB and BC are also equal. The measure of angle ADB is one half of the measure of arc AB, and the measure of angle CDB is one half of the measure of BC, so ADB = CDB. Hence, each is one-half of the measure of angle ADC. Angles ADC and ABC add up to 180° because the arcs in which they are inscribed add up to 360°. Writing x for the measure of ADC and 180 – x for the measure of ABC we have x = (180 – x) – 40, which gives x = 70°. The measure of CDB is one-half of this, which is 35°.

5. Think about how far along the ground the wheel moves when it makes one revolution moving to the right.
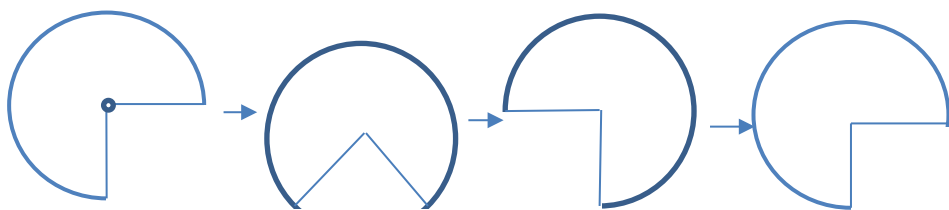
First, the right side of the wheel comes down until it touches the ground at a point that is $\sqrt{2}$ feet to the right of where the bottom of the wheel initially was (and still is) touching the ground. Second, the left side of the wheel will lift up off the ground, but initially this is just rotating the wheel around the point where the right side touches the ground without moving where that point is touching the ground. Third, the wheel starts rolling along the ground. It moves a distance of $\frac{3}{4}2\pi r = \frac{3}{2}\pi$ feet before completing its rotation in the same orientation in which it started. So in one complete revolution the wheel has moved $\sqrt{2} + \frac{3}{2}\pi \approx 1.414 + 4.712 = 6.126$ feet. After 16 complete revolutions it will have moved about $16 \times 6.126 = $

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98.016 feet. After 100 feet it will be in the early part of the 17th revolution. The answer is 16.

Note that it is only because the question only asked about full revolutions and not partial revolutions that it did not need to tell you whether the wheel was rolled to the left or the right.

2.3 A Quick Review of Triangles and Polygons

1. FEH is an inscribed angle in the circle subtending minor arc FH. The degree measure of minor arc FH is 90 degrees. (It is two-eighths of the circle.) So the measure of angle FEH is 45 degrees.

2. The semiperimeter of this triangle is $(4 + 5 + 7)/2 = 8$ cm, so by Heron's formula its area is $\sqrt{8 \cdot (8 - 4) \cdot (8 - 5) \cdot (8 - 7)} = \sqrt{8 \cdot 4 \cdot 3} = \sqrt{16 \cdot 6} = 4\sqrt{6}$ cm^2. The answer is $a = 4$.

3. The angles in a quadrilateral add up to $360°$. If a four-term arithmetic sequence has a sum of 360, then the smallest and largest numbers add to 180 and the two middle numbers add up to 180. This implies that the second-largest angle is $72°$. So the common difference in the arithmetic sequence is $108 - 72 = 36°$. The measure of the largest angle is then $108 + 36 = 144°$.

4. Given that the measure of angle A is $30°$ and the measure of angle C is $75°$, adding up the angles in triangle ABC implies that angle AB is also $75°$. This means that ABC is an isosceles triangle with AC = AB. To find the length of segment AB, notice that AED and BED are congruent. (They are similar because both have a right angle in the middle, and BAD = ABD because the triangle is isosceles, and the third angles are equal by adding up. They are congruent and not just similar because they share side DE.) This imples that BE = AE = 6 cm so AB = 12 cm. The answer is that AC = 12 cm as well.

5. Subtracting $(1, 1)$ from each point, the area is the same as the area of a triangle with vertices $(0, 0)$, $(3, -2)$, and $(2, 3)$. The shoelace formula says that the area of this triangle is $\frac{1}{2}(3 \times 3 - (-2 \times 2)) = \frac{13}{2}$.

6. We can compute the area as the difference between the area of hexagon ABCDEF and the area of hexagon A″B″C″D″E″F″. Think of hexagon ABCDEF as consisting of six equilateral triangles each connecting one side of the triangle to the center of the circle. The each of these triangles has a side-length of 1, so the area of the hexagon is six times the area of the equilateral triangle. The area of an equilateral triangle with side-length s is $s^2\sqrt{3}/4$, so the area of ABCDEF is $\frac{3\sqrt{3}}{2}$.

To find the area of hexagon, A″B″C″D″E″F″, we just need to find its side length. The figure below shows a closeup illustrating how the points A′ and B′ are defined. We have just drawn two of the six triangles that make up hexagon ABCDEF, OAB and OBC. Point A′ is the midpoint of AB. Because OAB is isosceles, the line connecting O and A′ is perpendicular to AB. Hence, OA′A is a 30-60-90 right triangle with OA = 1, AA′ = ½ and OA′ = $\frac{\sqrt{3}}{2}$. Note

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that OA′ is the distance from the center of hexagon A′B′C′D′E′F′ to each of its vertices, which is also equal to the side length of that hexagon. When the side length of the hexagon is multiplied by $\frac{\sqrt{3}}{2}$, its area is multiplied by $\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$. This shows that the area of A′B′C′D′E′F′ is three-fourths of the area of ABCDEF.
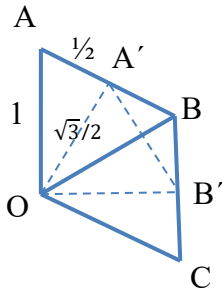
The relationship between A″B″C″D″E″F″ and A′B′C′D′E′F′ is the same as the relationship between A′B′C′D′E′F′ and ABCDEF. Hence, the area of A″B″C″D″E″F″ is three-fourths of the area of A′B′C′D′E′F′, which makes it 9/16 times the area of ABCDEF.

The area between ABCDEF and A″B″C″D″E″F″ will be 7/16 times the area of ABCDEF which is $\frac{7}{16} \cdot \frac{3\sqrt{3}}{2} = \frac{21\sqrt{3}}{32}$ cm^2.

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2.4 Advanced Circle Facts

1. The measure of OAB is $\frac{1}{2}(AB - CD) = \frac{1}{2}(60 - 20) = 20°$.

2. A formula for the radius of an inscribed circle is r = Area / Semiperimeter. Here the area is 6 and the semiperimeter is 6 so r = 1 cm.

3. The r = A/s formula gives that the inradius $r = \dfrac{\frac{1}{2}\cdot 9\cdot 12}{(9+12+15)/2} = \dfrac{9\cdot 12}{36} = 3$. In a right triangle the circumradius is simply one-half of the hypotenuse so R = 15/2. The ratio of the area of the circumscribed circle to the area of the inscribed circle will be $\dfrac{R^2}{r^2} = \dfrac{225/4}{9} = \dfrac{25}{4}$.

4. The area of a cyclic quadrilateral is $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$. Here s = 5 so the area is $\sqrt{(5-1)(5-2)(5-3)(5-4)} = \sqrt{24} = 2\sqrt{6}$.

5. The point inside a triangle that is equidistant from each of the three vertices is the center of the circumscribed circle, so D is the center of the circumscribed circle and DA is the circumradius R. By Heron's formula the area of this triangle is $A = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{9 \cdot 4 \cdot 3 \cdot 2} = 6\sqrt{6}$. The circumradius formula then gives $R = \dfrac{abc}{4A} = \dfrac{5\cdot 6\cdot 7}{24\sqrt{6}} = \dfrac{35}{4\sqrt{6}} = \dfrac{35\sqrt{6}}{24}$.

6. Let BX be the altitude from B drawn to AC. Using that AC = 10 cm and that the area of ABC is 25 cm², we find that BX = 5 cm. Because ABC is isosceles we know that X is the midpoint of AC and AX = XC = 5 cm. This implies that both AXB and BXC are isosceles right triangles, so both BAC and ACB are 45° angles and ABC is a right angle. The latter implies that AC is a diameter of the circle. This, in turn, implies that the midpoint X of AC is the center of the circle. And this gives us that the circle has a radius of 5 cm. Connecting X to E and D we see that XE = ED = DX = 5 cm, so XED is an equilateral triangle and the central angle EXD is 60°. Angle EBC is an inscribed angle that subtends arc ED, so its measure is one half of this or 30°.
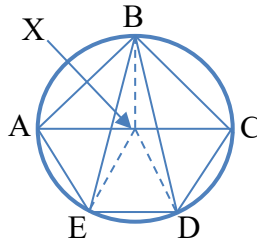
3.1 Basic Sequences and Series

1. The first term is 13 and the difference is 7, so the 11th term is $13 + 10 \times 7 = 83$.

2. The numbers 5, 10, 15, …, 100 are an arithmetic sequence. Since there are 20 terms, we can make 10 pairs: (5, 100), (10, 95), …, (50, 55). Each pair has sum 105, so the sum of the 10 pairs is $10 \times 105 = 1{,}050$.

3. A formula for the sum of an arithmetic sequence with an odd number of terms is that the sum is equal to the middle term times the number of terms. Here, there are nine terms and the middle term is –5 so the sum is –45.

4. The difference from the 8th to 11th term is $224 - 23 = 201$. The 11th term is the 8th term plus 3 times the difference d from term to term in the series, so $d = 201 \div 3 = 67$. The 26th term is the 11th term plus 15 increments or $224 + 15 \times d$, which is $224 + 15 \times 67 = 224 + 1{,}005 = 1{,}229$.
(You could also find the 1st term by subtracting $10 \times d$ from the 11th term, and then finding the 26th term by adding 25 times the distance: 1st term $= 224 - 10 \times 67 = 224 - 670 = -446$, so 26th term $= -446 + 25 \times 67 = 1{,}229$.)

5. In each month, the zombie follows an arithmetic sequence 1, 2, 3, … up to the number of days in the month. In 2017 (as in any non-leap year), there is one month with 28 days, four months with 30 days and seven months with 31 days.
<u>28 day month</u>: First term 1, $d = 1$, $N = 28$. There are 14 pairs (1, 28), (2, 27), …, (14, 15). Each pair has sum 29 so the zombie eats $14 \times 29 = 406$ brains.
<u>30 day month</u>: First term 1, $d = 1$, $N = 30$. There are 15 pairs (1, 30), (2, 29), …, (15, 16). Each pair has sum 31 so the zombie eats $15 \times 31 = 465$ brains.
<u>31 day month</u>: We could sum another arithmetic sequence, but it's easier to realize that the zombie eats 31 more brains than in a 30-day month. $465 + 30 = 495$.
The total eaten in the year is $406 + (4 \times 465) + (7 \times 496) = 406 + 1{,}860 + 3{,}472 = 5{,}738$.
One can do this with fewer multiplications by counting what the answer would be with 12 30-day months and then adjusting: $(12 \times 465) - 59 + (7 \times 31) = 5{,}738$.

6. This sequence has $N = 2017$ and $d = 4.8 - 3.2 = 1.6$. The next several terms in the sequence are 6.4 (3rd term), 8.0 (4th term), 9.6 (5th term) and 11.2 (6th term). Since $5 \times 1.6 = 8$, every fifth term starting from the 4th term on will be a whole number.
The number 2015 is the last number before 2017 that is divisible by 5 with $2{,}015 \div 5 = 403$. Since one out of every five terms is a whole number, 403 of the first 2,015 terms are whole numbers.
To complete the problem, we have to figure out if either of the 2016th or 2017th term is a whole number. Since the 4th term is a whole number, the 9th, 14th, 19th, … terms are whole numbers. That is, every term ending in 4 or 9 is a whole number, so terms 2016 and 2017 are not whole numbers. So there are 403 whole numbers in the first 2017 terms.

7. Extending the sequence back towards 0 gives the values 17, 10, 3, -4, -11. Every term is of form $3 + 7m$ for some integer m.

If N is odd and 3 is the middle value, then the pairs of values are (-4, 10), (-11, 17) and so on, where each pair has sum of 6, with N/2 pairs and the value 3 left over. The total of the sequence is the $6 \times (N/2) + 3 = 3N + 2$. So if 3 is the middle value, we want to have as few terms in the sum as possible meaning -11, -4, 3, 10, 17 (since we know 17 is in the sequence). This sequence has five numbers with average value 3 so its total is $5 \times 3 = 15$.

If N is odd and the middle value is 10, then the sum is 10 times the number of terms which is at least 30.

If N is odd and the middle term is 17 or more, then the sum will be at least 17.

If N is odd and the middle term is negative, then the overall sum will be negative. Therefore, if N is odd, the smallest possible value for the sum of the terms is 15.

If N is even and 17 is paired with a number -18 or below, each pair will have a negative sum and the overall sum will be negative.

If N is even and 17 is paired with a number -4 or above, then each pair will add to 13 or more and so the overall sum will be greater than 15. (It is not possible for 17 to be paired with -11 if N is even because then the value 3 will be left over.) Considering these cases, the minimum value for the sum of the sequence is 15, which occurs if the sequence is -11, -4, 3, 10, 17.

8. The difference between 318 and 303 is 15, which has integer factors 1, 3, 5, and 15. Since every term in the sequence is an integer, these four numbers 1, 3, 5, and 15 are the only possible values for the difference d between terms in the sequence. Consider each case in turn:

$d = 1$: The sequence doesn't include 9, so the smallest possible sum would come if it started with 10. The 20th term would be $10 + (19 \times 1) = 29$.

$d = 3$: 303 and 318 are multiples of 3, so if $d = 3$, every term is a multiple of 3. The sequence doesn't include 9, so the way to make the 20th term as small as possible would be to start at 12. The 20th term would be $12 + 19 \times 3 = 69$.

$d = 5$: 303 is 3 more than a multiple of 5, so every term is 3 more than a multiple of 5. Since every term is positive, the sequence could start at 3 with 20th term $3 + 19 \times 5 = 98$.

$d = 15$: This is obviously a bad idea. The 20th term would be the first term plus 29 $\times$ 15 which is way more than 29.

Comparing these possibilities, the smallest possible value for the 20th term of the sequence is 29.

Meet 4 Solutions

3.2 Modular Arithmetic Part 1

1. There are 24 hours per day. $11 \times 24 = 264$, so in 275 hours, it will be 11 days and 11 hours from now. If it is currently 3:30 pm, then 11 additional hours makes it 2:30 am.

2. Since $37 = 9 \times 4 + 1$, there are four teams of nine after the first 36 kids have been assigned to teams. So the 37th child will be on team 1.

3. In modular arithmetic problems involving addition and multiplication, you can reduce all numbers before doing the calculation. Here, $13 \equiv 1 \pmod 4$, $5 \equiv 1 \pmod 4$, and $12 \equiv 0 \pmod 4$ so $13 + (5 \times 12) \equiv 1 + (1 \times 0) \equiv 1 \pmod 4$.

4. You could either start by remembering that since 7 is prime, $3^6 = 1 \bmod 7$, or you could figure this out with successive multiplications as follows:
$3^1 = 3 \bmod 7$; $3^2 = 3 \times 3 = 9 = 2 \bmod 7$; $3^3 = 3^2 \times 3 = 2 \times 3 = 6 \bmod 7$; $3^4 = 3^3 \times 3 = 6 \times 3 = 18 = 4 \bmod 7$; $3^5 = 4 \times 3 = 5 \bmod 7$, and $3^6 = 5 \times 3 = 1 \bmod 7$.
Since $3^6 = 1 \bmod 7$, $3^6 \times 3^6 = 3^{12} = 1 \bmod 7$, and so $3^{13} = 3^{12} \times 3 = 1 \times 3 = 3 \bmod 7$.

5. Bill swims one lap in three-fourths of a minute, so he would swim 680 laps in $680 \times 3/4 = 170 \times 3 = 510$ minutes. 510 minutes divided by 60 minutes per hour gives 8 hours and 30 minutes. So if he starts at noon and swims continuously at this pace, he would finish at 8:30 with the minute hand pointing to 6 on a traditional clock.

6. We want to find all integer values of x between 0 and 9 so that the last digit of $3x + 3$ is a 7. As x ranges from 0 to 9, $3x + 3$ takes on the multiples of 3 from 3 to 30. Only one of these multiples of $3 - 27 -$ ends in a 7. So $3x + 3 = 27$ holds if $3x = 24$ or $x = 8$. This is the only solution among the values from 0 to 9.

7. A very neat way to solve this problem is to have the expression remind you that Fermat's little theorem implies that $2^{16} = 1 \bmod 17$. So if we multiple both sides of the equation $2^{13} x = 1 \bmod 17$ by 2^3 we get $2^{16} x = 2^3 \bmod 17$ which is equivalent to $x = 8 \bmod 17$. (This is because $2^{16} = 1$ and $2^3 = 8$ in mod 17 arithmetic.) This doesn't immediately tell us what x is, but does tell us what x is in mod 17 arithmetic. But the smallest positive whole number equal to 8 mod 17 is 8 so $x = 8$.

To do this without the cleverness, note that we want $2^{13} x = 1 \bmod 17$. To simplify this equation we reduce $2^{13} \bmod 17$: $2^4 = 16 = -1 \bmod 17$. So $2^{12} = (-1)^3 = -1 \bmod 17$ and $2^{13} = -1 \times 2 = -2 \bmod 17$. The equation is thus equivalent to $-2x = 1 \bmod 17$. An easy way to solve this is to observe that 1 is also equal to $-16 \bmod 17$, so we are looking for an x which solves $-2x = -16 \bmod 8$. The only solution to this is $x = 8 \bmod 17$.

3.2 Modular Arithmetic Part 2

1. $23(x-4) + 2(x+45) = (23x-92) + (2x+90) = 25x - 2 = x - 2 \bmod 12$, where in the final line we used that $25 = 1 \bmod 12$. When $x = 5$ this is equal to $3 \bmod 12$.

2. The key clue is that $7x = 5 \bmod 13$. One neat way to solve this is to start counting by 7's and notice that $7 \times 2 = 14 = 1 \bmod 13$, so multiplying both sides by 5 we find that $7 \times 10 = (7 \times 2) \times 5 = 1 \times 5 = 5 \bmod 13$, so $x = 10$.
Another way to do the problem is just to count by 13's from 5, i.e. 5, 18, 31, 44, 57, 70, and wait until you see a number that you recognize as a multiple of 7. The first one is 70, which is 7×10 so $x = 10$ is the answer.

3. $5 = 1 \bmod 4$, so any power of 5 is also equal to $1 \bmod 4$.
   $7 = -1 \bmod 4$, so the even powers of 7 are equal to $1 \bmod 4$ and the odd powers of 7 are equal to $-1 \bmod 4$.
This means that $5^{5555} = 1 \bmod 4$ and $7^{7777} = -1 \bmod 4$, so the sum is divisible by 4, leaving a remainder of 0.

4. For Cecilia to arrive home at 5 minutes past some hour it must be that $23x = 5 \bmod 60$. The question is asking us to find the smallest possible solution to this equation. Because 60 is also a multiple of 5, the fact that $23x = 5 \bmod 60$ means that $23x = 60k + 5$ for some integer k which implies that x is a multiple of 5. Walking five laps takes 115 minutes which is equal to $-5 \bmod 60$. If she walks 2×5 laps the total number of minues will be $2 \times -5 = -10 \bmod 60$ and so on. She will need to go around the block 11 sets of 5 times or 55 laps in all to arrive at 5 minutes after the hour.

5. $37(2x-1) = 74x - 37$. The equation $74x - 37 = 1 \bmod 10$ is equivalent to $4x - 7 = 1 \bmod 10$, which is equivent to $4x = 8 \bmod 10$. This is true for all numbers that end in a 2 or 7 when written in base 10, so the eleven smallest integers are 2, 7, 12, 17, 22, 27, 32, 37, 42, 47, 52, where the last of these is $2 + (5 \times 10)$ representing the first term plus ten increments of 5. This is a sequence of 11 terms with average equal to the middle value of 27, so the sum is $11 \times 27 = 297$.

6. To think about the first term compute that for $x = 1$ it is 2^2, for $x = 2$ it is $2^4 = (2^2)^2$, for $x = 3$ it is $2^8 = (2^4)^2$, and so on. The pattern is that changing x to $x + 1$ squares the first term.
In mod 5, the first term, $2^{2^x} = 4 = -1 \bmod 5$ when $x = 1$. Squaring this, the first term becomes 1 when $x = 2$, is also 1 when $x = 3$, also 1 when $x = 4$ and so on.
So the solutions with $x > 1$ will be all values of x with $3^x = -1 \bmod 5$. 3^x follows a cycle in mod 5 starting with 3, then -1 (since $3 \times 3 = 9 = -1 \bmod 5$), then 2 (since $-1 \times 3 = -3 = 2 \bmod 5$), then 1, and then 3 again. Every 4th term starting with the second is $-1 \bmod 5$. So all x in $\{2, 6, 10, \ldots, 98\}$ produce the desired result. This is one out of every 4 of the first 100 numbers, so 25 of them.

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3.3 Advanced Sequences and Series

1. The difference from one term to the next is increasing by 1, so the series is
$$1, 1+2, 1+2+3, 1+2+3+4, \text{ and so on.}$$

The 10th term is then $1+2+3+4+5+6+7+8+9+10$, which is the sum of an arithmetic sequence with ten terms. We can make 5 pairs starting with $(1, 10)$ with sum 11 each, so the total is $5 \times 11 = 55$.

2. Bounce 1 = $81 \times (2/3)$, Bound 2 = $81 \times (2/3)^2$, Bounce 3 = $81 \times (2/3)$ and Bounce 4 = $81 \times (2/3)^4$. Since $81 = 3^4$, $81 \times (2/3)^4 = 2^4 = 16$. So the 4th bounce is 16 feet high.

3. $2^2 = 4 \times 1^2$. $4^2 = 4 \times 2^2$, $6^2 = 4 \times 3^2$ and so on. So $2^2 + 4^2 + 6^2 + \ldots = 20^2$ is 4 times the sum of the first 10 squares. Using the formula, this sum is $\frac{4 \cdot 10 \cdot 11 \cdot 21}{6}$ Cancelling a 3 from the 6 and the 21 and a 2 from 4 and the 6 this is $2 \times 10 \times 11 \times 7 = 110 \times 14 = 1540$.

4. We can approximate this answer by using a starting value of $650 rather than $649.99.
After 1 year, the value is about $650 \times 0.8 = \$65 \times 8 = \$520$.
After 2 years, the value is about $520 \times 0.8 = \$52 \times 8 = \$416$.
After 3 years, the value is about $416 \times 0.8 = \$41.60 \times 8 = \$332.60$
After 4 years, the value is about $332.60 \times 0.8 = \$33.26 \times 8 = $ about $266.
Her iPhone first will be worth less than $300 on January 1st after 4 years, which is in 2021.

5. Looking at the differences between adjacent terms, $7 - 3 = 4$, $13 - 7 = 6$, $21 - 13 = 8$, $31 - 21 = 10$ and so on. So we can think of the terms as being defined by Term 1 = 3; Term 2 = 3+4; Term 3 = $3 + 4 + 6$; Term 4 = $3 + 4 + 6 + 8$ and so on. This gives that Term 100 is $3 + 4 + 6 + 8 + \ldots + 200$ since the last term in the sum is always two times the number of the term. That is, term 100 = $1 + (2 + 4 + 6 + 8 + \ldots + 200)$, where the second part is an arithmetic sequence of 100 terms with average equal to $(2+200) / 2 = 101$. So the 100th term is equal to $1 + 100 * 101 = 10101$.

Another way to have done the problem would have been to have written down the successive differences:
```
3   7   13   21   31   43
  4   6   8   10   12
    2   2   2   2
```
Seeing that the second difference is 2 means that the sequence is generated by a quadratic function with the first term being x^2, i.e. the xth term is $x^2 + ax + b$ for some constants a and b.

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Making a table showing the terms of the sequence and subtracting off x^2 from each, there is an obvious pattern to what is left over. What is left over is $x + 1$, so the xth term of the sequence is $x^2 + x + 1$.

Value of x:	1	2	3	4	5	6
xth term:	3	7	13	21	31	43
x^2:	1	4	9	16	25	36
left over:	2	3	4	5	6	7

So the 100th term is $100^2 + 100 + 1 = 10101$.

6. Square 1 has area 1 cm^2. Square 2 has area ¼ cm^2. Square 3 has area 1/16 cm^2. So the sum of the areas of all the squares is the sum of a geometric series with first term equal to 1 and multiplier ¼, so this sum of the areas is $1 / (1 - 1/4) = 1/(3/4) = 4/3$ cm^2.

7. After the first drop of 100 feet down, the ball goes up a certain distance and then back down that same distance. The total distance before it stops is the sum of (1) 100 foot initial drop; (2) twice the sum of a geometric series with first term 60 and multiplier 0.6 (since there is one geometric series for distance traveled down and the same geometric series for distance traveled down). This sum of all distance traveled is $100 + 2 \times 60 / (1 - .6) = 100 + 2 \times 60/(0.4) = 100 + 2 \times 60 \times (5/2) = 100 + 300 = 400$ feet. The 100th bounce up will travel 100×0.6^{100} feet, which is actually a very small number, so approximating to the nearest 10th of a foot, the ball will have traveled 400.0 feet in its first 100 bounces.

Meet 4 Solutions

4.1 Percent Applications

1. $.25 \times 244 = 244 \div 4 = 61$.

2. $89 \times (1 - 0.3) = 89 \times 0.7 = (80 \times 0.7) + (9 \times 0.7) = 56 + 6.3 = \$62.30$.

3. As a fraction, $16\frac{2}{3}\% = 1/6$. The discounted price is 5/6 of the original price. The original price $= 1 / (5/6)$ proportion of the discounted price. Since $1/ (5/6) = 6/5 = 1.2$, the original price is an increase of 20% over the discounted price.

4. $1.375 \times .5 \times 176 = 1.375 \times 88 = (11/8) \times 88 = 11 \times 11 = 121$.

5. The President's Day price is $0.9 \times (0.8\, M) = 0.72\, M$ where MSRP is the original price. If $0.72\, M = \$360$, then $72M = 36{,}000$ or $M = 36{,}000/72$. The top and bottom of this fraction are obviously both divisible by 36 giving $= 1000/2 = \$500$ as the answer.

6. If there is no discount the regular price including tax would be $\$449 \times 1.05$. So Arvind can pay proportion $\$420 / (1.05 \times 449) = 400/449$ of the regular price including tax. You could convert this to a percentage with long division or with use of a calculator. Or you could observe that 10% of 449 is $449 - 44.9 = 404.1$. 11% of 449 is $449 - 44.9 - 4.49 = 399.61$, which is less than \$400. So he would have enough money to afford the Xbox at an 11% discount.

Double checking this, at 11% discount without tax will give an initial price of \$399.61, plus tax just less than $.05 \times 400 = \$20$, so this confirms that he would pay less than \$420 in this case.

7. To produce the largest average score in Helen's class relative to Kate's class, the students in Helen's class should score 100 (the highest possible score at least 90) or 89 (the highest possible score below 90), while the students in Kate's class should score 80 or 90.

Suppose K_{90} is the number of students in Kate's class who scored a 90 and K_{80} is the number of student's in Kate's class who scored 80. We know $K_{80} + K_{90} = 20$, and also $H_{100} = 1.1\, K_{90}$ and $H_{89} + H_{100} = 1.25 \times 20 = 25$. For all of these numbers to be integers, K_{90} must be divisible by 10, so $K_{90} = 0$, 10 or 20, but assume that at least some students scored 90 or above so $K_{90} = 10$ or 20.

If half of the students in Kate's class scored 90 and half scored 89, then 11 students in Helen's class scored 100 and 14 scored 89, so Helen's class is $11 \times 10 - 14 \times 1 = 96$ points ahead of an average of 90. In this case, the Kate's class averages 85 and Helen's class averages $90 + 96/25 = 93.84$. The difference in these averages of 8.84 is an increase of $8.84 \times 100 / 85 = 10.4\%$.

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If all students in Kate's class scored 90, then 22 students in Helen's class scored 100 and 3 scored 89, which puts Helen's class 33 points below a perfect score for average score 98.68. This difference of 8.68 points is less than a 10% increase over the average in Kate's class. So the maximum difference in average scores between the classes is 10.4%.

It would have been better if the problem specified that some students in Kate's class got at least 90. We wouldn't normally say that 0 is 10% larger than 0, but you could argue that this is not ruled out by statements that are made. If all students in Helen's class scored exactly 89 and all students in Kate's class got exactly 10, then the average in Helen's class would be 11.25% larger.

4.2 Compound Interest

1. After two years the tuition will be $45000 \times (1.05)^2$. You can do this easily with a calculator or use that $1.05 = 21/20$ so $1.05^2 = 441/400 = 1 + 40/400 + 1/400$. This gives $45000 \times (1.05)^2. = 45000 (1 + 1/10 + 1/400) = 45000 + 4500 + 450/4 = $49,612.50$.

2, After 15 years the account balance will be $1000 \times (1.04)^{15} = 1800.94.

3. The monthly interest rate is one twelfth of 3%, which is one quarter of a percent or 0.0025. Eight years is 96 months so after 8 years the final balance will be $(1.0025)^{96} \approx 1.270868$ times the initial balance. Rounding to the nearest tenth of a percent, the percentage increase is 27.1.

4. Let r be the annual interest rate in percent. We are told that $32000 = 25000 \times (1 + r/100)^5$. This gives $(1 + r/100)^5 = 32/25$ so $1 + r/100 = (32/25)^{1/5} \approx 1.050611$. So the interest rate is 5.06%.

5. Let Martin's salary three years ago be s. This year his salary would be $(1.08)^3$s. The fact that he paid $3854.72 in social security taxes clearly means that his salary was below $117,000. (He would pay over $8000 if his salary was that high.) So his social security taxes this year were $(1.08)^3 \times 0.0765 \times s = 3854.72$. This gives $s = \frac{3854.72}{1.08^3 \cdot 0.0765} = $40,000$.

6. The largest possible value for the pension occurs if the pension increases by the same amount in each of the three years. The annual increase is one percentage point less than the increase in the CPI. So if the CPI goes up by 2% in each year pension will be $30000 (1.01)^3 = 30,909.03$.
The smallest possible value occurs if the increases are as uneven as possible. In this problem, that would be two years with a 1% percent CPI increase and one year with a 4% increase. With these increases the pension would be $30000 (1.03) = $30,900.00$.
The difference is $9.03.

To see that the difference is smallest when the increases are equal, suppose that the increases are x and $a - x$ so the average increase is a. We then have that $(1 + x)(1 + (a - x)) = 1 + 2\frac{x+(a-x)}{2} + x(a - x) = 1 + 2a + x(a - x)$. The quadratic equation $x(a - x)$ is maximized when x is halfway between the two roots, which gives $x = a/2$ and gets smaller the farther away from this value we move. To see that the same principle will apply when there are three or more years of increases as well, note that you can always think about holding one year's increase fixed and making the product of the other two as large or small as possible. This will show that you can always improve the total if the three returns are not all equal.

A - 98

5.1 Functions

1. There is clearly a discount for a half-dozen rather than buying six donuts individually and for a dozen rather than two half-dozen donuts. So the cheapest way to buy 15 donuts is to buy a dozen for $3.49 and three individual donuts for 75 cents each for a total of $3.49 + 3 \times \$.75 = \$5.74$.

2. She pays $40 for the calls and the first 500 texts and an additional $1.56 for the remaining 156 texts for a total of $41.56.

3. The total cost is $19.95 + (50 \times \$0.134) + (110 \times \$0.144)$ or $19.95 + \$6.70 + \$15.84 = \$42.49$.

4. A resident who earns $120,000 from self-employment owes $120,000 \times 0.153 = \$18,360$ in taxes. Since Josh owes $6,120 in self-employment tax, he must have earned an original ("gross") amount less than $120,000. So he pays 15.3% per dollar on every dollar of self-employment income, meaning that his before-tax income was $6120/.153 = \$40,000$.

5. Beyond the 500$^{\text{th}}$ text, the two plans are equivalent, each charging 2.2 cents per text. The only difference in the plans is an additional $5 fixed charge for Plan B and an additional charge of 2.2 cents per text for texts 101 to 500 for Plan A. So Plan A costs more than Plan B if she pays for at least 500/2.2 or a bit more than 227 additional texts under Plan A than Plan B. This means paying for the first 100 (which cost the same on both plans) and an additional 228 texts for a total of at least 328.

5.2 Two Equations in Two Unknowns

1. If $x + 2y = 73$ and $x - y = 13$, then subtracting the second equation from the first gives $3y = 73 - 13$ or $3y = 60$ with solution $y = 20$. Plugging back into the first equation gives $x + 40 = 73$ so $x = 33$.

2. If Genicia were the same age as the twins, then the total of their ages would be $29 - 5 = 24$, so each would be $24 \div 3 = 8$ years old. So she is $8 + 5 = 13$ and they are each 8 years old. You can also do this via algebra by writing g for Gencia's age, b for her brothers' age and solving $g = b + 5$ and $g + 2b = 29$.

3. Apart from the mutant chicken, they have animals with 36 heads and 108 legs. Writing x for the number of chickens and y for the number of animals that are cows or dogs, this is a two equation problem with $x + y = 36$ and $2x + 4y = 108$. One way to solve this is by substitution. If they have x two-legged chickens then there are $36 - x$ four-legged animals. Plugging into the second equation for a total of 108 legs gives $2x + 4(36 - x) = 108$ or $2x + 144 - 4x = 108$ or $36 - 2x = 0$, so x $= 18$. This means that they have 18 two-legged chickens plus the mutant chicken for a total of 19 chickens in all.

4. Think of this as a two equation system by writing x for AB: $x - A = 24$ and $2A + 4x = 456$. Multiplying the first equation by 4 gives $4x - 4A = 96$. Subtracting this from the second equation gives $6A = 456 - 96 = 360$ or $A = 60$. Substituting $A = 60$ into the first equation gives $x = 84$. So $AB = 84$ and $A = 60$. Dividing gives $B = 1.4$.

5. Let the current ages be x for Hannah and y for Hana. Let g be Anna's current grade.
The first sentence says that $x - 5 = y + (8 - g) + 3$. This can be written in a more standard form as
(1) $x - y + g = 16$.
The second sentence says that $g + (x + y)/2 = (x + y)/3 + 13$. Multiplying this equation by 6 gives $6g + 3x + 3y = 2(x+y) + 78$ which simplifies to
(2) $x + y + 6g = 78$.
The last sentence says that three years from now the difference between half of Hana's age and Anna's grade will be 5 so $\frac{1}{2}(y + 3) - (g + 3) = 5$. Multiplying the equation by 2 gives $y + 3 - 2(g + 3) = 10$ or $y + 3 - 2g - 6 = 10$ so
(3) $y = 2g + 13$.
Substituting this expression for y into the first two equations and simplifying gives:
(1') $x - g = 29$
(2') $x + 8g = 65$
Subtracting the first from the second gives $9g = 36$ so $g = 4$. Plugging back into (1') gives $x = 33$. So Hannah is 33 years old. (And Anna is in 4th grade and Hana is 21.)

6. Let the lengths of the two bases be b and y and let the lengths of the other two sides by x and z as shown in the diagram below. We know that b is the shortest

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side, but do not immediately know the order of the lengths of the other sides. The Pythagorean theorem tells us that $x^2 + (y - b)^2 = z^2$ so we know that $z > x$. But, this still leaves three possible orderings: (1) $b < x < y < z$; (2) $b < x < z < y$; and (3) $b < y < x < z$. We will try each of these possibilities in order.

b

x

z

y
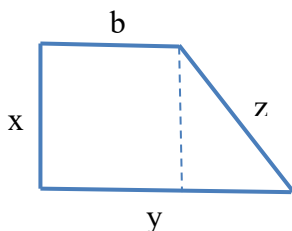
Case 1: $b < x < y < z$
In this case we have $x = b + d$, $y = b + 2d$, and $z = b + 3d$. The Pythagorean Theorem equation then becomes $(b + d)^2 + (2d)^2 = (b + 3d)^2$. Expanding both sides and collecting terms this is $b^2 + 2bd + 5d^2 = b^2 + 6bd + 9d^2$. This can never hold when b and d are both positive because the right side is bigger than the left side. It could hold if $d = 0$, but this was intended to be ruled out by the wording saying that one of the bases is "the shortest side". (It would also be ruled out by the trapezoid having just two right angles rather than four which you get if $d = 0$, but the wording here is not as clear.)

Case 3: $b < y < x < z$
In this case we have $x = b + 2d$, $y = b + d$, and $z = b + 3d$. The Pythagorean Theorem equation then becomes $(b + 2d)^2 + d^2 = (b + 3d)^2$. Expanding both sides and collecting terms this is $b^2 + 4bd + 5d^2 = b^2 + 6bd + 9d^2$. As in case 1, this won't work for b, $d > 0$.

Case 2: $b < x < z < y$
In this case we have $x = b + d$, $y = b + 3d$, and $z = b + 2d$. The Pythagorean Theorem equation then becomes $(b + d)^2 + (3d)^2 = (b + 2d)^2$. Expanding both sides and collecting terms this is $b^2 + 2bd + 10d^2 = b^2 + 4bd + 4d^2$. This gives $6d^2 = 2bd$ which will hold if $b = 3d$. We then have $b = 3d$, $x = 4d$, $y = 6d$, and $z = 5d$. The fact about the perimeter tells us that $3d + 4d + 5d + 6d = 18d = 12$ so $d = 2/3$.

Plugging this into the area formula for a trapezoid, the bases have lengths $3d = 2$ and $6d = 4$. The height is $x = 4d = 8/3$. So the area is $½·(2 + 4)·(8/3) = 8$ cm^2.

1. Working backwards if y = 7 and 2x – y = 3 then 2x – 7 = 3 or 2x = 10 which means x = 5. Then x + 2y + z = 32 where x = 5 and y = 7, so 5 + (2 × 7) + z = 32 or 5 + 14 + z = 32 or z = 13.

2. Denoting the ages by A, B and C the equations are:
   (1) A + B   = 20
   (2)   B + C = 40
   (3) A   + C = 34
Adding all three equations gives 2A + 2B + 2C = 94, so dividing by 2 gives A + B + C = 47.
(It is also possible to solve for the individual ages noting that since B+C=40 and A+C = 34, it must be that B = A + 6, so then if A + B = 20, 2A + 14 = 20 so A = 7, B = 13, C = 27. But this is slower and you don't need the full solution to answer the question.)

3. Denote the scores by x for Rhianne, y for Ryan, and z for Rian so
   (1) x = y + 23
   (2) z = 15+y;
   (3) (x + z) / 2 = 87 or x + z = 174.
The form of equations (1) and (2) suggests that this would be a good problem to do by substitution, writing everything in terms of one variable. We want to find y, so it is probably quicker to put everything in terms of y. (1) is x = y + 23. (2) gives z = y + 15. Substituting both of these into (3) gives (y + 23) + (y + 15) = 174 or 2y + 38 = 174 so 2y = 136 and y = 68. The answer is that Ryan got 68. (You could substitute back to get x = 91 and z = 83. Checking that these numbers satisfy all three equations is a good way to check your work.)

We wanted to do this using algebra, but it would have been easier to do mentally thinking of it as a problem about averages. If Rhianne had 23 points more than Ryan and Rian had 15 points more, then their average was 19 points ahead of him. 87 – 19 = 68.

4. Denote the prices as b for a cheeseburger, f for a small order of fries and c for a large coke. The prices for the three orders give us:
   (1) b + f   = 1.72
   (2) 2b + f + c = 4.20
   (3)   f + c = 2.22
We want to know the cost of Angel's order, which is b + 2f + c. We can notice that Angel's order is equal to the combination of Annika's order (1) and Melissa's order (3), so it must cost 1.72 + 2.22 = $3.94.

5. Denoting the costs of these metals by A, M, and Z, the problem says 0.95 A + 0.05 M = 88, 0.925 A + 0.025 M + 0.05 Z = 85, and 0.90A + 0.05 M + 0.05 Z = 89. Multiplying by 100 we can write these as:

A - 102

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(1) 95A   + 5M   = 8800
(2) 92.5A + 2.5M + 5Z = 8500
(3) 90A   + 5M   + 5Z = 8900

Subtracting the second equation from the third gives $2.5M - 2.5A = 400$ or $5M = 5A + 800$. Substituting this result for 5M in the first equation gives $95A + 5A + 800 = 8800$ or $100A = 8000$ with solution $A = 80$, meaning that the price of aluminum is 80 cents per pound.

6. Denoting the scores by A, G, H, and L (using the first initial of each name).
   (1) A + G + H + L = 444
   (2) A   − L = 24
   (3) A = (G + H)/2 + 9

(Note that in (2) we are taking the statement to indicate that Anna had the higher score. Questions often say "positive difference" if they mean that either score could be higher.)

We can rewrite (2) as $L = A - 24$.
We can rewrite (3) as $2A = G + H + 18$ after multiplying both sides by 2.

Then combining (1) and the new version of (2): $A + G + H + A - 24 = 444$ or $2A + G + H = 468$.
Combining this equation and the new version of (3) gives $2(G+H) + 18 = 468$ or $G + H = 225$ and then by (3) $A = (G+H)/2 + 9 = 112.5 + 9 = 121.5$. Since Anna's score was 24 points higher than Lib's score, $L = 97.5$. So Lib is the only score below 100 and the other three scores count for the team for a total of 225 for G and H plus 121.5 for Anna giving a team total of 346.5.

5.4 Time-Distance-Speed Problems

1. He walks $3/60 = 1/20$ miles per minute. To go ¼ mile, then he walks $(1/4) / (1/20) = 20 \div 4 = 5$ minutes to school.

2. $\frac{2 \cdot 15 \cdot 45}{(15+45)} = \frac{2 \cdot 15 \cdot 45}{4 \cdot 15} = \frac{45}{2} = 22.5$.

3. If Kamal works for H hours, Yotam works for H + 1 hours since Yotam started an hour earlier. Kamal completes half the job in an hour while Yotam completes ¼ of the job in an hour so

   $(1/2) H + (1/4) (H + 1) = 1$,   or

   $3/4 H + 1/4 = 1$,   or

   $3/4 H = 3/4$,   with solution H = 1.

So Kamal works for 1 hour, Yotam works for 2 hours, and they finish at 2 pm.

4. He runs a total of 20 minutes at 10 miles per hour – for distance $10 \times (20/60) = 10/3$ miles run at 10 miles per hour. He runs an additional mile in 12 minutes.

In all, he runs for $10/3 + 1 = 13/3$ miles in $20 + 12 = 32$ minutes. So his average speed in miles per minute is $(13/3) \div 32$ and his average speed per hour is $60 \times (13/3) \div 32 = 20 \times 13 \div 32 = 260 \div 32$ or $130 \div 16 = 8^1/_8 = 8.125$ miles per hour.

5. Suppose that she drives for x minutes at 30 miles per hour and x + 5 minutes at 20 miles per hour. Her speed per minute is 30/60 miles per minute at 30 miles per hour and 20/60 miles per minute at 20 miles per hour. So $\frac{30x}{60} = \frac{20(x+5)}{60}$.
Multiplying both side of the equation by 60 gives $30x = 20(x + 5)$ or $30x = 20x + 100$ or $10x = 100$ with solution x = 10. Ten minutes is one-sixth of an hour, so in 10 minutes at 30 mph she drives $30/6 = 5$ miles.

6. Suppose he travels K km per hour for H hours for a total of HK kilometers. The sentence about going 5 km/h faster says that HK = (H –2) (K + 5). The right side expands to HK – 2K + 5H – 10, so this is equivalent to

   (1) 5H – 2K = 10.

The sentence about going 10 km/h faster says that HK = (H –3) (K + 10). The right side expands to HK – 3K + 10H – 30, so this is equivalent to

   (2) 10H – 3K = 30.

Doubling the first equation gives 10H – 4K = 20, and comparing this with the second equation 10H – 3K = 30 indicates K = 10. Substituting K = 10 into the equation 5H – 2K = 10 gives 5H = 30 or H = 6. So Tom biked 6 hours at 10 km per hour for a total of 60 km.

To check then answer, if he increased his speed to 15 km/h he would have traveled 60 km in 4 hours. If he increased his speed to 20 km/h per hour, he would go 60 km in 3 hours, so this answer does satisfy both conditions.

Meet #5

2.1 Surface Areas and Volumes

1. When computing areas and volumes you want to put everything in the same units. Here, working in feet is easiest. Nine inches is three-quarters of a foot, so the volume in cubic feet is $6 \times 6 \times \frac{3}{4} = 27$ cubic feet.

2. Two of the walls are 12 by 7 and two are 9 by 7. The total surface area of the walls is $2(12 \times 7) + 2(9 \times 7) = (24 + 18) \times 7 = 42 \times 7 = 294$ square feet.

3. We will compute the total volume of the wall and divide by the total volume of each brick to determine how many bricks are needed. In cubic inches the volume of the wall is $240 \times 48 \times 8$. The volume of each block is $4 \times 6 \times 8$. Don't multiply things out before dividing. Instead just think of the division as a fraction and cancel as many terms as you can. $\frac{240 \cdot 48 \cdot 8}{4 \cdot 6 \cdot 8} = \frac{240 \cdot 48}{4 \cdot 6} = \frac{240 \cdot 8}{4} = 240 \cdot 2 = 480.$

4. Use proportional reasoning. If the blue cube has a volume that is one-eighth of the volume of the red cube, then the blue cube's side length is half as long. If the side length is half as long, then the surface area will be one-fourth as large. The answer is $\frac{1}{4} \cdot 726 = 181.5 \text{ cm}^2$.

5. The bottom of the tent is $6 \times 8 = 48$ square feet. The triangles at the front and back are each $\frac{1}{2} \times 6 \times 4 = 12$ square feet. The sides of the tent are rectangles that are 8 feet long and 5 feet high (because the edge of the side is the hypotenuse of a 3-4-5 right triangle). So each side is $8 \times 5 = 40$ square feet.
The total amount of canvass is $48 + (2 \times 12) + (2 \times 40) = 152$ square feet.
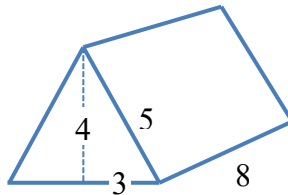
6. If dropping the three cubes into the measuring cup increases the total volume by three-eighths of a liter, then each of the three cubes has a volume of one eighth of a liter. A cube that is 10 cm by 10 cm by 10 cm has a volume of one liter. Using proportional reasoning, each of the cubes in this question has a volume that is one eighth as large as this, so they have a side length that is one half as long as this. Each cube has a side length of 5 cm.

2.2 Shapes Related to Circles

1. If the radius of the larger ball is twice as large, then its surface area is $2^2 = 4$ times as large. The answer is $4 \times 36\pi = 144\pi$ square inches.

2. Tbe base of the cone has a radius of 5 cm and the height of the cone is 12 cm, so the slant height of the cone is $s = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$ cm. The surface area of the cone is $\pi r^2 + \pi rs = 25\pi + 65\pi = 90\pi$ cm^2.

3. The ice cream takes the form of a cone with radius 3cm and height 12 cm plus one-half of a ball with radius 3 cm. The volume is therefore $\frac{1}{3}\pi r^2 h + \frac{1}{2} \cdot \frac{4}{3}\pi r^3 = \frac{108}{3}\pi + \frac{54}{3}\pi = 54\pi$ cm^3.

4. The cylindrical part of the silo has a radius of 10 feet. So the half of a sphere on top has its height equal to its radius of 10 feet. So the silo consists of a cylinder with a radius of 10 feet and a height of $60 - 10 = 50$ feet plus the half sphere. The surface area of the outside part of the cylinder is $2\pi rh = 2 \cdot 10 \cdot 50\pi = 1000\pi$ square feet. The surface area of the hemisphere is $\frac{1}{2} \cdot 4\pi 10^2 = 200\pi$ square feet. The total area to be painted is $1200\pi \approx 3600 + 120 + 48 + 1.2 + 0.7 \approx 3770$ square feet. 3770 / 350 is between 10 and 11 so he needs to buy 11 gallons.

5. It is better here to figure out the problem with an arbitrary radius r rather than plugging in 2.64 and ending up with a lot of messy expressions. The three balls together will have a volume of $3 \cdot \frac{4}{3}\pi r^3 = 4\pi r^3$. The cylinder will have a radius of r and a height of 6r so its volume is $\pi r^2 h = \pi r^2 \cdot 6r = 6\pi r^3$. The empty space is the difference between the volume of the cylinder and the volume of the balls, $6\pi r^3 - 4\pi r^3 = 2\pi r^3$. The ratio of empty space to the volume of the can is $\frac{2\pi r^3}{6\pi r^3} = \frac{1}{3} = 0.33$ to the nearest hundredth.

6. Vibha starts with a cone with a radius of 3 cm and a height of 4 cm. Its slant height is $s = \sqrt{3^2 + 4^2} = 5$ cm. As the drill bit goes into the cone from the top you can think of it as first obliterating a cone-shaped piece at the top leaving us with a frustum, and then drilling a hole through that frustum.
The original cone has a surface area of $\pi r^2 + \pi rs = 24\pi$ cm^2.
When the drill bit obliterates the top of the cone it is eliminating the slanted part of a cone that has a radius of ½ cm and a slant height of $\frac{5}{6}$ cm. (The slant height is 5/6 because the upper part of the cone is similar to the full cone and has a radius that is one-sixth as large so by proportional reasoning the height and slant height are each also one sixth as large. Hence the surface area that is obliterated is $\pi r's' = \pi \cdot \frac{1}{2} \cdot \frac{5}{6} = \frac{5}{12}\pi$ cm^2.

Meet 5 Solutions

As the drill continues down through the frustum it completely eliminates the top surface leaving only a hole where there was temporarily the top of a frustum. As the drill comes out the bottom of the cone it obliterates a circular region with an area of $\frac{1}{4}\pi$ cm^2. The drill bit also creates a new surface on the inside of the cone. It is like the outer surface of a cylinder with a radius of $\frac{1}{2}$ cm and a height of $(5/6)\cdot 4$ $= 10/3$ cm. The area of this added surface is $2\pi r'h' = 2\pi\frac{1}{2}\cdot\frac{10}{3} = \frac{10}{3}\pi$ cm^2.

The total surface area of the resulting shape is then $24\pi - \frac{5}{12}\pi - \frac{1}{4}\pi + \frac{10}{3}\pi = \left(24 + \frac{8}{3}\right)\pi = \frac{80}{3}\pi$ cm^2.

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2.3 Prisms and Pyramids

1. The volume of the pyramid is $\frac{1}{3} \times base\ area \times h = \frac{1}{3} 230^2 \cdot 150 = 230^2 \cdot 50 = 52900 \cdot 50 = 2{,}645{,}000$ cubic meters.

2. The volume of a rectangular prism is the product of the lengths of its three dimensions. If two lengths are 8 cm and 10 cm, the other dimension is $\frac{720}{8\cdot10} = 9$ cm. The surface area is $2\ (8\cdot10 + 8\cdot9 + 9\cdot10) = 2\ (80 + 72 + 90) = 484$ cm^2.

3. A right square prism only has two side lengths: the length s of the side of the square; and the height h. The volume is $s^2 h$. In this problem s must be either 3 cm or 5 cm with the h being the other. The two possibilities for the volume are $3^2 \cdot 5 = 45$ cm^2 or $5^2 \cdot 3 = 75$ cm^2. Their positive difference is 30 cm^2.

4. The center of the pyramid is directly above the center of the rectangle. Using the Pythagorean Theorem, the distance cach of the corners of the base rectangle and the point in the center of the base rectangle is $\sqrt{3^2 + 4^2} = 5$ cm. Using the Pythaorean Theorem this time on each vertical triangle connecting a corner of the base to the center of the base and the top of the pyramid we have $5^2 + h^2 = 13^2$. This gives that the height of the pyramid is $h = 12$ cm.
The volume of the pyramid is $\frac{1}{3} \cdot 6 \cdot 8 \cdot 12 = 192$ cm^3.

5. Let s be the slant height of each of the four sides of the pyramid. (By slant height we mean the length of the segment connecting the middle of a side of the base to the top vertex.) The surface area of the pyramid is $10^2 + 4 \left(\frac{1}{2} \cdot 10 \cdot s\right) = 100 + 20s$ square inches. This gives $s = 7$ inches.
To find the height of the pyramid apply the Pythagorean Theorem to the triangle that connects the center of one side of the base to the center of the base and the top of the Pyramid. The slant height is the hypotenuse of this triangle so $5^2 + h^2 = 7^2$. This gives $h = \sqrt{24} = 2\sqrt{6}$ inches.
The volume of the pyramid is $\frac{1}{3} \cdot 10^2 \cdot 2\sqrt{6} = \frac{200\sqrt{6}}{3}$ cubic inches.

6. To find the height apply the Pythagorean Theorem to a triangle connecting one of the vertices of the base to the center the base and the top of the pyramid. The segments connecting each vertex of the base to the center of the base are 10 cm, so $10^2 + h^2 = 20^2$. This gives $h = \sqrt{300} = 10\sqrt{3}$ cm. The hexagonal base can be divided in to six equilateral triangles with side length of 10 cm. Using the $s^2 \frac{\sqrt{3}}{4}$ formula for the area of an equilateral triangle, the area of the base is $6 \cdot 100 \cdot \frac{\sqrt{3}}{4} = 150\sqrt{3}$ cm^2. The volume of the pyramid is $\frac{1}{3} \cdot 150\sqrt{3} \cdot 10\sqrt{3} = 1500$ cm^3.

A - 108

2.4 Polyhedra

1. A cube has 12 edges and 8 vertices. The positive difference is 4.

2. It has 12 edges. If you have trouble picturing the octahedron, one way to count edges is to say that each triangular face has 3 edges, but calculating 8×3 is double counting the edges because each belongs to two faces. $8 \times 3 \div 2 = 12$.

3. Each of the 12 pentagonal faces has 5 surface diagonals. (You can connect each of the five vertices to either of the two nonadjacent vertices and we have to divide by 2 to correct for double counting.) Surface diagonals are obtained by connecting each vertex to any of the other 19 vertices that do not share a face. Nine other vertices share a face with any given vertex. (Three share an edge with the given vertex and hence are on two separate faces and there are two additional points on each of the three faces that the given vertex belongs to.) So there are $20 \times (19 - 9) \div 2 = 100$ space diagonal. The positive difference is $100 - 60 = 40$.

4. The original dodecahedron has 20 vertices, 30 ($=12 \times 5 \div 2$) edges, and 12 faces. The new shape has one 10-sided face coming from each face of the original dodecahedron and one triangular face coming from each vertex that was cut off, so it has $20 + 12 = 32$ faces. It has 60 vertices. You can see this either by adding up the vertices on each face and dividing by 3 to avoid triple counting, $(12 \times 10 + 20 \times 3)/3 = 60$, or by noting that when cutting off each of the original vertices you create 20 new vertices. It has 90 edges. You can see this either by again counting the sides of the faces and avoiding double counting, $(12 \times 10 + 20 \times 3)/2 = 90$, or thinking of the shape as having as its edges the middle parts of the 30 original edges plus 60 new edges because 3 are created when you cut off each of the 20 vertices of the original shape. To recap, V = 60, E = 90, and F = 32.

5. Let x be the length of the third edge in cm. The surface area in cm^2 is $2(4x + 8x + 32) = 24x + 64$. The volume is $32x$. These are equal when $8x = 64$ so $x = 8$. The length of a space diagonal is $\sqrt{4^2 + 8^2 + 8^2} = \sqrt{144} = 12$ cm.

6. The base of his pyramid is a triangle with area $\frac{1}{2} \times 2 \times 2 = 2$ cm^2. The height of his pyramid is 2 cm. Its volume is $\frac{1}{3} \times 2 \times 2 = \frac{4}{3}$ cm^3.

Meet 5 Solutions

2.5 Units of Measurement

1. The area in square yards is $99 \times 50 / 9 = 11 \times 50 = 550$ square yards.

2. On the map, Josh's path is a 3-4-5 right triangle. Its total length on the map is 12 inches. In real life, this is $12 \times 2.5 = 30$ miles.

3. One cubic meter is $100^3 = 10^6$ cubic centimeters. The volume in cubic meters is $\frac{14,875}{10^6} = 0.014875 \approx 0.015$ cubic meters to the nearest thousandth.

4. 385 yards is $\frac{385}{5280/3} = \frac{385}{1760} = \frac{77}{176 \times 2} = \frac{7}{32}$ miles. Using the distributive property we get $26\frac{7}{32} \times 32,000 = \left(26 \times 32,000 + \frac{7}{32} \times 32,000\right) = 832,000 + 7,000 = 839,000$ miles. This is quite a bit farther than the trip to the moon and back.

5. Let c be the side-length of the cube in centimeters. Its volume in cubic centimeters is c^3. Its surface area in square millimeters is $6 \times (10c)^2 = 600c^2$. Solving $c^3=600c^2$ gives $c = 600$ cm $= 6$ m. Its volume in cubic meters is $6^3 = 216$ m^3. We are told to approximate one meter as 39.36 inches $= 39.36 \div 36 = 1\frac{336}{3600} = 1\frac{28}{300} = 1\frac{7}{75}$ yards. The volume in cubic yards is $216 \times (1\frac{7}{75})^3 = 216 \times 1.09\overline{3}^3 \approx 216 \times 1.307 \approx 216 + 64.8 + 1.5 \approx 282$ cubic yards.

6. The shortened cone is similar to the original cone, but with all dimensions being 0.9 times as large. Hence, its volume is 0.9^3 times the volume of the original cone. The original cone had a volume of 1 liter $= 1000$ cubic centimeters $= 1,000,000$ cubic millimeters. The answer is $0.9^3 \times 1,000,000 = .729 \times 1,000,000 = 729,000$ cubic millimeters.

2.6 Advanced Topics

1. The volume of the original cone is $\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 36 \cdot 9 = 108\pi$. The upper part of the cone that is cut off to make the frustum is similar to the original cone, but with each dimension being two-thirds as large. So its volume is $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$ of the volume of the original cone. The volume of the frustum is $108\pi - \frac{8}{27} \cdot 108\pi = 108\pi - 32\pi = 76\pi$ cm^2.

2. The area of a spherical triangle is given by $A = \frac{S-180}{180}\pi R^2$ where S is the sum of the measures of the interior angles and R is the radius of the sphere. Here, this is $\frac{3}{180}\pi \cdot 3950^2 = \frac{3950 \cdot 395}{6}\pi = \frac{1975 \cdot 395}{3}\pi = \frac{(2000-25)(400-5)}{3}\pi = \frac{800{,}000-(10{,}000+10{,}000)+125}{3}\pi = \frac{780{,}125}{3}\pi = 260{,}041.\overline{6}\pi \approx 816{,}945$ square miles.

3. Jack cuts the sphere into five pieces. The smallest are the piece of the northern hemisphere between the 10°W and 70°W longitude lines and the opposite piece between the 110°W and 170°W lines. Each has four faces. Two are quarter circles (the left and right vertical faces if the North Pole is at the top) with a radius of 6 inches. One (on the bottom in the same orientation) is a sector that is one-sixth of a circle of radius 6 inches. Finally, there is one curved face with a surface area that is $\frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}$ of the surface area of the sphere. The total surface area is $2 \cdot \frac{1}{4} \cdot 36\pi + \frac{1}{6} \cdot 36\pi + \frac{1}{12} \cdot 4\pi \cdot 36 = 36\pi$ cubic inches.

4. Using $\frac{4}{3}\pi R^3 = 288\pi$ we get $R^3 = 3 \cdot 72 = 216$, so the radius R of the sphere is 6 cm. The sum of the three surface areas will be equal to the surface area of the original sphere plus the areas of the four new circular faces. The radius r of the new circular faces is the horizontal distance between a point that is three cm above the center of the sphere and the points on the sphere. Applying the Pythagorean Theorem to the triangle connecting the center of the sphere to the point three cm above the center and to points on the sphere at the same height we have $3^2 + r^2 = 6^2$, so $r^2 = 27$. The total surface area is $4\pi R^2 + 4 \cdot \pi r^2 = 4 \cdot 36\pi + 4 \cdot 27\pi = 252\pi$ cm^2.

5. The outer curved surface has a surface area of $\frac{3}{4} \cdot 4\pi \cdot 15^2 = 675\pi$. Each of the two flat surfaces is a semicircle with radius 15 cm minus a semicircle with radius 3 cm. So together their areas sum to $\pi \cdot 15^2 - \pi \cdot 3^2 = 216\pi$. Finally, the small curved inner surface has a surface area that is one-quarter of the surface area of a sphere of radius 3 cm. $\frac{1}{4} \cdot 4\pi \cdot 3^2 = 9\pi$. The total surface area is $(675 + 216 + 9)\pi = 900\pi$ cm^2.

3.1 Venn Diagrams

1. The numbers in the intersection of sets A and B are 21 and 42 with sum 63.

2. Because 17 of 30 teams have won the Super Bowl, there are 13 teams that have not won the Super Bowl. How many of these teams have also not had the first overall pick? We are told that 6 of the 13 teams that have had the first overall draft pick have won the Super Bowl. This implies that the remaining 7 teams have not won the Super Bowl but have had the first overall pick. So $13 - 7 = 6$ teams have not won a Super Bowl or had the first overall draft pick since 2000.

It is probably easier to do this problem by writing down a Venn Diagram with one set for teams that won the Super Bowl and one for teams that had the first draft pick. We are told that 6 teams are in the middle region for having both won the Super Bowl and had the first draft pick. Since 17 teams in total won the Super Bowl, we know there are 11 teams in the left region. Similarly, 13 teams in total had the first draft pick, so there are 7 in the right region. Adding up $11 + 6 + 7 = 24$ teams have either won the Super Bowl or had the 1st draft pick. This leaves 6 of the 30 teams outside both sets. These are the teams that have not won a Super Bowl and not had the first overall draft pick.
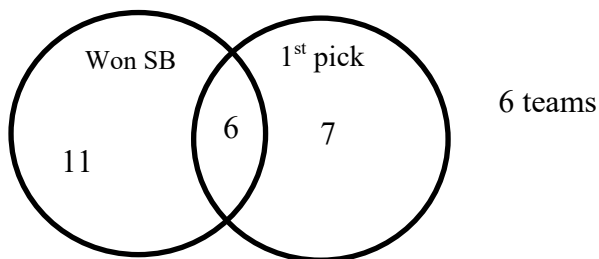
3. Of the 222 students invited to the USAJMO, 207 took the AMC 10A, so 15 took the AMC 10B but not the AMC 10A. Since 176 of the invitees took the AMC B, this means that $176 - 15 = 161$ of the invitees took both tests.

Alternatively, we can think of the problem as asking for $|A \cap B|$ where A is the set of USAJMO qualifiers who took the AMC 10A and B is the set of USAJMO qualifiers who took the AMC 10B. We know that $|A \cup B| = |A| + |B| - |A \cap B|$. This can be rearranged to give $|A \cap B| = |A| + |B| - |A \cup B|$. We are told that $|A \cup B|=222$, $|A|=207$, and $|B|=176$. This gives $|A \cap B| = 207 + 176 - 222 = 161$.

4. Look at the Venn Diagram below. Vivaan is in the part inside A and B, but outside C. Vivaan is the only student who knows how to draw a five-set Venn Diagram, so the other parts of set B are all empty. Write x, y, and z for the number of students in the three remaining regions.

We are told that 9 of the 11 students who like Venn Diagrams know how to spell Venn, so y = 9 and x = 1. (Vivaan is also one of the 11.)

We are told that Vivann and eight other students misspell Venn, so 11 of the 20 students spell Venn correctly and are in set C. This give y + z = 11. With y = 9 we have z = 2.

The number of students in exactly one set is x + z + 0 = 1 + 2 + 0 = 3.

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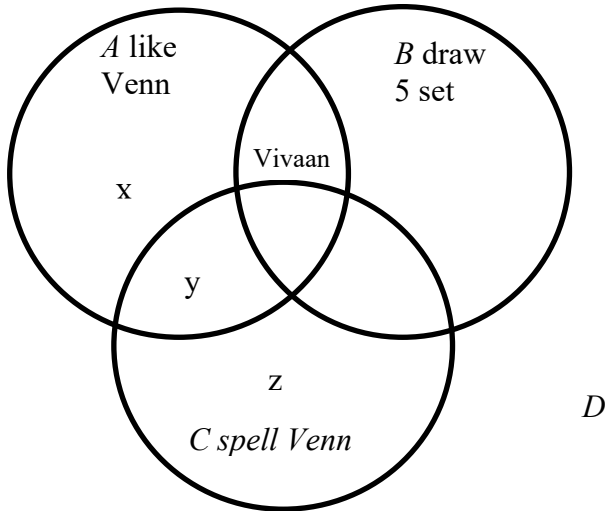
3.2 Basic Set Theory

1. Three people are in set A alone, three people are in Set B alone and three people are in both sets. So $|A \cup B| = 9$, $|A \cap B| = 3$, and the difference is 6. Some students may notice that people are in set A if they dated Taylor Swift and in set B if they are singers.

2. The intersection of B and C is {E, I, L, N, O}. So the union of that set with A contains those five elements plus G for a total of six elements.

3. A contains the odd numbers from 1 to 15, B contains multiples of 5 from 5 to 30. $C \cap D = \{2, 3, 5, 13\}$. Of these numbers, 3, 15 and 13 are in A, while 2 is not in A or B, so the answer is {3, 5, 13}.

4. Every number in $A \cap B$ is also in $A \cup B$, so the problem is simply asking for the intersection of all three sets.

The two-digit prime numbers that are equal to 3 mod 5 must all end in 3, since numbers that end in 8 are divisible by 2. So the two-digit prime numbers equal to 3 mod 5 are 13, 23, 43, 53, 73, and 83 – this is the intersection of A, B, and C.

5. The intersection of E and T is the set of three digit multiples of 11 from 110 to 990. To be a perfect square, a multiple of 11 must be of form $11^2 \cdot x^2$. The only three-digit numbers that are of this form are $11^2 \cdot 1^2 = 121$ and $11^2 \cdot 2^2 = 484$, because for $11^2 \cdot 3^2 = 33^2 = 1089$ is already too big to be a three-digit number. So the intersection of E, T, and S contains only these two numbers 121 and 484.

Both 121 and 484 are palindromic numbers, so both are in P, which means that both are in the union of P and Z and since they are perfect squares they are in the intersection of (P U Z) and S.

So the final set includes just 121 and 484 and the sum of these elements is 605.

3.3 Inclusion-Exclusion Counting

1. Let E be the set of Kyle's friends in the English class and let M be the set the Math class. By inclusion-exclusion counting we have $|E \cup M| = |E| + |M| - |E \cap M| = 5 + 7 - 1 = 11$.

In words, there are a total of 12 students in the two sets (including duplicates) of whom exactly one student appears twice. So there are ten students who appear in just one of the two sets plus an eleventh who appears in both for a total of 11 friends who are in at least one of the classes.

2. There are ten two-digit counting numbers (from 70 to 79) with a seven in the tens digit. There are nine two-digit counting numbers {17, 27, 37, 47, 57, 67, 77, 87, 97} with a 7 in the ones-digit. There is exactly one number with a 7 in both the tens and ones digit (77). So using inclusion-exclusion counting, there are a total of $10 + 9 - 1 = 18$ different two-digit counting number containing at least one 7.

3. Since $27 \times 37 = 999$, there are exactly 37 multiples of 27 and 27 multiples of 37 between 1 and 1000. Exactly one number in the given range, 999, is a multiple of both. Using inclusion-exclusion counting, there are a total of $27 + 37 - 1 = 63$ different numbers from 1 to 1000 that are multiples of 27 or 37.

4. Let S_{12} be the set of positive three-digit integers where the first two digits are the same. We can form such an integer by picking any digit from 1 to 9 to be the first (and second) digit and any digit from 0 to 9 for the last digit so $|S_{12}| = 9 \times 10 = 90$. Writing S_{13} for the set where the first and third digits are the same, an identical argument gives that $|S_{13}| = 90$. And writing S_{23} for the set where the second and third digits are the same, $|S_{23}| = 90$. (The argument here is slightly different: we pick some digit from 1 to 9 to be the first digit and then a digit from 0 to 9 to be both the second and third digit.)

The number of positive three digit integers with at least two digits that are the same is $|S_{12} \cup S_{13} \cup S_{23}|$. Inclusion-exclusion tells us that we can compute this as

$$|S_{12} \cup S_{13} \cup S_{23}| = |S_{12}| + |S_{13}| + |S_{23}| - |S_{12} \cap S_{13}| - |S_{12} \cap S_{23}| - |S_{13} \cap S_{23}| + |S_{12} \cap S_{13} \cap S_{23}|$$

Each of the first three terms are 90. Each of the sets like $S_{12} \cap S_{13}$ is the set of all three digit positive integers in which all three digits are identical, because if the first digit is equal to the second and equal to the third, then all the digits are the same. There are nine such numbers, 111, 222, …, 999. Finally, the last set, $S_{12} \cap S_{13} \cap S_{23}$ is also the set of positive three digit integers where all three digits are the same, which again is 9 numbers So, the inclusion-exclusion formula gives

$$|S_{12} \cup S_{13} \cup S_{23}| = 90 + 90 + 90 - 9 - 9 - 9 + 9 = 252.$$

Meet 5 Solutions

5. Define A to be the set of positive integers less than 100 that are a multiple of 7. Define B to be the set of positive integers less than 100 with a 7 in the tens place. And define C to be the set of positive integers less than 100 with a 7 in the ones place.

The number of positive numbers less than 100 that are "seveny" is $|A \cup B \cup C|$. By inclusion-exclusion counting this is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Since $7 \times 14 = 98$, there are 14 multiples of 7 that are less than 100 and $|A|=14$. There are ten two-digit numbers, 70, 71, …,79 with a 7 as the tens digit so $|B|=10$. There are also ten numbers, 7, 17, …, 97 with a 7 in the ones place so $|C|=7$.

There are two numbers, 70 and 77, in $A \cap B$. There are two numbers, 7 and 77 in $A \cap C$. There is just one number, 77, in $B \cap C$. And $A \cap B \cap C$ also contains exactly one number, 77. The inclusion-exclusion formula therefore gives that there are $|A \cup B \cup C| = 14 + 10 + 10 - 2 - 2 - 1 + 1 = 30$ positive numbers less than 100 that are "seveny".

6. Define A to be the set students who have taken Algebra. Define B to be those who have taken Number Theory. Define C to be those who have taken Counting and Probability. The number of students who have taken at least one class is $|A \cup B \cup C|$. By inclusion-exclusion counting this is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

The number of students who have taken all three courses is $|A \cup B \cup C| = 11$. If we add the number of students in each set like $A \cap B$, we count every student who has taken exactly two courses once, but we could all students who have taken all three courses three times. Hence, the number of students who have taken exactly two courses is $|A \cap B| + |A \cap C| + |B \cap C| - 3|A \cap B \cap C|$. We are told that this is equal to 21, so $|A \cap B| + |A \cap C| + |B \cap C| = 21 + 3|A \cap B \cap C| = 54$.

The inclusion exclusion formula then gives

$$|A \cup B \cup C| = 42 + 34 + 38 - 54 + 11 = 71$$

Another way to do this problem is to count the number of courses taken: $42 + 34 + 38 = 114$ courses. Of these 114 courses, 42 are accounted for by the 21 students who took exactly two of them and another 33 are accounted for by the 11 students who took all three of them. This leaves $114 - 42 - 33 = 39$ courses taken by students who took only one course for a total of $21 + 11 + 39 = 71$ students who took at least one course.

A - 116

Meet 5 Solutions

7. There are four three-digit multiples of 3 that have 3's in each of the first two digits:

(330, 333, 336, 339).

There are four three-digit multiples of 3 that have 3's in the first and last digits:

(303, 333, 363, 393)

There are three three-digit multiples of 3 that have 3's in the second and last digits:

(333, 633, 933)

With sets this small, it is probably easiest to combine them to make a single set of three-digit numbers with at least two 3's: (303, 330, 333, 336, 339, 363, 393, 633, 933). So, there are nine numbers of this form.

8. Since $7 \times 11 \times 13 = 1,001$ there is exactly one number from 1 to 1,001 that is a multiple of all three of 7, 11 and 13. A number is a multiple of 7 and 11 if and only if it is a multiple of $7 \times 11 = 77$. The fact that $7 \times 11 \times 13 = 1,001$ also implies that there are:

(1) 13 multiples of $7 \times 11 = 77$ between 1 and 1,001
(2) 11 multiples of $7 \times 13 = 91$ between 1 and 1,001
(3) 7 multiples of $11 \times 13 = 143$ between 1 and 1,001

The only overlap of these sets is that 1,001 appears in each of them. It is NOT a multiple of "exactly two" of 7, 11, and 13. So eliminating 1,001 from all three sets there are $12 + 10 + 6 = 28$ numbers that are multiples of exactly two of 7, 11, and 13.

A - 117

3.4 Review of Modular Arithmetic

1. Numbers that end in 2 or 7 are 2 more than a multiple of 5. Between 0 and 1,000, there are 100 of each: (2, 12, 22, ..., 992) from $(10 \times 0) + 2$ to $(10 \times 99) + 2$ and (7, 17, 27, ..., 997) which are the numbers from $(10 \times 0) + 7$ to $(10 \times 99) + 7$. So there are a total of 200 such numbers.

2. Since $5 \times 7 = 35$, this will be true whenever the number leaves a particular remainder in mod 35 arithmetic. Checking the numbers that are five more than a multiple of 7,

$$7 \times 0 + 5 = 5 = 0 \ (\text{mod } 5), \qquad 7 \times 1 + 5 = 12 = 2 \ (\text{mod } 5),$$
$$7 \times 2 + 5 = 19 = 4 \ (\text{mod } 5), \qquad 7 \times 3 + 5 = 26 = 1 \ (\text{mod } 5),$$
$$7 \times 4 + 5 = 33 = 3 \ (\text{mod } 5)$$

So the first number with this property is 33. After this, all numbers of form $33 + 35x$ will also have this property. To the first two are 33 and $33 + 35 = 68$, and the answer is $33 + 68 = 101$.

One could do this problem more quickly by noting that a number is 3 more than a multiple of 5 if it is **two less** than a multiple of 5. And a number is 5 more than a multiple of 7 if it is **two less** than a multiple of 7. When we think about looking for a number that is two less than a multiple of 5 and two less than a multiple of 7 it is fairly obvious that an answer is two less than 5×7, i.e. $35 - 2 = 33$.

3. The smallest element of $(A \cap B) \cup C$ will be either the smallest element of $A \cap B$ or the smallest element of C, depending on which is smaller.

The numbers that are 1 less than a multiple of 17 are 16, 33, 50, 67, ..., of which 67 is the smallest that is 2 more than a multiple of 5, so 67 is the smallest element of $A \cap B$.

Since $18 = 2 \times 3^2$ and $34 = 2 \times 17$, a number is divisible by 18 and 34 if and only if it is divisible by $2 \times 3^2 \times 17 = 17 \times 18$. Hence, the smallest element if C is 304. This is larger than the smallest element of $A \cap B$, so the answer is 67.

4. The number 3 is obviously the smallest number that is both 3 more than a multiple of 5 and 3 more than a multiple of 13. The full set of positive numbers that are 3 more than a multiple of 5 and 3 more than a multiple of 13 are the numbers of the form $3 + 65x$ for some nonnegative integer x: 3, 68, 133, ... To think about how many of these are less than 1000, note that $65 \times 15 = 975$, so the numbers from $3 + (65 \times 0)$ to $3 + (65 \times 15)$ will be those that are less than 1,000. This is a set of 16 numbers.

5. We can find the intersection of the four sets in any order we like. $A \cap B$ seems like the easiest one to find, so we start there. The first number that is equal to 5

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mod 10 and 10 mod 25 is 35. The least common multiple of 10 and 25 is 50, so numbers of form $35 + 50x$ are in $A \cap B$.

Which of these numbers are also a multiple of 17? Writing out the first few numbers, 35, 85, 135, ... we immediately notice that 85 is a multiple of 17. Because 17 and 50 have no common factors, the full set of numbers that satisfy all three conditions, i.e. the numbers in $A \cap B \cap D$, will be numbers that are of the form $85 + (50 \times 17)y$ for some integer y. The only numbers that are less than 1000 are 85 and $85+850 = 935$. Hence, there are two elements in the desired set.

6. If y is 5 mod 22 and y is between 0 and 100, then y is one of 5, 27, 49, 71, 93. We will count the number of $x \leq y$ that give a valid ordered pair separately for each of these values of y.

First, consider $y = 5$. Because $y = 5 \pmod 8$ we will $x + y = 3 \pmod 8$ if and only if $x = 6 \pmod 8$. There are **0** values of x that are both ≤ 5 and 6 more than a multiple of 8.

If $y = 27$, then $x + y = 3 \pmod 8$ if x is a multiple of 8. Three are **4** such x that are between 0 and 27: 0, 8, 16, and 24.

If $y = 49$, then $x + y = 3 \pmod 8$ if $x = 2 \pmod 8$. There are **6** such numbers between 0 and 49: 2, 10, 18, 26, 34, and 42.

If $y = 71$, then $x + y = 3 \pmod 8$ if $x = 4$ (4 mod 8). There are **9** such numbers between 0 and 71: the numbers from $8 \times 0 + 4$ to $8 \times 8 + 4$.

If $y = 93$, then $x + y = 3 \pmod 8$ if $x = 6 \pmod 8$. There are **11** such numbers from 0 to 93: the numbers from $8 \times 0 + 6$ to $8 \times 10 + 6$.

So in all, there are $0 + 4 + 6 + 9 + 11 = 30$ pairs that satisfy these requirements.

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1. Between 100 and 200 there are:
  -- 9 multiples of 11 from $11 \times 10 = 110$ to $11 \times 18 = 198$.
  -- 8 multiples of 13 from $13 \times 8 = 104$ to $13 \times 15 = 195$.
  -- 6 multiples of 17 from $17 \times 6 = 102$ to $17 \times 11 = 187$.
  -- 5 multiples of 19 from $19 \times 6 = 114$ to $19 \times 10 = 190$.

There are only two numbers that are in two of these sets: $11 \times 13 = 143$ and $11 \times 17 = 187$. (We can't use any higher multiple of 11 because $11 \times 19 = 209$ and the smallest multiple of two numbers bigger than 11 is $13 \times 17 = 221$.) No numbers are multiples of three more numbers.

So there are a total of $9 + 8 + 6 + 5 - 2 = 26$ numbers between 100 and 200 that are divisible by at least one of 11, 13, 17, 19.

If you want to think of this using the inclusion-exclusion formula, let A be the multiples of 11 between 100 and 200, let B be the multiples of 13 between 100 and 200, and so on. We then have

$$\begin{aligned}
|A \cup B \cup C \cup D| = &|A| + |B| + |C| + |D| \\
&- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\
&+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\
&- |A \cap B \cap C \cap D|.
\end{aligned}$$

The terms in the first row of the RHS are $9 + 8 + 6 + 5$. In the second row $|A \cap B| = 2$ and all other terms are zero. All terms in the third and fourth rows are zero. So answer is $9 + 8 + 6 + 5 - 2 = 26$.

2. x is the number of numbers that are in A, B, and C but not in D. Any such number must be a two-digit multiples of 7 (set A), a two-digit multiple of 9 (set B), and a two-digit multiples of 11 (set C), and must NOT be a perfect squares (not in set D). Since 7, 9 and 11 have no factors in common, the least common multiple of them is $7 \times 9 \times 11 = 693$. Since this is a three-digit number, there are no two-digit numbers that are multiples of 7, 9 and 11 so $x = 0$..

y is the number of numbers that are in set A, but not in B, C, or D. A is the two-digit multiples of 7, which range from $7 \times 2 = 14$ to $7 \times 14 = 98$, o there are 13 of them. Of these $7 \times 9 = 63$ is in set B, $7 \times 11 = 77$ is in set C and $7 \times 7 = 49$ is in set D. So there are $13 - 3 = 10$ numbers that are in set A but not in B, C, or D and $y = 10$.

z is the numbers that are in sets B and D, but not in A or C. B is the set of two-digit multiples of 9 or $\{18, 27, 36, 45, 54, 63, 73, 81, 90, 99\}$ of which 36 and 81

Meet 5 Solutions

are perfect squares so are also in D. Since 36 and 81 are not multiples of 7 or 11, neither is in A or C and $z = 2$.

The question asks us to find $x + y + z$, which is $0 + 10 + 2 = 12$.

3. We can count this by inclusion-exclusion counting by letting A be the set of ways to put 6 balls in the four bags with NO ball in bag A, letting B be the set of ways to put 6 balls in the four bags with NO ball in bag B, and so on. The question is then asking us to find $N - |A \cup B \cup C \cup D|$, where N is the total number of ways to put 6 balls in four bags and the second term on the right side is the number of "bad" ways that leave some bag empty.

$N = 4^6 = 4096$ because we are just choosing which bag to each ball in.

The inclusion exclusion formula tells us that
$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$$
$$- |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D|$$
$$+ |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$$
$$- |A \cap B \cap C \cap D|.$$

Most terms in this expression are surprisingly easy to find. For example, in the first row each term is $3^6 = 729$ because the number of ways to put the balls in bags with bag A empty is just the number of ways to choose in which of the three other bags to put each ball. The terms in the second row are all equal to $2^6 = 64$, because to allocate the balls with both bags A and B empty we choose whether to put each ball in bag C or D. The terms in the third row are all equal to 1, and the term in the fourth row is 0, because having all four bags empty is clearly impossible. This gives
$$|A \cup B \cup C \cup D| = 4 \times 729 - 6 \times 64 + 4 \times 1 + 1 \times 0 = 2916 - 384 + 4 = 2536.$$

The answer is $4096 - 2536 = 1560$.

The above answer is elegant and uses the material in the section of the text, but most students will probably get to this answer by just breaking things up into cases and counting them separately. Here's how that would go.

There are either 3 balls in one bag and 1 in each of the others or 2 bags that have 2 balls each and 2 bags that have 1 ball each.

First count the number of 3-1-1-1 combinations. If bag A has three balls, then there are $_6C_3 = 20$ ways to choose the 3 balls that can go in A. For each of these 20 combinations, there are $3 \times 2 \times 1 = 6$ ways to put the remaining balls in bags B, C, and D, so a total of $20 \times 6 = 120$ ways to put 3 balls in bag A and 1 in each of the others. Since there are four possible bags that could have 3 balls, there are a total of $4 \times 120 = 480$ ways to produce 3-1-1-1 combinations.

A - 121

Next, count the number of 2-2-1-1 combinations. If bag A has two balls, then there are $_6C_2 = 15$ ways to choose them. If bag B also has two ball, then there are $_4C_2 = 6$ ways to choose them for each possible choice of two balls in bag A. Finally, when there are two balls in A and two balls in B, there are 2 ways to put the remaining balls in bags C and D. So there are a total of $15 \times 6 \times 2 = 180$ ways to put two balls in A, two balls in B, and one ball each in C and D. Since there are $_4C_2 = 6$ possible choice of two bags to each have two balls, there are a total of $6 \times 180 = 1080$ ways to produce 2-2-1-1 combinations.

In total then, there are $480 + 1080 = 1560$ ways to put the balls into bags with at least one ball in each bag.

1. There are 5×4×3×2×1 = 120 possibilities.

2. There are 8×7 = 56 combinations of two toppings if the order matters, but assuming that the order does not matter (so that "Pepperoni and Onion" is the same as "Onion and Pepperoni"), this is a combinations problem and there are 56 ÷ 2 = 28 different two-topping pizzas.

3. There are 101×100×99 = 10,100 * 99 = 999,900 possible orders. Since the order determines the prizes, each order produces a different outcome, so 999,900 possible outcomes. This is a permutations problem.

4. This is a combinations problem. The number of ways to choose 3 students from a group of N is $_NC_3 = N(N-1)(N-2)/6$. We are told that this is 220, so we know that $N(N-1)(N-2) = 1320$. If a product of three similar numbers is a little bigger than 1000, the biggest number will need to be around 10, but a little bigger. Trying different values of N in this range, we find that $12 \times 11 \times 10 = 1320$, so there are 12 students on the team.

5. "ELLISON" contains a double "L" and five single letters. We can think about the reorderings of this that also have a double L as corresponding to choosing an ordering of six tiles: one with an "E", one with "LL", one with "I", and so on. Hence, this is just a standard permutations problem. We can choose any of the six tiles to be first, than any one of the remaining five to be second, and so on. This gives a total of 6×5×4×3×2×1 = 720 possible sequences.

6. If Kevin must ride with his mother, then his mother could take 1, 2, or 3 other students.
There are 5 ways to choose 1 other students from the other five on the team.
There are 5×4/2 = 10 ways to choose 2 other students from the other five on the team.
There are 5×4×3/(3×2) = 10 ways to choose 3 other students from the other five on the team.
So, there are a total of 5 + 10 + 10 = 25 ways to assign the team members to the two cars.

Meet 5 Arithmetic: 4.1 Combinatorics (Part 2)

1. There are $2^7 = 128$ different possible sequences.

2. There are two choices for each topping: "Yes" or "No", so a total of $2^8 = 256$ different pizzas. Note that if all eight toppings are "No", the pizza is plain.

3. The number of subsets of a set of size n is 2^N, so we want to find the largest whole number N with $2^N < 1,000,000$. The first ten powers of 2 in order are: $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, $2^6 = 64$, $2^7 = 128$, $2^8 = 256$, $2^9 = 512$, and $2^{10} = 1,024$. The fact that 2^{10} is close to 1000 often comes in handy for making approximations. If you know that $2^{10} = 1,024$, then squaring this we have $2^{20} = (2^{10})^2 = 1,024^2 > 1,000^2 = 1,000,000$, so we know that N = 20 is too big. It should also be obvious that $2^{19} = 2^{10} \times 2^9 = 1,024 \times 512$ is less than one million even without doing the multiplication and finding that it is 524,288. So, the answer is 19.

4. There are 36 different choices of ice cream for a one scoop serving. Since a one scoop serving can be (1) in a cup, (2) in a cone without chocolate; (3) in a cone dipped in chocolate, there are 108 different possible orders for a one-scoop serving. Since it is not possible to specify which flavor is on top, there are $_{36}C_2 = 36\times35/2 = 18\times35 = 630$ different combinations of two different flavors of ice cream, plus an additional 36 combinations of two scoops of the same flavor of ice cream. Since there are three ways that each combination of flavors can be served, this makes a total of $3 \times (630 + 36) = 1998$ different possible orders for a two-scoop serving. In all, this makes $108 + 1998 = 2,106$ different orders.

5. Consider the possible orders in each category one at a time:

Sandwiches: There are 5 possibilities where all three children choose the same sandwich (one for each sandwich). There are $5\times4 = 20$ possible ways that they can choose two of one sandwich and one of another sandwich. There are $_5C_3 = 10$ ways that they can choose three different sandwiches for a total of $5 + 20 + 10 = 35$ different sandwich orders.

French Fries: A useful observation is that "No" fries can be treated as a third possible size of fries. There are 3 possibilities for all three children to make the same order (which could be "Large", "Small", "No" fries), $3\times2 = 6$ possibilities for them to order 2 of one size and 1 of another size and only 1 possible way to order three separate items, for a total of $6 + 3 + 1 = 10$.

Drinks: There are 6 possibilities for all to order the same drink, $6\times5 = 30$ for two of one type of drink and 1 of another type of drink, and $_6C_3 = 20$ ways to choose three separate drinks for a total of $6 + 30 + 20 = 56$.

Combining the options from these three categories gives a total of $35 \times 10 \times 56 = 19,600$ possible orders.

Meet 5 Solutions

1. There are 6×6 = 36 possible outcomes of which (1, 4), (2, 3), (3, 2), (4, 1) give a sum of 5, so the probability is 4/36 = 1/9.

2. There are 20×20 = 400 possible outcomes of which 20 have the same number twice, so the probability that the numbers match is 20/400 = 1/20.

3. Again, there are 36 possible outcomes. To count the number of outcomes where the product of the two numbers is a perfect square we make an organized list. The product is between 1 and 36, so the possible squares 1, 4, 9, 16, 25, and 36.

1 is the product the outcome is (1, 1)
4 is the product if the outcome is (1, 4), (2, 2) or (4, 1)
9 is the product if the outcome is (3, 3)
16 is the product if the outcome is (4, 4)
25 is the product of the outcome is (5, 5)
36 is the product if the outcome is (6, 6)

So there are a total of 8 ways that the product could be a perfect square, and the probability that the product is a perfect square is 8/36 = 4/18 = 2/9.

4. There are 2^3 = 8 outcomes. Heads occurs twice in a row for the orders (HHH, HHT, THH), so there are 5 ways and probability 5/8 that Heads does not occur twice in a row.

5. The prime numbers from 1 to 6 are 2, 3, and 5. There are three outcomes where the largest number is 2: {(1, 2), (2, 1), (2, 2)}. There are five outcomes where the largest number is 3: {(1, 3), (3, 1), (2, 3), (3, 2), (3, 3)}. There are nine outcomes where the largest number is 5: {(1, 5), (5, 1), (2, 5), (5, 2), (3, 5), (5, 3), (4, 5), (5, 4), (5, 5)}. In total this gives 3 + 5 + 7 = 17 outcomes where the largest number is prime, for a probability of 17/36.

6. There are 10×9/2 = 45 possible values for the ordered pair (smaller number, larger number). For the difference between the two numbers to be smaller than 2, they must be consecutive. There are nine pairs of consecutive numbers: (1, 2), (2, 3), …, (9, 10)}. This gives that the probability of getting consecutive numbers is 9/45 = 1/5. So the probability that the difference is two or more is 1 − 1/5 = 4/5.

7. There are 400 possible outcomes. To count the number of outcomes for which the product is a perfect n^{th} power for some n > 1 we need to organize the counting in some way.

One way to organize the list is by the possible values of n. For n = 2 we are looking for a perfect square, i.e. we want the product to be 1^2, 2^2, …, or 20^2. For

A - 125

any value of x we can get a product of x^2 using (x, x). For some of the other values of x there are other ways to get a product of x^2.

2^2 – (1, 4), or (4, 1)

3^2 – (1, 9) or (9, 1)

4^2 – (1, 16), (2, 8), (8, 2), (16, 1)

6^2 – (2, 18), (3, 12), (4, 9), (9, 4), (12, 3), (18, 2)

8^2 – (4, 16), (16, 4)

10^2 – (5, 20), (20, 5)

12^2 – (8, 18), (9, 16), (16, 9), (18, 8)

All together this gives $20 + 2 + 2 + 4 + 6 + 2 + 2 + 4 = 42$ possibilities.

For n=3 we are looking for a perfect cube (that we have not already counted as a perfect square. There are only a few because 1^3 and 4^3 are squares and soon the numbers get too big.

2^3 – (1, 8), (2, 4), (4, 2), (8, 1)

3^3 – (3, 9), (9, 3)

6^3 – (12, 18), (18, 12)

n=4 gives no new possibilities because they are all perfect squares. n=5 gives ono one outcome

2^5 – (2, 16), (4, 8), (8, 4), (16, 2)

n=6 gives no new possibilities because they are all squares (and cubes). n=7 gives one outcome

2^7 – (8, 16), (16, 8).

n=8 again gives no new possibilities because they are squares, and there are no possibilities with bigger n because the numbers are too big.

In total, we have $42 + 8 + 4 + 2 = 56$ outcomes where the product is an nth power for $n > 1$. The probability of getting one of these combinations is $56/400 = 7/50$.

Another way to do the counting would be to think about prime factorizations. Among the numbers from 1 to 20, there are eight primes (2, 3, 5, 7, 11, 13, 17, 19), three powers of 2 (4, 8, 16), a square of a prime (9), four products of two primes (6, 10, 14, 15), three products of a squared prime and another prime (12, 18, 20) and lastly the number 1.

Thinking about each possible value for the smaller number in a pair other than (x, s) we find the possibilities are:

1 with 4, 8, 9, 16;

2 with 4, 8, 16, 18;

3 with 9, 12;

4 with 8, 9, 16

5 with 20;

8 with 16, 18;

9 with 16;

12 with 18;

A - 126

These 18 unordered pairs give 36 possible ordered pairs which is another way to find that there are 20 + 36 = 56 possible outcomes.

Meet 5 Arithmetic: 4.3 Probability Problems with Two Events

1. Each die is a multiple of 3 if it is a 3 or a 6, which occurs with probability $2/6 = 1/3$. So the probability that this occurs on two consecutive rolls is $1/3 \times 1/3 = 1/9$.

2. The probability that Amy has a green marble is $3/12 = 1/4$. If she chooses a green marble, there are 11 marbles left and 4 of them are blue, so the "conditional probability" that Blanche chooses a blue marble after Amy chooses a green marble is $4/11$. The probability that both events occur together is $1/4 \times 4/11 = 1/11$.

3. One good way to compute the probability that at least one number is prime is by negation – we compute the probability that NO numbers are prime, and then subtract this from one. The prime numbers from 1 to 10 are 2, 3, 5, and 7. There is probability $6/10$ that the first ball is not prime. The first ball is put back into the bag after it is drawn, so the probability of two non-primes is $6/10 \times 6/10 = 36/100 = 9/25$. So the probability of at least one prime is $1 - 9/25 = 16/25$.

4. This is an "and" problem where the probability changes after each ball is taken out se-o we need to use conditional probabilities. The probability that all numbers are even is $(5/10) \times (4/9) \times (3/8)$. Cancelling common factors of 5, 4, and 3 from the numerators and denominators this is $(1/2) \times (1/3) \times (1/2) = 1/12$.

5. The product of three numbers is even if at least one of them is even. Each die is equally likely to be even or odd. The only way to get an odd product is if all three are odd, which occurs with probability $1/2 \times 1/2 \times 1/2 = 1/8$, so the probability that the product is even is the difference $1 - 1/8 = 7/8$.

6. Think about the possible orderings of 5 B's and 5 G's. There are 10 choose 5 separate orders for a total of $(10 \times 9 \times 8 \times 7 \times 6) / (5 \times 4 \times 3 \times 2) = 252$. For no two girls to be next to each other, they either each have to be two places apart (i.e. in places 1, 3, 5, 7, 9 or 2, 4, 6, 8, 10) or start in place 1, end in place 10 and have two gaps of 2 places and one gap of three places (i.e. in places 1, 4, 6, 8, 10 or 1, 3, 6, 8, 10 or 1, 3, 5, 8, 10 or 1, 3, 5, 7, 10). This is a total of six possibilities for a total probability of $6/252 = 1/42$.

1. If Anna's coin comes up 1, it will always be a factor of the number on top of Bernice's die. If Anna's coin comes up 2, it is a factor of the number on the top of Bernice's die if that number is even. The number on top of Bernice's die is even with probability ½. Using conditional probabilities conditioning on Anna's coin flip gives a probability of $1/2 \times 1 + 1/2 \times 1/2 = 3/4$.

2. There are **36** possible rolls of two dice. There are 11 that have at least one number equal to 5, so Ben can cross out **25** of the 36 possibilities. He can circle exactly **1** possibility (5, 5). The answer is **1/11**, because he has circled one of the eleven possibilities that is not crossed out.

3. A total of 30 of 36 possible rolls of two dice have a sum less than 10. Of these, (1, 1) produces a sum of 2, (1,2) and (2, 1) produce a sum of 3, (1,4), (2,3), (3,2) and (4,3) produce a sum of 5 and (1, 6), (2, 5), (3, 4), (4, 3), (5, 2) and (6, 1) produce a sum of 7. So, he circles 13 pairs and concludes that the probability is 13/30.

4. The product of the numbers is a multiple of 8 if the prime factorization of the product includes 2^3.
If the first number is 8, there is probability 1 that the product is a multiple of 8.
If the first number is 4, there is probability ½ that the product is a multiple of 8.
If the first number is 2 or 6 (which are multiples of 2^1), there is probability ¼ that the product is a multiple of 8.
If the first number is 1, 3, 5 or 7, there is probability 1/8 that the product is a multiple of 8.

This yields probability $(1/8 \times 1) + (1/8 \times 1/2) + (1/4 \times 1/4) + (1/2 \times 1/8) = 1/8 + 1/16 + 1/16 + 1/16 = 5/16$ that the product of the numbers is a multiple of 8.

5. The product of two numbers ends in 1 if (1) they both end in 1; (2) one ends in 3 and the other ends in 7; (3) both end in 9. If the first number ends in 1, there are 9 remaining numbers that also end in 1 for conditional probability 9/99 that the product will end in 1. Similarly, if the first number ends in 9, there are 9 remaining numbers that also end in 9. If the first number ends in 3 or 7, there are 10 remaining numbers that make the product end in 1, so conditional probability 10/99 that the product will end in 1. The combined probability is then $(1/10 \times 9/99) + (1/10 \times 9/99) + (2/10 \times 10/99) + 6/10 \times 0 = (9 + 9 + 20)/990 = 38/990 = 19/495$.

6. There are 21 numbers that are 1 mod 5, 21 numbers that are 2 mod 5 and 20 numbers that are 3, 4, and 0 mod 5 among the numbers from 1 to 102. Consider three cases:

Meet 5 Solutions

(1) The first number is a multiple of 5, so is equal to 0 mod 5. The sum is a multiple of 5 if the second number is also 0 mod 5. Probability (20/102) for the 1st number, and then (19/101) for the 2nd number.

(2) The first number is equal to 3 (or 4) mod 5. The sum is a multiple of 5 if the second number is 2 mod 5 (or 1 mod 5 if the first is 4 mod 5). Probability 2× (21/102) for the 1st and then (20/101) for the 2nd number.

(3) The first number is 1 (or 2) mod 5. The sum is a multiple of 5 if the second number is 4 mod 5 (or 3 mod 5 if the first is 2 mod 5). Probability 2×(21/102) for the first and (20/101) for the 2nd number.

Taking the sum across these cases gives a probability of $=$

$$\frac{20 \times (19 + 42 + 42)}{102 \times 101} = \frac{20 \times 103}{102 \times 101}$$

or $\quad$ $(10 \times 103)/(51 \times 101) = 1030 / 5151.$

7. The probability of getting at least four heads in a row can be computed as the probability of getting one of four mutually exclusive events. One is a sequence of four or more heads that starts with the first coin. The second is a sequence of four or more heads with the first of these heads being the second coin. And so on up to a sequence where the first of the four consecutive heads is the fourth coin.

The probability that the first four flips are all heads is 1/16.

The probability that we get a sequence of at least four or more heads starting with the second coin is the probability that the flips for the first through fifth coins are THHHH. This is 1/32
.
The probability that we get a sequence of four or more heads starting with the third coin is the probability that the second through sixth flips are THHHH. The probability of this is also 1/32.

The probability that we get a sequence of four or more heads starting with the fourth coin is the probability that the third through seventh flips are THHHH. The probability of this is also 1/32.

The answer is $1/16 + 3 \times (1/32) = 5/32.$

Another way to do the problem which is more directly using conditional probability is to condition on all possible values for the third through fifth flips. Conditional on any of the sequences that have two or more T's, i.e. TTT, TTH, THT, HTT, or on HTH getting four or more heads in a row is impossible.

This leaves three cases for which need to compute probabilities.

Case 1: Flips 3 to 5 are HHT. This case occurs with probability 1/8. Conditional on this case arising we get four heads in a row with probability ¼ because we must have that the first and second flips are H.

Case 2: Flips 3 to 5 are THH. Again this occurs with probability 1/8. As in the first case, conditional on this case we get four heads in a row with probability ¼.

Case 3: Flips 3 to 5 are HHH. Again this occurs with probability 1/8. Conditional on this case we get four heads in a row if the second or sixth flip is H, i.e. if the 2nd and 6th flips are HH, HT, or TH. This occurs with probability ¾.

Summing across the cases gives $1/8 \times (¼ + ¼ + ¾) = 5/32$.

Meet 5 Arithmetic: 4.5 Averages

1. The girls score an average of 10 points higher than the boys, so the total score is 10×14 = 140 points ahead of an average of 80. With 20 students in the class, this means that the class average is $80 + (140 \div 20) = 80 + 7 = 87$.

2. Using ahead and behind counting, she is 12 × (.75 - .8) = 0.6 free throws behind an 80% success rate. Every free throw that she makes in the second half puts her 1 – 0.8 or 0.2 free throws ahead of an 80% success rate. So if she makes three free throws in the second half she will be 3 × 0.2 = 0.6 ahead of an 80% success rate in the second half, which exactly cancels out the 0.6 free throws that she is behind that success rate from the first half. So she makes 3 free throws in the second half.

3. He is 4 points ahead of a 93 average after the 4th test. If he gets a maximum score of 100 on the 6th test, then he is a total of 11 points ahead of a 93 average not counting the fifth test. So his lowest possible score on the 5th test is 93 – 11 or 82.

4. After dropping a score of 64, Martin is exactly even with a score of 92 for the semester. So, before dropping that score he was 28 points behind an average of 92. If 28 points behind an average of 92 gives an average of 88 and there are n tests in all with 28/n = 4 or n = 7.

5. Since 97×20 is not a multiple of 11, at least one student must have a score that is not a multiple of 11. The nearest multiple of 11 to 97 is 99. If nineteen students score 99, that would yield a total score that is 19 × 2 = 38 points ahead of an average of 97. Then if the last student scores 97 – 38 = 59, the class average would be 97 with 19 students having a score that is a multiple of 11. The probability would then be 19/20.
(There are also other ways to have 19 students have scores that are multiples of 11. For example, 1 student scores 70, 1 students scores 88, and 18 score 99.)

6. 35% of the group are green men and 8% of the group are green women, so a minimum of 43% of them are green. Assuming that the percentages are exact and not rounded, 8% means 2 out of every 25 people, so the size of the group must be a multiple of 25. Similarly, 35% means 7 out of every 20 people, so the size of the group must be a multiple of 20. The least common multiple of 20 and 25 is 100, so the size of the group must be a multiple of 100 after we account for these separate restrictions. There could be 100 Martians if 35 are green men, 35 are red men, 8 are green women, 12 are red women, 7 are red other-gendered, and 3 are green other-gendered.

Meet 5 Algebra: 5.1 Solving Quadratic Equations

1. This is easiest to solve by factoring since $x^2 - 5x + 6 = (x - 3)(x - 2)$. So (x-3) (x-2) = 0 which means either $x - 3 = 0$ or $x - 2 = 0$, so the solutions to the equation are x = 2 and x = 3.

2. This problem can be done without ever finding the roots. Using the facts about sums and products of roots, the sum of the roots is 10 and the product of the roots is 16. So $r + s + rs = 10 + 16 = 26$. Factoring also works fine: $(x^2 - 10x + 16) = (x - 8)(x - 2) = 0$ gives solutions r = 2 and s = 8. So $r + s + rs = 2 + 8 + 16 = 26$.

3. This does not factor easily so instead we use the quadratic formula with a = 2, b = -6, c = 3. This gives solutions $x = \dfrac{6 \pm \sqrt{6^2 - 4 \times 2 \times 3}}{4} = \dfrac{6 \pm \sqrt{12}}{4} = \dfrac{6 \pm 2\sqrt{3}}{4}$.

In reduced form, the solutions are $x = \dfrac{3 - \sqrt{3}}{2}$ and $x = \dfrac{3 + \sqrt{3}}{2}$ and the larger one is $\dfrac{3 + \sqrt{3}}{2}$

4. Here we can apply the quadratic formula with a = 2, b = 13, c = -7, so the solutions are

$$x = \frac{-13 \pm \sqrt{13^2 - 2 \times 4 \times (-7)}}{4} = \frac{-13 \pm \sqrt{169 + 56}}{4} = \frac{-13 \pm \sqrt{225}}{4}$$

The difference in the two solutions is given purely by the plus/minus term or $\dfrac{2\sqrt{225}}{4} = 15/2$.

5. The prime factorization of 999 is $111 \times 9 = 3 \times 9 \times 37 = 3^3 \times 37$. The two factors of 999 that are closest together are $3^3 = 27$ and 37. This means that we can factor the equation using integers: $x^2 + 10x - 999 = (x + 37)(x - 27) = 0$ with solutions x = +27 and x = -37.

6. Define $y = (x+1)^2$ so that this is a quadratic equation $y^2 - 5y^2 + 4 = 0$, which can be factored to produce the equation $(y - 4)(y - 1) = 0$. The solutions are y = 1 and y = 4. Given this value of y, we can solve for all possible values of x given $(x + 1)^2 = 1$ OR $(x + 1)^2 = 4$.
First, $(x + 1)^2 = 1$ means x + 1 = -1 or x + 1 = 1 which gives solutions x = -2 and x = 0.
Second, $(x + 1)^2 = 4$ means x + 1 = -2 or x + 1 = 2 which gives solutions x = -3 and x = 1. So the full set of solutions is x = -3, -2, 0, or 1.

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7. If both solutions are integers, then the quadratic must factor. If the factorization is (2x + A)(x + B), then A has to be an even number. So the factorization must be 2(x - D)(x + E) = 0 where D and E are positive integers and D × E = 30.

The prime factorization of 30 is 2 × 3 × 5 so it has 2×2×2 = 8 factors – in order 1, 2, 3, 5, 6, 10, 15, 30 so the possible pairs of values of D and E are (1, 30), (2, 15), (3, 10), (5, 6), (6, 5), (10, 3), (15, 2), and (30, 1). 8 possible pairs of solutions. For any pair (D, E), the value of B will be 2(E – D). Lining up the solutions as we have it is clear that E – D is different for each solution, so there are 8 possible values for b.

1. If a quadratic equation has roots x=5 and x=8, then its equation is
$y = k(x − 5)(x − 7)$ for some value of k. In this problem we are given that the
quadratic equation has a coefficient of 1 on the x^2 term, so k must be 1. The
quadratic is $(x − 5)(x − 8) = x^2 − 13x + 40$. (a, b) = (13, 40).

2. The points (2, 0) and (7, 0) are in the graph. Since quadratics are symmetric
about their lowest point, the lowest value occurs half way between x=2 and x=7.
The lowest value is when x = (2+7)/2 or x = 4.5.

3. Because the quadratic passes through the points (3, 0) and (6, 0), its equation
must be $y = a(x − 3)(x − 6)$. We are also given that the quadratic passes through the
point (0, -6). This gives that a(-3)(-6) = -6, which implies x = -1/3.

Plugging in x=7 we then have z = (-1/3)(7 − 3)(7 − 6) = -4/3.

Meet 5 Algebra: 5.3 Problems that Involve Quadratic Equations

1. $x^2 - 4x - 18 = 3$ can be written as $x^2 - 4x - 21 = 0$ and then factored as $(x-7)(x+3) = 0$, with solutions $x = 7$ and $x = -3$. The positive-valued solution is $x = 7$.

2. This is a right triangle, so by the Pythagorean Theorem, $(2x+1)^2 = x^2 + (2x-1)^2$ or $4x^2 + 4x + 1 = x^2 + 4x^2 - 4x + 1$. This simplifies to $4x^2 + 4x + 1 = 5x^2 - 4x + 1$ or $8x = x^2$. The solution is $x = 8$.
Note that the other solution to the quadratic, $x=0$, is not a valid solution because it would make the length of one side of the triangle negative.

3. Denoting the ages by D, G, and K, we are given $D = 2$, $G = K + 74$, and $K^2 = DG - 49$. Putting everything in terms of Kate's age K the last equation gives $K^2 = 2(K + 74) - 49$ sr $K^2 - 2K + 99 = 0$.

We can factor this equation as $(K - 11)(K + 9) = 0$. The solution must be positive, so $K = 11$, i.e. Kate is 11 years old.

4. If the width of the rectangle is W, then the length is $4W + 2$. The area is $LW = 4W^2 + 2W = 12$. Solving for W, we have $2W^2 + W - 6 = 0$. Using the quadratic formula with $a = 2$, $b = 1$, $c = -6$ the solutions are

$$W = \frac{-1 \pm \sqrt{1 + 4 * 2 * 6}}{4} = \frac{-1 \pm 7}{4}.$$ The positive-valued root is $W = 6/4 = 1.5$.

Then the length is $1.5 \times 4 + 2 = 8$ cm. With sides of 1.5 and 8 cm, the rectangle has perimeter of $2(1.5 + 8) = 19$ cm.

5. Suppose the pig walks at x feet per second. It takes $\dfrac{720}{x}$ seconds for the pig to get home and $\dfrac{720}{x-3}$ seconds for Max to get home, so we know $\dfrac{720}{x} + 54 = \dfrac{720}{x-3}$.
Multiplying both sides of the equation by $x(x-3)$ gives a quadratic equation $720(x-3) + 54x(x-3) = 720x$. Cancelling the 720x terms from both sides and then dividing both sides by 54 this simplifies to
$-40 + x(x - 3) = 0$ or $x^2 - 3x - 40$. After factoring, this is $(x - 8)(x + 5) = 0$ with positive-valued solution $x = 8$. Max walks 3 feet per second slower, which is $8 - 3 = 5$ feet per second.

6. If the number is x then $\dfrac{1}{x} - \dfrac{1}{4x-1} = 10$. Multiplying both sides by $x(4x - 1)$ gives the quadratic equation $4x - 1 - x = 10x(4x - 1)$ or $3x - 1 = 40x^2 - 10x$, which we can write as $40x^2 - 13x + 1 = 0$.

The question does not ask for the possible values of the number. It asks us to find $1/r + 1/s$ where r and s are the two possible values. Note that $1/r + 1/s = (r+s)/rs$.

A - 136

We can compute this using the standard formulas for the sum and product of the roots of a quadratic. $r + s = 13/40$ and $rs = 1/40$. Dividing we find that the sum of the reciprocals of the roots is 13.

Using sum and product of roots formulas is usually the easiest way to solve problems like this. In this case, it is also easy to just factor the quadratic. It factors as $(5x - 1)(8x - 1) = 0$ with solutions $x = 1/5$ and $x = 1/8$. The sum of the reciprocals of these two solutions is $5 + 8 = 13$.

Meet 5: Algebra 5.4 Advanced Topic: Complex Numbers

1. This is a quadratic equation with coefficients A = 1, B = -2, C = 5. Applying the quadratic formula, the solutions are $\frac{2\pm\sqrt{4-20}}{2} = 1 \pm 2i$ so (a, b) = (1, 2).

2. This is a quadratic equation with coefficients a = 1, b and c unknown. The same rules about sums and products of roots apply to quadratics with complex roots. So b is the negative of the sum of the roots and c is the product of the root. This gives b = -((3 + 2i) + (3 − 2i)) = -6. And c = (3 + 2i)(3 − 2i) = 9 + 6i − 6i − 4i² = 9 + 4 = 13. The answer is (-6, 13).

3. One way to solve this problem is with sum and product rules. This is often the best approach when a question asks for some function of the roots rather than asking you to find the roots. In this problem, $r^2s + s^2r = (r+s)$ rs. By the sum and product rules r + s = 11 and rs = 24. So the answer is 11 × 24 = 264.

In this problem the roots of the quadratics are whole numbers so finding the roots and computing the expression is also easy. We can solve by factoring $x^2 - 11x + 24 = 0$ as (x − 8) (x − 3) = 0 with solutions x = 3, 8. So the sum $3^2 \times 8 + 8^2 \times 3 = 9{\times}8 + 64{\times}3 = 72 + 192 = 264$.

4. Again, use sum and product rules. $\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs}$. The sum of the roots is -13. The product of the roots is 72. So the answer is -13/72.

This problem would be much harder if you tried to find the roots. You'd need to use the quadratic formula to find that the roots were $\frac{-13\pm\sqrt{-119}}{2}$, and then go through a messy calculation taking reciprocals, rationalizing denominators, etc. After all this work you would get -13/72 as the answer (if you didn't make any mistakes).

5. The trick to this problem is to recognize that that we can express $r^2 + s^2$ in terms of r + s and rs by writing $r^2 + s^2 = r^2 + 2rs + s^2 - 2rs = (r + s)^2 - 2rs$. Plugging in 97 for rs and -17 for r + s this is 289 − 2×97 = 289 − 194 = 95.

6. $i^3 - 2i^2 + 3i - 2 = i \times (i^2)$ - $2i^2 + 3i - 2 = (-1) i - 2(-1) + 3i - 2 = 2i$. So (a, b) = (0, 2).

7. Multiply numerator and denominator of 1 / (2 + i) by (2 − i) to get
$$\frac{2-i}{(2+i)(2-i)} = \frac{2-i}{5}.$$ Next simplify $(2 + i)^2$ to get $2^2 + 4i + i^2 = 3 + 4i$. Finally, subtracting the first result from the second gives the answer (3 + 4i) − (2 − i)/ 5 for final result 13/5 + 21/5 i.

## 2.1 Special Triangles

1. Because ACB is a 45° angle, triangle ADC is also a 45-45-90 triangle. The hypotenuse AC has length 2, so the length of leg AD is $\sqrt{2}$.

Another way to find AD is to compute the area of ABC in two ways: as ½AB×AD, and as ½BC×AD.

2. Triangles AOB, BOC, and COA are all congruent (by S-S-S) so angles AOB, BOC, and COA are all equal. By adding up, each is 120°. The isosceles triangle theorem implies that angles OBC and OCD are each 30°. Angle BDO is a right angle (by adding up in triangle BDA) so BDO is a 30-60-90 triangle. The long leg of this triangle, BD, has length ½, so the short leg, OD has length $\frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$.

A generally useful theorem that would let you solve this problem more quickly is that the medians of a triangle are concurrent and "trisect" each other, i.e. they meet in a point that this 2/3 of the way along each median. In an equilateral triangle the incenter, circumcenter, and centroid are all the same. The extensions of AO, BO, and CO are medians of the triangle. AD has length $\frac{\sqrt{3}}{2}$ because ADB is a 30-60-90 triangle. And the length of OD is 1/3 of the length of AD by the medians trisect theorem.
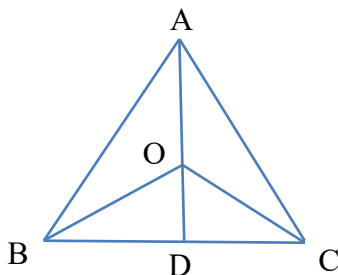
3. Triangle EDC is a right triangle with CE = 2 DE. Hence, EDC is a 30-60-90 triangle and ED = DC/$\sqrt{3}$. Using DC = AB = AE, the ratio of ED to EA is also $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.
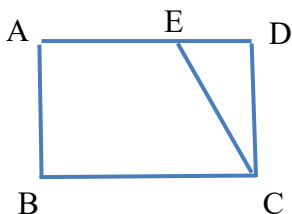
4. Euler's Theorem says that the square of the distance between the incenter and circumcenter is given by $d^2 = R^2 - 2Rr$, where R is the radius of the circumscribed circle and r is the radius of the inscribed circle. In a right triangle, the circumcenter is the midpoint of the hypotenuse, so in this problem $R = \sqrt{2}$. A formula for the incenter is $r = K/s$ where K is the area of the triangle and s in the semiperimeter. Here, $K = 2$ and $s = 2 + \sqrt{2}$ so $r = \frac{2}{2+\sqrt{2}} = \frac{2(2-\sqrt{2})}{(2+\sqrt{2})(2-\sqrt{2})} = 2 - \sqrt{2}$. Plugging into the Euler's formula gives $d^2 = R^2 - 2Rr = 2 - 2\sqrt{2}(2 - \sqrt{2}) = 6 - 4\sqrt{2}$.

This problem can be done almost as easily without knowledge of Euler's formula by noting that the incenter G and the circumcenter O both lie on the bisector of the right angle (call this A). Hence, we can find GO as GO = AO – AG. AO is the circumradius (or the distance from A to BC) and AG is $\sqrt{2}r$.

5. Draw horizontal lines connecting the opposite vertices of the octagon as shown in the figure below. Write the length of the square as 1+2x, where x is the common length of the two very small segments that this creates along the side of the square in the middle of the figure.

The lines also create two tiny 45-45-90 triangles in which each leg has length x. We can compute the width of the octagon as twice the width of the square plus twice the leg of this tiny triangle: 2(1+2x) + 2x = 2+6x. We can also compute the width of the octagon in the standard way using 45-45-90 triangles (see the dotted lines) as $1 + 2(1/\sqrt{2}) = 1 + \sqrt{2}$. Setting the two equal to each other gives $x = (\sqrt{2} - 1)/6$ and the length of the side of the square is $1 + \frac{\sqrt{2}-1}{3} = \frac{2+\sqrt{2}}{3}$.

X
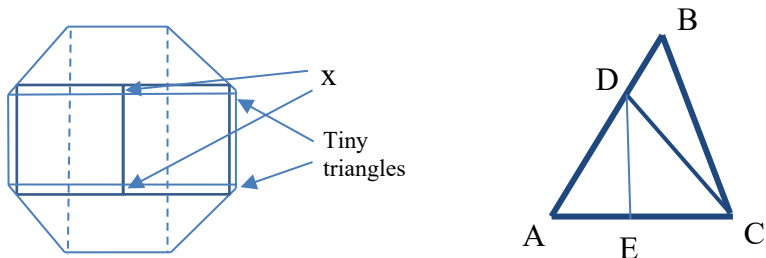
Tiny triangles

6. Draw a perpendicular segment DE from D to AC. Assume that DE has length 1. (If it has length x then every other distance we find will be multiplied by x and the x's will cancel out when we compute ratios. AED is a 30-60-90 triangle so AE=$1/\sqrt{3}$ and AD = $2/\sqrt{3}$. DEC is a 45-45-90 triangle so EC = 1 and DC = $\sqrt{2}$. Combining these observations, we have found that the side lengths of a 45-60-75 are in the ratio $\frac{2}{\sqrt{3}} : \sqrt{2} : 1 + \frac{1}{\sqrt{3}}$ from smallest to largest. (The smallest side is opposite the 45° angle and the largest opposite the 75° angle.

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Triangle ABC is also a 45-60-75 triangle. Its shortest side, AC, has length $1 + \frac{1}{\sqrt{3}}$.

Side AB is opposite the 75° angle so its length is $\frac{1 + \frac{1}{\sqrt{3}}}{\frac{2}{\sqrt{3}}} AC = \frac{\sqrt{3}+1}{2} AC$. We saw

earlier that the length of AB is $\frac{\frac{2}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} AC = \frac{2}{\sqrt{3}+1} AC$. So by subtraction the length

of DB is $\left(\frac{\sqrt{3}+1}{2} - \frac{2}{\sqrt{3}+1} \right) AC$. We can rationalize the second fraction by multiplying

the top and bottom by $\sqrt{3} - 1$. The ratio is $\frac{\sqrt{3}+1}{2} - \frac{2(\sqrt{3}-1)}{3-1} = \frac{3-\sqrt{3}}{2}$.

2.2 Analytic Geometry: Lines (pt 1)

1. The slope of the line is a. Given two points we compute the slope as the change in y over the change in x: $\frac{-8-5}{10-0} = \frac{13}{10}$. As a decimal this gives a=-1.3.

2. The slope of the line is $\frac{6-12}{10-4} = -1$, so we know that in slope-intercept form the equation is y = -x + d for some constant d. To the fact that the line goes through (4, 12) we plug in 4 for x and 12 for y and find that d = 16, so in slope-intercept form the equation is y = -x + 16. Moving all terms to the LHS the equation in the desired form is x + y − 16 = 0 so (a, b, c) = (1, 1, 16).

3. The line goes through both the points (0, b) and (3, 17) with b > 0. Its slope, $\frac{17-b}{3}$, is a prime number. The smallest positive value of b for which $\frac{17-b}{3}$ is prime is b = 2.

4. The lines y = 0 and y = 3x + 6 intersect when 3x + 6 = 0 which gives x = -2. The point of intersection is (-2, 0). Similarly, the lines y=0 and y = 4 − 2x intersect when 4 − 2x = 0 which gives x = 2 and (2, 0) as the point of intersection. Finally, the lines y = 3x + 6 and y = 4 − 2x intersect when 3x + 6 = 4 − 2x which gives x = -2/5 and y = 24/5. We can easily compute the area of a triangle with vertices (-2, 0), (2, 0), and (-2/5, 24/5) by thinking of the segment connecting (-2, 0) and (2, 0) as the base. The length of that base is 4. The height is 24/5. So, the area is $\frac{1}{2} \cdot 4 \cdot \frac{24}{5} = \frac{96}{10}$. As a decimal this is 9.6.

5. The point (x, 3x + 9) lies on the line 3x + 5y = 9 if 3x + 5(3x + 9) = 9. This gives 18x + 45 = 9 which holds for x = -2. When x = -2, 3x + 9 = 3, so the point of intersection is (-2, 3). The distance from this point to the "positive x axis" is the distance to the closest point on the positive x axis, which is (0, 0). Using the distance formula, this distance is $\sqrt{(-2)^2 + 3^2} = \sqrt{13}$.

6. Drawing in the very light lines shown below connecting A and C to the midpoint of AD, we realize that they divide the trapezoid into three equilateral triangles. So if we put A at (0, 0) on a coordinate grid and D at (20, 0), the coordinates of B and C will be $(5, 5\sqrt{3})$ and $(15, 5\sqrt{3})$.

A - 142
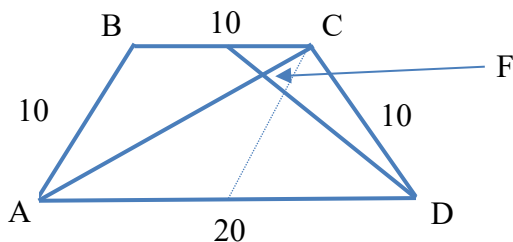
Point E is the midpoint of BC and hence has coordinates $(10, 5\sqrt{3})$. The equation for the line connecting A and C is $y = \frac{\sqrt{3}}{3} x$. The line connecting D and E has slope $\frac{-5\sqrt{3}}{10} = -\frac{\sqrt{3}}{2}$ and would pass through the point $(0, 10\sqrt{3})$ if extended another 10 units in the x direction, so its equation is $y = -\frac{\sqrt{3}}{2} x + 10\sqrt{3}$. The x-coordinate of the point where AC and DE intersect satisfies $\frac{\sqrt{3}}{3} x = -\frac{\sqrt{3}}{2} x + 10\sqrt{3}$ which gives x = 12. The y-coordinate of the intersection is $\frac{\sqrt{3}}{3} 12 = 4\sqrt{3}$, so on our grid the point F is $(12, 4\sqrt{3})$. We can find the area of ADF easily by the base-height formula treating AD as the base. It is $\frac{1}{2} \cdot 20 \cdot 4\sqrt{3} = 40\sqrt{3}$.

7. The line must pass through the points $(0, b)$, $(2, 3)$ and (b, m) and have slope m. For $b \neq 2$, this will be true if and only if the slope of the segments connecting each of the consecutive pairs of points is m. This gives $m = \frac{m-3}{b-2}$ and $m = \frac{3-b}{2}$. The second of these is equivalent to $b = 3 - 2m$. Plugging into the first gives $m = \frac{m-3}{1-2m}$. This gives the quadratic $2m^2 = 3$. The product of the solutions to this is $-\frac{3}{2}$.

For b=2 the line passes through $(0, 2)$, $(2, 3)$ and $(2, m)$. This would only be possible if m=3, but the slope of the line connecting $(0, 2)$ and $(2, 3)$ is ½, so this does not give an additional solution, and the answer to the question is -3/2.

1. The midpoint can be obtained by averaging the x and y coordinates of the endpoints. Here this gives $\left(\frac{1+2}{2}, \frac{0+5}{2}\right) = \left(\frac{3}{2}, \frac{5}{2}\right)$.

2. In standard form the equation for the line is $3x + y - 10 = 0$. The point-to-line distance formula gives that the distance is $\frac{3 \cdot 6 + 1 \cdot 2 - 10}{\sqrt{3^2 + 1^2}} = \frac{10}{\sqrt{10}} = \sqrt{10}$.

3. Point E is $\left(\frac{-2+2}{2}, \frac{0+6}{2}\right) = (0,3)$. Point F is $\left(\frac{10+11}{2}, \frac{6+0}{2}\right) = \left(\frac{21}{2}, 3\right)$. The point on EF that is twice as far from E and F is the point that you get by taking a weighted average of the coordinates of E and F placing weight 2/3 on F. This gives $\left(\frac{1}{3} \cdot 0 + \frac{2}{3} \cdot \frac{21}{2}, 3\right) = (7, 3)$.

4. Recognizing that this is a right triangle, we can put it on a coordinate grid with A = (0, 5), B = (0, 0), and C = (12, 0). The coordinates of D would then be $\left(\frac{0+12}{2}, \frac{0+5}{2}\right) = \left(6, \frac{5}{2}\right)$. The distance between this point and (0, 0) is $\sqrt{6^2 + \left(\frac{5}{2}\right)^2} = \sqrt{36 + \frac{25}{4}} = \sqrt{\frac{169}{4}} = \frac{13}{2}$.

While coordinate geometry gives a straightforward way to do the problem, one can do it more quickly if one remembers that the circumcenter of a right triangle is the midpoint of the hypotenuse. In this problem this means that BD is the circumradius of the triangle, which is one-half of the length of its hypotenuse.

5. The circumcenter of a right triangle is the midpoint of its hypotenuse. Averaging the coordinates of B and C we find that O = (5, 6).

6. The set of all points in a plane that are equidistant from two distinct points is a line perpendicular to the segment connecting the two points passing through its midpoint. Here, the segment connecting the two points has slope 7/4 and midpoint (3, 13/2). So the line y = ax + b passes through (3, 13/2) and has slope -4/7. Its y intercept is $13/2 + (-3) \cdot (-4/7) = 6\frac{7}{14} + \frac{24}{14} = 8\frac{3}{14}$.

7. If we put the cube on a coordinate grid so that one vertex is at (0, 0, 0) and the the three connecting vertices are (1, 0, 0), (0, 1, 0), and (0, 0, 1), then the pyramid is has vertices (0, 0, 0), (1/3, 0, 0), (0, 1/3, 0), and (0, 0, 1/3). Three of its sides are 45-45-90 right triangles and the fourth is an equilateral triangle.

If we place the pyramid on a table top with any of the right triangles on the bottom, then the other vertex is directly above the right angle of the base and the

height is 1/3. Thinking about what the pyramid look like it seems unlikely that this would be the way to minimize the height.

One way to determine the height if the equilateral triangle is placed on the bottom is to note that the vertex that was at $(0, 0, 0)$ in the original orientation will now be the highest point. It will be directly above the center of the equilateral side, whose coordinates in the original orientation were $(1/9, 1/9, 1/9)$. (It is the point you get by averaging the coordinates of the vertices.) So, the distance is $\sqrt{\frac{1}{81} + \frac{1}{81} + \frac{1}{81}} = \sqrt{\frac{3}{81}} = \frac{\sqrt{3}}{9}$.

2.3 Analytic Geometry: Circles

1. The equation $(x - a)^2 + (y - b)^2 \leq r^2$ describes all points in a circle of radius r centerd at (a, b). In this case, the circle is centered at (1, 0) and has area is $\pi r^2 = 9\pi$.

2. We are told that a circle centered at (3, 0) with radius r is tangent to the line y = 3x. A line that is tangent to a circle is perpendicular to the line connecting the center of the circle to the point of tangency. Here, the tangent has slope 3, which implies that the line connecting the center of the circle to the point of tangency has slope -1/3.

Hence, the point of tangency is at the intersection of the line y = 3x and the line with slope -1/3 passing through (3, 0). The equation for the latter line is y = -(1/3)x + 1. The lines intersect where 3x = (-1/3)x + 1 which gives x = 3/10. The y-coordinate is 3x = 9/10.

3. The circle has center (a, 0) and radius r. If it passes through the point (1, 0) its center must be 1+r or 1-r. The fact that it also passes through (3, 4) and (10, -3) implies that the center must be at 1 + r (otherwise (1, 0) is the rightmost point on the circle).

The points (3, 4), and (10, -3) are both at a distance of r from the point (1+r, 0). The distance formula this implies that $(r - 2)^2 + 4^2 = (r - 9)^2 + (-3)^2$. Expanding the quadratic terms we get $r^2 - 4r + 20 = r^2 - 18r + 90$. This simplifies to $14r = 70$, so $r = 5$.

One could also do this problem fairly easily by noting that the radius of the circle must be the circumradius of the triangle with vertices (1, 0), (3, 4), and (10, -3). We can find the circumradius via the formula $R = \frac{abc}{4\,Area}$. We can find the side lengths using the distance formula and the Area using the shoelace formula covered in section 2.4.

4. Let O be the center of the circle which is located at (0, 0). Since the circle has radius 3, OAB is an equilateral triangle and angle BOA is 60°. Since CB is tangent to the circle, it is perpendicular to radius OB. Hence, COB is 90°. By adding up in triangle OBC, angle ACB is 30°.

A - 146

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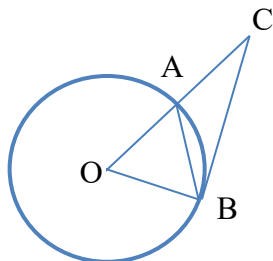
5. Completing the square, $x^2 + 4x + y^2 \leq 12$ if and only if $(x + 2)^2 + y^2 \leq 16$, so the inequality describes a circle of radius 4 centered at the point X = (-2, 0). Note that this circle passes through the points A = $(0, 2\sqrt{3})$ and B = (2, 0).

We can compute the area of the region in the first quadrant by subtracting the area of triangle XAO from the area of sector XAB where O is the origin (0, 0). Because XO = 2 and OA = $2\sqrt{3}$, triangle XAO is a 30-60-90 triangle with angle AXO a 60° angle. Using this, the area of the sector is one sixth of the area of the circle, $\frac{1}{6}\pi 4^2 = \frac{8}{3}\pi$. The area of triangle AXO is $\frac{1}{2} \cdot 2 \cdot 2\sqrt{3} = 2\sqrt{3}$. The area of the region is $\frac{8}{3}\pi - 2\sqrt{3}$.
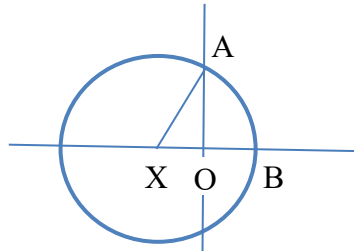
6. The circle is centered at the point (a/2, 1). If the circle is tangent to the two lines, then its center must be equidistant from the two lines. Using the point-to-line distance formula and setting the distances from the two lines equal to each other we find that a must satisfy $\frac{|7\left(\frac{a}{2}\right)-6\cdot 1|}{\sqrt{7^2+6^2}} = \frac{|2\left(\frac{a}{2}\right)+9\cdot 1-8|}{\sqrt{2^2+9^2}}$. Note that the denominators of the two sides are equal so the equation reduces to $|7a - 12| = |2a + 2|$.

We look for solutions in three ranges. For a < -1 both linear terms would be negative, so a would need to satisfy $7a - 12 = 2a + 2$. The solution to this equation, a = 14/5 is not in the range we are considering. For -1 < a < 12/7 we would need a solution to $12 - 7a = 2a + 2$. This has a valid solution, a = 10/9. Finally, for a > 12/7 we would again need a solution to $7a - 12 = 2a + 2$. The solution, a=14/5, is a valid solution in this case. Hence, there are two possible values for a, 10/9 and 14/5.

7. Points with x ≥ 0 satisfy the inequality if and only if $(x - 1)^2 + y^2 \leq 4$. Points with x < 0 satisfy the inequality if and only if $(x + 1)^2 + y^2 \leq 4$. Hence, the area is the area of the shape formed by two overlapping circles as shown elow: one centered at (1, 0) with radius 2, and the other centered at (-1, 0) also with radius 2. We can find the area of the portion of the region with x < 0 as one often does with circular shapes. It is the area of the portion of the full circle to the left of the lines shown (which connect the center X to the points A and B on the y axis where the circles intersect) plus the area of triangle XAB.

Points A and B have coordinates $(0, \pm\sqrt{3})$, so XOA and XOB are both 30-60-90 right triangles. Therefore the portion of the circle to the left of AXB is 2/3 of the

A - 147

full circle and has area $\frac{2}{3}\pi \cdot 4 = \frac{8}{3}\pi$. Triangle ABX has area $\frac{1}{2} \cdot 2\sqrt{3} \cdot 1$. Adding the two together, and then doubling to count also the area of the region with $x > 0$ we get the at the full area is $\frac{16}{3}\pi + 2\sqrt{3}$.

A

X   O

B
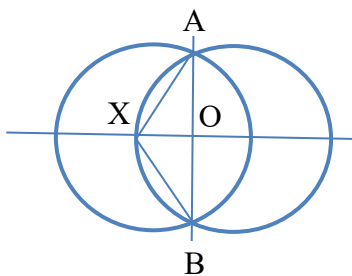
2.4 Areas of Polygons on a Grid

1. The pentagon is small enough that one can most easily just count the points. There are 9 interior points in each row for y = 1, 2, 3, and 4. And the diagonal edge in the upper right passes through (8.5, 5), so there are 8 lattice points with y=5. This gives 44 points in total.

To practice a technique that can be useful with more complicated figures one could instead use Pick's Theorem; Area = # Interior + ½ # Boundary Pts – 1. Think of the pentagon in this problem as a rectangle with vertices (0, 0), (10, 0), (0, 6), and (10, 6) with a triangle bounded by (7, 6), (10, 6), and (10, 4) cut off the corner. Its area is therefore 60 – ½ ·3·2=57. The number of interior points is then 57 + 1 – ½ # Boundary Points. There are 11 lattice points along the bottom of the pentagon, 8 along the top, 5 more on the left side, 4 more on the right side, and none on the diagonal (other than the vertices which we've already counted). This gives 57 + 1 – ½ ·28 = 44 as we found above.

2. We can divide this octagon into a central square bounded by (±3, ±3), four 2-by-6 rectangles like that bounded by (-3, 3), (-3, 5), (3, 3), and (3, 5), and four corner triangles like that bounded by (3, 3), (3, 5), and (5, 3). Computing the area of each part gives the area as 36 + 4·12+ 4·2 = 92.

3. The area of this pentagon is most easily computed using the shoelace method. Graphing the points and looking at the pentagon to put them in the proper order, they can be ordered going counterclockwise around the pentagon as (1, 1), (-3, 2), (-2, -3), (1, -1), and (5, -1).

(1, 1)

$1 \cdot 1 - 1 \cdot (-3) =$ 5

(-3, 2)

$(-3) \cdot (-3) - 2 \cdot (-2) = 13$

(-2, -3)

$(-2) \cdot (-1) - 1 \cdot (-3) =$ 5

(1, -1)

$1 \cdot (-1) - (-1) \cdot 5 =$ 4

(5, -1)

$5 \cdot 1 - 1 \cdot (-1) =$ 6

(1, 1)

$33 \div 2 = 16.5$

Computing the shoelace product (don't forget to put (1, 1) at the end as well as the beginning) and dividing by two at the end we find that the area is 16.5.

Another good way to compute the area of this pentagon is to divide it into two triangles and a trapezoid by adding vertical lines at x=1 and x=-2.

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4. One good way to do this problem is just to make a good drawing and count the points. We can also use Pick's Theorem. Dividing the big hexagon with vertical lines at x=-3 and x=3 we see that it has area $54 + 108 + 54 = 216$. It has 24 boundary lattice points, so it contains $216 - \frac{1}{2} \cdot 24 + 1 = 205$ interior lattice points.

The smaller interior hexagon is similar to the first one, but one-third as large, so by proportional reasoning its area is $216 \div 3^2 = 24$. It has 8 boundary lattice points, so it contains $24 - \frac{1}{2} \cdot 8 + 1 = 21$ interior lattice points.

Lattice points are interior to the larger hexagon and exterior to the smaller one if they are interior to the first hexagon and neither interior to nor on the boundary of the smaller one. There are $205 - 8 - 21 = 176$ points meeting both conditions.

5. The region satisfying all three inequalities is a triangle. One vertex is the intersection point of the lines $3x + 2y = 14$ and $x - 2y = -6$. A second is the intersection point of the lines $3x + 2y = 14$ and $x + 6y = -6$. The third is the intersection of $x - 2y = -6$ and $x + 6y = -6$.

Solving each pair of equations we find that the vertices of the triangle are (2, 4), (6, -2), and (-6, 0). Using the shoelace method or dividing the triangle with a horizontal line connecting (-6, 0) with (14/3, 0), we find that the area is 32.

6. The inequalities say that the points (x, y) must be strictly inside a rhombus with vertices (-7, 5), (13, 5), (3, 0) and (3, 10) and strictly outside the rhombus with vertices (-3, 5), (9, 5), (3, 2), and (3, 8). All such points have $y > 0$, so the additional requirement that $|xy| > 0$ rules out all points with $x \leq 0$. The number of possible x and y values is sufficiently small, and the positions of the rhombuses simple enough to compute (all sides have slope $\pm 1/2$), so that a good approach to just do an organized count of the points.

At x=1 the inner rhombus passes through (1, 3) and (1, 7). The outer rhombus passes through (1, 1) and (1, 9). So there are exactly two grid points in between them ((1, 2) and (1, 8)). At x=2 the inner rhombus passes through (2, 2½) and (2, 7½) and the outer rhombus passes through (2, ½) and (2, 9½). So there are four lattice points in between them: (2, 1), (2, 2), (2, 8), and (2, 9). Continuing this calculation, we realize that this will be a repeating pattern until the small rhombus ends. There will be two lattice points with x=3, four with x=4, and so on up to x=9. There is no inner rhombus for x = 10, 11, or 12. The outer rhombus passes through (10, 3½) and (10,6½), (11, 4) and (11, 6), and (12, 4½) and (12, 5½). This gives another three, one, and one lattice points. In total, 31 lattice points satisfy the conditions.

x =	1	2	3	4	5	6	7	8	9	10	11	12	Total
# points	2	4	2	4	2	4	2	4	2	3	1	1	31

A - 150

2.5 The Triangle Inequality

1. There are 9 possible values. The third side can be 2, 3, 4, 5, 6, 7, 8, 9, or 10 cm long. It cannot be 11 or more cm because that would violate $5 + 6 > x$. It cannot be 1 or less because that would violate $5 + x > 6$. Each of the numbers from 2 through 10 does work because for each of those values the sum of the two smallest sides is bigger than the largest side.

2. To make the area as small as possible it is intuitive that we want the sides to be as short as possible. The sides cannot be 1, 2, and 3 inches long because this violates the triangle inequality. Side lengths of 2, 3, and 4 do satisfy the triangle inequality. To calculate the area of this triangle we use Heron's formula:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\frac{9}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}} = \frac{\sqrt{135}}{4}.$$ To approximate this to the

nearest tenth, we can test numbers between 11 and 12 to find $11.6 < \sqrt{135} < 11.7$, so $\frac{\sqrt{135}}{4} \approx 2.9$ to the nearest tenth.

To see that choosing the smallest sides is the way to get the smallest area one could suppose that the sides are x-1, x, and x+1. Heron's formula then gives that

the area is $\sqrt{\frac{3}{2}x(\frac{x}{2}-1)\frac{x}{2}(\frac{x}{2}+1)}$, which is obviously increasing in x.

3. The three smallest prime numbers are 2, 3, and 5, but these won't work as side lengths because of they fail the triangle inequality. Thinking about this, we realize that we cannot have 2 as a side length at all: all other primes are odd so the smaller prime plus 2 won't be bigger than the bigger prime.

Since none of the side lengths can be 2, the smallest possible lengths are 3, 5, and 7. These do satisfy the triangle inequality so the answer is $3 + 5 + 7 = 15$ inches.

4. Make an ordered list considering each possible value of x.

For x=1, the only possible value for y is 5.
For x=2, y can be 4, 5, or 6.
For x=3, y can be 3, 4, 5, 6, or 7.
For x=4, y can be 2, 3, 4, 5, or 6.
For x=5, y can be 1, 2, 3, 4, or 5.
For x=6, y can be 2, 3, or 4.
For x=7, y can be 3.

There are no solutions with x=8, 9, or 10 because $x + y \leq 10$ implies $y \leq 2$, which violates the triangle inequality. In total we have 23 solutions.

5. We will want to count the number of different possible triangles in some organized manner. One good method is to organize the list by the length of the longest side. (Within each value I thought about triangles ordered by the possible values for the shortest side.)

Longest side 2: 2-2-2
Longest side 3: 2-2-3, 2-3-3 (Note that 3-3-3 would be similar to 2-2-2.)
Longest side 5: 2-5-5, 3-3-5, 3-5-5
Longest side 7: 2-7-7, 3-5-7, 3-7-7, 5-5-7, 5-7-7

In total, there are 11 possible triangles.

6. Note that two of the three triangle inequality constraints will be satisfied for any values of x and y that we plug in:

$$x+y + 2x-y = 3x > 2x$$
$$x+y + 2x = 3x+y > 2x-y$$

The one remaining triangle inequality constraint is that $2x - y + 2x = 4x - y > x+y$. This is equivalent to $2y < 3x$ or $y < 3/2\ x$.

Note also that if 2x and 2x-y are both whole numbers, then y is whole number. The fact that x+y is a whole number implies that x is a whole number as well.

With these two observations we have reduced the problem to finding the largest possible value for y/x given that x and y are whole numbers with $y < 3/2\ x$ and each of x+y, 2x, and 2x-y being at least 1 and at most 9.

The fact that $2x < 10$ implies that x is 1, 2, 3, or 4. The largest fraction $< 3/2$ we can make using any of these as a denominator is 4/3. Checking y=4 and x=3 we see that it does give a valid solution: the sides are 7, 5, and 8. The answer is 4/3.

1. The diagonal connecting (-1,-1) and (5, 5) has length $6\sqrt{2}$. The diagonal connecting (0, 4) and (4, 0) has length $4\sqrt{2}$. The area of the rhombus is $\frac{1}{2}d_1d_2 = \frac{1}{2}(6\sqrt{2})(4\sqrt{2}) = 24$.

2. The area is $\frac{1}{2}\overline{AB} \cdot \overline{AC} \sin(A) = \frac{1}{2}8 \cdot 8 \cdot \sin(30°) = \frac{1}{2}8 \cdot 8 \cdot \frac{1}{2} = 16$. Alternately, draw altitude BD from B to AC. ABD is a 30-60-90 right triangle so $\overline{BD} = \frac{1}{2}8 = 4$. Treating AC as the base the base-height formula gives the area as $\frac{1}{2}8 \cdot 4 = 16$.

3. Treat the regular octagon as the union of eight triangles each connecting the center of the circle to two vertices. The area is then $8 \times \frac{1}{2}1 \cdot 1 \cdot \sin(45°) = 4\frac{\sqrt{2}}{2} = 2\sqrt{2}$. Again, students who don't know trigonometry can find the area of the triangles by drawing an altitude from side of length one to the other. This creates a 45-45-90 triangle that can be used to find the height.

4. The diagonals of a parallelogram bisect each other. So think of one diagonal as a horizontal segment of length 17cm and the other diagonal as a segment of length 13cm through the midpoint of this section. The area is the area of two triangles: one above the horizontal segment and one below. The largest possible areas for the triangles will be if the height is as large as possible. This obviously occurs when the 13cm diagonal goes straight up and down. Its area is then $\frac{1}{2}13 \cdot 17 = \frac{221}{2}$. More formally for those who know trigonometry the area of any parallelogram is $\frac{1}{2}d_1d_2\sin(x)$ where x is the angle between the diagonals. The largest possible value for $\sin(x)$ is 1.

5. There are two possible locations for point C: clockwise or counterclockwise around the circle from B. In one case the measure of angle OAC is 135º. In the other case it is 45º. The area of triangle OAC is $\frac{1}{2}15 \cdot 8 \cdot \sin(135°)$ or $\frac{1}{2}15 \cdot 8 \cdot \sin(45°)$. Each of these is equal to $\frac{1}{2}15 \cdot 8 \cdot \frac{\sqrt{2}}{2}$ which is the same as the area of triangle OAB. So the answer is 0. The areas can also be found by using the $\frac{1}{2}$ base $\times$ height formula treating OA as the base and using 45-45-90 triangles to find the heights.

6. There are four possible locations for points D, E, F, and G corresponding to the top and bottom vertices of the four 4-5-6 triangles coming off AB shown in the figure at left below. One of X and Y will be above segment AB and the other will be below. Call the one above the line X. Depending on how one assigns the names D, E, F, and G to the four possible points the intersection point X will be at one of

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the three points marked in the figure (one of which is well above the part that we've graphed. The area of ABXY is the sum of the area of ABX and the area of ABY and each of these can be found using the ½ base × height formula, so the smallest possible area will occur when X and Y re as close to AB as possible. This occurs when X is the point we've labeled in the left figure. To find the area of ABX we want to find the height which is the length of the altitude from X to AB. To find this draw in altitudes EK and altitude XJ as in the magnified figure on the right. One way to find EK is to note that Heron's formula implies that the area of triangle ABE is $\frac{15}{4}\sqrt{7}$ so the ½ base × height fomula implies that the height EK is $\frac{5}{4}\sqrt{7}$. (One can also find this by writing z for the length of KB and noting that the length of EK can be found either as $4^2 - z^2$ or as $5^2 - (6-z)^2$ and solving for the z that equates these.) The Pythagorean Theorem gives that AK = 15/4. Because J is the midpoint of AB AJ=3 and XJ can be found using similar triangles as $\frac{3}{15/4}\frac{5}{4}\sqrt{7} = \sqrt{7}$. The area of triangle ABX is $\frac{1}{2}6 \cdot \sqrt{7}$ and the area of the parallelogram AXBY is twice this or $6\sqrt{7}$.

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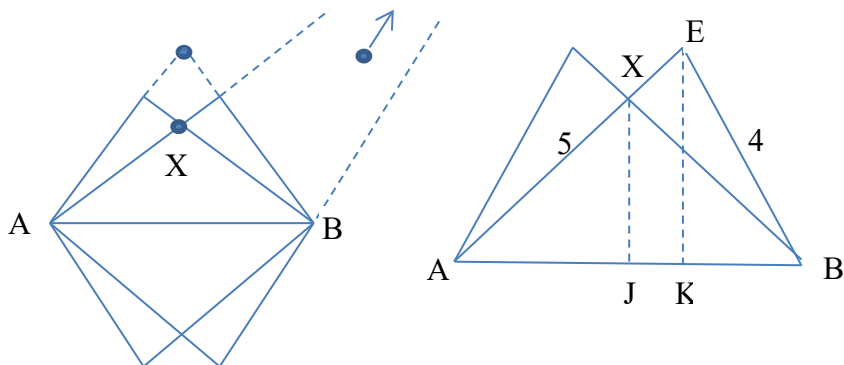
3.1: More Modular Arithmetic: The Fermat-Euler Theorem

1. This is $\varphi(100) = \varphi(2^2 5^2) = 2^1(2-1)5^1(5-1) = 40$.

2. We want to simplify 7^{49} mod 65. GCF(7, 65)=1 so $7^{\varphi(65)} \equiv 1 \ (mod \ 65)$. $\varphi(65) = (5-1)(13-1) = 48$, so $7^{49} = 7^1 7^{48} \equiv 7^1 \equiv 7 \ (mod \ 65)$. The remainder is 7.

3. Fermat-Euler implies that $3^{40} \equiv 1 \ (mod \ 10)$, so $123^{321} \equiv 3^{321} \equiv 3^1 \equiv 3 \ (mod \ 10)$. This is the units digit.

4. To find the tens digit we simplify mod 100. Using the fact that $a^{40} \equiv 1 \ (mod \ 100)$ whenever GCF(a, 100)=1 we have $1449^{124} \equiv 49^{124} \equiv 49^4 \ (mod \ 100)$. $49^2 = 2401 \equiv 1 \ (mod \ 100)$ so $49^4 \equiv 1^2 \equiv 1 \ (mod \ 100)$. This implies that the number ends in 01 and the tens digit is zero.

5. This is a problem where it helps to use negative numbers. $2015 \equiv 6 \equiv -1 \ (mod \ 7)$ so $2015^{(7^{13})} \equiv (-1)^{(7^{13})} \equiv -1 \ (mod \ 7)$. The last equality comes from the fact that 7^{13} is an odd number and -1 raised to any odd power is -1. The fact that $2015^{(7^{13})} \equiv -1 \ (mod \ 7)$ means that it is one less than a multiple of 7 so the remainder is 6.

6. Computing 14th powers is not easy, but notice that $343 = 7^3$. This lets us use the standard method for reducing numbers mod 100. $343^{14} \equiv (7^3)^{14} \equiv 7^{42} \equiv 7^2 \equiv 49 \ (mod \ 100)$. The remainder is 49.

7. One technique for simplifying modulo a composite number like 65 that sometimes works very well is to simplify them relative to each factor of the composite. Here, $5^{121} \equiv 0 \ (mod \ 5)$ is very easy. And from Fermat's little theorem we know $5^{12} \equiv 1 \ (mod \ 13)$ so $5^{121} \equiv 5^1 \equiv 5 \ (mod \ 13)$. The only number that satisfies both of these modular equations is 5. So the remainder is 5. (To solve pairs of equations like this note that the solutions to the first are 5, 10, 15,…, 60, and just try these in succession until you find something that is equal to 5 mod 13.) Another method that turns out to work well here is just to start computing powers of 5 mod 65. $5^2 = 25$. $5^3 = 125 \equiv -5 \ (mod \ 65)$, $5^4 \equiv -5 \times 5 \equiv -25 \ (mod \ 65)$, and $5^5 \equiv -25 \times 5 \equiv -125 \equiv 5 \ (mod \ 65)$. Once we see that this cycles with a period of 4 we have $5^{121} \equiv 5^1 \equiv 5 \ (mod \ 65)$.

8. For any positive integer x the expression $\frac{x}{1000} - \left\lfloor \frac{x}{1000} \right\rfloor$ will just be the three-digit remainder when x is divided by a 1000 preceded by a decimal point, e.g. $\frac{1234}{1000} - \left\lfloor \frac{1234}{1000} \right\rfloor = 0.234$. So the main thing we need to do is to simplify 111^{2002} mod 1000. The Fermat-Euler theorem gives that $a^{400} \equiv 1 \ (mod \ 1000)$ whenever a is not a multiple of 2 or 5, so $111^{2002} \equiv 111^2 = 12321 \equiv 321 \ (mod \ 1000)$. The answer is 0.321.

9. As in the Euclidean algorithm we know $\text{GCF}(x, x+11) = \text{GCF}(x, 11)$, which will always be equal to 1 or 11. $\text{GCF}(x, 90)$ is a factor of 90, so the two can be equal if and only if both are equal to 1. Hence, the question is essentially asking us to count how many numbers from 10, 11, ..., 99 have no common factors with 11 or 90. A good place to start is by counting the number that have no common factor with 90. Because the set $\{10, \ldots, 99\}$ is the same in modular arithmetic as the set 10, ..., 89, 0, 1, ..., 9} this is exactly the same as $\varphi(90) = \varphi(2^1 3^2 5^1) = (2-1) \cdot 3 \cdot (3-1) \cdot (5-1) = 24$. Almost all of these also have no common factor with 11. The only exceptions are numbers less than 90 that are multiples of 11 and have no 2's, 3's, or 5's, in their prime factorization. The only such numbers are 11 and 77. So the answer is 22.

3.2: Repeating Decimals

1. The repeat length is a factor of 16 so the 99th digit is the same as the 3rd. Using long division we find $12/17 = 0.705...$. The answer is 5.

2. We know that the expansion for $1/11$ has a repeat length of 2 and $1/7$ has a repeat length of 6 so $1/77$ has a repeat length of LCM(2, 6) = 6. The 100th digit is therefore the same as 4th. Using long division we find $1/77 = 0.0129...$. The answer is 9.

3. $\frac{4}{35} = \frac{1}{10} \cdot \frac{40}{35} = \frac{1}{10} \cdot \frac{8}{7}$. Remembering the pattern for sevenths we know $\frac{8}{7} = 1.\overline{142857}$. So the expansion for $\frac{4}{35}$ is $\frac{4}{35} = 0.1\overline{142857}$. The 50th digit is the same as 2nd which is a 1.

4. 53 is a prime number so we know $10^{52} \equiv 1 \pmod{53}$ and the answer will be some factor of 52, i.e. 1, 2, 4, 13, 26, or 52. To figure out which one it is we just start computing powers of 10 mod 53. Again, it is easier to sometimes use negative numbers. And it is good to take advantage of things you've computed previously to get to the powers you want more quickly. $10^2 = 100 \equiv 47 \equiv -6 \pmod{53}$. $10^4 \equiv -6 \times -6 \equiv 36 \pmod{53}$. $10^6 \equiv -6 \times 36 = -216 \equiv -4 \pmod{53}$. $10^{12} \equiv -4 \times -4 \equiv 16 \pmod{53}$. $10^{13} \equiv 16 \times 10 \equiv 160 \equiv 1 \pmod{53}$. So the answer is 13.

5. In the previous question we learned that $10^{13} \equiv 1 \pmod{53}$. This implies that the repeat length for $6/53$ is 13. So the 18th digit is the same as the 5th. Using long division we find $\frac{6}{53} = 0.11320 ...$. So the answer is 0.

6. Recall that $\frac{1}{37} = \frac{27}{999}$ has a repeat length of three. Writing $\frac{1}{5^k 37} = \frac{1}{10^k} \cdot \frac{2^k \cdot 27}{999}$ we see that the decimal for $1/(5^k \cdot 37)$ will have k patternless digits and then a repeat length of 3. This means that $k = 1$ will work if the third digit after the decimal point in the expansion for $\frac{2^1 \cdot 27}{999}$ is 1, $k = 2$ will work if the 2nd digit after the decimal point in $\frac{2^2 \cdot 27}{999}$ is a 1, and so on. To find the answer just keep checking until you find an answer that works. Computing the numerators just involves multiplying by two over and over so it isn't super hard, but it is tricky because once the numerators get bigger than 999 you have to subtract the appropriate multiple of 999 to find the decimal and a 1 doesn't appear in the necessary place until we get to $k = 8$: $\frac{2^1 \cdot 27}{999} = 0.\overline{054}, \frac{2^2 \cdot 27}{999} = 0.\overline{108}, \frac{2^3 \cdot 27}{999} = 0.\overline{216}, \frac{2^4 \cdot 27}{999} = 0.\overline{432}, \frac{2^5 \cdot 27}{999} = 0.\overline{864}, \frac{2^6 \cdot 27}{999} = 1.\overline{729}, \frac{2^7 \cdot 27}{999} = 3.\overline{459}, \frac{2^8 \cdot 27}{999} = 6.\overline{718}$. The answer is 8.

7. The repeat length for $1/n$ will be a factor of 12 if $10^k \equiv 1 \pmod{n}$ for some factor k of 12. For the first several n in S it is easy to see that this is true. $10^2 = 100 \equiv 1 \pmod{11}$. so the repeat length for $1/11$ is two. $10^3 = 1,000 \equiv$

$-221 \ (mod \ 1221)$ so $10^6 \equiv 221^2 = 48841 \equiv 1 \ (mod \ 1221)$ so the repeat length for 1/1221 is six. $10^6 = 1,000,000 \equiv (8 \times 123321) + 13432 \equiv 13432 \ (mod \ 123321)$. Squaring this and diving by 123321 (using a calculator) we find that $10^{12} \equiv 1 \ (mod \ 123321)$. The answer turns out to be 12344321. Why the first few numbers work is mysterious with this approach. To understand why they do, note that 1/n has a repeat length that is a factor of 12 if n is a factor of $10^{12} - 1$. Among the factors (which one can find by repeatedly using formulas for the differences of squares and cubes and sums of cubes if one knows them) are 11, 111, and 101. $1221 = 11 \times 111$. $123321 = 101 \times 1221$.

8. Note that $\frac{5}{9} = \frac{35}{63}$ and that $63 = 8^2 - 1$. The base 8 expansions for fractions with 63 as the denominator are similar to the base 10 expansions of fractions with 99 as the denominator. By analogy we have $\frac{35}{63} = 35_{(10)} \times 0.0101 ..._{(8)} = 43_{(8)} \times 0.0101 ..._{(8)} = 0.4343 ..._{(8)}$. The 50th digit is a 3. (To understand why these decimals repeat at length two think about the formula for the sum of an infinite geometric sequence. $\frac{1}{8^2-1} = \frac{1/8^2}{1-1/8^2} = \frac{1}{8^2}\left(1 + \frac{1}{8^2} + \frac{1}{8^4} + \cdots\right) = \frac{1}{8^2} + \frac{1}{8^4} + \frac{1}{8^6} + \cdots$.)

1. The difference between terms is (28-7)/9 = 21/9. The 100th term is 99 terms after the first so it is $7 + \frac{21}{9} 99 = 7 + (21 \times 11) = 238$.

2. The series $3 + 10 + \ldots + 94$ is an arithmetic term with first term 3, difference 7, and 14 terms. One fairly easy way to do the arithmetic is to think about the pairs formed by the first and last terms, the second and next-to-last, etc.: $(3 + 94) \times 7 = 679$.

3. In the first week she does 57 problems leaving 243 undone. The next week she does 1/3 of these or 81 leaving 162 undone. The next week she does 54 leaving 108 undone, etc. Continuing this we get $57 + 81 + 54 + 36 + 24 = 252$. A quicker way to get the answer is to focus on the number of problems not done. After the first week it is 243. Four weeks later it is $\left(\frac{2}{3}\right)^4 \cdot 243 = \frac{16}{81} \cdot 243 = 48$. There were 300 problems so she must have done 252.

4. One good way to do this is to use a trick similar to summing an arithmetic series by adding the numbers in pairs. The upper left and lower right numbers sum to $1 + 200 = 201$. The second number in the top row and the second to last in the bottom row sum to $2 + 199 = 201$. Similarly the third number in the top row and the third to last in the bottom row sum to 201, and so one. And moving down the row the first number in the second row and the last number in the next to last row sum to 201, etc. There are 5100 numbers in the matrix (51 rows of 100 numbers each). So we have 2550 pairs. $2050 \times 201 = 512{,}750$. One can also do the problem by using the formula for summing an arithmetic series twice. The first column sums to $51 \times 51 = 2601$. Each term in the second column is one bigger so the second sums to $2601 + 51$. The third sums to $2601 + 2 \times 51$. Continuing we get that the entire sum is $(100 \times 2601) + 51 + (2 \times 51) + (3 \times 51) + \ldots + (99 \times 51) = 260{,}100 + 5{,}100 \cdot \frac{99}{2} = 260{,}100 + (99 \times 2550) = 260{,}100 + 255{,}000 - 2{,}550 = 512{,}750$.

5. In problems like this the best approach is often just to start computing terms until you see a pattern. In this problem it takes a while. The sequence is 191, 891, 1458, 450, 225, 36, 81, 576, 882, 1152, 225, 36, 81, … The sequence has started to cycle with the 11th term equal to the 5th. 2010 is a multiple of 6 (2010 is even and has the sum of its digits equal to 3) so with a repeat length of 6 the 2017th term will be the same as the 7th, which is 81.

6. This is a classic telescoping series. Note first that $\frac{1}{1} \cdot \frac{1}{3} = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3}\right)$, $\frac{1}{3} \cdot \frac{1}{5} = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$, and so on. Hence, $\frac{1}{1} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{5} + \cdots + \frac{1}{2015} \cdot \frac{1}{2017} = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{2015} - \frac{1}{2017}\right) = \frac{1}{2}\left(1 - \frac{1}{2017}\right) = \frac{1008}{2017}$. One can also often do problems like

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this by just adding the first few terms and guessing the first pattern. The first term is $\frac{1}{3}$. The sum of the first two is $\frac{2}{5}$. The sum of the first three is $\frac{3}{7}$. After this point (or maybe one more) one may realize that the pattern is that the numerators are 1, 2, 3, … and the denominators are 3, 5, 7, …

4.2: Review of Standard Combinations Problems

1. This is $_{10}C_4 = \frac{10!}{6! \cdot 4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 3 \cdot 7 = 210$. When computing combinations there are always opportunities for cancelling terms that make the arithmetic easier. Here, I used the 4 and the 2 on the bottom to cancel the 8 on the top and cancelled the 3 on the bottom to turn the 9 on the top into a 3. Many calculators also have buttons that compute combinations. You should get one and learn to use it if you'll be participating in competitions that allow calculators.

2. There are $6! = 720$ possible combinations. He switches to the 2nd possible combination on January 1, 2017, so it will be 719 days later than this that he would need a 721st combination to avoid breaking his resolution. One good way to then figure this out is to say that 730 days later would be January 1, 2019. Eleven days before this is December 21, 2018. (One day before would be December 31, 2018.)

3. If the teams were distinguishable, e.g. if one was the A team and one was the B team, the answer would be $_{10}C_5$. But in this problem it doesn't matter if you put ABCDE on the first team and FGHIJ on the second team or vice versa. So if you count ways by thinking of the coach as picking five students out of ten to be on the first team you are double counting each way of dividing up the kids. The answer is one-half of $_{10}C_5$ which is $\frac{1}{2} \cdot \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{2} \cdot \frac{2 \cdot 3 \cdot 7 \cdot 6}{1} = 126$.

4. This is $_{102}C_2 + {}_{102}C_3 = \frac{102 \cdot 101}{2} + \frac{102 \cdot 101 \cdot 100}{3 \cdot 2 \cdot 1} = 51 \cdot 101 + 17 \cdot 101 \cdot 100 = 5{,}151 + 171{,}700 = 176{,}851$.

5. We want to count the number of values of N for which $_{N}C_4$ a prime number. The answer is 1: only $N = 5$ works. To see this we start by trying the smallest possible values of N (which must be at least 4). $_4C_4 = 1$ is not prime. $_5C_4 = 5$ is prime. $_6C_4 = 15$ is not prime. $_7C_4 = 35$ is not prime. At this point or soon thereafter you might guess that no bigger numbers will work. To see why this is indeed correct note that $_NC_4 = \frac{N \cdot (N-1) \cdots (N-3)}{4 \cdot 3 \cdot 2 \cdot 1}$. When N is at least 6 all of the terms in the numerator are at least 3. None that one will be a multiple of 4 that will cancel the four in the denominator and another will be an even number that cancels the 2 in the denominator and still leaves some number greater than one in the numerator. The numerator now consists of a product of at least three numbers that are all at least two. Only one can cancel the 3 in the denominator. So we're left with a number with at least two proper factors.

6. We can think of Mr. Yellin as just choosing four kids to be on Amy's team from among the 8 available kids. (The other four automatically go to Elizabeth's team.) The number of ways to do this is just $_8C_4 = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 2 \cdot 5 = 70$.

A - 161

7. We want to find an N for which $\frac{N\cdot(N-1)\cdot(N-2)}{3\cdot2\cdot1} = 1330$. One way to do this is by guess-and-check. Pick some N that seems roughly the right size and then adjust the guess up or down if $_NC_3$ is too small or two big. A great way to make a good first guess is to think of the equation as $N(N-1)(N-2) = 7980$. The left side is close to $(N-1)^3$. The right side is close to $8000 = 20^3$. So guessing $N - 1 = 20$ is the obvious guess. Another good approach would be to factor the number on the right side: $1330 = 10 \times 133 = 2 \times 5 \times 7 \times 19$. From this we realize that one of N, N $- 1$, or $N - 2$ must be 19 and another must be a multiple of 7. The way to do this is to choose $N = 21$ so $N - 2 = 19$. The answer is 21.

4.3: Combinations Problems with Three or More Groups

1. This is $_7C_{1,3,2,1} = \frac{7!}{1!\cdot2!\cdot3!\cdot1!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2}{2\cdot3\cdot2} = 7\cdot6\cdot5\cdot2 = 420.$

2. Thinking of the coach as also choosing 2 students to be kids who do not get to go this is $_{12}C_{4,6,2} = \frac{12!}{4!\cdot6!\cdot2!} = \frac{12\cdot11\cdot10\cdot9\cdot8\cdot7}{4\cdot3\cdot2\cdot1\cdot2} = 11\cdot10\cdot9\cdot2\cdot7 = 13860.$

3. They are choosing 3 kids to go in Mr. Ellison's car, 3 kids to go in Julia's mother's car and 2 kids to go in the third car from 8 available kids. The number of ways to do this is $_8C_{3,3,2} = \frac{8!}{3!\cdot3!\cdot2!} = \frac{8\cdot7\cdot6\cdot5\cdot4}{3\cdot2\cdot2} = 8\cdot7\cdot5\cdot2 = 560.$

4. Thinking of the choice as also choosing 60 squares to be empty this is $_{64}C_{60,3,1} = \frac{64!}{60!\cdot3!\cdot1!} = \frac{66\cdot63\cdot62\cdot61}{3\cdot2\cdot1} = 11\cdot63\cdot62\cdot61 = 2{,}541{,}504.$

5. The only way to divide a class of 22 into groups of size 4 or 5 is to have three groups of size 4 and two groups of size 5. If the groups were distinguishable the answer would be $_{22}C_{5,5,4,4,4}$, but in this problem this would be greatly counting each division $3! \times 2!$ times: any reordering of the three four-person groups is the same division; and switching the two five-person groups is also the same division. So the answer is $\frac{1}{3!\cdot2!} \, _{22}C_{5,5,4,4,4} = \frac{1}{3!\cdot2!} \frac{22!}{5!\cdot5!\cdot4!\cdot4!\cdot4!} = 470{,}531{,}961{,}900.$ (To do the arithmetic you want to do a lot of cancelling and use a calculator. If no calculator is available you want to try to keep easy-to-multiply numbers like 20 and 10 whole in the numerator and cancel as many harder-to-mulitply numbers as you can.)

6. The best way to do this is to first count the total number of ways in which the tickets can be allocated and then to subtract off the number of ways in which you can allocate the tickets with each kid going to at most one event (which means you are essentially choosing 2 kids to have the jobs of going to each of the three events and four kids to not go anywhere. Thinking about it this way the answer is $(_{10}C_2)^3$
$- \, _{10}C_{2,2,2,4} = \left(\frac{10\cdot9}{2}\right)^3 - \frac{10!}{4!\cdot2!\cdot2!\cdot2!} = 45^3 - 10\cdot9\cdot7\cdot6\cdot5 = 45(45^2 - 420) = 45 \cdot 1605 = 72{,}225.$

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4.4: Using Combinations to Count Orderings

1. This is $_5C_{2,3} = \frac{5!}{2!3!} = \frac{5\cdot4}{2} = 10$ because you need to choose two of the five places to be 2's and three to be 3's.

2. Three places must be R's, two must be E's, one must be an O, and one a D. So this is $_7C_{3,2,1,1} = \frac{7!}{3!2!1!1!} = \frac{7\cdot6\cdot5\cdot4}{2} = 420$.

3. One good way to think about this is to think of the DD as a single object. The problem is then just the number of possible orders for 5 objects. $5! = 120$.

4. The first three letters of a palindrome must be A, N and H in some order. Each arrangement of those three letters corresponds to a palindrome. So the number of palindromes is $3! = 6$.

5. We can count these by counting the total number of 5-digit numbers with two 1's, a 2, a 3, and a 4 and subtracting the number in which the two 1's are together. The former is $_5C_{2,1,1,1} = \frac{5!}{2!\,1!1!1!} = 60$. The second is $4! = 24$ by the same argument as in problem 3. So the answer is $60 - 24 = 36$.

6. We can count all rearrangements and then subtract off the ones that begin with a zero (which aren't valid 8-digit numbers). The first is $_8C_{3,1,1,1,1,1}$. The second is $_7C_{3,1,1,1,1}$. The difference is $\frac{8!}{3!} - \frac{7!}{3!} = 8\cdot7\cdot6\cdot5\cdot4 - 7\cdot6\cdot5\cdot4 = 7\cdot7\cdot6\cdot5\cdot4 = 49\cdot120 = (50-1)\cdot120 = 6000 - 120 = 5880$.

7. The middle letter of the palindrome must be an A. The first four letters will be an A, two B's, and a C in some order. The number of ways to choose such an order is $_4C_{2,1,1} = \frac{4!}{2!1!1!} = 12$.

4.5: Counting Ordered Ways to Make Numbers Add Up

1. The order of the numbers matters because one die is being rolled three times and the question asks about sequences. So this is asking for the number of ordered sums like $1 + 1 + 6, 1 + 6 + 1, 6 + 1 + 1, 2 + 5 + 1, \ldots$. This is the classic counting problem equivalent to choosing two of the seven spaces between eight markers in which to put a divider. The answer is $_7C_2 = 21$.

2, Adding one to each number, this is the same as the number of ways to write 15 as a sum of 3 positive integers. This is $_{14}C_2 = 91$.

3. We can count all ways to write 12 as a sum of three positive integers and then subtract off the number of ways to write 12 as a sum of three positive integers with at least one not being a one-digit number. The first of these is $_{11}C_2 = 55$. The second of these is 3. (The only ways are $10 + 1 + 1, 1 + 10 + 1$, and $1 + 1 + 10$.) So the answer is $55 - 3 = 52$.

4. Each such path can be described by the number of vertical segments which we use on the first, second, third, fourth, and fifth vertical lines of the grid. There are three vertical segments in total, so the number of paths is the same as the number of ways to write 3 as a sum of 5 nonnegative integers. Adding one to each this is the same as the number of ways to write 8 as a sum of 5 positive integers. This is $_7C_4 = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35.$

5. The number of such paths is the product of the number of paths of length 6 from A to B and the number of paths of length 6 from B to C. The first of these is the number of ways to write 2 as a sum of five nonnegative integers, which is the same as the number of ways to write 7 as a sum of five positive integers, which is $_6C_4 = 15$. Similarly, the second is the number of ways to write 3 as a sum of four nonnegative integers, which is the same as the number of ways to write 7 as a sum of four positive integers, which is $_6C_3 = 20$. So the answer is $15 \times 20 = 300$.

6. The fact that the number must be less than 20,000 means that it starts with a 1. The fact that it is one less than a multiple of 10 means that it ends with a 9. So the number is of the form 1abc9 with $1 < a < b < c < 9$. Thinking about the differences between adjacent digits, $a - 1, b - a, c - b$, and $9 - c$, we can see that the number of ways to do this is the same as the number of ways to choose four positive integers that sum to 8. This is $_7C_3 = 35$.

7. We need to count separately the total number of ways to write 9 as a sum of two or more positive integers and the number of ways to write 9 as a sum of two or more positive odd integers. The first is not so hard. We can think of this as choosing whether or not to put a divider between each of 9 tokens with the restriction that you must use at least one divider. This is $2^8 - 1 = 255$. The second is harder. Note that the number of terms in the sum must be odd for the sum to be

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odd. So the sum will have to have 3, 5, 7, or 9 terms. To count the number of three-term sums we think of starting with $1 + 1 + 1$ and then adding a total of three 2's to the terms. The number of ways to do this is the number of ways to choose three nonnegative numbers that add up to 3, which is the same as the number of ways to choose three positive integers that add up to 6, which is $_5C_2 = 10$. Similarly, to count the five-term sums we think of starting with $1 + 1 + 1 + 1 + 1$ and then adding a total of two 2's to the five terms. This is the number of ways to choose 5 nonnegative integers that add up to 2, which is the number of ways to choose 5 positive integers that add up to 7, which is $_6C_4 = 15$. Using similar reasoning the number of seven-term sums is $_7C_6 = 7$. And the number of 9-term sums is obviously 1. So the answer is $\frac{10+15+7+1}{255} = \frac{33}{255} = \frac{11}{85}$.

4.6: Counting Problems Without An Easy Answer: Organize Your List

1. A good way to organize the list is first by the largest number in the subset and then holding the largest number fixed by the smallest number.

Largest number 3: {1, 2, 3}
Largest number 4: {1, 3, 4}
Largest number 5: {1, 4, 5}, {2, 3, 5}
Largest number 6: {1, 5, 6}, {2, 4, 6}
Largest number 7: {1, 6, 7}, {2, 5, 7}, {3, 4, 7}

There are 9 subsets with the desired property.

2. One good way to order the list is by the number of times that 2 is used.

No 2's: 5+7, 7+5, 3+3+3+3
One 2: Six orderings of 2+3+7, three orderings of 2+5+5
Two 2's: Twelve orderings of 2+2+3+5.
Three 2's: Ten orderings of 2+2+2+3+3.
Four 2's: None
Five 2's: None
Six 2's: 2+2+2+2+2+2

This gives a total of 35 ways to write 12 as a sum of primes.

3. All of the nonzero products will be powers of 2. Let's order them by considering positive, negative, and zero products separately. When counting the positive products a good subdivision is by the number of negative numbers used.

Positive products (0 negatives): 4
Positive products (2 negatives): 2, 8, 16
Zero products: 0
Negative products: -1, -2, -4, -8, -32

This gives a total of 10 possible products.

4. Think of the grid as on a coordinate axis with the lower left point at (0, 0) and the upper right at (2, 3). We can organize the count by looking separately at each of the possible side lengths of the square. Note that squares parallel to the coordinate axis will have side lengths that are integers, but squares can also potentially go diagonally. Diagonal squares have lengths that are of the form $\sqrt{a^2 + b^2}$ where a and b are very small integers.

Side length 1: 6 possibilities. The lower left is at (x, y) with x=0 or 1 and y=0, 1 or 2.

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Side length $\sqrt{2}$: 2 possibilities. The bottom can be at (0, 1) or (1, 1).
Side length $\sqrt{3}$ or other larger noninteger: 0 possibilities.
Side length 2: 2 possibilities. The lower left can be at (0, 0) or (0, 1).

This gives a total of 10 squares.

5. If the product of the side lengths is a multiple of 49, then at least two of the sides must have lengths that are 7, 14, or 21.

I'll organize the list of possible triangles on the basis of which numbers from this list are side lengths:

7-7-7: One possibility
14-14-14: One possibility
21-21-21: One possibility
7-14-14: One possibility
7-21-21: One possibility
14-14-21: One possibility
14-21-21: One possibility
7-7 only: 12 possible. The third side must be from 1 to 13 and can't be 7.
7-14 only: 12 possible. The third side must be from 8 to 20 and can't be 14.
7-21 only: 9 possible. The third side must be 15 to 24 and can't be 21.
14-14 only: 21 possible. The third side must be from 1 to 24 and can't be
        7, 14, or 21.
14-21 only: 15 possible. The third side must be from 8 to 24 and can't be
        14, or 21.
21-21 only: 21 possible. The third side must be from 1 to 24 and can't be
        7, 14, or 21.

This gives 97 possibilities.

6. We will count all six digit weakly increasing sequences that start and end with a power of two, and then subtract off the number of all such sequences in which all digits are distinct.

Before starting an organized list we note a preliminary result. The number of six digit sequences that start with a and end with b (for b>a) and in which each digit is at least as large as the previous one is $\binom{b - a + 4}{4}$. To see this note that the differences between each pair of adjacent digits are 5 numbers each at least 0 that add up to b – a. The number of sets of such differences is the same as the number of ways to choose five numbers that are at least 1 that add up to b – a + 5. The formula in section 4.5 gives that this is $\binom{b - a + 4}{4}$.

Using this formula we can count the number of possibilities starting and ending with any pair of powers of two (ignoring for now the need to have a double digit).

Start 1. End 1. Number of choices is $\binom{4}{4} = 1$

Start 1. End 2. Number of choices is $\binom{5}{4} = 5$

Start 1. End 4. Number of choices is $\binom{7}{4} = 35$

Start 1. End 8. Number of choices is $\binom{11}{4} = 330$

Start 2. End 2. Number of choices is $\binom{4}{4} = 1$

Start 2. End 4. Number of choices is $\binom{6}{4} = 15$

Start 2. End 8. Number of choices is $\binom{10}{4} = 210$

Start 4. End 4. Number of choices is $\binom{4}{4} = 1$

Start 4. End 8. Number of choices is $\binom{8}{4} = 70$

Start 8. End 8. Number of choices is $\binom{4}{4} = 1$

Adding this up gives 669 possible sequences.

Some of the sequences above do not have a doubled digit, so we now subtract off the number of sequences that do not have a doubled digit. This can happen only in two cases. To count these we just think of picking the four digits in the middle from the digits that are larger than the starting digit and smaller than the ending digit.

Start 1. End 8. Number of choices is $\binom{6}{4} = 15$

Start 2. End 8. Number of choices is $\binom{5}{4} = 5$

So, the total number of six digit numbers meeting all of the conditions is $669 - 20 = 649$.

4.7: Probability Problems Involving Combinations

1. The number of two-element subsets is $_{20}C_2 = 190$. There are four possible subsets that are a pair of twin primes: $\{3, 5\}$, $\{5, 7\}$, $\{11, 13\}$, and $\{17, 19\}$. So the probability that one of these is chosen is $4/190 = 2/95$.

2. Think of the bag as containing 17 objects labeled $\{G1, G2, G3, S1, …, S14\}$. The number of two element subsets is $_{17}C_2 = 136$. The number of subsets that consist of one gold ball and one silver ball is $3 \times 14 = 42$. The probability that one of these subsets is chosen is $42/136 = 21/68$.

3. The number of possible 4-student teams is $_{10}C_4 = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2} = 210$. The number of ways to choose a team with 3 boys and 1 girl is $_6C_3 \times {_4}C_1 = 20 \times 4 = 80$ because we must choose 3 of the 6 boys and 1 of the 4 girls to put on the team. The probability that such a team will be chosen is therefore $80/210 = 8/21$.

4. The number of possible combinations is $_{69}C_5 \times {_{26}}C_1 = \frac{69 \cdot 68 \cdot 67 \cdot 66 \cdot 65}{5 \cdot 4 \cdot 3 \cdot 2} \times 26$. Cancelling common factors from the numerator and denominator and then multiplying out this is $23 \cdot 17 \cdot 67 \cdot 33 \cdot 13 \cdot 26 = 292{,}201{,}338$. The probability of winning this lottery if you choose any one combination is $1/292{,}201{,}338$.

5. It is easier to compute the probability that the four balls are all different colors. The number of subsets containing four balls is $_{20}C_4 = \frac{20 \cdot 19 \cdot 18 \cdot 17}{4 \cdot 3 \cdot 2 \cdot 1} = 5 \cdot 19 \cdot 3 \cdot 17$. The number of subsets containing balls of four different colors is $9 \cdot 5 \cdot 5 \cdot 1$. Hence, the probability that a four ball subset chosen at random will contain balls of four different colors is $\frac{9 \cdot 5 \cdot 5 \cdot 1}{5 \cdot 19 \cdot 3 \cdot 17} = \frac{3 \cdot 5}{19 \cdot 17} = \frac{15}{323}$.

The set has at least two balls of some color if and only if the balls are not of four different colors. So the probability that this happens is $1 - \frac{15}{323} = \frac{308}{323}$.

6. The number of ways to choose 6 of the 20 tiles is $_{20}C_6 = \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 19 \cdot 17 \cdot 8 \cdot 15$. The number of sets of tiles that can be rearranged to spell VIKRAM is $1 \cdot 3 \cdot 2 \cdot 3 \cdot 4 \cdot 2$ because you need the one V, one of the 3 I's, one of the 2 K's, and so on. The probability is $\frac{3 \cdot 2 \cdot 3 \cdot 4 \cdot 2}{19 \cdot 17 \cdot 8 \cdot 15} = \frac{3 \cdot 2}{19 \cdot 17 \cdot 5} = \frac{6}{1615}$.

7. The number of ways to arrange the 6 letters is $6! = 720$. $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. By inclusion-exclusion counting, the number of arrangements with a double letter is the number with a double O, plus the number with a double N, minus the number with both a double O and a double N. The number with a double N is $2 \cdot 5! = 240$, because we can get such a combination by choosing any ordering of CONNR and then putting the other O immediately before or after wherever the first O was placed. The number with a double N is also 240. And the number with both a

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double O and a double N is $2 \cdot 2 \cdot 4! = 96$ because we can generate those orderings by picking some ordering of CONR and then adding the other O and the other N before or after where the other copy of each letter are. This gives $240 + 240 - 96$ = 384 combinations with a double letter. The probability is $\frac{384}{720} = \frac{64}{120} = \frac{8}{15}$.

4.8: Binomial Problems and Pascal's Triangle

1. The probability that heads will come up an odd number of times is $(_5C_1 + _5C_3 + _5C_5) / 2^5 = (5 + 10 + 1)/32 = \frac{1}{2}$.

Another way to see that the answer must be $\frac{1}{2}$ is to note that by symmetry the probability that the number of Heads is odd is equal to the probability that the number of Tails is odd. When you toss a coin 5 times there is always an odd number of Heads or Tails (and never both) so the two probabilities sum to one, and each must be $\frac{1}{2}$.

2. The probability that heads will come up two or fewer times is $(_7C_0 + _7C_1 + _7C_2) / 2^7 = (1 + 7 + 21)/128 = 29/128$.

3. The probability that Ted gets exactly two hits in five at bats is $_5C_2 (0.4)^2 (0.6)^3 = 10 \cdot \frac{16}{100} \cdot \frac{216}{1000} = \frac{3456}{10000} = 0.3456$.

4. There are $_5C_2 + _5C_3 + _5C_4 + _5C_5 = 10 + 10 + 5 + 1$ sequences like HTHHT that have at least 2 heads. The number of sequences that start with an H and have at least one other H is 15: H can be followed by any sequence of four outcomes other than TTTT. So the conditional probability is 15/26.

5. We can compute this as the probability that both come up heads 0 times, plus the probability that both come up heads one time, and so on. The probability that both coins have zero heads is $(_6C_0/2^6)^2 = 1^2/2^{12}$. The probability both coins have one head is $(_6C_1/2^6)^2 = 6^2/2^{12}$. Continuing this pattern the probability is $(1^2 + 6^2 + 15^2 + 20^2 + 15^2 + 6^2 + 1^2) /2^{12} = (1 + 36 + 225 + 400 + 225 + 36 + 1) / 2^{12} = 924 / 2^{12} = 231 / 2^{10} = 231/1024$.

6. This conditional probability is the number of sequences with at three or more numbers of at least 3 divided by the number of sequences with one or more numbers that is at least 3. We can compute the numerator as the sum of the number of sequences with exactly three numbers that are at least 3, the number with exactly four numbers that are at least 3, and so on. The number of sequences with exactly three numbers that are at least 3 is $_5C_3 \, 2^2 \, 4^3$, because we need to pick three places in the sequence to be the numbers that are at least 3, and then pick which of $\{1, 2\}$ to put in the spaces that are supposed to be small numbers and which of $\{3, 4, 5, 6\}$ to put in the spaces that are supposed to be large numbers. Hence, the numerator is $_5C_3 \, 2^2 \, 4^3 + _5C_4 \, 2^1 \, 4^4 + _5C_5 \, 2^0 \, 4^5 = 10 \cdot 2^8 + 5 \cdot 2^9 + 1 \cdot 2^{10} = (10 + 10 + 4) \cdot 2^8 = 3 \cdot 2^{11}$.
The denominator can be computed most quickly as the total number of sequences minus the number with all numbers less than 3. This is $6^5 - 2^5 = 2^5 (3^5 - 1) = 2^5 (243 - 1) = 2^6 \cdot 121$.
The probability is $3 \cdot 2^{11} / 121 \cdot 2^6 = 3 \cdot 2^5 / 121 = 96/121$.

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7. This problem can be done very quickly if one recognizes a symmetry: the probability that the median is 1s equal to the probability that the median is 6; the probability that the median is 2 is equal to the probability that the median is 5; and the probability that the median is 3 is equal to the probability that the median is 4. (To see this, think about the numbers on the bottom of the dice. The median of the numbers on top is x if the median of the numbers on the bottom is $7 - x$.)

This fact implies that the probability that the median is 4 plus the probability that the median is 5 plus the probability that the median is 6 is exactly equal to one-half. So, the answer is 4.

8. We need to have 5 prime number, 3 composite numbers, and 2 numbers equal to 1. There are 8 possible prime numbers (2, 3, 5, 7, 11, 13, 17, 19) and 11 possible composite numbers (all of the rest other than 1). The probability is $_{10}C_{5,3,2} \cdot 8^5 \cdot 11^3$
$\cdot 1^2 / 20^{10} = \frac{10!}{5!3!2!} \cdot \frac{2^{15}11^3}{2^{20}5^{10}} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{2} \cdot \frac{11^3}{2^5 5^{10}} = \frac{11^3 \cdot 9 \cdot 7}{2^2 5^9} = \frac{83,853}{7,812,500}.$

5.1: Proportional Reasoning

1. The volume of a pyramid is $1/3 \cdot$ (Area of base) $\cdot$ Height. Pyramid B's base is has a side-length that is 2/3 as long as the side of the base of pyramid A, so the area of the base is $(2/3)^2 = 4/9$ as large. The volume is $(4/9) \cdot 72 = 32$ cm^3.

2. Triangle ADE is similar to ABC. Its sides are half of the length of the sides of ABC, so its area is one-fourth as large. Trapezoid DECB is the leftover part of triangle ABC after triangle ADE is taken away, so its area is three-fourths of the area of ABC. The ratio is $(1/4) / (3/4) = 1/3$.

3. The $40 item is $40/16 = 2.5$ times as expensive. Its sales tax would be 2.5 times as large. $(5/2) \times 0.92 = 5 \times 0.46 = \$2.30$.

4. Having 3 robots instead of 5 will make the job take 5/3 as long. Painting 9 rooms instead of 5 will make the job take 9/5 as long. With both changes the job will take $5 \times (5/3) \times (9/5) = 15$ days.

5. In three hours, the large hose will have supplied enough water to fill three quarters of the pool with water. If the two hoses working together have filled the pool completely in this time, the smaller hose has filled one-quarter of the pool in three hours. This implies that it would take four times as long, or 12 hours, to fill the whole pool.

6. On the way to Vivian's house, Carol is going 4 mph for half of the distance and 10 mph for the other half. On the way back she is going 10 mph for half of the distance and 25 mph for the other half. Note that $10 = (5/2) \times 4$ and $25 = (5/2) \times 10$. Because she is traveling at $(5/2)$ times the speed in each part of the way back, the travel time will be only 2/5 as long. $40 \times (2/5) = 16$ minutes. Cathy leaves Vivian's house at 5:00 and spends 16 minutes traveling and 5 minutes changing the tire, so she arrives home at 5:21 pm.

7. When two people are running in opposite directions around a track, they pass each every time they have run a combined distance of one lap. If Kevin and Elena pass each other for the 10th time after 8 minutes (= 480 seconds) it takes them 48 seconds to run a total of 400 meters. In 48 seconds Kevin has run $(48/80) = 3/5$ of a lap. So Elena has run the other 2/5 of the lap and is running two-thirds as fast as Kevin. Kevin's speed is 400m / 80 seconds $= 5$ m/s. Elena's average speed is 2/3 as fast, which is $5 \times (2/3) = 10/3$ m/s.

5.2: Polynomial Division with Remainders

1. Doing the long division below we find that the remainder is -1.

$$
\begin{array}{r}
2x + 2 \\
x + 2 \overline{)2x^2 + 6x + 3} \\
\underline{2x^2 + 4x} \\
2x + 3 \\
\underline{2x + 4} \\
-1
\end{array}
$$

2. This problem can be done without long division by noting that $x + 3$ a factor of $x^3 + 13x^2 - 14x + a$ if the cubic has a value of 0 when evaluated at $x = -3$. Plugging in -3 the polynomial is $-27 + 117 + 42 + a$, so we need $a = -132$.

As a long division problem the calculation below shows that the remainder is a + 132. This is zero for $a = -132$.

$$
\begin{array}{r}
x^2 + 10x + 44 \\
x + 3 \overline{)x^3 + 13x^2 - 14x + a} \\
\underline{x^3 + 3x^2} \\
10x^2 - 14x \\
\underline{10x^2 + 30x} \\
-44x + a \\
\underline{-44x - 132} \\
a + 132
\end{array}
$$

3. Again, to do this without long division, note that when the quadratic is evaluated at $x = 3$ its value will be c. Plugging in, the value of the quadratic at 3 is $5 \cdot 9 + 7 \cdot 3 + 3 = 69$. This problem could also be done with long division, but since the latest version of Microsoft Word's equation editor has made it much harder to type out division problems, I won't do it for this one.

4. Doing the long division, we find that $x^3 - 5x^2 + 9x + 8 = (x^2 - 2x + 3)(x - 3) + 17$, so whenever $x^2 - 2x + 3 = 0$, the cubic will be equal to 17.

$$
\begin{array}{r}
x - 3 \\
x^2 - 2x + 3 \overline{)x^3 - 5x^2 + 9x + 8} \\
\underline{x^3 - 2x^2 + 3x} \\
-3x^2 + 6x + 8 \\
\underline{-3x^2 + 6x - 9} \\
17
\end{array}
$$

5. If the cubic $x^3 - 9x^2 + 23x - 15$ satisfies f(3)=0, then it has $x - 3$ as a factor. Doing the long division we find that $x^3 - 9x^2 + 23x - 15 = (x^2 - 6x + 5)(x - 3)$. The right side factors further as $(x - 5)(x - 1)(x - 3)$. The largest x for which this is zero is $x = 5$.

A - 175

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$$\begin{array}{r}
x^2 + 10x + 44 \\
x - 3 \overline{)x^3 - 9x^2 + 23x - 15} \\
\underline{x^3 - 3x^2} \\
-6x^2 + 23x \\
\underline{-6x^2 + 18x} \\
5x - 15 \\
\underline{5x - 15} \\
0
\end{array}$$

6. Let x be the length in inches of a side of the square. The first sentence tells us that $x^2 = 1 + \frac{1}{2}(4x) = 2x + 1$, so $x^2 - 2x - 1 = 0$. The product $\overline{AB} \cdot \overline{AC} \cdot \overline{AD}$ is equal to $\sqrt{2}x^3$. By long division, as shown below, we find $x^3 = (x^2 - 2x - 1)(x + 2) + 5x + 2$, so the product we are looking for is equal to $\sqrt{2}(5x + 2)$. The positive solution to $x^2 - 2x - 1 = 0$ is $\frac{2+\sqrt{4+4}}{2} = 1 + \sqrt{2}$, so the answer is $\sqrt{2}(5(1 + \sqrt{2}) + 2) = 10 + 7\sqrt{2}$.

$$\begin{array}{r}
x + 2 \\
x^2 - 2x - 1 \overline{)x^3} \\
\underline{x^3 - 2x^2 - x} \\
2x^2 + x \\
\underline{2x^2 - 4x - 2} \\
5x + 2
\end{array}$$

7. Using long division we find we find that $6x^3 + 13x^2 - 15x + 75 = (2x^2 - 5x - 7)(3x + 14) + 76x + 173$, so the sum of the possible values of the cubic when $2x^2 - 5x - 7 = 0$, will be 76 times the sum of all possible values of x plus 2×173. Using the standard formula for the sum of the roots of a quadratic, the sum of all possible solutions to $2x^2 - 5x - 7$ is 5/2. Hence, the answer is $(5/2) \cdot 76 + 346 = 190 + 346 = 536$

$$\begin{array}{r}
3x + 14 \\
2x^2 - 5x - 7 \overline{)6x^3 + 13x^2 - 15x + 75} \\
\underline{6x^3 - 15x^2 - 21x} \\
28x^2 + 6x + 75 \\
\underline{28x^2 - 70x - 98} \\
76x + 173
\end{array}$$

5.3: Systems of Equations

1. Substituting the second equation into the first gives $3x - 2(4x + 2) = 5$. This simplifies to $-5x - 4 = 5$, which gives $x = -1.8$. Plugging into the second equation gives $y = 4 \cdot 1.8 + 2 = -5.2$. The answer is $(-1.8, -5.2)$.

2. Denote the degree measure of each angle by the lower-case version of the middle letter, i.e. write a for the measure of EAB, b for the measure of ABC, c for the measure of BCD, and so on. We are told that
$$a = b = c + 20$$
$$c = d = e + 20$$
Putting everything in terms of e we have: $a = e + 40$, $b = e + 40$, $c = e + 20$, $d = e + 20$. The angles of any pentagon add up to 540 degrees. This gives
$$(e + 40) + (e + 40) + (e + 20) + (e + 20) + e = 540.$$
$5e = 420$ gives $e = 84$, so the measure a of EAB is $84 + 40 = 124°$.

3. Suppose ab is two digit number that is 600% larger than the sum of its digits. Such a number satisfies $10a + b = 7 (a + b)$ or $3a = 6b$. The solutions to this with a in the range from 1 to 9 and b from 0 to 9 are $(a, b) = (2, 1)$, $(4, 2)$, $(6, 3)$, and $(8, 4)$. The sum of all of the two digit numbers is $21 + 42 + 63 + 84 = 210$.

4. Write x for the first number and y for the second. The sentences describe the system of equations:
$$(2x + 3) + (y - 1) = 5$$
$$2x - y = 4$$
The first equation simplifies to
$$2x + y = 3.$$
Adding the equations we get $4x = 7$, so $x = 7/4$ and $y = -1/2$. The absolute value of the larger number is $7/4$.

5. Write x for Bob's speed in km/h and t for the number of hours that Bob walks. The first set of facts that we are given implies that Anna walks for 1.5 hours less than Bob. The total distance that each walks is the same, so
$$(t - 1.5)(x + 2) = tx, \text{ which implies that } -1.5x + 2t - 3 = 0.$$
Similarly, the second fact gives us that
$$(t - 2)(x + 3) = tx, \text{ which implies that } \quad -2x + 3t - 6 = 0.$$
Multiplying the first equation by 4 and the second by 3, the system is:
$$-6x + 8t = 12$$
$$-6x + 9t = 18$$
Subtracting the first equation from the second gives $t = 6$ hours. Plugging into the first equation then gives $x = 6$ km/h. The length of the walk is $tx = 36$ km.

6. Write x for the price of a bingle, y for the price of a bangle, and z for the price of a bongle. "A bingle costs twice as much as a bongle" tells us that
$$x = 2z.$$

A - 177

"A bangle costs as much as two bingles and one bongle" tells us that
$$y = 2x + z = 5z.$$
So, the prices for bingles, bangles, and bongles are 2z, 5z, and z.

Write a for the number of bingles bought, b for the number of bangles bought, and c for the number of bongles bought. We are told that
$$a = b - 5$$
and
$$c = b + 4.$$
So, the number of bingles, bangles and bongles are b − 5, b, and b+4.

The fact that the customer buys 23 items gives $(b - 5) + b + (b + 4) = 23$, so b=8. The customer buys 3 bingles, 8 bangles, and 12 bongles.

Adding up the total prices of all of these items gives
$$3 (2z) + 8 (5z) + 12z = 87$$
Simplifying 58z = 87 gives z = 3/2. The cost of a bangle is 5z = 15/2 = $7.50.

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5.4: Maximizing Quadratic Functions

1. The quadratic $6x - x^2$ has roots $x = 0, 6$. The quadratic is maximized for x halfway in between these two. Plugging $x = 3$ into the quadratic, its value is $18 - 9 = 9$.

2. If two real numbers have a sum of 17 they are x and $17 - x$ for some x. The question is thus asking for the largest possible value for the product $x(17 - x) = 17x - x^2$. It is maximized when x is halfway between 0 and 17 or 17/2. The value is $17/2 \times 17/2 = 289/4 = 72.25$.

3. The problem assumes that John will fence three sizes of the rink. If x is the length of one of the sides coming out from the cliff, the other sides will have length x and $200 - 2x$. The area is $x(200 - 2x)$. The roots of this quadratic are $x = 0$, 100. The quadratic is maximized for $x = 50$ where its value is $50 \times 100 = 5,000$ square feet.

4. The product of the Fahrenheit and Celsius temperatures is $C((9/5)C + 32)$. This product will be negative when temperatures are in the small interval just below $32°F$ where the Fahrenheit temperature is still positive and the Celsius temperature is negative. One root of this quadratic is obviously $C = 0$. The other occurs when $(9/5) C = -32$ which gives $C = -160/9$. The point halfway between these roots is $C = -80/9$. The value of the product there is $-\frac{80}{9} \times \left(\frac{9}{5} \cdot \frac{-80}{9} + 32\right) = -\frac{80}{9} \times 16 = -\frac{1280}{9}$.

5. The total number of likes that Kara will get if she posts x photos is $x(43 - 3x)$. The roots of this quadratic are $x = 0$ and $x = 43/3 = 14\frac{1}{3}$. The quadratic will be maximized when x is halfway in between these, which is $7\frac{1}{6}$. The number of photos posted must be a whole number. The whole number that makes a quadratic as large as possible is the whole number that is as close as possible to the number where the quadratic is maximized. Here, the closest whole number is 7. She should post 7 photos.

6. Let $y = x^2 + 11$. The product in the question is $(y - 48) y$. This quadratic has roots $y = 0$, 48, so it is as small as possible when $y = 24$. This is not possible when x is an integer. So we must choose the value of x that makes $x^2 + 11$ as close to 24 as possible. The closest value occurs when x is 4 or -4. For these x the value of the polynomial is $(16 - 37) \times (16 + 11) = -21 \times 27 = -567$.

7. The question does not specify whether A is the right angle of the triangle. However, it does tell us that AFDG is a square with point G on DE, which implies that ADG is a right angle. This would not be possible if A were the right angle of ABC with DE parallel to the hypotenuse. So it must be that B or C is the right

angle and the figure looks like the one below. (It does not matter if you make B or C the right angle. The problem works out the same either way.)
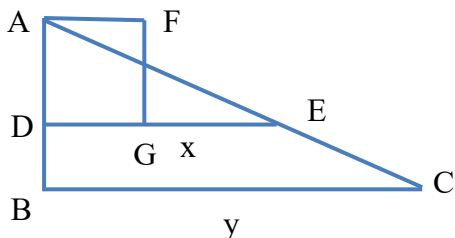
By similar triangles, the length of AD is x/y. The area of square AFDG is $(x/y)^2$. The length of DF $\sqrt{2}$ (x/y). Define $z = x/y$. The question asks for the largest possible value for the difference $\sqrt{2}z - z^2$. The roots of this quadratic are 0 and $\sqrt{2}$, so the quadratic is maximized for $z = \sqrt{2}/2 \approx 0.707$.

The square root of 2 is irrational, so there are no whole numbers x and y (let along single digit ones) for which $x/y = \sqrt{2}/2$. Instead, we just want to pick one-digit whole numbers to make x/y as close to this value as possible. It is generally useful for students to know constants like $\sqrt{2}$, $\sqrt{3}$, and π to several decimal places. In this case, if students know that $\sqrt{2} = 1.41421356 \ldots$, they will know that $\sqrt{2}/2 \approx 0.707$. The closest we can come to this with a single digit denominator is $5/7 \approx 0.7142857$. So ordered pair (x, y) is (5, 7).

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