

14.452 Economic Growth: Lectures 5 and 6, Neoclassical Growth

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Foundations of Neoclassical Growth

- Solow model: constant saving rate.
- Ramsey or Cass-Koopmans model: differs from the Solow model only because it explicitly models the consumer side and endogenizes savings. This model specifies the *preference orderings* of individuals and derives their decisions from these preferences. It also
 - Enables better understanding of the factors that affect savings decisions.
 - Enables to discuss the “optimality” of equilibria
 - Clarifies whether the (competitive) equilibria of growth models can be “improved upon”.
- Beyond its use as a basic growth model, also a workhorse for many areas of macroeconomics.

Preliminaries

- Consider an economy consisting of a unit measure of infinitely-lived households.
- I.e., an uncountable number of households: e.g., the set of households \mathcal{H} could be represented by the unit interval $[0, 1]$.
- Emphasize that each household is infinitesimal and will have no effect on aggregates.
- Can alternatively think of \mathcal{H} as a countable set of the form $\mathcal{H} = \{1, 2, \dots, M\}$ with $M = \infty$, without any loss of generality.
- Advantage of unit measure: averages and aggregates are the same
- Simpler to have \mathcal{H} as a finite set in the form $\{1, 2, \dots, M\}$ with M large but finite.
- Acceptable for many models, but with overlapping generations require the set of households to be infinite.

Time Separable Preferences

- Standard assumptions on preference orderings so that they can be represented by utility functions.
- In addition, **time separable preferences**: each household i has an *instantaneous (Bernoulli) utility function* (or felicity function):

$$u_i(c_i(t)),$$

- $u_i : \mathbb{R}_+ \rightarrow \mathbb{R}$ is increasing and concave and $c_i(t)$ is the consumption of household i .
- Note instantaneous utility function is *not* specifying a complete preference ordering over all commodities—here consumption levels in all dates.
- Instead, household i preferences at time $t = 0$ are obtained by combining this with *exponential* discounting.

Infinite Horizon and the Representative Household

- Given by the von Neumann-Morgenstern expected utility function

$$\mathbb{E}_0^i \sum_{t=0}^T \beta_i^t u_i(c_i(t)), \quad (1)$$

where $\beta_i \in (0, 1)$ is the discount factor of household i , where $T < \infty$ or $T = \infty$, corresponding to finite planning horizon (e.g., with overlapping generations) or infinite planning horizon.

- The second is often assumed because the standard approach in macroeconomics is to impose the existence of a *representative household*—costs of this to be discussed below.
- Exponential discounting and time separability also ensure “time-consistent” behavior.
- A solution $\{x(t)\}_{t=0}^T$ (possibly with $T = \infty$) is *time consistent* if:
 - whenever $\{x(t)\}_{t=0}^T$ is an optimal solution starting at time $t = 0$, $\{x(t)\}_{t=t'}^T$ is an optimal solution to the continuation dynamic optimization problem starting from time $t = t' \in [0, T]$.

Challenges to the Representative Household

- An economy *admits a representative household* if preference side can be represented *as if* a single household made the aggregate consumption and saving decisions subject to a single budget constraint.
- This description concerning a representative household is purely positive
- Stronger notion of “normative” representative household: if we can also use the utility function of the representative household for welfare comparisons.
- Simplest case that will lead to the existence of a representative household: suppose each household is identical.
- But we can do a little better than that.
- I now discuss foundations of representative household (which are quite restrictive), and in the rest of the course I will impose the existence of a representative household, though this should always be taken with a grain of salt (and a lot of doubt).

Representative Household II

- If instead households are not identical but assume can model *as if* demand side generated by the optimization decision of a representative household:
- More realistic, but:
 - ① The representative household will have positive, but not always a normative meaning.
 - ② Models with heterogeneity: often not lead to behavior that can be represented as if generated by a representative household.

Theorem (Debreu-Mantel-Sonnenschein Theorem) Let $\varepsilon > 0$ be a scalar and $N < \infty$ be a positive integer. Consider a set of prices $\mathbf{P}_\varepsilon = \{p \in \mathbb{R}_+^N : p_j / p_{j'} \geq \varepsilon \text{ for all } j \text{ and } j'\}$ and any continuous function $\mathbf{x} : \mathbf{P}_\varepsilon \rightarrow \mathbb{R}_+^N$ that satisfies Walras' Law and is homogeneous of degree 0. Then there exists an exchange economy with N commodities and $H < \infty$ households, where the aggregate demand is given by $\mathbf{x}(p)$ over the set \mathbf{P}_ε .

Representative Household IV

- That excess demands come from optimizing behavior of households puts no restrictions on the form of these demands.
 - E.g., $\mathbf{x}(p)$ does not necessarily possess a negative-semi-definite Jacobian or satisfy the weak axiom of revealed preference (requirements of demands generated by individual households).
- Hence without imposing further structure, impossible to derive specific $\mathbf{x}(p)$'s from the maximization behavior of a single household.
- Severe warning against the use of the representative household assumption.
- Partly an outcome of very strong income effects:
 - special but approximately realistic preference functions, and restrictions on distribution of income rule out arbitrary aggregate excess demand functions.

Gorman Aggregation

- Recall an indirect utility function for household i , $v_i(p, y^i)$, specifies (ordinal) utility as a function of the price vector $p = (p_1, \dots, p_N)$ and household's income y^i .
- $v_i(p, y^i)$: homogeneous of degree 0 in p and y .

Theorem (Gorman's Aggregation Theorem) Consider an economy with a finite number $N < \infty$ of commodities and a set \mathcal{H} of households. Suppose that the preferences of household $i \in \mathcal{H}$ can be represented by an indirect utility function of the form

$$v^i(p, y^i) = a^i(p) + b(p)y^i, \quad (2)$$

then these preferences can be aggregated and represented by those of a representative household, with indirect utility

$$v(p, y) = \int_{i \in \mathcal{H}} a^i(p) di + b(p)y,$$

where $y \equiv \int_{i \in \mathcal{H}} y^i di$ is aggregate income.

Linear Engel Curves

- Demand for good j (from Roy's identity):

$$x_j^i(p, y^i) = -\frac{1}{b(p)} \frac{\partial a^i(p)}{\partial p_j} - \frac{1}{b(p)} \frac{\partial b(p)}{\partial p_j} y^i.$$

- Thus linear Engel curves.
- “Indispensable” for the existence of a representative household.
- Let us say that there exists a *strong representative household* if redistribution of income or endowments across households does not affect the demand side.
- Gorman preferences are sufficient for a strong representative household.
- Moreover, they are also *necessary* (with the same $b(p)$ for all households) for the economy to admit a strong representative household.
 - The proof is easy by a simple variation argument.

Importance of Gorman Preferences

- Gorman Preferences limit the **extent of income effects** and enables the aggregation of individual behavior.
- Integral is “Lebesgue integral,” so when \mathcal{H} is a finite or countable set, $\int_{i \in \mathcal{H}} y^i di$ is indeed equivalent to the summation $\sum_{i \in \mathcal{H}} y^i$.
- Stated for an economy with a finite number of commodities, but can be generalized for infinite or even a continuum of commodities.
- Note all we require is there exists a monotonic transformation of the indirect utility function that takes the form in (2)—as long as no uncertainty.
- Contains some commonly-used preferences in macroeconomics.
- Gorman preferences also imply the existence of a **normative representative household**, i.e., a representative household such that if it prefers an allocation to another, then the former allocation Pareto dominates the latter.

Preferences, Technology and Demographics I

- Infinite-horizon, continuous time.
- Representative household with instantaneous utility function

$$u(c(t)), \quad (3)$$

Assumption $u(c)$ is strictly increasing, concave, twice continuously differentiable with derivatives u' and u'' , and satisfies the following Inada type assumptions:

$$\lim_{c \rightarrow 0} u'(c) = \infty \text{ and } \lim_{c \rightarrow \infty} u'(c) = 0.$$

- Suppose representative household represents set of identical households (normalized to 1).
- Each household has an instantaneous utility function given by (3).
- $L(0) = 1$ and

$$L(t) = \exp(nt). \quad (4)$$

Preferences, Technology and Demographics II

- All members of the household supply their labor inelastically.
- Objective function of each household at $t = 0$:

$$U(0) \equiv \int_0^{\infty} \exp(-(\rho - n)t) u(c(t)) dt, \quad (5)$$

where $c(t)$ = consumption per capita at t , and ρ = subjective discount rate, and effective discount rate is $\rho - n$.

- Continues time analogue of $\sum_{t=0}^{\infty} \beta_i^t u_i(c_i(t))$.
- Objective function (5) embeds:
 - Household is fully altruistic towards all of its future members, and makes allocations of consumption (among household members) cooperatively.
 - Strict concavity of $u(\cdot)$
- Thus each household member will have an equal consumption

$$c(t) \equiv \frac{C(t)}{L(t)}$$

Preferences, Technology and Demographics III

- Utility of $u(c(t))$ per household member at time t , total of $L(t) u(c(t)) = \exp(nt) u(c(t))$.
- With discount at rate of $\exp(-\rho t)$, obtain (5).

ASSUMPTION 4'.

$$\rho > n.$$

- Ensures that in the model without growth, discounted utility is finite (otherwise infinite utility and not well behaved equilibrium). Will strengthen it in model with growth.
- Start model without any technological progress.
- Factor and product markets are competitive.
- Production possibilities set of the economy is represented by

$$Y(t) = F[K(t), L(t)],$$

- Standard constant returns to scale and Inada assumptions still hold.

Preferences, Technology and Demographics IV

- Per capita production function $f(\cdot)$

$$\begin{aligned} y(t) &\equiv \frac{Y(t)}{L(t)} \\ &= F\left[\frac{K(t)}{L(t)}, 1\right] \\ &\equiv f(k(t)), \end{aligned}$$

where, as before,

$$k(t) \equiv \frac{K(t)}{L(t)}. \quad (6)$$

- Competitive factor markets then imply:

$$R(t) = F_K[K(t), L(t)] = f'(k(t)). \quad (7)$$

and

$$w(t) = F_L[K(t), L(t)] = f(k(t)) - k(t)f'(k(t)). \quad (8)$$

Preferences, Technology and Demographics V

- Denote asset holdings of the representative household at time t by $\mathcal{A}(t)$. Then,

$$\dot{\mathcal{A}}(t) = r(t) \mathcal{A}(t) + w(t) L(t) - c(t) L(t)$$

- $r(t)$ is the risk-free market flow rate of return on assets, and $w(t) L(t)$ is the flow of labor income earnings of the household.
- Defining per capita assets as

$$a(t) \equiv \frac{\mathcal{A}(t)}{L(t)},$$

we obtain:

$$\dot{a}(t) = (r(t) - n) a(t) + w(t) - c(t). \quad (9)$$

- Household assets can consist of capital stock, $K(t)$, which they rent to firms and government bonds, $B(t)$.

Preferences, Technology and Demographics VI

- With uncertainty, households would have a portfolio choice between $K(t)$ and riskless bonds.
- With incomplete markets, bonds allow households to smooth idiosyncratic shocks. But for now no need.
- Thus, market clearing \Rightarrow

$$a(t) = k(t).$$

- No uncertainty depreciation rate of δ implies

$$r(t) = R(t) - \delta. \tag{10}$$

The Budget Constraint

- The differential equation

$$\dot{a}(t) = (r(t) - n) a(t) + w(t) - c(t)$$

is a flow constraint

- Not sufficient as a proper budget constraint unless we impose a lower bound on assets.
- Three options:
 - 1 Lower bound on assets such as $a(t) \geq 0$ for all t
 - 2 Natural debt limit.
 - 3 No Ponzi Game Condition.
- The last two equivalent as long of the natural debt limit is specified properly. Let us focus on the latter.

The No Ponzi Game Condition

- Infinite-horizon no Ponzi game condition is:

$$\lim_{t \rightarrow \infty} a(t) \exp \left(- \int_0^t (r(s) - n) ds \right) \geq 0. \quad (11)$$

- Transversality condition ensures individual would never want to have positive wealth asymptotically, so no Ponzi game condition can be strengthened to (though not necessary in general):

$$\lim_{t \rightarrow \infty} a(t) \exp \left(- \int_0^t (r(s) - n) ds \right) = 0. \quad (12)$$

Understanding the No Ponzi Game Condition

- Why?
- Write the single budget constraint of the form:

$$\int_0^T c(t) L(t) \exp\left(\int_t^T r(s) ds\right) dt + \mathcal{A}(T) \quad (13)$$

$$= \int_0^T w(t) L(t) \exp\left(\int_t^T r(s) ds\right) dt + \mathcal{A}(0) \exp\left(\int_0^T r(s) ds\right).$$

- Differentiating with respect to T and dividing $L(t)$ gives (9).
- Now imagine that (13) applies to a finite-horizon economy .
- Flow budget constraint (9) by itself does not guarantee that $\mathcal{A}(T) \geq 0$.
- Thus in finite-horizon we would simply impose (13) as a boundary condition.
- The no Ponzi game condition is the infinite horizon equivalent of this (obtained by dividing by $L(t)$ and multiplying both sides by $\exp\left(-\int_0^T r(s) ds\right)$ and taking the limit as $T \rightarrow \infty$).

Definition of Equilibrium

Definition A competitive equilibrium of the Ramsey economy consists of paths $[C(t), K(t), w(t), R(t)]_{t=0}^{\infty}$, such that the representative household maximizes its utility given initial capital stock $K(0)$ and the time path of prices $[w(t), R(t)]_{t=0}^{\infty}$, and all markets clear.

- Notice refers to the entire path of quantities and prices, not just steady-state equilibrium.

Definition A competitive equilibrium of the Ramsey economy consists of paths $[c(t), k(t), w(t), R(t)]_{t=0}^{\infty}$, such that the representative household maximizes (5) subject to (9) and (11) given initial capital-labor ratio $k(0)$, factor prices $[w(t), R(t)]_{t=0}^{\infty}$ as in (7) and (8), and the rate of return on assets $r(t)$ given by (10).

Household Maximization I

- Maximize (5) subject to (9) and (12).
- First ignore (12) and set up the current-value Hamiltonian:

$$\hat{H}(a, c, \mu) = u(c(t)) + \mu(t) [w(t) + (r(t) - n)a(t) - c(t)],$$

- *Maximum Principle* \Rightarrow “candidate solution”

$$\begin{aligned}\hat{H}_c(a, c, \mu) &= u'(c(t)) - \mu(t) = 0 \\ \hat{H}_a(a, c, \mu) &= \mu(t)(r(t) - n) \\ &= -\dot{\mu}(t) + (\rho - n)\mu(t)\end{aligned}$$

$$\lim_{t \rightarrow \infty} [\exp(-(\rho - n)t) \mu(t) a(t)] = 0.$$

and the transition equation (9).

- Notice transversality condition is written in terms of the current-value costate variable.

Household Maximization II

- For any $\mu(t) > 0$, $\hat{H}(a, c, \mu)$ is a concave function of (a, c) and strictly concave in c .
- The first necessary condition implies $\mu(t) > 0$ for all t .
- Therefore, *Sufficient Conditions* imply that the candidate solution is an optimum (is it unique?)
- Rearrange the second condition:

$$\frac{\dot{\mu}(t)}{\mu(t)} = -(r(t) - \rho), \quad (14)$$

- First necessary condition implies,

$$u'(c(t)) = \mu(t). \quad (15)$$

Household Maximization III

- Differentiate with respect to time and divide by $\mu(t)$,

$$\frac{u''(c(t)) c(t) \dot{c}(t)}{u'(c(t)) c(t)} = \frac{\dot{\mu}(t)}{\mu(t)}.$$

- Substituting into (14) gives

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))} (r(t) - \rho) \quad (16)$$

where

$$\varepsilon_u(c(t)) \equiv -\frac{u''(c(t)) c(t)}{u'(c(t))} \quad (17)$$

is the elasticity of the marginal utility $u'(c(t))$ or the inverse of the *intertemporal elasticity of substitution*.

- Consumption will grow over time when the discount rate is less than the rate of return on assets.

Household Maximization IV

- Integrating (14),

$$\begin{aligned}\mu(t) &= \mu(0) \exp\left(-\int_0^t (r(s) - \rho) ds\right) \\ &= u'(c(0)) \exp\left(-\int_0^t (r(s) - \rho) ds\right),\end{aligned}$$

- Substituting into the transversality condition,

$$\begin{aligned}0 &= \lim_{t \rightarrow \infty} \left[\exp(-(\rho - n)t) a(t) u'(c(0)) \exp\left(-\int_0^t (r(s) - \rho) ds\right) \right] \\ 0 &= \lim_{t \rightarrow \infty} \left[a(t) \exp\left(-\int_0^t (r(s) - n) ds\right) \right].\end{aligned}$$

- Thus the “strong version” of the no-Ponzi condition, (12) has to hold.

Household Maximization V

- Since $a(t) = k(t)$, transversality condition is also equivalent to

$$\lim_{t \rightarrow \infty} \left[\exp \left(- \int_0^t (r(s) - n) ds \right) k(t) \right] = 0$$

- Notice term $\exp \left(- \int_0^t r(s) ds \right)$ is a present-value factor: converts a unit of income at t to a unit of income at 0.
- When $r(s) = r$, factor would be $\exp(-rt)$. More generally, define an average interest rate between dates 0 and t given by $\frac{1}{t} \int_0^t r(s) ds$.

Equilibrium Prices

- Equilibrium prices given by (7) and (8).
- Thus market rate of return for consumers, $r(t)$, is given by (10), i.e.,

$$r(t) = f'(k(t)) - \delta.$$

- Substituting this into the consumer's problem, we have

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))} (f'(k(t)) - \delta - \rho) \quad (18)$$

Optimal Growth I

- In an economy that admits a representative household, optimal growth involves maximization of utility of representative household subject to technology and feasibility constraints:

$$\max_{[k(t), c(t)]_{t=0}^{\infty}} \int_0^{\infty} \exp(-(\rho - n)t) u(c(t)) dt,$$

subject to

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t),$$

and $k(0) > 0$.

- Versions of the First and Second Welfare Theorems for economies with a continuum of commodities: solution to this problem should be the same as the equilibrium growth problem.
- But straightforward to show the equivalence of the two problems.

Optimal Growth II

- Again set up the current-value Hamiltonian:

$$\hat{H}(k, c, \mu) = u(c(t)) + \mu(t) [f(k(t)) - (n + \delta)k(t) - c(t)],$$

- Candidate solution from the *Maximum Principle*:

$$\hat{H}_c(k, c, \mu) = 0 = u'(c(t)) - \mu(t),$$

$$\begin{aligned} \hat{H}_k(k, c, \mu) &= -\dot{\mu}(t) + (\rho - n)\mu(t) \\ &= \mu(t) (f'(k(t)) - \delta - n), \end{aligned}$$

$$\lim_{t \rightarrow \infty} [\exp(-(\rho - n)t) \mu(t) k(t)] = 0.$$

- *Sufficiency Theorem* \Rightarrow unique solution (\hat{H} and thus the maximized Hamiltonian strictly concave in k).

Optimal Growth III

- Repeating the same steps as before, these imply

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))} (f'(k(t)) - \delta - \rho),$$

which is identical to (18), and the transversality condition

$$\lim_{t \rightarrow \infty} \left[k(t) \exp \left(- \int_0^t (f'(k(s)) - \delta - n) ds \right) \right] = 0,$$

which is, in turn, identical to (12).

- Thus the competitive equilibrium is a Pareto optimum and that the Pareto allocation can be decentralized as a competitive equilibrium.

Proposition In the neoclassical growth model described above, with standard assumptions on the production function (assumptions 1-4'), the equilibrium is Pareto optimal and coincides with the optimal growth path maximizing the utility of the representative household.

Steady-State Equilibrium I

- Steady-state equilibrium is defined as an equilibrium path in which capital-labor ratio, consumption and output are constant, thus:

$$\dot{c}(t) = 0.$$

- From (18), as long as $f(k^*) > 0$, *irrespective* of the exact utility function, we must have a capital-labor ratio k^* such that

$$f'(k^*) = \rho + \delta. \quad (19)$$

- Pins down the steady-state capital-labor ratio only as a function of the production function, the discount rate and the depreciation rate.
- Modified golden rule*: level of the capital stock that does not maximize steady-state consumption, because earlier consumption is preferred to later consumption.

Steady-State Equilibrium II

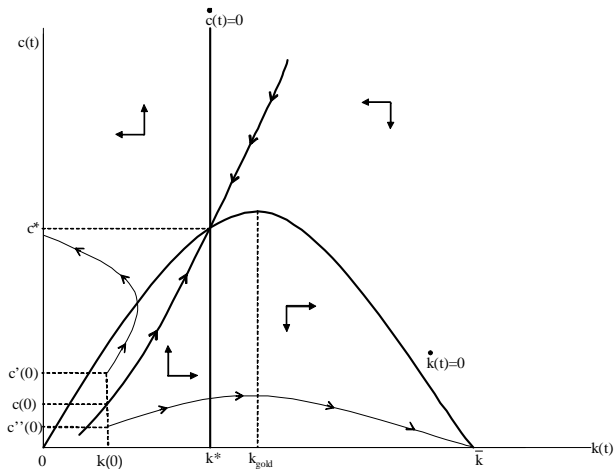


Figure: Steady state in the baseline neoclassical growth model

Steady-State Equilibrium III

- Given k^* , steady-state consumption level:

$$c^* = f(k^*) - (n + \delta)k^*, \quad (20)$$

- Given Assumption 4', a steady state where the capital-labor ratio and thus output are constant necessarily satisfies the transversality condition.

Proposition In the neoclassical growth model described above, with Assumptions 1, 2, assumptions on utility above and Assumption 4', the steady-state equilibrium capital-labor ratio, k^* , is uniquely determined by (19) and is independent of the utility function. The steady-state consumption per capita, c^* , is given by (20).

- Comparative statics again straightforward.

Steady-State Equilibrium IV

- Instead of the saving rate, it is now the discount factor that affects the rate of capital accumulation.
- Loosely, lower discount rate implies greater patience and thus greater savings.
- Without technological progress, the steady-state saving rate can be computed as

$$s^* = \frac{\delta k^*}{f(k^*)}. \quad (21)$$

- Rate of population growth has no impact on the steady state capital-labor ratio, which contrasts with the basic Solow model.
 - result depends on the way in which intertemporal discounting takes place.
- k^* and thus c^* do *not* depend on the instantaneous utility function $u(\cdot)$.
 - form of the utility function only affects the transitional dynamics
 - not true when there is technological change,

Transitional Dynamics I

- Equilibrium is determined by two differential equations:

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t) \quad (22)$$

and

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))} (f'(k(t)) - \delta - \rho). \quad (23)$$

- Moreover, we have an initial condition $k(0) > 0$, also a boundary condition at infinity,

$$\lim_{t \rightarrow \infty} \left[k(t) \exp \left(- \int_0^t (f'(k(s)) - \delta - n) ds \right) \right] = 0.$$

Transitional Dynamics II

- Appropriate notion of *saddle-path stability*:
 - consumption level (or equivalently μ) is the control variable, and $c(0)$ (or $\mu(0)$) is free: has to adjust to satisfy transversality condition
 - since $c(0)$ or $\mu(0)$ can jump to any value, need that there exists a one-dimensional manifold tending to the steady state (*stable arm*).
 - If there were more than one path equilibrium would be indeterminate.
- Economic forces are such that indeed there will be a one-dimensional manifold of stable solutions tending to the unique steady state.
- See Figure.

Transitional Dynamics III

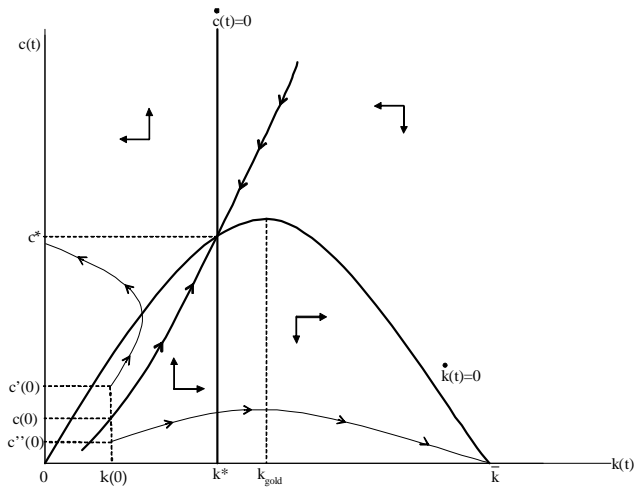


Figure: Transitional dynamics in the baseline neoclassical growth model

Transitional Dynamics: Sufficiency

- Why is the stable arm unique?
- Three different (complementary) lines of analysis
 - 1 Sufficiency Theorem
 - 2 Global Stability Analysis
 - 3 Local Stability Analysis
- *Sufficiency Theorem*: solution starting in $c(0)$ and limiting to the steady state satisfies the necessary and sufficient conditions, and thus unique solution to household problem and unique equilibrium.

Proposition In the neoclassical growth model described above, with Assumptions 1, 2, assumptions on utility above and Assumption 4', there exists a unique equilibrium path starting from any $k(0) > 0$ and converging to the unique steady-state (k^*, c^*) with k^* given by (19). Moreover, if $k(0) < k^*$, then $k(t) \uparrow k^*$ and $c(t) \uparrow c^*$, whereas if $k(0) > k^*$, then $k(t) \downarrow k^*$ and $c(t) \downarrow c^*$.

Global Stability Analysis

- Alternative argument:
 - if $c(0)$ started below it, say $c''(0)$, consumption would reach zero, thus capital would accumulate continuously until the maximum level of capital (reached with zero consumption) $\bar{k} > k_{gold}$. This would violate the transversality condition. Can be established that transversality condition necessary in this case, thus such paths can be ruled out.
 - if $c(0)$ started above this stable arm, say at $c'(0)$, the capital stock would reach 0 in finite time, while consumption would remain positive. But this would violate feasibility (a little care is necessary with this argument, since necessary conditions do not apply at the boundary).

Local Stability Analysis I

- Linearize the set of differential equations, and looking at their eigenvalues.
- Recall the two differential equations:

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t)$$

and

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))} (f'(k(t)) - \delta - \rho).$$

- Linearizing these equations around the steady state (k^*, c^*) , we have (suppressing time dependence)

$$\begin{aligned} \dot{k} &= \text{constant} + (f'(k^*) - n - \delta)(k - k^*) - c \\ \dot{c} &= \text{constant} + \frac{c^* f''(k^*)}{\varepsilon_u(c^*)} (k - k^*). \end{aligned}$$

Local Stability Analysis II

- From (19), $f'(k^*) - \delta = \rho$, so the eigenvalues of this two-equation system are given by the values of ζ that solve the following quadratic form:

$$\det \begin{pmatrix} \rho - n - \zeta & -1 \\ \frac{c^* f''(k^*)}{\varepsilon_u(c^*)} & 0 - \zeta \end{pmatrix} = 0.$$

- Since $c^* f''(k^*) / \varepsilon_u(c^*) < 0$, there are two real eigenvalues, one negative and one positive.
- Thus local analysis also leads to the same conclusion, but can only establish local stability.

Neoclassical Growth Model in Discrete Time

- Economically, nothing is different in discrete time.
- Mathematically, a few details need to be sorted out.
- Sometimes discrete time will be more convenient to work with, and sometimes continuous time.
- See recitation for details of the discrete time model.

Technological Change and the Neoclassical Model

- Extend the production function to:

$$Y(t) = F[K(t), A(t)L(t)], \quad (24)$$

where

$$A(t) = \exp(gt) A(0).$$

- A consequence of Uzawa Theorem.: (24) imposes purely labor-augmenting—Harrod-neutral—technological change.
- Continue to adopt all usual assumptions, and Assumption 4' will be strengthened further in order to ensure finite discounted utility in the presence of sustained economic growth.

Technological Change II

- Define

$$\begin{aligned}\hat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)} \\ &= F\left[\frac{K(t)}{A(t)L(t)}, 1\right] \\ &\equiv f(k(t)),\end{aligned}$$

where

$$k(t) \equiv \frac{K(t)}{A(t)L(t)}. \quad (25)$$

- Also need to impose a further assumption on preferences in order to ensure balanced growth.

Technological Change III

- Define balanced growth as a pattern of growth consistent with the *Kaldor facts* of constant capital-output ratio and capital share in national income.
- These two observations together also imply that the rental rate of return on capital, $R(t)$, has to be constant, which, from (10), implies that $r(t)$ has to be constant.
- Again refer to an equilibrium path that satisfies these conditions as a balanced growth path (BGP).
- Balanced growth also requires that consumption and output grow at a constant rate. Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))} (r(t) - \rho).$$

Technological Change IV

- If $r(t) \rightarrow r^*$, then $\dot{c}(t)/c(t) \rightarrow g_c$ is only possible if $\varepsilon_u(c(t)) \rightarrow \varepsilon_u$, i.e., if the elasticity of marginal utility of consumption, which determines the intertemporal elasticity of substitution, is asymptotically constant.
- Thus balanced growth is only consistent with utility functions that have asymptotically constant elasticity of marginal utility of consumption, so that the way that individuals substitute consumption today vs. consumption tomorrow does not change with the level of consumption (i.e., as the economy grows).

Proposition Balanced growth in the neoclassical model requires that asymptotically (as $t \rightarrow \infty$) all technological change is purely labor augmenting and the elasticity of intertemporal substitution, $\varepsilon_u(c(t))$, tends to a constant ε_u .

Example: CRRA Utility I

- Recall the Arrow-Pratt coefficient of relative risk aversion for a twice-continuously differentiable concave utility function $U(c)$ is

$$\mathcal{R} = -\frac{U''(c)c}{U'(c)}.$$

- Constant relative risk aversion (CRRA) utility function satisfies the property that \mathcal{R} is constant.
- Integrating both sides of the previous equation, setting \mathcal{R} to a constant, implies that the family of CRRA utility functions is given by

$$U(c) = \begin{cases} \frac{c^{1-\theta}-1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \ln c & \text{if } \theta = 1 \end{cases},$$

with the coefficient of relative risk aversion given by θ .

- CRRA utility functions are useful precisely because they imply constant intertemporal elasticity of substitution.
- Details: see recitation.

Technological Change V

- Given the restriction that balanced growth is only possible with a constant elasticity of intertemporal substitution, start with

$$u(c(t)) = \begin{cases} \frac{c(t)^{1-\theta} - 1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\ \ln c(t) & \text{if } \theta = 1 \end{cases},$$

- Elasticity of marginal utility of consumption, ε_u , is given by θ .
- When $\theta = 0$, these represent linear preferences, when $\theta = 1$, we have log preferences, and as $\theta \rightarrow \infty$, infinitely risk-averse, and infinitely unwilling to substitute consumption over time.
- Assume that the economy admits a representative household with CRRA preferences

$$\int_0^{\infty} \exp(-(\rho - n)t) \frac{\tilde{c}(t)^{1-\theta} - 1}{1-\theta} dt, \quad (26)$$

Technological Change VI

- $\tilde{c}(t) \equiv C(t) / L(t)$ is per capita consumption.
- Refer to this model, with labor-augmenting technological change and CRRA preference as given by (26) as the *canonical model*
- Euler equation takes the simpler form:

$$\frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} = \frac{1}{\theta} (r(t) - \rho). \quad (27)$$

- Steady-state equilibrium first: since with technological progress there will be growth in per capita income, $\tilde{c}(t)$ will grow.

Technological Change VII

- Instead define

$$\begin{aligned}c(t) &\equiv \frac{C(t)}{A(t)L(t)} \\ &\equiv \frac{\tilde{c}(t)}{A(t)}.\end{aligned}$$

- This normalized consumption level will remain constant along the BGP:

$$\begin{aligned}\frac{\dot{c}(t)}{c(t)} &\equiv \frac{\dot{\tilde{c}}(t)}{\tilde{c}(t)} - g \\ &= \frac{1}{\theta} (r(t) - \rho - \theta g).\end{aligned}$$

Technological Change VIII

- For the accumulation of capital stock:

$$\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta)k(t),$$

where $k(t) \equiv K(t) / A(t)L(t)$.

- Transversality condition, in turn, can be expressed as

$$\lim_{t \rightarrow \infty} \left\{ k(t) \exp \left(- \int_0^t [f'(k(s)) - g - \delta - n] ds \right) \right\} = 0. \quad (28)$$

- In addition, equilibrium $r(t)$ is still given by (10), so

$$r(t) = f'(k(t)) - \delta$$

Technological Change IX

- Since in steady state $c(t)$ must remain constant:

$$r(t) = \rho + \theta g$$

or

$$f'(k^*) = \rho + \delta + \theta g, \quad (29)$$

- Pins down the steady-state value of the normalized capital ratio k^* uniquely.
- Normalized consumption level is then given by

$$c^* = f(k^*) - (n + g + \delta)k^*, \quad (30)$$

- Per capita consumption grows at the rate g .

Technological Change X

- Because there is growth, to make sure that the transversality condition is in fact satisfied substitute (29) into (28):

$$\lim_{t \rightarrow \infty} \left\{ k(t) \exp \left(- \int_0^t [\rho - (1 - \theta)g - n] ds \right) \right\} = 0,$$

- Can only hold if $\rho - (1 - \theta)g - n > 0$, or alternatively :

ASSUMPTION 4:

$$\rho - n > (1 - \theta)g.$$

- Remarks:
 - Strengthens Assumption 4' when $\theta < 1$.
 - Alternatively, recall in steady state $r = \rho + \theta g$ and the growth rate of output is $g + n$.
 - Therefore, equivalent to requiring that $r > g + n$.

Technological Change XI

Proposition Consider the neoclassical growth model with labor augmenting technological progress at the rate g and preferences given by (26). Suppose that Assumptions 1, 2, assumptions on utility above hold and $\rho - n > (1 - \theta)g$. Then there exists a unique balanced growth path with a normalized capital to effective labor ratio of k^* , given by (29), and output per capita and consumption per capita grow at the rate g .

- Steady-state capital-labor ratio no longer independent of preferences, depends on θ .
 - Positive growth in output per capita, and thus in consumption per capita.
 - With upward-sloping consumption profile, willingness to substitute consumption today for consumption tomorrow determines accumulation and thus equilibrium effective capital-labor ratio.

Transitional Dynamics with Technological Change

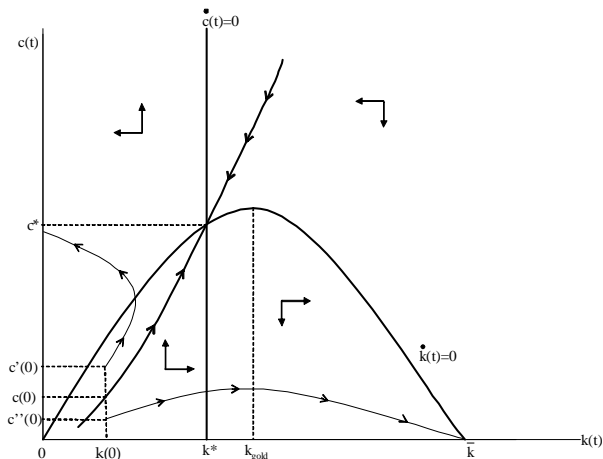


Figure: Transitional dynamics in the neoclassical growth model with technological change.

Technological Change XII

- Steady-state effective capital-labor ratio, k^* , is determined endogenously, but steady-state growth rate of the economy is given exogenously and equal to g .

Proposition Consider the neoclassical growth model with labor augmenting technological progress at the rate g and preferences given by (26). Suppose that Assumptions 1, 2, assumptions on utility above hold and $\rho - n > (1 - \theta)g$. Then there exists a unique equilibrium path of normalized capital and consumption, $(k(t), c(t))$ converging to the unique steady-state (k^*, c^*) with k^* given by (29). Moreover, if $k(0) < k^*$, then $k(t) \uparrow k^*$ and $c(t) \uparrow c^*$, whereas if $k(0) > k^*$, then $k(t) \downarrow k^*$ and $c(t) \downarrow c^*$.

Example: CRRA and Cobb-Douglas

- One solvable case: CRRA (or even better log) preferences and Cobb-Douglas production function, given by $F(K, AL) = K^\alpha (AL)^{1-\alpha}$, so that

$$f(k) = k^\alpha.$$

- See recitation.

The Role of Policy I

- Growth of per capita consumption and output per worker (per capita) are determined exogenously.
- But level of income, depends on $1/\theta$, ρ , δ , n , and naturally the form of $f(\cdot)$.
- Proximate causes of differences in income per capita: here explain those differences only in terms of preference and technology parameters.
- Link between proximate and potential fundamental causes:
 - e.g. intertemporal elasticity of substitution and the discount rate can be as related to cultural or geographic factors.
- But an explanation for cross-country and over-time differences in economic growth based on differences or changes in preferences is unlikely to be satisfactory.
- More appealing: link incentives to accumulate physical capital (and human capital and technology) to the institutional environment.

The Role of Policy II

- Simple way: through differences in policies.
- Introduce linear tax policy: returns on capital net of depreciation are taxed at the rate τ and the proceeds of this are redistributed back to the consumers.
- Capital accumulation equation remains as above:

$$\dot{k}(t) = f(k(t)) - c(t) - (n + g + \delta)k(t),$$

- But interest rate faced by households changes to:

$$r(t) = (1 - \tau) (f'(k(t)) - \delta),$$

The Role of Policy III

- Growth rate of normalized consumption is then obtained from the consumer Euler equation, (27):

$$\begin{aligned}\frac{\dot{c}(t)}{c(t)} &= \frac{1}{\theta} (r(t) - \rho - \theta g). \\ &= \frac{1}{\theta} ((1 - \tau) (f'(k(t)) - \delta) - \rho - \theta g).\end{aligned}$$

- Identical argument to that before implies

$$f'(k^*) = \delta + \frac{\rho + \theta g}{1 - \tau}. \quad (31)$$

- Higher τ , since $f'(\cdot)$ is decreasing, reduces k^* .
- Higher taxes on capital have the effect of depressing capital accumulation and reducing income per capita.
- But have not so far offered a reason why some countries may tax capital at a higher rate than others.

Comparative Dynamics I

- Comparative statics: changes in steady state in response to changes in parameters.
- Comparative dynamics look at how the entire equilibrium path of variables changes in response to a change in policy or parameters.
- Look at the effect of a change in tax on capital (or discount rate ρ)
- Consider the neoclassical growth in steady state (k^*, c^*) .
- Tax declines to $\tau' < \tau$.
- From Propositions above, after the change there exists a unique steady state equilibrium that is saddle path stable.
- Let this steady state be denoted by (k^{**}, c^{**}) .
- Since $\tau' < \tau$, $k^{**} > k^*$ while the equilibrium growth rate will remain unchanged.

Comparative Dynamics II

- Figure: drawn assuming change is unanticipated and occurs at some date T .
- At T , curve corresponding to $\dot{c}/c = 0$ shifts to the right and laws of motion represented by the phase diagram change.
- Following the decline c^* is above the stable arm of the new dynamical system: consumption must drop immediately
- Then consumption slowly increases along the stable arm
- Overall level of normalized consumption will necessarily increase, since the intersection between the curve for $\dot{c}/c = 0$ and for $\dot{k}/k = 0$ will necessarily be to the left side of k_{gold} .

Comparative Dynamics III

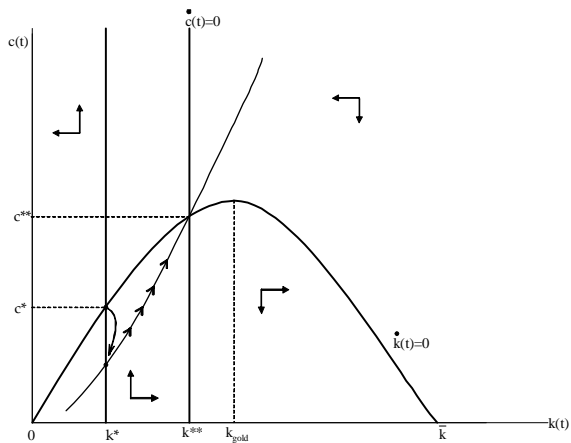


Figure: The dynamic response of capital and consumption to a decline in capital taxation from τ to $\tau' < \tau$.

Beyond Full Rationality

- The mathematical analysis so far depends intricately on infinite horizon maximization with a time-separable utility function and without any mistakes or behavioral biases.
- Do the conclusions also strongly and critically depend on these assumptions?
- If the answer is yes, one may be worried about the general applicability of these results.
- Fortunately, it turns out that the qualitative implications of the neoclassical growth model are a little more robust than that.

Comparative Statics in the BNGM

- Let us consider a “behavioral” version of the model considered so far, where we relax the assumption of full rationality, no mistakes and no behavioral biases — hence “behavioral neoclassical growth model (BNGM)”.
- This becomes more interesting if we also relax the “representative household” assumption, so that there are many households each subject to idiosyncratic shocks (and idiosyncratic biases and mistakes).
- The full mathematical details of this case are somewhat tricky, but the main ideas and results are worked over viewing.
- The details are in Acemoglu and Jensen (2019).

BNGM

- Continuum of households, no aggregate risk, and each household subject to labor and potentially capital income risk, and with different preferences and biases (e.g., hyperbolic discounting, time-inconsistency, systematic mistakes, non-separable preferences, temptation etc.) as well as possibly borrowing and lending constraints.
- The production side is identical to what we have analyzed so far, except aggregate “distortions” and taxes on labor and capital.
- The key object is the *market correspondence*:

$$\mathcal{M}(k) = f(k) + (1 - \delta)k - G(k) - \int c_{w(k), R(k)}^i(\lambda^i(k)) di,$$

where $\lambda(k) : i \mapsto \lambda^i(k)$ is an asset distribution of assigning a (possibly random) asset level $\lambda^i(k)$ to each household i such that $\int_0^1 \lambda^i(k) di$, $G(k)$ government spending/waste from taxes/distortions, and the last term is the integral of consumption across households is a function of asset distribution and factor prices.

BNGM (continued)

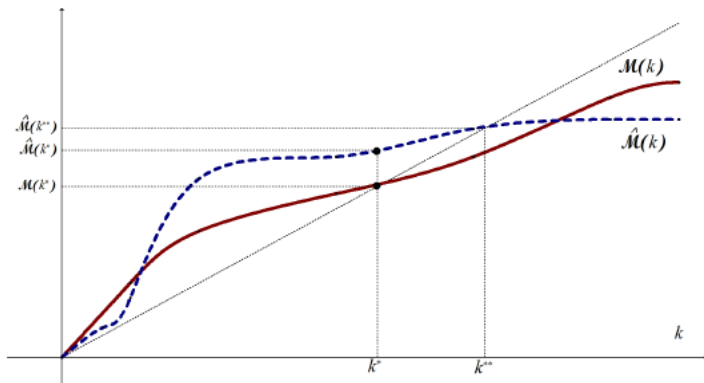
- A steady-state equilibrium involves an aggregate capital-labor ratio k^* such that

$$k^* \in \mathcal{M}(k^*).$$

- Here the asset distribution has to be an equilibrium object (determined as part of the same fixed point problem).
- Under fairly weak assumptions on preferences (which covers essentially all existing behavioral and limited rationality models), the relevant correspondence has a number of convenient regularity conditions, and in particular, it can have upward jumps but no downward jumps.

Steady-State Equilibrium

- Then the determination of the steady-state can be represented as:



Main Result

- Consider a change in the environment (productivity shock, preference shock, expectation shock or policy change).
- Define the *direct effect* of this change as the impact holding all equilibrium objects (in particular the capital-labor ratio which determines the other equilibrium objects) as given.

Theorem

If the direct effect of a change in the environment at k^ is positive [negative] at the unique (or greatest) steady-state equilibrium, then the unique (or greatest) steady state increases [decreases] after the shock.*

- Despite complex behavioral preferences/biases leading to rich general equilibrium effects, they cannot overturn the partial equilibrium effect.

A Quantitative Evaluation I

- Does the neoclassical growth model help us understand/account for cross-country growth or income differences? Not growth differences, since all countries have the same growth rate. What about income differences?
- Consider a world consisting of a collection \mathcal{J} of closed neoclassical economies (with the caveats of ignoring technological, trade and financial linkages across countries)
- Each country $j \in \mathcal{J}$ admits a representative household with identical preferences and no population growth,

$$\int_0^{\infty} \exp(-\rho t) \frac{C_j^{1-\theta} - 1}{1-\theta} dt. \quad (32)$$

- Equation (32) imposes that all countries have the same discount rate ρ .
- All countries also have access to the same production technology given by the Cobb-Douglas production function

A Quantitative Evaluation II

- The accumulation equation is

$$\dot{K}_j = I_j - \delta K_j.$$

- The only difference across countries is in the budget constraint for the representative household,

$$(1 + \tau_j) I_j + C_j \leq Y_j, \quad (34)$$

- τ_j is the tax on investment: varies across countries because of policies or differences in institutions/property rights enforcement.
- $1 + \tau_j$ is also the relative price of investment goods (relative to consumption goods): one unit of consumption goods can only be transformed into $1 / (1 + \tau_j)$ units of investment goods.
- The right-hand side variable of (34) is still Y_j : assumes that $\tau_j I_j$ is wasted, rather than simply redistributed to some other agents.

A Quantitative Evaluation III

- Euler equation of the representative household

$$\frac{\dot{C}_j}{C_j} = \frac{1}{\theta} \left(\frac{(1 - \alpha)}{(1 + \tau_j)} \left(\frac{AH_j}{K_j} \right)^\alpha - \delta - \rho \right).$$

- Steady state: because A is assumed to be constant, $\dot{C}_j / C_j = 0$. Thus,

$$K_j = \frac{(1 - \alpha)^{1/\alpha} AH_j}{[(1 + \tau_j)(\rho + \delta)]^{1/\alpha}}$$

- Thus countries with higher taxes on investment will have a lower capital stock, lower capital per worker, and lower capital output ratio.
- Substituting into (33), and comparing two countries with different taxes (but the same human capital):

$$\frac{Y(\tau)}{Y(\tau')} = \left(\frac{1 + \tau'}{1 + \tau} \right)^{\frac{1 - \alpha}{\alpha}} \quad (35)$$

A Quantitative Evaluation V

- For differences in τ 's across countries there is no obvious answer:
 - popular approach: obtain estimates of τ from the relative price of investment goods (as compared to consumption goods)
 - data from the Penn World tables suggest there is a large amount of variation in the relative price of investment goods.
- E.g., countries with the highest relative price of investment goods have relative prices almost eight times as high as countries with the lowest relative price.
- Plausible value for α from the labor share in national income is $\alpha = 2/3$, so equation (35) implies:

$$\frac{Y(\tau)}{Y(\tau')} \approx 8^{1/2} \approx 3.$$

- Thus, even very large differences in taxes or distortions are unlikely to account for the large differences in income per capita that we observe.

A Quantitative Evaluation VI

- Parallels discussion of the Mankiw-Romer-Weil approach:
 - differences in income per capita unlikely to be accounted for by differences in capital per worker alone.
 - need sizable differences in the efficiency with which these factors are used, absent in this model.
- But many economists have tried (and still try) to use versions of the neoclassical model to go further.
- Motivation is simple: if instead of using $\alpha = 2/3$, we take $\alpha = 1/3$

$$\frac{Y(\tau)}{Y(\tau')} \approx 8^2 \approx 64.$$

- Thus if there is a way of increasing the responsiveness of capital or other factors to distortions, predicted differences across countries can be made much larger.

A Quantitative Evaluation VII

- To have a model in which $\alpha = 1/3$, must have additional accumulated factors, while still keeping the share of labor income in national product roughly around $2/3$.
- E.g., include human capital, but human capital differences appear to be insufficient to explain much of the income per capita differences across countries.
- Or introduce other types of capital or perhaps technology that responds to distortions in the same way as capital.

Conclusions

- Major contribution: open the black box of capital accumulation by specifying the preferences of consumers.
- Also by specifying individual preferences we can explicitly compare equilibrium and optimal growth.
- Paves the way for further analysis of capital accumulation, human capital and endogenous technological progress.
- Did our study of the neoclassical growth model generate new insights about the sources of cross-country income differences and economic growth relative to the Solow growth model? Largely no.
- This model, by itself, does not enable us to answer questions about the fundamental causes of economic growth.
- But it clarifies the nature of the economic decisions so that we are in a better position to ask such questions.