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Abstract

We introduce a *game in preference form*, which consists of a game form and a preference structure, and define *preference rationalizability* that allows for each player's ex-post preferences over outcomes to depend on opponents' actions. We show that preference rationalizability is invariant to redundant types and states as long as all players have simplex restrictions on their ex-post preferences. We analyze the relationship between preference-form games and conventional payoff-form games. In particular, even if all players have simplex restrictions, we argue that there are multiple payoff-form games that correspond to a given preference-form game, and show that only one of them has the set of interim correlated rationalizable actions equal to the set of preference rationalizable actions in the preference-form game. We also discuss cases where the simplex assumption is violated.

1 Introduction

Since the ordinalist revolution of the 1930's, economists have generally taken preferences to be the primitive object of analysis in consumer choice, with utility functions being understood as convenient tools to represent preferences. In game theoretic analysis, it has been conventional since von Neumann and Morgenstern (1944) to represent a game in "payoff form", where action profiles are associated with "payoffs" or "utilities" that players receive. This is usually understood to be a reduced-form model of a situation where action profiles give rise to outcomes and players' preferences over (lotteries over) outcomes can be represented using the payoffs/utilities. However, we will argue that this reduced-form representation causes confusion for foundational questions such as defining solution concepts. For this reason, we introduce a class of games in "preference form", which consist of a "game form", mapping action profiles into outcomes, and a "preference

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structure", specifying players' expected-utility preferences. We characterize the implications of common certainty of rationality in this setting and show that it corresponds to a solution concept that we label "preference rationalizability". We show how preference-form games can be mapped into standard payoff-form games by the introduction of appropriate state spaces. However, we show that while preference rationalizability is not sensitive to the choice of state space in the model (and thus captures common certainty of rationality independent of the state space), standard definitions of rationalizability, i.e. correlated rationalizability (Brandenburger and Dekel (1987)) for complete-information game and interim correlated rationalizability (Dekel, Fudenberg and Morris (2007)) for incomplete-information games, are sensitive in the payoff-form formulation.

The following example will be used to motivate preference-form games and our definition of preference rationalizability. Consider a setting with two players, Ann and Bob, and three possible outcomes, go to an old (known) restaurant (z_{old}), try a new one (z_{new}) or stay at home (z_{stay}).[1] Suppose that we (the modelers) know that both players happen to have the same expected-utility preferences over lotteries: they strictly prefer to go to the new restaurant to the old restaurant and strictly prefer either restaurant to staying at home. More specifically, they are each indifferent between (i) the old restaurant and (ii) a lottery consisting of 4/5 chance of going to the new restaurant and a 1/5 chance of staying at home. We can summarize these preferences by saying that they have utility indices $(12, 15, 0)$ over the three outcomes $(z_{\text{old}}, z_{\text{new}}, z_{\text{stay}})$.[2] While we are certain of the players' preferences over lotteries (i.e., what they would choose if given a choice among lotteries), we do not believe that we know the players' "true" preferences, i.e., their preferences in all contingencies we could imagine. Let us suppose that we believe that contingent on all possibilities we are prepared to allow for, each player strictly prefers the old restaurant to staying at home, weakly prefers the new restaurant to staying at home and weakly prefers the old restaurant to the lottery consisting of a 1/10 chance of going to the new restaurant and a 9/10 chance of staying at home. Thus there are upper and lower bounds on how good or bad the new restaurant could be. We can summarize these preferences by saying that each player has ex-post preferences that can be represented by utility indices that are a convex combination of $(12, 120, 0)$ and $(12, 0, 0)$. While we are sure that Ann and Bob have the same unconditional preferences, we are not sure that they have the same ex-post preferences.

Now suppose that the players face the following strategic environment: if both say "yes", they go to the new restaurant for sure; if both say "no", they go to the old one; if one says "yes" and

[1] Outcome z_{stay} does not play a substantive role in the discussion in the introduction but will be needed to conform with our general setup in Section 2, where we exclude completely indifferent ex-post preferences.

[2] Utility indices are determined only up to affine transformations and the choice of utility indices in the introduction is for expository convenience, i.e., to avoid fractions later.

one says "no", there is a 2/3 chance they will end up going to the new restaurant and 1/3 to the old restaurant. The following table represents this situation

	yes	no
yes	z_{new}	$(\frac{2}{3}, z_{\text{new}}; \frac{1}{3}, z_{\text{old}})$
no	$(\frac{2}{3}, z_{\text{new}}; \frac{1}{3}, z_{\text{old}})$	z_{old}

, (1)

where rows correspond to Ann's action, columns correspond to Bob's action and the entries in the table represent lotteries over the outcomes.

We just defined an example of a preference-form game by specifying a preference structure (in the paragraph before the previous one) describing what is known about players' preferences over outcomes; and game form (in the previous paragraph and summarized in table (1)). Since the new restaurant is strictly preferred to the old, saying "yes" seems like a dominant strategy for, say, Ann. However, we will argue that if Ann is unsure if Bob will say "yes" or "no", then action "no" is also consistent with common certainty of rationality. To see why, observe that choosing an action for Ann is equivalent to choosing an (Anscombe-Aumann) act that maps actions of Bob to lotteries over outcomes. Thus saying "yes" is equivalent to going to the new restaurant for sure if Bob says "yes", and with only probability 2/3 if Bob says "no"; saying "no" is equivalent to going to the new restaurant with probability 2/3 if Bob says "yes", and going to the old restaurant if Bob says "no". The preference structure did not pin down Ann's preferences over such acts. In particular, suppose that Ann assigned probability 1/2 to Bob saying "yes" and her ex-post preferences being represented by $(12, 30, 0)$, and probability 1/2 to Bob saying "no" and her ex-post preferences being represented by $(12, 0, 0)$. In this case, the expected payoff to saying "yes" would be $(30 + 1/3 \times 12)/2 = 17$, while the payoff to saying "no" would be $(2/3 \times 30 + 1/3 \times 12 + 12)/2 = 18$. By symmetry, each action is then consistent with common certainty of rationality and thus preference rationalizable in the sense that we will define it.

While it is important to our approach that ex-post preferences are not pinned down by the preference structure, we do impose restrictions on ex-post preferences. In the above example, recall that we made assumptions that ex-post preferences are represented by convex combinations of utility indices $(12, 120, 0)$ and $(12, 0, 0)$. It turns out that if we impose no restriction on ex-post preferences, then our framework and solution concept behave quite badly. For the first half of the paper, we therefore focus on a well-behaved class of "simplex" restrictions on ex-post preferences. We impose the requirement that the set of ex-post preferences are represented by the convex hull of a finite set of independent utility indices on outcomes. Note that our example satisfies this restriction. In the second half of the paper, we examine what happens as the simplex restrictions are relaxed.

A key feature of the above example and our analysis in general is that while we make assumptions about players' (perhaps interdependent) preferences over lotteries, we assume that these do not pin down all possible ex-post preferences. An alternative response would be to assume a rich set of incomplete-information states, as in Harsanyi (1967/68), and assume that ex-post preferences are pinned down by a sufficiently rich state space. There are two reasons why we prefer not to go down this route. First, we do not think that the modeler will know how to specify the state space. If states reflect outcome uncertainty (i.e., different states give rise to distinct outcomes), we can imagine that states are in principle observable. But to the extent that the uncertainty is about preferences over fixed physical outcomes, it is not clear that states have an observable counterpart. Second, we believe our modelling choice follows a long and productive "small worlds" modelling strategy in the literature on the epistemic foundations of game theory: Savage (1954) argued that states (in his single person decision theory) should not be interpreted as being exhaustive and this view has been widely adopted in giving decision theoretic foundations to game theoretic analysis. Thus Aumann (1987) made an influential argument that correlation between the actions of two players in a third player's mind...

> ....has no connection with any overt or even covert collusion between 1 and 2; they may be acting entirely independently. Thus it may be common knowledge that both 1 and 2 went to business school, or perhaps to the same business school; but 3 may not know what is taught there. In that case 3 would think it quite likely that they would take similar actions, without being able to guess what those actions might be.[3]

Aumann was not proposing that we should explicitly include the possibility that 1 and 2 attended the same business school in our description of states. Rather, he was arguing that there might be a richer structure behind his small worlds model of correlated equilibrium. Similarly, we can imagine that there is some reason why Bob saying "yes" is correlated in Ann's mind with a more desirable experience at the new restaurant but we do not know how to model it. To allow Aumann's unmodelled correlation and not allow our correlation of actions with preferences seems inconsistent.

Nonetheless, the "payoff-form" framework is standard and it is natural to ask if our richer framework is needed or if there is an appropriate way of embedding our analysis in the standard framework. We investigate this issue in detail. Under simplex restrictions, we argue that there are two natural ways to represent an arbitrary preference-form game by a payoff-form game. We can illustrate these two representations with our example. One representation is a complete-information

[3]This interpretive issue arises in Aumann's (1987) justification of correlated equilibrium and also in work arguing that correlated rationalizability captures the implications of common certainty of rationality in complete-information games (see Brandenburger and Dekel (1987), Tan and Werlang (1988) and, for an opposing view, Bernheim (1987)).

payoff-form game with expected payoffs

	yes	no
yes	15, 15	14, 14
no	14, 14	12, 12

In this game, saying “yes” is a dominant strategy and thus saying “yes” is the unique correlated rationalizable action. An alternative approach—which relies on simplex restrictions—is to consider the incomplete-information payoff-form game with four states where a state specifies for each of the two players which of the two extreme points of the possible ex-post preferences correspond to that player’s true preference. Thus there would be states for each player where payoffs of that player would be given by matrices corresponding to the two extreme points:

state 1	yes	no
yes	120	72
no	72	12

,

state 2	yes	no
yes	0	4
no	4	12

.

In this class of payoff-form games with incomplete information, Dekel, Fudenberg and Morris (2007) argued that interim correlated rationalizability is the solution concept that reflects common certainty of rationality. Both actions are interim correlated rationalizable in the example. In general, we show that the set of (Bayes-Nash) equilibria are the same in the preference-form game and its minimal (i.e., the former) and expansive (i.e., the latter) payoff-form versions, but that the set of preference-rationalizable actions equals the set of interim correlated rationalizable actions in the expansive payoff-form representation only, and includes (strictly in some games) the set of interim correlated rationalizable actions in the minimal payoff-form representation. Thus if we stick to the payoff-form formulation, interim correlated rationalizability characterizes the behavioral implications of common certainty of rationality and beliefs and higher-order beliefs over the (exogenously given) state space, but does not provide an operational definition of the state space. We show that if we assume simplex restrictions on ex-post preferences, we can use those restrictions to define the unique state-space representation under which interim correlated rationalizable actions coincide with preference rationalizable actions.

In the second half of the paper, we examine what happens to our analysis if we relax the simplex assumption. If we relax this assumption slightly and allow for non-simplex convex polytope restrictions, then preference rationalizability may not be invariant to redundant types or states, nor may it be equivalent to interim correlated rationalizability in any corresponding payoff-form game. Moreover, if we drop such restrictions and allow for any state-dependent utility indices, then preference rationalizability becomes oddly behaved. For example, the set of preference rationalizable actions may not be upper hemicontinuous with respect to slight changes in game forms. Also, the iteration procedure always stops in one step, and hence preference rationalizability depends only on preferences over outcomes ("first-order preference"), and is independent of how these preferences depend on each other's type ("higher-order preferences"). This is in sharp contrast to Bergemann, Morris and Takahashi (2011), where we show that if ex-post utility indices are restricted to a bounded set, then there exists a game form in which two types have disjoint sets of preference rationalizable actions if and only if they have different hierarchies of preferences (preferences and higher-order preferences) over outcomes.[4]

In the example we presented, the preference structure was degenerate with each player having one possible type so that there was common certainty of preferences over lotteries. Our main results allow for rich structures of interdependent preferences, where players have a set of types and each type has expected-utility preferences over Anscombe-Aumann acts over the other players' types. A companion paper, Bergemann, Morris and Takahashi (2011), identifies a canonical space of all relevant interdependent preferences à la Mertens and Zamir (1985), and shows how they can be observationally distinguished. In this sense, we allow for arbitrary preference uncertainty in our framework. Our main results focus on the case where there is no outcome uncertainty, so that there is common certainty of the mapping from action profiles to lotteries over outcomes. However, we also discuss how to incorporate outcome uncertainty into the analysis.

The rest of the paper is organized as follows. Section 2 introduces decision-theoretic notations to discuss preferences represented by state-dependent utility indices. Section 3 defines preference-form games and preference rationalizability as a solution concept. Section 4 discusses relationships between preference rationalizability in preference-form games and interim correlated rationalizability in payoff-form games. Section 5 discusses cases with non-simplex restrictions on ex-post utility indices. Section 6 concludes.

2 State-Dependent Utilities

Throughout the paper, we consider expected-utility preferences over Anscombe-Aumann acts. In this section, we provide notations for a single decision-maker, which will be incorporated to strategic environments in the next section.

Let Z be a finite set of outcomes with $|Z| \geq 2$. We say that U is a *restriction* on state-dependent utility indices if U is a non-empty convex subset of $\mathbb{R}^Z$ such that (A) no $u \in U$ is constant, and

[4]Dekel, Fudenberg and Morris (2006, 2007) show an analogous result for payoff-form games: given a state space, there exists a profile of state-dependent payoff functions in which two types have disjoint sets of interim correlated rationalizable actions if and only if they have different hierarchies of beliefs over states.

(B) for any $u, v \in U$, if there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $v(z) = \alpha u(z) + \beta$ for any $z \in Z$, then $u = v$. A restriction U is a *simplex* if (C) U is equal to the convex hull of a finite set $\{u^1, u^2, \dots, u^K\}$ such that $u^2 - u^1, u^3 - u^1, \dots, u^K - u^1$ are linearly independent. An element of U can be identified with the expected-utility preference over lotteries it represents. Property (A) ensures that it does not contain complete indifference. Property (B) ensures that there is a unique cardinal representation of a given preference. Property (C) is a substantial assumption on preferences. To present our results in the simplest environment, we maintain this assumption in Sections 2–4, and discuss non-simplex restrictions only in Section 5.

Fix a non-empty finite state space X, and let $F(X)$ be the set of all (Anscombe-Aumann) acts $f\colon X \to \Delta(Z)$. Given X and any restriction U on utility indices, let $P^U(X)$ be the set of preferences $\succeq$ over $F(X)$ that are represented by state-dependent utility indices that belong to U, i.e., the set of preferences $\succeq$ over $F(X)$ such that, for some $q \in \Delta(X)$ and $\{u(\cdot \mid x)\}_{x \in X} \subset U$,

$$\forall f, f' \in F(X), \quad f \succeq f' \Leftrightarrow \sum_{x,z} q(x) f(z \mid x) u(z \mid x) \geq \sum_{x,z} q(x) f'(z \mid x) u(z \mid x).$$

A state x is $\succeq$*-null* if $\succeq$ is indifferent among outcomes contingent on x, or equivalently, $q(x) = 0$. By Properties (A) and (B) of U, q and $u(\cdot \mid x)$ are uniquely determined up to $\succeq$-null states. For $\succeq$-non-null state x, we can define the conditional preference $\succeq_x$ over $\Delta(Z)$, which is represented by $u(\cdot \mid x)$.

Note that the decision-maker's subjective belief q over X depends on cardinal utility representations in U. However, $P^U(X)$ is independent of cardinal utility representations. That is, for two restrictions $U = \operatorname{conv}\{u^1, \dots, u^K\}$ and $V = \operatorname{conv}\{v^1, \dots, v^K\}$, if for each k, there exist $\alpha^k > 0$ and $\beta^k \in \mathbb{R}$ such that $v^k(z) = \alpha^k u^k(z) + \beta^k$ for any $z \in Z$, then we have $P^U(X) = P^V(X)$.[5] Thus $P^U(X)$ corresponds to a set of preferences independent of representation with probabilities and cardinal utilities. For ease of exposition, however, we will sometimes refer to probabilities and cardinal utilities.

Note also that, given simplex restriction U, $P^U(X)$ is isomorphic to $\Delta(X \times \operatorname{ext} U)$.[6] It is because each $\succeq \in P(X)$ is represented by $q \in \Delta(X)$ and $\{u(\cdot \mid x)\}_{x \in X} \subset U$ uniquely up to $\succeq$-null states x, and each $u(\cdot \mid x)$ is expressed as a convex combination of $\operatorname{ext} U$ uniquely.

We have the following examples of $P^U(X)$: Examples 1–4 for simplex restrictions, and Example 5 for a non-simplex restriction.

Example 1 (state independence). Let $U = \{u\}$ with non-constant $u \in \mathbb{R}^Z$. Then $P^U(X)$ is the set of preferences with state-independent utility index u. Each $\succeq \in P^U(X)$ is then represented by $q \in \Delta(X)$.

[5] For a subset $X \subseteq \mathbb{R}^Z$, $\operatorname{conv} X$ denotes the convex hull of X.

[6] For a convex subset $X \subseteq \mathbb{R}^Z$, $\operatorname{ext} X$ denotes the set of extreme points of X.

Example 2 (ordinal monotonicity). For $Z = \{z_1, \ldots, z_m\}$, let $U = \text{conv}\{u^2, \ldots, u^m\}$, where

$$u^k(z) = \begin{cases} 0 & \text{if } z = z_1, \ldots, z_{k-1}, \\ 1 & \text{if } z = z_k, \ldots, z_m \end{cases}$$

for each $k = 2, \ldots, m$. Then each $\succeq \in P^U(X)$ is represented by $q \in \Delta(X)$ and $\{u(\cdot \mid x)\}_{x \in X}$ such that $1 = u(z_m \mid x) \geq u(z_{m-1} \mid x) \geq \cdots \geq u(z_1 \mid x) = 0$ for any $x \in X$. That is, $P^U(X)$ is the set of preferences such that the conditional preference $\succeq_x$ satisfies $z_m \succeq_x z_{m-1} \succeq_x \cdots \succeq_x z_1$ for all x, and $z_m \succ_x z_1$ for some (non-null) x.[7]

Example 3 (worst outcome). For $Z = \{z_1, \ldots, z_m\}$, let $U = \text{conv}\{u^2, \ldots, u^m\}$, where

$$u^k(z) = \begin{cases} 0 & \text{if } z \neq z_k, \\ 1 & \text{if } z = z_k \end{cases}$$

for each $k = 2, \ldots, m$. Then each $\succeq \in P^U(X)$ is represented by $q \in \Delta(X)$ and $\{u(\cdot \mid x)\}_{x \in X}$ such that for each $x \in X$, we have $1 \geq u(z \mid x) \geq u(z_1 \mid x) = 0$ for all $z \neq z_1$ and $u(z \mid x) > 0$ for some $z \neq z_1$. That is, $P^U(X)$ is the set of preferences such that the conditional preference $\succeq_x$ satisfies $z \succeq_x z_1$ for all (x, z), and $z \succ_x z_1$ for some (x, z).[8]

Example 4 (bounded valuation). For $Z = \{z_1, \ldots, z_m\}$, let $U = \text{conv}\{u^2, u^3, \ldots, u^m\}$, where

$$u^2(z) = \begin{cases} 0 & \text{if } z \neq z_2, \\ 1 & \text{if } z = z_2 \end{cases}$$

and

$$u^k(z) = \begin{cases} 0 & \text{if } z \neq z_2, z_k, \\ 1 & \text{if } z = z_2, \\ v & \text{if } z = z_k \end{cases}$$

for $k = 3, \ldots, m$. Then each $\succeq \in P^U(X)$ is represented by $q \in \Delta(X)$ and $\{u(\cdot \mid x)\}_{x \in X}$ such that for each $x \in X$, we have $1 = u(z_2 \mid x) > u(z_1 \mid x) = 0$, $u(z \mid x) \geq 0$ for any $z \neq z_1, z_2$, and $\sum_{z \neq z_1, z_2} u(z \mid x) \leq v$.[9] That is, $P^U(X)$ is the set of preferences such that there exists a non-null

[7] In related contexts, Ledyard (1986, Section 4.3) requires a strict version of ordinal monotonicity, i.e., $z_m \succ_x z_{m-1} \succ_x \cdots \succ_x z_1$ for non-null x; and Börgers (1993) considers another strict version where utility indices are required to be state-independent. See also Lo (2000). Bogomolnaia and Moulin (2001) introduce a related notion of ordinal efficiency in the random assignment problem.

[8] This restriction is adopted in Bergemann, Morris and Takahashi (2011).

[9] The normalization based on these utility indices is in the spirit of quasi-linear utilities, where the probability difference between z_1 and z_2 is used as a numeraire. That is, each outcome $z \neq z_1, z_2$ is evaluated by how many percentage points of z_2 versus z_1 the decision-maker is willing to give up in order to increase one percentage point of the outcome z versus z_1.

state, and for every such non-null state x, the conditional preference $\succeq_x$ satisfies $z_2 \succ_x z_1$, $z \succeq_x z_1$ for any $z \neq z_1, z_2$, and

$$(1-(m-2)pv)z_1 + (m-2)pvz_2 \succeq_x (1-(m-2)p)z_1 + p(z_3 + \cdots + z_m)$$

for $0 < p \leq (m-2)\min(1/v, 1)$.

Example 5 ("square"). For $Z = \{z_1, z_2, z_3, z_4\}$, let $U = \mathrm{conv}\{u^{\{2\}}, u^{\{2,3\}}, u^{\{2,4\}}, u^{\{2,3,4\}}\}$, where

$$u^k(z_l) = \begin{cases} 0 & \text{if } l \notin k, \\ 1 & \text{if } l \in k \end{cases}$$

for each $k \subseteq \{2,3,4\}$. Note that U is not a simplex restriction. Each $\succeq \in P^U(X)$ is represented by $q \in \Delta(X)$ and $\{u(\cdot \mid x)\}_{x \in X}$ such that for each $x \in X$, we have $u(z_1 \mid x) = 0$, $u(z_2 \mid x) = 1$, and $u(z_3 \mid x), u(z_4 \mid x) \in [0,1]$. That is, $P^U(X)$ is the set of preferences such that the conditional preference $\succeq_x$ satisfies $z_2 \succeq_x z_3 \succeq_x z_1$ and $z_2 \succeq_x z_4 \succeq_x z_1$ for all x, and $z_2 \succ_x z_1$ for some x.

3 Preference-Form Games and Preference Rationalizability

3.1 A Game in Preference Form

We describe a strategic environment by a combination of a game form (or a mechanism) and what we call a preference structure. Let I be a non-empty finite set of players, and Z be a finite set of outcomes with $|Z| \geq 2$, which incorporate both public outcomes and allocations of private outcomes. (To distinguish between public and private outcomes explicitly, we could let $Z = Z_0 \times \prod_i Z_i$ and assume that player i cares only about outcomes in the $Z_0 \times Z_i$ dimension.) A *game form* consists of $((A_i)_{i \in I}, O)$, where A_i is a non-empty finite set of actions available to player i, and $O\colon A = \prod_i A_i \to \Delta(Z)$ is the outcome function. Given this game form, players play an action profile $a \in A$ simultaneously, and an outcome z realizes with probability $O(z \mid a)$. A *preference structure* consists of $(T_i, \pi_i)_{i \in I}$, where each player i has a non-empty finite set T_i of possible types, $T_{-i} = \prod_{j \neq i} T_j$, and $\pi_i\colon T_i \to P^{U_i}(T_{-i})$ assigns each type $t_i \in T_i$ with his preference $\pi_i(t_i)$ over $F(T_{-i})$, the set of outcomes contingent on the other players' types. We assume that ex-post utility indices belong to a simplex $U_i = \mathrm{conv}\{u_i^1, \ldots, u_i^{K_i}\}$. As a special case, if $|T_i| = 1$ for all $i \in I$, then a preference structure specifies a utility index $u_i \in U_i$ (or a preference over lotteries) for each player i. We call a pair of a game form and a preference structure a *game in preference form* or a *preference-form game*, and denote it by $\Gamma = (I, Z, (A_i, T_i, U_i, \pi_i)_{i \in I}, O)$.[10]

[10] In Section 3.6, we extend our framework to explicitly incorporate "states" that are relevant for outcomes or preferences.

3.2 Equilibria

An equilibrium notion is defined as follows. Let $\sigma_i \colon T_i \to \Delta(A_i)$ be a behavior strategy of player i. Then each pair of player i's action $a_i \in A_i$ and his opponents' behavior strategies $\sigma_{-i} = (\sigma_j)_{j \neq i}$ induces an act $O(\cdot \mid a_i, \sigma_{-i})$ over T_{-i}, where an outcome z realizes with probability $O(z \mid a_i, \sigma_{-i}) = \sum_{a_{-i}} O(z \mid a_i, a_{-i}) \prod_{j \neq i} \sigma_j(a_j \mid t_j)$ contingent on t_{-i}. We say that $\sigma = (\sigma_i)_{i \in I}$ is a (Bayes-Nash) *equilibrium* of Γ if, for any $i \in I$ and $t_i \in T_i$, if $\sigma_i(a_i \mid t_i) > 0$, then $\pi_i(t_i)$ weakly prefers $O(\cdot \mid a_i, \sigma_{-i})$ to $O(\cdot \mid a'_i, \sigma_{-i})$ for any $a'_i \in A_i$.

3.3 Preference Rationalizability

In a preference-form game $\Gamma = (I, Z, (A_i, T_i, U_i, \pi_i)_{i \in I}, O)$, we define preference rationalizability as follows:

$$PR_i^0(t_i) = A_i,$$

$$PR_i^{n+1}(t_i) = \left\{ a_i \in A_i \;\middle|\; \begin{array}{l} \text{there exists } \succeq_i \in P^{U_i}(A_{-i} \times T_{-i}) \text{ s.t.} \\ \text{(i) } (a_{-i}, t_{-i}) \text{ is } \succeq_i\text{-non-null} \Rightarrow a_{-i} \in PR_{-i}^n(t_{-i}) \\ \text{(ii) } \operatorname{mrg}_{T_{-i}} \succeq_i = \pi_i(t_i) \\ \text{(iii) } O(\cdot \mid a_i, \cdot) \succeq_i O(\cdot \mid a'_i, \cdot) \text{ for any } a'_i \in A_i \end{array} \right\},$$

$$PR_i(t_i) = \bigcap_{n=0}^{\infty} PR_i^n(t_i),$$

where $\operatorname{mrg}_{T_{-i}} \succeq_i \in P^{U_i}(T_{-i})$ denotes the *marginal preference* of $\succeq_i \in P^{U_i}(A_{-i} \times T_{-i})$, which is the restriction of $\succeq_i$ to outcomes that are contingent on T_{-i} but constant along the A_{-i} dimension. Equivalently, if $\succeq_i$ is represented by $q_i \in \Delta(A_{-i} \times T_{-i})$ and $\{u_i(\cdot \mid a_{-i}, t_{-i})\}_{a_{-i}, t_{-i}} \subset U_i$, then the marginal preference $\operatorname{mrg}_{T_{-i}} \succeq_i$ is the preference represented by the marginal belief

$$\bar{q}_i(t_{-i}) := \sum_{a_{-i}} q_i(a_{-i}, t_{-i})$$

and the average utility indices

$$\bar{u}_i(z \mid t_{-i}) := \sum_{a_{-i}} \frac{q_i(a_{-i}, t_{-i})}{\bar{q}_i(t_{-i})} u_i(z \mid a_{-i}, t_{-i}).$$

We say that any element in $PR_i(t_i)$ is (interim) *preference rationalizable for type* t_i. We also write $PR_i(t_i; \Gamma)$ to emphasize the underlying game Γ.

In words, each type t_i of player i rationalizes his action a_i by forming a preference $\succeq_i$ over outcomes contingent on the opponents' actions and types. Condition (i) requires that the preference $\succeq_i$ responds to outcomes only on the opponents' action-type pairs that have survived in the previous step of iteration. Condition (ii) requires that $\succeq_i$ is consistent with type t_i's own preference over outcomes contingent on the opponents' types. Condition (iii) requires that a_i is optimal given $\succeq_i$. As long as these conditions are satisfied, different players and different types (of the same player) can have different preferences. Also, even the same type can use different preferences to rationalize different actions.

In this definition, notice that type t_i's utility index and the opponents' actions may be correlated with each other. To understand this correlation, suppose that nature chooses an implicitly defined state, which affects type t_i's utility index, or for short, nature chooses type t_i's utility index. Then we treat the nature as another player, and apply the idea of correlated rationalizability to this environment. This is why we allow for type t_i to believe that the opponents' actions and the nature's action are correlated through some channels outside of our explicit model.

We can rewrite the iteration process using a variant of behavior strategies. Namely, type t_i rationalizes his action a_i by "correlated" behavior strategy $\sigma_{-i}\colon T_{-i} \to \Delta(A_{-i})$ and "action-type-contingent" utility indices $\{v_i(\cdot \mid a_{-i}, t_{-i})\}_{a_{-i},t_{-i}} \subset U_i$ such that

(i′) $\sigma_{-i}(a_{-i} \mid t_{-i}) > 0 \Rightarrow a_{-i} \in PR^n_{-i}(t_{-i})$,

(ii′) $\sum_{a_{-i}} \sigma_{-i}(a_{-i} \mid t_{-i}) v_i(z \mid a_{-i}, t_{-i}) = u_i(z \mid t_i, t_{-i})$ for any $t_{-i} \in T_{-i}$ and $z \in Z$,

(iii′) a_i maximizes $\sum_{a_{-i},t_{-i},z} q_i(t_{-i} \mid t_i)\sigma_{-i}(a_{-i} \mid t_{-i}) O(z \mid a'_i, a_{-i}) v_i(z \mid a_{-i}, t_{-i})$,

where $\pi_i(t_i)$ is represented by $q_i(\cdot \mid t_i) \in \Delta(T_{-i})$ and $\{u_i(\cdot \mid t_i, t_{-i})\}_{t_{-i}} \subset U_i$. Here, if $|I| \geq 3$, then a correlated behavior strategy $\sigma_{-i}(a_{-i} \mid t_{-i})$ is a more permissive notion than a profile $(\sigma_j)_{j\neq i}$ of "independent" behavior strategies $\sigma_j\colon T_j \to \Delta(A_j)$, as it allows for correlation of action profile a_{-i} across the opponents, and is contingent on type profile t_{-i}. Also, utility index $v_i(\cdot \mid a_{-i}, t_{-i})$ may depend on the opponents' action-type profile (a_{-i}, t_{-i}). Thus, from type t_i's perspective, it is as if the opponents observe each other's type t_{-i} and play an action profile a_{-i} in a coordinated manner, and then the nature chooses a utility index for type t_i after observing a_{-i}.

The following conditions are also equivalent: there exist "utility-index-contingent" behavior strategies $\sigma^k_{-i}\colon T_{-i} \to \Delta(A_{-i})$ for $k = 1, \ldots, K_i$ such that

(i″) $\sigma^k_{-i}(a_{-i} \mid t_{-i}) > 0$ for some $k = 1, \ldots, K_i \Rightarrow a_{-i} \in PR^n_{-i}(t_{-i})$,

(ii″) a_i maximizes $\sum_{a_{-i},t_{-i},z,k} q_i(t_{-i} \mid t_i)\sigma^k_{-i}(a_{-i} \mid t_{-i}) O(z \mid a'_i, a_{-i}) \lambda^k_i(t_i, t_{-i}) u^k_i(z)$,

where $\lambda^k_i(t_i, t_{-i})$ is the coefficient of u^k_i to express $u_i(\cdot \mid t_i, t_{-i})$ as a convex combination of $\operatorname{ext} U_i = \{u^1_i, \ldots, u^{K_i}_i\}$. In this expression, σ^k_{-i} depends not only on the opponents' type profile t_{-i}, but also player i's own utility index u^k_i. Thus, we can interpret this expression similarly to the previous one, but along the opposite time line: it is as if the nature chooses a utility index u_i^k for type t_i, and then the opponents of player i play a correlated action profile contingent on the nature's choice.

One can also use duality and define rationalizability by eliminating dominated actions iteratively. Namely, type t_i eliminates action a_i if there exists $\alpha_i \in \Delta(A_i)$ such that $O(\cdot \mid \alpha_i, \cdot) \succ_i O(\cdot \mid a_i, \cdot)$ for any $\succeq_i \in P^{U_i}(A_{-i} \times T_{-i})$ that satisfies

(i) (a_{-i}, t_{-i}) is $\succeq_i$-non-null $\Rightarrow a_{-i} \in PR^n_{-i}(t_{-i})$,

(ii) $\mathrm{mrg}_{T_{-i}} \succeq_i = \pi_i(t_i)$.

Two comments are in order regarding the (in)dependency of preference rationalizability on utility index restrictions $(U_i)_{i\in I}$. First, note that sets of preferences, $P^{U_i}(T_{-i})$ and $P^{U_i}(A_{-i} \times T_{-i})$, are independent of cardinal utility representations, i.e., invariant to positive affine transformations of elements of U_i. Thus the set of preference rationalizable actions is also independent of cardinal utility representations.

Second, however, the set of preference rationalizable actions may vary if utility index restrictions are changed in other ways. For example, preference rationalizability becomes more permissive if each U_i is expanded to $V_i \supseteq U_i$. One might use this fact to argue against preference rationalizability. We, however, think that restrictions $(U_i)_{i\in I}$ are an indispensable aspect of the strategic environment, and our preference-form games allow the modeler to state these restrictions explicitly. The modeler can take U_i as a singleton if she is certain about player i's utility index; otherwise, she should take a relatively large U_i to allow for various unmodelled factors that can affect player i's utility indices.

3.4 Two Examples

The following two examples illustrate how to construct preference-form games, and how to apply preference rationalizability to these games.

Example 6 (Winner's Curse). The first example is a discretized version of common-value auction, where we discuss winner's curse that arises from correlation between a bidder's valuation and the opponent's bid. Unlike the conventional approach, however, we do not specify such correlation explicitly; instead, we let the solution concept—preference rationalizability—capture various forms of correlation that are implicit in the model.

There are two bidders in an action. Each bidder can bid high (action H) or low (L). If both bid low, they get nothing (outcome z_0); if bidder i bids high while bidder $j \neq i$ bids low, then bidder i gets an object and pays a fixed price (z_i); if both bid high, the winner is determined by a fair coin toss. The game form is given by

$$O = \begin{array}{c|c|c|} & H & L \\ \hline H & (\frac{1}{2}, z_1; \frac{1}{2}, z_2) & z_1 \\ \hline L & z_2 & z_0 \\ \hline \end{array} .$$

We impose restrictions on utility indices such that for each bidder i, $u_i(z_0) = u_i(z_j) = 0$ and $u_i(z_i) \in [-v, v]$, where $v \geq 1$. We consider a simple preference structure, where each bidder i has a single type $T_i = \{t_i\}$, and his preference is represented by u_i such that $u_i(z_0) = u_i(z_j) = 0$ and $u_i(z_i) = 1$. It is easy to see that H is preference rationalizable. Then under what condition is L also preference rationalizable? Suppose that bidder i believes that his own valuation is positively correlated with the opponent's bid so that with probability $(v + 1)/(2v)$, "bidder j bids H and $u_i(z_i \mid H) = v$", and with probability $(v - 1)/(2v)$, "bidder j bids L and $u_i(z_i \mid H) = -v$". This belief over j's bids and i's own utility indices represents i's preference, which rationalizes L for bidder i if and only if

$$0 \geq \frac{v+1}{2v} \times \frac{1}{2} \times v + \frac{v-1}{2v} \times (-v),$$

i.e., $v \geq 3$. In this case, bidder i bids L because he is afraid of suffering from winner's curse when he wins the object. Conversely, if $1 \leq v < 3$, then no correlation structure between bidder i's valuation and the opponent's bid can prevent him from playing H.

Example 7 (Bilateral Trade). Consider the following game form:

$$O = \begin{array}{c|c|c|} & \text{yes} & \text{no} \\ \hline \text{yes} & z^* & z_1 \\ \hline \text{no} & z_2 & z_0 \\ \hline \end{array} .$$

We interpret this as a trading problem, where players are considering a trade where each player delivers an object—of a known value to him but perhaps an unknown value to the other player—to the other. Thus actions correspond to say yes or no to a proposed trade. Outcome z^* corresponds to the full trade taking place, and z_0 corresponds to "no trade". Outcome z_i is a situation where player i (who said yes) sends his object but—because the other agent says no—it is destroyed.

Each player i is restricted to have utility indices u_i such that $u_i(z_0) = u_i(z_j) = 0$, $u_i(z_i) = -1$, and $u_i(z^*) \in [-v, v]$ with $v \geq 0$. Thus the cost of sending an object is 1, and getting an object from the other player has a value between $-v + 1$ and $v + 1$.

Fix a preference type space $(T_i, \pi_i)_{i=1}^2$. We can represent type t_i's preference $\pi_i(t_i)$ by a combination of the probability $q_i(T_j' \mid t_i)$ of $T_j' \subseteq T_j$ given t_i and the expected utility $u_i(z^* \mid t_i, T_j')$ of z^* conditional on t_i and $T_j' \subseteq T_j$.

Now we can ask what actions are preference rationalizable for different types. Clearly, $N \in PR_i(t_i)$ for any $t_i \in T_i$. Let $T_i^n = \{t_i \in T_i \mid Y \in PR_i^n(t_i)\}$. Then

$$T_i^{n+1} = \{t_i \in T_i \mid q_i(T_j^n \mid t_i)u_i(z^* \mid t_i, T_j^n) \geq 1 - q_i(T_j^n \mid t_i)\},$$

if $0 \leq v \leq 1$, and

$$T_i^{n+1} = \left\{t_i \in T_i \mid q_i(T_j^n \mid t_i)\frac{v + u_i(z^* \mid t_i, T_j')}{2} \geq 1 - q_i(T_j^n \mid t_i)\frac{v + u_i(z^* \mid t_i, T_j')}{2v}\right\}$$

if $v > 1$. To see this, in the case of $0 \leq v \leq 1$, we have $Y \in PR_i^{n+1}(t_i)$ if and only if type t_i prefers saying yes to no by conjecturing that t_j says yes if $t_j \in T_j^n$ and no if $t_j \notin T_j^n$. In contrast, in the case of $v > 1$, we have $Y \in PR_i^{n+1}(t_i)$ if and only if type t_i prefers saying yes to no by conjecturing that with probability $(v + u_i(z^* \mid t_i, T_j'))/(2v)$, "player j says yes if $t_j \in T_j^n$ and no if $t_j \notin T_j^n$, and $u_i(z^*) = v$", and with probability $(v - u_i(z^* \mid t_i, T_j'))/(2v)$, "player j always says no and $u_i(z^*) = -v$".

Thus, intuitively, trade takes place if and only if, for each player, (a) trade has a non-negative expected value, (b) trade has a non-negative expected value conditional on trade having a non-negative expected value for the other player, (c) trade has positive expected value conditional on (b) being true for the other player, and so on... However, the highest expected value of trade is computed differently between the case of $0 \leq v \leq 1$ and the case of $v > 1$. In particular, in the case of $v > 1$, trade can take place as an outcome of preference rationalizable actions even if it does not in equilibrium.

3.5 Redundant Types

The definition of preference rationalizability respects the "small world" view in the sense that a preference structure is not understood to capture everything in the world. Rather, the modeler admits that there may be always states and types that are implicit in the model but relevant for players' preferences.

As a consequence of this "small world" view, we can show that "enlarging" the model does not affect the set of preference rationalizable actions. More formally, consider two preference structures $(T_i, \pi_i)_{i \in I}$ and $(\hat{T}_i, \hat{\pi}_i)_{i \in I}$ with the same simplex restrictions $(U_i)_{i \in I}$. We say that a profile $(\varphi_i)_{i \in I}$ of mappings $\varphi_i \colon T_i \to \hat{T}_i$ is *preference-preserving* if for any $i \in I$ and $t_i \in T_i$,

$$\forall f, f' \in F(\hat{T}_{-i}), \quad \hat{\pi}_i(\varphi_i(t_i)) \text{ prefers } f \text{ to } f' \Leftrightarrow \pi_i(t_i) \text{ prefers } f \circ \varphi_{-i} \text{ to } f' \circ \varphi_{-i},$$

where $f \circ \varphi_{-i}$ is an act over T_{-i} that assigns a lottery $f((\varphi_j(t_j))_{j \neq i})$ contingent on t_{-i}, and $f' \circ \varphi_{-i}$ is defined similarly. If these preference-preserving mappings are not one-to-one, then we can see a preference-form game $\hat{\Gamma} = (I, Z, (A_i, \hat{T}_i, U_i, \hat{\pi}_i)_{i\in I}, O)$ as a "smaller" model than another preference-form game $\Gamma = (I, Z, (A_i, T_i, U_i, \pi_i)_{i\in I}, O)$.

Proposition 1. *If $(U_i)_{i\in I}$ are simplex restrictions and $(\varphi_i)_{i\in I}$ are preference-preserving mappings from $(T_i, \pi_i)_{i\in I}$ to $(\hat{T}_i, \hat{\pi}_i)_{i\in I}$, then $PR_i(t_i; \Gamma) = PR_i(\varphi_i(t_i); \hat{\Gamma})$ for any $i \in I$ and $t_i \in T_i$.*

Proof. By induction, suppose $PR_i^n(t_i; \Gamma) = PR_i^n(\varphi_i(t_i); \hat{\Gamma})$ for any $i \in I$ and $t_i \in T_i$. Then fix any $i \in I$ and $t_i \in T_i$. For any $a_i \in PR_i^{n+1}(t_i; \Gamma)$, there exists $\succeq_i \in P^{U_i}(A_{-i} \times T_{-i})$ such that (i) $a_{-i} \in PR_{-i}^n(t_{-i}; \Gamma)$ whenever (a_{-i}, t_{-i}) is $\succeq_i$-non-null, (ii) $\mathrm{mrg}_{T_{-i}} \succeq_i = \pi_i(t_i)$, and (iii) $O(\cdot \mid a_i, \cdot) \succeq_i O(\cdot \mid a_i', \cdot)$ for any $a_i' \in A_i$. Let $\hat{\succeq}_i \in P^{U_i}(A_{-i} \times \hat{T}_{-i})$ be given by

$$\forall f, f' \in F(A_{-i} \times \hat{T}_{-i}), \quad f \,\hat{\succeq}_i\, f' \Leftrightarrow f \circ (\mathrm{id}_{A_{-i}} \times \varphi_{-i}) \succeq_i f' \circ (\mathrm{id}_{A_{-i}} \times \varphi_{-i}).$$

Then (i) by the induction hypothesis, we have $a_{-i} \in PR_{-i}^n(\hat{t}_{-i}; \hat{\Gamma})$ whenever $(a_{-i}, \hat{t}_{-i})$ is $\hat{\succeq}_i$-non-null, (ii) by the definition of preference-preserving mappings, we have $\mathrm{mrg}_{T_{-i}} \hat{\succeq}_i = \hat{\pi}_i(\hat{t}_i)$, and (iii) $O(\cdot \mid a_i, \cdot) \,\hat{\succeq}_i\, O(\cdot \mid a_i', \cdot)$ for any $a_i' \in A_i$. Thus $a_i \in PR_i^{n+1}(\varphi_i(t_i); \hat{\Gamma})$.

Conversely, fix any $i \in I$ and $t_i \in T_i$. Let $\pi_i(t_i)$ be represented by $q_i(\cdot \mid t_i) \in \Delta(T_{-i})$ and $\{u_i(\cdot \mid t_i, t_{-i})\}_{t_{-i}\in T_{-i}} \subset U_i$, and $\hat{\pi}_i(\hat{t}_i)$ by $\hat{q}_i(\cdot \mid \hat{t}_i) \in \Delta(\hat{T}_{-i})$ and $\{\hat{u}_i(\cdot \mid \hat{t}_i, \hat{t}_{-i})\}_{\hat{t}_{-i}\in \hat{T}_{-i}} \subset U_i$. Also let $u_i(\cdot \mid t_i, t_{-i})$ and $\hat{u}_i(\cdot \mid \hat{t}_i, \hat{t}_{-i})$ be expressed as convex combinations of $\mathrm{ext}\, U_i$: $u_i(\cdot \mid t_i, t_{-i}) = \sum_k \lambda_i^k(t_i, t_{-i}) u_i^k$ and $\hat{u}_i(\cdot \mid \hat{t}_i, \hat{t}_{-i}) = \sum_k \hat{\lambda}_i^k(\hat{t}_i, \hat{t}_{-i}) u_i^k$. Since φ_{-i} preserves player i's preferences $\pi_i(t_i)$ and $\hat{\pi}_i(\varphi_i(t_i))$ and U_i is a simplex, for any $\hat{t}_{-i} \in \hat{T}_{-i}$ and $k = 1, \dots, K_i$, we have

$$\hat{q}_i(\hat{t}_{-i} \mid \varphi_i(t_i)) = \sum_{t_{-i}\in \varphi_{-i}^{-1}(\hat{t}_{-i})} q_i(t_{-i} \mid t_i),$$

$$\hat{q}_i(\hat{t}_{-i} \mid \varphi_i(t_i)) \hat{\lambda}_i^k(\varphi_i(t_i), \hat{t}_{-i}) = \sum_{t_{-i}\in \varphi_{-i}^{-1}(\hat{t}_{-i})} q_i(t_{-i} \mid t_i) \lambda_i^k(t_i, t_{-i}). \tag{2}$$

For any $a_i \in PR_i^{n+1}(\varphi_i(t_i); \hat{\Gamma})$, there exist $\hat{\sigma}_{-i}^k \colon \hat{T}_{-i} \to \Delta(A_{-i})$ for $k = 1, \dots, K_i$ such that (i″) if $\hat{\sigma}_{-i}^k(a_{-i} \mid \hat{t}_{-i}) > 0$ for some k, then $a_{-i} \in PR_{-i}^n(\hat{t}_{-i}; \hat{\Gamma})$, and (ii″) a_i maximizes

$$\sum_{a_{-i}, \hat{t}_{-i}, z, k} \hat{q}_i(\hat{t}_{-i} \mid \varphi_i(t_i)) \hat{\sigma}_{-i}^k(a_{-i} \mid \hat{t}_{-i}) O(z \mid a_i', a_{-i}) \hat{\lambda}_i^k(t_i, \hat{t}_{-i}) u_i^k(z).$$

Let

$$\sigma_{-i}^k(a_{-i} \mid t_{-i}) = \hat{\sigma}_{-i}^k(a_{-i} \mid \varphi_{-i}(t_{-i}))$$

for any $a_{-i} \in A_{-i}$, $t_{-i} \in T_{-i}$, and $k = 1, \dots, K_i$. Then (i″) if $\sigma_{-i}^k(a_{-i} \mid t_{-i}) = \hat{\sigma}_{-i}^k(a_{-i} \mid \varphi_{-i}(t_{-i})) >$

0, then $a_{-i} \in PR^n_{-i}(\varphi_{-i}(t_{-i}); \hat{\Gamma}) = PR^n_{-i}(t_{-i}; \Gamma)$, and (ii$''$) a_i maximizes

$$\begin{aligned}
&\sum_{a_{-i}, \hat{t}_{-i}, z, k} \hat{q}_i(\hat{t}_{-i} \mid \varphi_i(t_i)) \hat{\sigma}^k_{-i}(a_{-i} \mid \hat{t}_{-i}) O(z \mid a'_i, a_{-i}) \hat{\lambda}^k_i(t_i, \hat{t}_{-i}) u^k_i(z) \\
&= \sum_{a_{-i}, \hat{t}_{-i}, z, k} \sum_{t_{-i} \in \varphi^{-1}_{-i}(\hat{t}_{-i})} q_i(t_{-i} \mid t_i) \sigma^k_{-i}(a_{-i} \mid t_{-i}) O(z \mid a'_i, a_{-i}) \lambda^k_i(t_i, t_{-i}) u^k_i(z) \\
&= \sum_{a_{-i}, t_{-i}, z, k} q_i(t_{-i} \mid t_i) \sigma^k_{-i}(a_{-i} \mid t_{-i}) O(z \mid a'_i, a_{-i}) \lambda^k_i(t_i, t_{-i}) u^k_i(z)
\end{aligned}$$

by (2). Thus $a_i \in PR^{n+1}_i(t_i; \Gamma)$. ∎

Proposition 1 is analogous to Dekel, Fudenberg and Morris (2007, Proposition 1), who show that in an incomplete-information game with payoff-relevant states, the set of interim correlated rationalizable actions is invariant to belief-preserving mappings, and hence depends only on hierarchies of beliefs (beliefs and higher-order beliefs) over payoff-relevant states. Indeed, one can derive Proposition 1 from Dekel, Fudenberg and Morris (2007, Proposition 1), given that preference rationalizability in a preference-form game is equivalent to interim correlated rationalizability in an incomplete-information game with appropriately chosen payoff-relevant states (as shown in our Proposition 4 below).

3.6 Explicit States

We can extend our framework by incorporating states that influence outcomes or preferences. Formally, let Ω be a non-empty finite set of states. An Ω-based game form consists of $((A_i)_{i \in I}, \Omega, O)$ with $O\colon A \times \Omega \to \Delta(Z)$, and an Ω-based preference structure consists of $(T_i, \pi_i)_{i \in I}$ with $\pi_i\colon T_i \to P^{U_i}(T_{-i} \times \Omega)$. An Ω-based preference-form game is then given by $\Gamma^\Omega = (I, Z, \Omega, (A_i, T_i, U_i, \pi_i)_{i \in I}, O)$.

Note that Ω includes not only states that affect outcomes but also those that affect preferences. It is necessary to incorporate states that influence outcomes to reflect the modeler's knowledge of the physical structure of the game. In the introduction, we made an argument for not using states that are not known to the players in modelling preference uncertainty. However, we allow for the possibility here, and later show that adding states that affect only preferences does not affect the set of preference rationalizable actions.

Given this preference-form game, we can define preference rationalizability by

$$PR_i^0(t_i) = A_i,$$

$$PR_i^{n+1}(t_i) = \left\{ a_i \in A_i \;\middle|\; \begin{array}{l} \text{there exists } \succeq_i \in P^{U_i}(A_{-i} \times T_{-i} \times \Omega) \text{ s.t.} \\ \text{(i) } (a_{-i}, t_{-i}, \omega) \text{ is } \succeq_i\text{-non-null for some } \omega \in \Omega \Rightarrow a_{-i} \in PR_{-i}^n(t_{-i}) \\ \text{(ii) } \mathrm{mrg}_{T_{-i} \times \Omega} \succeq_i = \pi_i(t_i) \\ \text{(iii) } O(\cdot \mid a_i, \cdot, \cdot) \succeq_i O(\cdot \mid a_i', \cdot, \cdot) \text{ for any } a_i' \in A_i \end{array} \right\},$$

$$PR_i(t_i) = \bigcap_{n=0}^{\infty} PR_i^n(t_i).$$

In this formulation, the modeler can incorporate as many states as she wishes as long as she knows how these states affect outcomes and players' preferences. However, those states affect the set of preference rationalizable actions only to the extent that they affect outcomes. If states affect only preferences and do not affect outcomes, then since such states are already incorporated implicitly in the solution concept, they do not affect the set of preference rationalizable actions. More formally, let $\Gamma^{\Omega} = (I, Z, \Omega, (A_i, T_i, U_i, \pi_i)_{i \in I}, O)$ and $\Gamma^{\hat{\Omega}} = (I, Z, \hat{\Omega}, (A_i, T_i, U_i, \hat{\pi}_i)_{i \in I}, \hat{O})$ be preference-form games with uncertainty Ω and $\hat{\Omega}$, respectively. We say that Ω is *redundant in* $\hat{\Omega}$ if there exists a mapping $\varphi_0 \colon \Omega \to \hat{\Omega}$ such that $O(\cdot \mid a, \omega) = \hat{O}(\cdot \mid a, \varphi_0(\omega))$ for any $a \in A$ and $\omega \in \Omega$, and φ_0 is preference-preserving, i.e., for any $i \in I$ and $t_i \in T_i$,

$$\forall f, f' \in F(T_{-i} \times \hat{\Omega}), \quad \hat{\pi}_i(t_i) \text{ prefers } f \text{ to } f' \Leftrightarrow \pi_i(t_i) \text{ prefers } f \circ (\mathrm{id}_{T_{-i}} \times \varphi_0) \text{ to } f' \circ (\mathrm{id}_{T_{-i}} \times \varphi_0),$$

where $\mathrm{id}_{T_{-i}}$ is the identity mapping on T_{-i}.

Proposition 2. *If $(U_i)_{i \in I}$ are simplex restrictions and Ω is redundant in $\hat{\Omega}$, then $PR_i(t_i; \Gamma^{\Omega}) = PR_i(t_i; \Gamma^{\hat{\Omega}})$ for any $i \in I$ and $t_i \in T_i$.*

Proof. This follows from applying Proposition 1 to the following two games with dummy player 0 (but without uncertainty): in the first game, the dummy player has the set of actions equal to Ω' and the set of types equal to Ω, and each type $\omega \in \Omega$ strictly prefers playing action $\varphi_0(\omega) \in \Omega'$; in the second game, both the set of actions and the set of types are equal to Ω', and each type ω' strictly prefers playing action ω'. ∎

4 Relationship with Payoff-Form Games

In this section, we compare our approach and the conventional one that uses games in payoff form. The main issue is that for a given preference-form game, one can construct (at least) two different payoff-form games, depending on how we specify payoff-relevant states. We show that rationalizability in one payoff-form game is equivalent to preference rationalizability in the preference-form game, but rationalizability in the other payoff-form game is more restrictive.

4.1 A Payoff-Form Game, Equilibria and ICR

Recall that a conventional incomplete-information game is given by $G = (I, (A_i, \Theta_i, g_i, T_i, \mu_i)_{i \in I})$, where Θ_i is a non-empty finite set of states that are relevant for player i's payoffs, $g_i \colon A \times \Theta_i \to \mathbb{R}$ is player i's state-dependent payoff function, and $(T_i, \mu_i)_{i\in I}$ is an information structure with $\mu_i \colon T_i \to \Delta(T_{-i} \times \Theta_i)$.[11] To emphasize the difference from a game in preference form, we call G a *game in payoff form* or a *payoff-form game*.

A profile $(\sigma_i)_{i\in I}$ of behavior strategies $\sigma_i \colon T_i \to \Delta(A_i)$ is an *equilibrium* of G if, for any $i \in I$ and $t_i \in T_i$, if $\sigma_i(a_i \mid t_i) > 0$, then a_i maximizes $\sum_{t_{-i},\theta_i} \mu_i(t_{-i}, \theta_i \mid t_i) g_i(a_i', \sigma_{-i}(t_{-i}), \theta_i)$.

Dekel, Fudenberg and Morris (2007) introduce the notion of *interim correlated rationalizability* (*ICR*) as follows:

$$
\begin{aligned}
ICR_i^0(t_i) &= A_i,\\
ICR_i^{n+1}(t_i) &= \left\{ a_i \in A_i \;\middle|\; \begin{array}{l} \text{there exists } \nu_i \in \Delta(A_{-i} \times T_{-i} \times \Theta_i) \text{ s.t.}\\ \text{(i) } \nu_i(a_{-i}, t_{-i}, \theta_i) > 0 \text{ for some } \theta_i \Rightarrow a_{-i} \in ICR_{-i}^n(t_{-i})\\ \text{(ii) } \sum_{a_{-i}} \nu_i(a_{-i}, t_{-i}, \theta_i) = \mu_i(t_{-i}, \theta_i \mid t_i) \text{ for all } t_{-i}, \theta_i\\ \text{(iii) } a_i \text{ maximizes } \sum_{a_{-i}, t_{-i}, \theta_i} \nu_i(a_{-i}, t_{-i}, \theta_i) g_i(a_i', a_{-i}, \theta_i) \end{array} \right\},\\
ICR_i(t_i) &= \bigcap_{n=0}^{\infty} ICR_i^n(t_i).
\end{aligned}
$$

We also write $ICR_i(t_i; G)$ to emphasize the underlying game G.

4.2 Two Constructions of Payoff-Form Games

Given a game in preference form $\Gamma = (I, Z, (A_i, T_i, U_i, \pi_i)_{i\in I}, O)$, there are at least two ways to construct games in payoff form. One way is to set $\Theta_i = T$ for each player i. In the example discussed in the introduction, where there was a singleton type space, this minimal corresponds to a complete information payoff-form game. For each type $t_i \in T_i$, whose preference $\pi_i(t_i) \in P^{U_i}(T_{-i})$ is represented by belief $q_i(\cdot \mid t_i) \in \Delta(T_{-i})$ and utility indices $\{u_i(\cdot \mid t_i, t_{-i})\}_{t_{-i} \in T_{-i}} \subset U_i$, we define

[11] Strictly speaking, our definition of incomplete-information games is slightly different from Dekel, Fudenberg and Morris (2007), where they use a single "public" set Θ without subscripts to capture payoff-relevant states to all players.

player i's state-dependent payoff function $g_i^T\colon A \times T \to \mathbb{R}$ by

$$g_i^T(a,(t_i,t_{-i})) = \sum_{z\in Z} O(z \mid a) u_i(z \mid t_i, t_{-i})$$

and his beliefs $\mu_i^T\colon T_i \to \Delta(T_{-i} \times T)$ by

$$\mu_i^T(t_{-i}, t' \mid t_i) = \begin{cases} q_i(t_{-i} \mid t_i) & \text{if } t' = (t_i, t_{-i}), \\ 0 & \text{if } t' \neq (t_i, t_{-i}). \end{cases}$$

We denote this payoff-form game by G^T.

Another way is to set $\Theta_i = \operatorname{ext} U_i = \{u_i^1, \ldots, u_i^{K_i}\}$ for each player i. This more expansive version corresponded to an incomplete information payoff-form game in the example in the introduction. Then we define each player's state-dependent payoff function $g_i\colon A \times \operatorname{ext} U_i \to \mathbb{R}$ by

$$g_i^U(a, u_i^k) = \sum_{z\in Z} O(z \mid a) u_i^k(z)$$

and his beliefs $\mu_i\colon T_i \to \Delta(T_{-i} \times \operatorname{ext} U_i)$ by

$$\mu_i^U(t_{-i}, u_i^k \mid t_i) = q_i(t_{-i} \mid t_i)\lambda_i^k(t_i, t_{-i}),$$

where $\lambda_i^k(t_i, t_{-i})$ is the coefficient of u_i^k to express $u_i(\cdot \mid t_i, t_{-i})$ as a convex combination of $\{u_i^1, \ldots, u_i^{K_i}\}$. We denote this payoff-form game by G^U.

At the first glance, both games G^T and G^U are similar to the preference-form game Γ. In fact, Γ, G^T, and G^U have the same set of equilibria. Let $E(\Gamma)$ be the set of equilibria in preference-form game Γ, and $E(G)$ be the set of equilibria in payoff-form game G,

Proposition 3. $E(G^T) = E(G^U) = E(\Gamma)$.

Proof. Note that for each $t_i \in T_i$, $a_i \in A_i$, and $\sigma_{-i} = (\sigma_j)_{j\neq i}$ with $\sigma_j\colon T_j \to \Delta(A_j)$, we have

$$\begin{aligned}
&\sum_{t_{-i},t'} \mu_i^T(t_{-i}, t' \mid t_i) g_i^T(a_i, \sigma_{-i}(t_{-i}), t') \\
&= \sum_{t_{-i},z} q_i(t_{-i} \mid t_i) O(z \mid a_i, \sigma_{-i}(t_{-i})) u_i(z \mid t_i, t_{-i}) \\
&= \sum_{t_{-i},k,z} q_i(t_{-i} \mid t_i) O(z \mid a_i, \sigma_{-i}(t_{-i})) \lambda_i^k(t_i, t_{-i}) u_i^k(z) \\
&= \sum_{t_{-i},k} \mu_i^U(t_{-i}, u_i^k \mid t_i) g_i^U(a_i, \sigma_{-i}(t_{-i}), u_i^k).
\end{aligned}$$

Thus type t_i solves the same maximization problem in G^T, G^U and Γ. ∎

However, G^T and G^U may have different sets of ICR actions. In general, ICR in G^T is more restrictive than ICR in G^U, which is equivalent to preference rationalizability in Γ.

Proposition 4. *If $(U_i)_{i \in I}$ are simplex restrictions, then $ICR_i(t_i; G^T) \subseteq ICR_i(t_i; G^U) = PR_i(t_i; \Gamma)$ for any $i \in I$ and $t_i \in T_i$.*

Proof. For the first part (set inclusion), by induction, suppose $ICR_i^n(t_i; G^T) \subseteq ICR_i^n(t_i; G^U)$ for any $i \in I$ and $t_i \in T_i$. Then for each $i \in I$, $t_i \in T_i$, and $a_i \in ICR_i^{n+1}(t_i; G^T)$, there exists $\nu_i^T \in \Delta(A_{-i} \times T_{-i} \times T)$ such that (i) $a_{-i} \in ICR_{-i}^n(t_{-i}; G^T)$ if $\nu_i^T(a_{-i}, t_{-i}, t') > 0$ for some $t' \in T$, (ii) $\sum_{a_{-i}} \nu_i^T(a_{-i}, t_{-i}, t') = \mu_i^T(t_{-i}, t' \mid t_i)$, and (iii) a_i maximizes

$$\sum_{a_{-i}, t_{-i}, t'} \nu_i^T(a_{-i}, t_{-i}, t') g_i^T(a_i', a_{-i}, t') = \sum_{a_{-i}, t_{-i}} \nu_i^T(a_{-i}, t_{-i}, (t_i, t_{-i})) g_i^T(a_i', a_{-i}, (t_i, t_{-i})).$$

Let $\nu_i^U \in \Delta(A_{-i} \times T_{-i} \times \operatorname{ext} U_i)$ be given by

$$\nu_i^U(a_{-i}, t_{-i}, u_i^k) = \nu_i^T(a_{-i}, t_{-i}, (t_i, t_{-i})) \lambda_i^k(t_i, t_{-i}),$$

where $\lambda_i^k(t_i, t_{-i})$ is the coefficient of u_i^k to represent $u_i(\cdot \mid t_i, t_{-i})$ as a convex combination of $\{u_i^1, \ldots, u_i^{K_i}\}$. Then (i) $a_{-i} \in ICR_{-i}^n(t_{-i}; G^T) \subseteq ICR_{-i}^n(t_{-i}; G^U)$ if $\nu_i^U(a_{-i}, t_{-i}, u_i^k) > 0$ for some $u_i^k \in \operatorname{ext} U_i$, (ii) $\sum_{a_{-i}} \nu_i^U(a_{-i}, t_{-i}, u_i^k) = \sum_{a_{-i}} \nu_i^T(a_{-i}, t_{-i}, (t_i, t_{-i})) \lambda_i^k(t_i, t_{-i}) = \mu_i^T(t_{-i}, (t_i, t_{-i}) \mid t_i) \lambda_i^k(t_i, t_{-i}) = q_i(t_{-i} \mid t_i) \lambda_i^k(t_i, t_{-i}) = \mu_i^U(t_{-i}, u_i^k \mid t_i)$, and (iii) a_i maximizes

$$\begin{aligned}
&\sum_{a_{-i}, t_{-i}} \nu_i^T(a_{-i}, t_{-i}, (t_i, t_{-i})) g_i^T(a_i', a_{-i}, (t_i, t_{-i})) \\
&= \sum_{a_{-i}, t_{-i}, z} \nu_i^T(a_{-i}, t_{-i}, (t_i, t_{-i})) O(z \mid a_i', a_{-i}) u_i(z \mid t_i, t_{-i}) \\
&= \sum_{a_{-i}, t_{-i}, z, k} \nu_i^T(a_{-i}, t_{-i}, (t_i, t_{-i})) O(z \mid a_i', a_{-i}) \lambda_i^k(t_i, t_{-i}) u_i^k(z) \\
&= \sum_{a_{-i}, t_{-i}, k} \nu_i^U(a_{-i}, t_{-i}, u_i^k) g_i^U(a_i', a_{-i}, u_i^k).
\end{aligned}$$

Thus $a_i \in ICR_i^{n+1}(t_i; G^U)$.

The second part (set equality) follows from the isomorphisms between $\Delta(T_{-i} \times \operatorname{ext} U_i)$ and $P^{U_i}(T_{-i})$ and between $\Delta(A_{-i} \times T_{-i} \times \operatorname{ext} U_i)$ and $P^{U_i}(A_{-i} \times T_{-i})$ when U_i is a simplex. ∎

The next example shows that the set inclusion may be strict.

Example 8. Consider the following preference-form game Γ. There are two players $I = \{1, 2\}$ and three outcomes $Z = \{z_1, z_2, z_3\}$. For each player i, his utility indices are restricted to $U_i = \operatorname{conv}\{u^2, u^3\}$, where $(u^2(z_1), u^2(z_2), u^2(z_3)) = (0, 1, 0)$ and $(u^3(z_1), u^3(z_2), u^3(z_3)) = (0, 0, 1)$. Each player i has only one type $T_i = \{t_i\}$, whose preference is represented by $(u_i(z_1), u_i(z_2), u_i(z_3)) = (0, 1/2, 1/2)$. The players play the following game form:

$$O = \begin{array}{c|c|c|} & a_2 & a_2' \\ \hline a_1 & z_3 & (\frac{1}{2}, z_1; \frac{1}{2}, z_2) \\ \hline a_1' & (\frac{1}{2}, z_1; \frac{1}{2}, z_2) & z_1 \\ \hline \end{array} .$$

Given such Γ, we define G^T as a complete-information game with the following payoffs

$$g^T(\cdot, t_1, t_2) = \begin{array}{c|c|c|} & a_2 & a_2' \\ \hline a_1 & \frac{1}{2}, \frac{1}{2} & \frac{1}{4}, \frac{1}{4} \\ \hline a_1' & \frac{1}{4}, \frac{1}{4} & 0, 0 \\ \hline \end{array} .$$

It is easy to see that a_i is the unique rationalizable action for player i.

On the other hand, G^U is an incomplete-information game, where $\Theta_1 = \Theta_2 = \{u^2, u^3\}$, payoffs are given by

$$g^U = \begin{array}{cc} (\theta_1, \theta_2) = (u^2, u^2) \begin{array}{c|c|c|} & a_2 & a_2' \\ \hline a_1 & 0, 0 & \frac{1}{2}, \frac{1}{2} \\ \hline a_1' & \frac{1}{2}, \frac{1}{2} & 0, 0 \\ \hline \end{array} & (u^2, u^3) \begin{array}{c|c|c|} & a_2 & a_2' \\ \hline a_1 & 0, 1 & \frac{1}{2}, 0 \\ \hline a_1' & \frac{1}{2}, 0 & 0, 0 \\ \hline \end{array} \\ (u^3, u^2) \begin{array}{c|c|c|} & a_2 & a_2' \\ \hline a_1 & 1, 0 & 0, \frac{1}{2} \\ \hline a_1' & 0, \frac{1}{2} & 0, 0 \\ \hline \end{array} & (u^3, u^3) \begin{array}{c|c|c|} & a_2 & a_2' \\ \hline a_1 & 1, 1 & 0, 0 \\ \hline a_1' & 0, 0 & 0, 0 \\ \hline \end{array} \end{array} ,$$

and each player i believes that $\theta_i = u^2$ or u^3 with equal probability. Then player i can rationalize a_i' by the conjecture $\nu_i \in \Delta(A_j \times \Theta_i)$ such that $\nu_i(a_j, u^2) = \nu_i(a_j', u^3) = 1/2$. In fact, both a_i and a_i' are ICR actions for player i.

As illustrated by the above example, ICR depends on how we specify state spaces. In game G^T, it is (somewhat implicitly) assumed that type profiles capture everything that is relevant for utility indices. That is, conditional on t_{-i}, type t_i's utility index is given by $u_i(\cdot \mid t_i, t_{-i})$ without any further uncertainty. On the other hand, in game G^U, even after type profile (t_i, t_{-i}) is fixed, there may remain uncertainty about type t_i's utility index, which can be correlated with the opponents' actions a_{-i} to the extent that each realization of the utility index belongs to U_i, and that the "average" of the utility index is equal to $u_i(\cdot \mid t_i, t_{-i})$.

4.3 A Treatment of Explicit States

Given a preference-form game $\Gamma^{\Omega} = (I, Z, \Omega, (A_i, T_i, U_i, \pi_i)_{i\in I}, O)$ with explicit states Ω, we can define a payoff-form game $G^{U,\Omega}$ by setting $\Theta_i = \operatorname{ext} U_i \times \Omega$, $g_i^{U,\Omega}\colon A \times \operatorname{ext} U_i \times \Omega \to \mathbb{R}$ by

$$g_i^{U,\Omega}(a, u_i^k, \omega) = \sum_{z\in Z} O(z \mid a, \omega) u_i^k(z)$$

and $\mu_i\colon T_i \to \Delta(T_{-i} \times \operatorname{ext} U_i \times \Omega)$ by

$$\mu_i^{U,\Omega}(t_{-i}, u_i^k \mid t_i) = q_i(t_{-i}, \omega \mid t_i)\lambda_i^k(t_i, t_{-i}, \omega),$$

where $\pi_i(\cdot \mid t_i)$ is represented by belief $q_i(\cdot \mid t_i) \in \Delta(T_{-i}\times\Omega)$ and utility indices $\{u_i(\cdot \mid t_i, t_{-i}, \omega)\}_{t_{-i}\in T_{-i},\omega\in\Omega} \subset U_i$, and $\lambda_i^k(t_i, t_{-i}, \omega)$ is the coefficient of u_i^k to express $u_i(\cdot \mid t_i, t_{-i}, \omega)$ as a convex combination of $\{u_i^1, \ldots, u_i^{K_i}\}$.

We can show that ICR in $G^{U,\Omega}$ is equivalent to preference rationalizability in Γ^{Ω}.

Proposition 5. *If $(U_i)_{i\in I}$ are simplex restrictions, then $ICR_i(t_i; G^{U,\Omega}) = PR_i(t_i; \Gamma^{\Omega})$ for any $i \in I$ and $t_i \in T_i$.*

Proof. Similarly to the second part of Proposition 4, it follows from the isomorphisms between $\Delta(T_{-i}\times\operatorname{ext} U_i\times\Omega)$ and $P^{U_i}(T_{-i}\times\Omega)$ and between $\Delta(A_{-i}\times T_{-i}\times\operatorname{ext} U_i\times\Omega)$ and $P^{U_i}(A_{-i}\times T_{-i}\times\Omega)$ when U_i is a simplex. ∎

Thus, if there is uncertainty in the outcome function, then our approach of using an Ω-based preference-form game is equivalent to using the corresponding payoff-form game with $\Theta_i = \operatorname{ext} U_i \times \Omega$. The choice of Ω is less problematic than that of Θ_i, as the relationship between action profiles and outcomes is more "objective" than players' preferences over outcomes.

4.4 A Remark on Singleton Preference Structures

If $|T_i| = 1$ for all $i \in I$, then a preference-form game simply consists of a game form $((A_i)_{i\in I}, O)$ and each player i's preference over lotteries, represented by a utility index $u_i \in U_i$. In this case, the corresponding payoff-form game G^T is indeed a complete-information game, and ICR is equivalent to correlated rationalizability. On the other hand, the other payoff-form game G^U still contains incomplete information (unless in addition to $|T_i| = 1$, we have $|U_i| = 1$ for every $i \in I$). Thus the distinction between complete- and incomplete-information games in payoff form is not obvious and requires some care from the viewpoint of preference-form games.

5 Non-Simplex Restrictions

We can easily extend the definitions of $P^U(X)$, preference-form games and preference rationalizability to cases with non-simplex restrictions $(U_i)_{i\in I}$ by following their definitions almost verbatim.[12] In this section, we discuss whether and how our results extend to such cases.

To simplify our expositions, except when we discuss redundant states in Proposition 6, we assume away explicit states Ω. Extensions to cases with such explicit states are immediate.

5.1 Convex Polytope Restrictions

We begin with the slightest deviation from the simplex assumption, and consider cases where each U_i is a convex polytope, i.e., a convex hull of finitely many utility indices $\{u_i^1, \ldots, u_i^{K_i}\}$, but $u_i^2 - u_i^1, \ldots, u_i^{K_i} - u_i^1$ are not linearly independent. (Without loss of generality, we assume that all u_i^k are extreme points of U_i.) For example, the "square" restriction in Example 5 is not a simplex, but a convex polytope.

Then we can show that Propositions 1 and 2 hold in one direction of set inclusion.

Proposition 6. *If $(U_i)_{i\in I}$ are convex polytope restrictions and $(\varphi_i)_{i\in I}$ are preference-preserving mappings from $(T_i, \pi_i)_{i\in I}$ to $(\hat{T}_i, \hat{\pi}_i)_{i\in I}$, then $PR_i(t_i; \Gamma) \subseteq PR_i(\varphi_i(t_i); \hat{\Gamma})$ for any $i \in I$ and $t_i \in T_i$; similarly, if Ω is redundant in Ω', then $PR_i(t_i; \Gamma^{\Omega}) \subseteq PR_i(t_i; \Gamma^{\hat{\Omega}})$ for any $i \in I$ and $t_i \in T_i$.*

Proof. These results follow since the first half of the proof of Proposition 1 does not rely on the simplex assumption. ∎

The following example shows that the other direction of set inclusion does not hold for redundant types (the first half of Proposition 6). A similar counterexample can be constructed for redundant states (the second half).

Example 9. Suppose that $I = \{1, 2\}$ and $Z = \{z_1, z_2, z_3, z_4\}$. For each $k \subseteq \{2, 3, 4\}$,

$$u^k(z_l) = \begin{cases} 0 & \text{if } l \notin k, \\ 1 & \text{if } l \in k. \end{cases}$$

Player 1's utility indices are restricted to the "square" $U_1 = \text{conv}\{u^{\{2\}}, u^{\{2,3\}}, u^{\{2,4\}}, u^{\{2,3,4\}}\}$, i.e., U_1 is the set of utility indices u_1 such that $u_1(z_1) = 0$, $u_1(z_2) = 1$, and $u_1(z_3), u_1(z_4) \in [0, 1]$. Player

[12] An exception is one of the equivalent definitions of preference rationalizability that uses utility-index-contingent behavior strategies satisfying (i″) and (ii″), where $\lambda_i^k(t_i, t_{-i})$ is not uniquely defined if U_i is not a simplex.

2 has a unique utility index $U_2 = \{u^{\{2\}}\}$. Consider the following game form:

$O =$		a_2	a_2'
	a_1	z_3	z_4
	a_1'	$(\frac{1}{3}, z_1; \frac{2}{3}, z_2)$	$(\frac{1}{3}, z_1; \frac{2}{3}, z_2)$

Suppose that in a "smaller" preference structure $(\hat{T}_i, \hat{\pi}_i)_{i=1,2}$, each player has one type $\hat{T}_i = \{\hat{t}_i\}$, player 1's preference is represented by the center $\bar{u}$ of U_1 such that $(\bar{u}(z_1), \bar{u}(z_2), \bar{u}(z_3), \bar{u}(z_4)) = (0, 1, 1/2, 1/2)$, and player 2's preference is represented by $u^{\{2\}}$. In this preference-form game $\hat{\Gamma}$, all actions are preference rationalizable. To see this, note that player 1 can rationalize a_1 by conjecturing that with probability 1/2, "player 2 plays a_2 and player 1's preference is represented by $u^{\{2,3\}}$", and with the remaining probability, "player 2 plays a_2' and player 1's preference is represented by $u^{\{2,4\}}$". Player 1 can also rationalize a_1' by conjecturing that player 2 plays a_2 for sure. Player 2 is indifferent between a_2 and a_2'.

Now consider a "larger" preference-form game Γ with preference structure $(T_i, \pi_i)_{i=1,2}$, where $T_1 = \{t_1\}$ and $T_2 = \{t_2, t_2'\}$, player 1 believes that with probability 1/2, "player 2's type is t_2 and player 1's preference is represented by $u^{\{2\}}$", and with the remaining probability, "player 2's type is t_2' and player 1's preference is represented by $u^{\{2,3,4\}}$", and player 2's preference is represented by $u^{\{2\}}$ for both types t_2 and t_2'. Then although $\varphi_1(t_1) = \hat{t}_1$ and $\varphi_2(t_2) = \varphi_2(t_2') = \hat{t}_2$ are preference-preserving mappings from $(T_i, \pi_i)_{i=1,2}$ to $(\hat{T}_i, \hat{\pi}_i)_{i=1,2}$, player 1 cannot rationalize a_1 in Γ.

Similarly, without the simplex assumption, the relationship between preference-form games and payoff-form games becomes weak. In particular, given a preference-form game Γ, we cannot uniquely define a payoff-form game G^U with $\Theta_i = \operatorname{ext} U_i$. It is because each $u_i(\cdot \mid t_i, t_{-i})$ can be expressed as a convex combination of $\operatorname{ext} U_i$ in multiple ways, and hence player i's belief

$$\mu_i^U(t_{-i}, u_i^k \mid t_i) = q_i(t_{-i} \mid t_i)\lambda_i^k(t_i, t_{-i})$$

is not uniquely defined. Nevertheless, Proposition 4 holds in one direction of set inclusion. (The same is true for Proposition 5.)

Proposition 7. *If $(U_i)_{i\in I}$ are convex polytope restrictions, then $ICR_i(t_i; G^T) \subseteq ICR_i(t_i; G^U) \subseteq PR_i(t_i; \Gamma)$ for any $i \in I$, $t_i \in T_i$ and any specification of G^U.*

Proof. The first set inclusion follows since the first half of the proof of Proposition 4 does not rely on the simplex assumption. The second set inclusion follows from the (many-to-one) homomorphisms from $\Delta(T_{-i} \times \operatorname{ext} U_i)$ to $P^{U_i}(T_{-i})$ and from $\Delta(A_{-i} \times T_{-i} \times \operatorname{ext} U_i)$ to $P^{U_i}(A_{-i} \times T_{-i})$ when U_i is a convex polytope. ∎

ICR in G^U may be strictly more restrictive than preference rationalizability in Γ for any specification of G^U. This is because G^U forces us to pick a particular expression of $u_i(\cdot \mid t_i, t_{-i})$ as a convex combination of $\operatorname{ext} U_i$, which imposes a restriction on conjectures that a type can use to rationalize actions. The next example illustrates this point.

Example 10. The following example is an expansion of Example 9. Suppose that $I = \{1, 2\}$ and $Z = \{z_1, z_2, z_3, z_4, z_5\}$. For each $k \subset \{2, 3, 4, 5\}$, u^k denotes the utility index given by $u^k(z_l) = 0$ for $l \notin k$ and $u^k(z_l) = 1$ for $l \in k$. Let $U_1 = \operatorname{conv}\{u^{\{2\}}, u^{\{2,3\}}, u^{\{2,4\}}, u^{\{2,3,4\}}\}$, i.e., U_1 is the set of utility indices u_1 such that $u_1(z_1) = u_1(z_5) = 0$, $u_1(z_2) = 1$, and $u_1(z_3), u_1(z_4) \in [0, 1]$; let $U_2 = \{u^{\{5\}}\}$. Consider the following game form:

$O =$

	a_2	a_2'	a_2''
a_1	$(\frac{1}{6}, z_2; \frac{1}{3}, z_3; \frac{1}{2}, z_5)$	$(\frac{1}{2}, z_1; \frac{1}{6}, z_2; \frac{1}{3}, z_3)$	$(\frac{1}{6}, z_1; \frac{1}{6}, z_2; \frac{1}{3}, z_4; \frac{1}{3}, z_5)$
a_1'	$(\frac{2}{3}, z_1; \frac{1}{3}, z_2)$	$(\frac{1}{6}, z_1; \frac{1}{3}, z_2; \frac{1}{2}, z_5)$	$(\frac{1}{3}, z_3; \frac{1}{3}, z_4; \frac{1}{3}, z_5)$
a_1''	$(\frac{3}{7}, z_2; \frac{4}{7}, z_5)$	$(\frac{3}{7}, z_2; \frac{4}{7}, z_5)$	$(\frac{4}{7}, z_1; \frac{3}{7}, z_2)$

.

Suppose that each player has one type: player 1's preference is represented by the center $\bar{u}$ of U_1 such that $(\bar{u}(z_1), \bar{u}(z_2), \bar{u}(z_3), \bar{u}(z_4), \bar{u}(z_5)) = (0, 1, 1/2, 1/2, 0)$, which is expressed as a convex combination $\lambda(u^{\{2\}} + u^{\{2,3,4\}}) + (1/2 - \lambda)(u^{\{2,3\}} + u^{\{2,4\}})$ for any $\lambda \in [0, 1/2]$; player 2's preference is represented by $u^{\{5\}}$. In this preference-form game Γ, all actions are preference rationalizability. To see this, note that player 1 can rationalize a_1 by conjecturing that with probability 1/2, "player 2 plays a_2 and player 1's preference is represented by $u^{\{2,3\}}$", and with the remaining probability, "player 2 plays a_2'' and player 1's preference is represented by $u^{\{2,4\}}$". Similarly, player 1 can rationalize a_1' by conjecturing that with probability 1/2, "player 2 plays a_2 and player 1's preference is represented by $u^{\{2\}}$", and with the remaining probability, "player 2 plays a_2'' and player 1's preference is represented by $u^{\{2,3,4\}}$". Player 1 can rationalize a_1'' by conjecturing that player 2 plays a_2 for sure. Player 2 can rationalize a_2 and a_2' by conjecturing that player 1 plays a_1'' for sure. Also, player 2 can rationalize a_2'' by conjecturing that player 1 randomizes between a_1 and a_1' with equal probabilities.

On the other hand, in any corresponding payoff-form game G^U with $\Theta_1 = \{u^{\{2\}}, u^{\{2,3\}}, u^{\{2,4\}}, u^{\{2,3,4\}}\}$, a_1'' is the unique ICR action for player 1. To see this, note that in G^U, player 1 assigns equal probabilities, denoted by $\lambda \in [0, 1/2]$, on $u^{\{2\}}$ and on $u^{\{2,3,4\}}$, and the complimentary probabilities $1/2 - \lambda$ on $u^{\{2,3\}}$ and on $u^{\{2,4\}}$. Then in the first step of the iterative definition of ICR, a_1 can be a best response if and only if $\lambda \leq 3/14$. On the other hand, a_1' can be a best response if and only if $\lambda \geq 2/7$. Therefore, for any λ, either a_1 or a_1' must be eliminated. Since player 2 can rationalize a_2'' only by conjecturing that player 1 randomizes between a_1 and a_1', in the second step of iteration, a_2'' must be eliminated. Given this, in the third step, both a_1 and a_1' are eliminated.

5.2 No Restriction

Now we consider an extreme case where there is no restriction on utility indices. That is, a preference structure is given by $(T_i, \pi_i)_{i \in I}$ with $\pi_i\colon T_i \to P(T_{-i})$, where $P(X)$ denotes the set of all preferences $\succeq$ over $F(X)$ represented by

$$f, f' \in F(X), \quad f \succeq f' \Leftrightarrow \sum_{x,z} f(z \mid x) w(x,z) \geq \sum_{x,z} f'(z \mid x) w(x,z)$$

for some $w\colon X \times Z \to \mathbb{R}$.[13] In this case, type t_i has a preference over $F(T_{-i})$ represented by $\bar{w}_i\colon T_{-i} \times Z \to \mathbb{R}$, and rationalizes an action by a preference over $F(A_{-i} \times T_{-i})$ represented by $w_i\colon A_{-i} \times T_{-i} \times Z \to \mathbb{R}$ such that $\sum_{a_{-i}} w_i(a_{-i}, t_{-i}, z) = \bar{w}_i(t_{-i}, z)$.

We argue that if there is no restriction on utility indices, preference rationalizability is oddly behaved. Indeed, the next example illustrates that the set of preference rationalizable actions is not upper hemicontinuous with respect to the outcome function. To see why, note that even if $\sum_{a_{-i}} w_i(a_{-i}, t_{-i}, z) = \bar{w}_i(t_{-i}, z)$ is fixed, the marginal rate of substitution $(w_i(a_{-i}, t_{-i}, z) - w_i(a_{-i}, t_{-i}, z''))/(w_i(a_{-i}, t_{-i}, z') - w_i(a_{-i}, t_{-i}, z''))$ is unbounded and depends on a_{-i} almost arbitrarily.

Example 11. Suppose that $I = \{1, 2\}$ and $Z = \{z_1, z_2\}$. Consider the following game form:

$O =$	a_2	a_2'
a_1	z_1	$(p, z_1; 1-p, z_2)$
a_1'	$(p, z_1; 1-p, z_2)$	z_2

Suppose that each player has only one type, who strictly prefers z_2 to z_1. In this preference-form game, a_i is preference rationalizable if and only if $p \neq 1/2$. (a_i' is always preference rationalizable.) To see this, note that if $p \neq 1/2$, then player i forms a preference $\succeq_i \in P(\{a_j, a_j'\})$ represented by $w_i(a_j, z_1) = w_i(a_j', z_1) = 0$, $w_i(a_j, z_2) = (1+p)/(2p-1)$ and $w_i(a_j', z_2) = (p-2)/(2p-1)$. Since $w_i(a_j, z_2) + w_i(a_j', z_2) > w_i(a_j, z_1) + w_i(a_j', z_1)$, $\succeq_i$ is consistent with player i's preference over lotteries. Also, since $w_i(a_j, z_1) + p w_i(a_j', z_1) + (1-p) w_i(a_j', z_2) > p w_i(a_j, z_1) + (1-p) w_i(a_j, z_2) + w_i(a_j', z_2)$, player i can rationalize a_i.

If $p = 1/2$, then any preference $\succeq_i \in P(A_j)$ prefers a_i' to a_i if $\succeq_i$ is consistent with player i's preference over lotteries. It is because $w_i(a_j, z_1) + (w_i(a_j', z_1) + w_i(a_j', z_2))/2 < (w_i(a_j, z_1) + w_i(a_j, z_2))/2 + w_i(a_j', z_2)$ whenever $w_i(a_j, z_2) + w_i(a_j', z_2) > w_i(a_j, z_1) + w_i(a_j', z_1)$.

Also, we can show that the iteration procedure of preference rationalizability always stops in the first step, and hence preference rationalizability for type t_i depends only on the restriction of $\pi_i(t_i)$ to lotteries. These results are derived from the following dual characterization.

[13] One can interpret $w(x,z)$ as the product of belief $q(x)$ and utility $u(z \mid x)$ although such decomposition is not unique.

Proposition 8. *Suppose that $U_i = \mathbb{R}^Z$ for any $i \in I$.*

1. *We have $a_i \notin PR_i^1(t_i)$ if and only if there exists $\alpha_i \in \Delta(A_i)$ such that*

   (a) *for any $z \in Z$, $O(z \mid \alpha_i, a_{-i}) - O(z \mid a_i, a_{-i})$ is independent of $a_{-i} \in A_{-i}$, and*

   (b) *$\pi_i(t_i)$ strictly prefers $O(\cdot \mid \alpha_i, a_{-i})$ to $O(\cdot \mid a_i, a_{-i})$ for some (and hence for all) $a_{-i} \in A_{-i}$.*

2. *$PR_i(t_i) = PR_i^1(t_i)$.*

3. *If $\pi_i(t_i)$ and $\pi_i(t_i')$ induce the same preference over lotteries, then $PR_i(t_i) = PR_i(t_i')$.*

Proof. For part 1, the if direction is immediate. To show the only-if direction, let $\pi_i(t_i)$ be represented by $\bar{w}_i \colon T_{-i} \times Z \to \mathbb{R}$. If $a_i \notin PR_i^1(t_i)$, then there is no $w_i \colon A_{-i} \times T_{-i} \times Z \to \mathbb{R}$ such that

$$\sum_{a_{-i}} w_i(a_{-i}, t_{-i}, z) = \bar{w}_i(t_{-i}, z) \qquad \text{for all } t_{-i}, z,$$

$$\sum_{a_{-i}, t_{-i}, z} (O(z \mid a_i, a_{-i}) - O(z \mid a_i', a_{-i})) w_i(a_{-i}, t_{-i}, z) \geq 0 \qquad \text{for all } a_i'.$$

By Farkas' lemma, there exist $h \colon T_{-i} \times Z \to \mathbb{R}$ and $\alpha_i \in \Delta(A_i)$ such that

$$h(z \mid t_{-i}) - (O(z \mid \alpha_i, a_{-i}) - O(z \mid a_i, a_{-i})) = 0 \qquad \text{for all } t_{-i}, a_{-i}, z,$$

$$\sum_{t_{-i}, z} h(z \mid t_{-i}) \bar{w}_i(t_{-i}, z) > 0.$$

(Indeed, $h(z \mid t_{-i})$ is independent of t_{-i}.) Thus $O(\cdot \mid \alpha_i, a_{-i}) - O(\cdot \mid a_i, a_{-i})$ is independent of a_{-i}, and $\pi_i(t_i)$ strictly prefers $O(\cdot \mid \alpha_i, a_{-i})$ to $O(\cdot \mid a_i, a_{-i})$.

For part 2, fix any player $i \in I$. For each $j \neq i$ and $t_j \in T_j$, if $a_j \in PR_j^1(t_j)$, then let $\sigma_j(\cdot \mid a_j, t_j)$ be the point mass on a_j. If $a_j \notin PR_j^1(t_j)$, then by part 1, there exists $\sigma_j(\cdot \mid a_j, t_j) \in \Delta(A_j)$ such that for any $z \in Z$, $O(z \mid \sigma_j(\cdot \mid a_j, t_j), a_{-j}) - O(z \mid a_j, a_{-j})$ is independent of a_{-j}. Without loss of generality, we assume that $\sigma_j(\cdot \mid a_j, t_j) \in \Delta(PR_j^1(t_j))$. For each $a_{-i} \in A_{-i}$ and $t_{-i} \in T_{-i}$, define $\sigma_{-i}(\cdot \mid a_{-i}, t_{-i}) \in \Delta(PR_{-i}^1(t_{-i}))$ by $\sigma_{-i}(a_{-i}' \mid a_{-i}, t_{-i}) = \prod_{j \neq i} \sigma_j(a_j' \mid a_j, t_j)$ for each $a_{-i}' \in PR_{-i}^1(t_{-i})$.

Pick any $t_i \in T_i$ and $a_i \in PR_i^1(t_i)$, which is a best response under state-dependent utility index $w_i \colon A_{-i} \times T_{-i} \times Z \to \mathbb{R}$ and that $\sum_{a_{-i}} w_i(a_{-i}, t_{-i}, z)$ represents $\pi_i(t_i)$. We will show that a_i is a best response in the second step of iteration.

Define a state-dependent utility index $w_i'\colon A_{-i} \times T_{-i} \times Z \to \mathbb{R}$ by

$$w_i'(a_{-i}', t_{-i}, z) = \sum_{a_{-i}} \sigma_{-i}(a_{-i}' \mid a_{-i}, t_{-i}) w_i(a_{-i}, t_{-i}, z)$$

for $a_{-i}' \in A_{-i}$, $t_{-i} \in T_{-i}$ and $z \in Z$. (i) Since $\sigma_{-i}(\cdot \mid a_{-i}, t_{-i}) \in \Delta(PR_{-i}^1(t_{-i}))$ for any $a_{-i} \in A_{-i}$ and $t_{-i} \in T_{-i}$, we have $a_{-i} \in PR_{-i}^1(t_{-i})$ whenever (a_{-i}, t_{-i}) is non-null under the preference represented by w_i', (ii)

$$\sum_{a_{-i}'} w_i'(a_{-i}', t_{-i}, z) = \sum_{a_{-i}, a_{-i}'} \sigma_{-i}(a_{-i}' \mid a_{-i}, t_{-i}) w_i(a_{-i}, t_{-i}, z) = \sum_{a_{-i}} w_i(a_{-i}, t_{-i}, z),$$

which represents $\pi_i(t_i)$, and (iii) for any $a_i' \in A_i$,

$$\begin{aligned}
&\sum_{a_{-i}'} O(z \mid a_i', a_{-i}') w_i'(a_{-i}', t_{-i}, z) \\
&= \sum_{a_{-i}, a_{-i}'} O(z \mid a_i', a_{-i}') \sigma_{-i}(a_{-i}' \mid a_{-i}, t_{-i}) w_i(a_{-i}, t_{-i}, z) \\
&= \sum_{a_{-i}} O(z \mid a_i', \sigma_{-i}(\cdot \mid a_{-i}, t_{-i})) w_i(a_{-i}, t_{-i}, z) \\
&= \sum_{a_{-i}} (O(z \mid a_i', a_{-i}) + h(z \mid a_{-i}, t_{-i})) w_i(a_{-i}, t_{-i}, z) \\
&= \sum_{a_{-i}} O(z \mid a_i', a_{-i}) w_i(a_{-i}, t_{-i}, z) + \sum_{a_{-i}} h(z \mid a_{-i}, t_{-i}) w_i(a_{-i}, t_{-i}, z),
\end{aligned}$$

for any $t_{-i} \in T_{-i}$ and $z \in Z$, where $h(z \mid a_{-i}, t_{-i}) := O(z \mid a_i', \sigma_{-i}(\cdot \mid a_{-i}, t_{-i})) - O(z \mid a_i', a_{-i})$ is independent of a_i' by the definition of $\sigma_{-i}(\cdot \mid a_{-i}, t_{-i})$. Since a_i is a best response with respect to w_i, it is also a best response with respect to w_i'.

Part 3 follows from part 2. ∎

Similarly to Example 11, Part 1 of Proposition 8 depends on the unbounded sensitivity of preferences with respect to other players' actions. For parts 2 and 3 of Proposition 8, it is important that under the no-restriction assumption, all outcomes are "public" in the sense that players care about the entire distribution over Z. These results no longer hold if $Z = Z_0 \times \prod_i Z_i$, the product of the set Z_0 of public outcomes and the sets Z_i of private outcomes, and player i is assumed to care only about the marginal distribution over $Z_0 \times Z_i$.

6 Conclusion

We introduced preference-form games, developed the notion of preference rationalizability, and showed that preference rationalizability is invariant to redundant types and states.

Throughout the paper, we restricted ourselves to the Anscombe-Aumann framework and assumed state-dependent expected utilities. One could extend our framework to the Savage framework or to non-expected-utility preferences, and relax various state independence or monotonicity conditions assumed in Epstein (1997), Lo (2000) and Chen and Luo (2010).[14] It is yet to be seen whether such an extension would generate fruitful theoretical results and applications.

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