

# 14.452 Economic Growth: Lectures 2 and 3

## The Solow Growth Model

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# Solow Growth Model

- Develop a simple framework for the *proximate* causes and the mechanics of economic growth and cross-country income differences.
- Solow-Swan model named after Robert (Bob) Solow and Trevor Swan, or simply the *Solow model*
- Before Solow growth model, the most common approach to economic growth built on the Harrod-Domar model.
- Harrod-Domar model emphasized potential dysfunctional aspects of growth: e.g, how growth could go hand-in-hand with increasing unemployment.
- Solow model demonstrated why the Harrod-Domar model was not an attractive place to start.
- At the center of the Solow growth model is the *neoclassical* aggregate production function.

# Households and Production I

- Closed economy, with a unique final good.
- Discrete time running to an infinite horizon, time is indexed by  $t = 0, 1, 2, \dots$
- Economy is inhabited by a large number of households, and for now households will not be optimizing.
- This is the main difference between the Solow model and the *neoclassical growth model*.
- To fix ideas, assume all households are identical, so the economy admits a *representative household*.

## Households and Production II

- Assume households save a constant exogenous fraction  $s$  of their disposable income
- Same assumption used in basic Keynesian models and in the Harrod-Domar model; at odds with reality.
- Assume all firms have access to the same production function: economy admits a **representative firm**, with a representative (or aggregate) production function.
- Aggregate production function for the unique final good is

$$Y(t) = F[K(t), L(t), A(t)] \quad (1)$$

- Assume capital is the same as the final good of the economy, but used in the production process of more goods.
- $A(t)$  is a *shifter* of the production function (1). Broad notion of technology.
- Major assumption: technology is **free**; it is publicly available as a non-excludable, non-rival good.

## Some Assumptions

**Assumption 1 (Continuity, Differentiability, Positive and Diminishing Marginal Products, and Constant Returns to Scale)** The production function  $F : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$  is twice continuously differentiable in  $K$  and  $L$ , and satisfies

$$F_K(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial K} > 0, \quad F_L(K, L, A) \equiv \frac{\partial F(\cdot)}{\partial L} > 0,$$

$$F_{KK}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial K^2} < 0, \quad F_{LL}(K, L, A) \equiv \frac{\partial^2 F(\cdot)}{\partial L^2} < 0.$$

Moreover,  $F$  exhibits constant returns to scale in  $K$  and  $L$ .

- Assume  $F$  exhibits *constant returns to scale* in  $K$  and  $L$ . I.e., it is *linearly homogeneous* (homogeneous of degree 1) in these two variables.

# Review

**Definition** Let  $K$  be an integer. The function  $g : \mathbb{R}^{K+2} \rightarrow \mathbb{R}$  is homogeneous of degree  $m$  in  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$  if and only if

$$g(\lambda x, \lambda y, z) = \lambda^m g(x, y, z) \text{ for all } \lambda \in \mathbb{R}_+ \text{ and } z \in \mathbb{R}^K.$$

**Theorem (Euler's Theorem)** Suppose that  $g : \mathbb{R}^{K+2} \rightarrow \mathbb{R}$  is continuously differentiable in  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$ , with partial derivatives denoted by  $g_x$  and  $g_y$  and is homogeneous of degree  $m$  in  $x$  and  $y$ . Then

$$\begin{aligned} mg(x, y, z) &= g_x(x, y, z)x + g_y(x, y, z)y \\ &\text{for all } x \in \mathbb{R}, y \in \mathbb{R} \text{ and } z \in \mathbb{R}^K. \end{aligned}$$

Moreover,  $g_x(x, y, z)$  and  $g_y(x, y, z)$  are themselves homogeneous of degree  $m - 1$  in  $x$  and  $y$ .

# Market Structure, Endowments and Market Clearing I

- We will assume that markets are competitive, so ours will be a prototypical *competitive general equilibrium model*.
- Households own all of the labor, which they supply inelastically.
- Endowment of labor in the economy,  $\bar{L}(t)$ , and all of this will be supplied regardless of the price.
- The *labor market clearing* condition can then be expressed as:

$$L(t) = \bar{L}(t)$$

for all  $t$ , where  $L(t)$  denotes the demand for labor (and also the level of employment).

- More generally, should be written in complementary slackness form.
- In particular, let the *wage rate* at time  $t$  be  $w(t)$ , then the labor market clearing condition takes the form

$$L(t) \leq \bar{L}(t), w(t) \geq 0 \text{ and } (L(t) - \bar{L}(t)) w(t) = 0$$

## Market Structure, Endowments and Market Clearing II

- But Assumption 1 and competitive labor markets make sure that wages have to be strictly positive.
- Households also own the capital stock of the economy and rent it to firms. Take initial holdings,  $K(0)$ , as given
- Denote the *rental price of capital* at time  $t$  be  $R(t)$ .
- Capital market clearing condition:

$$K^s(t) = K^d(t)$$

- Assume capital depreciates, with “exponential form,” at the rate  $\delta$ : out of 1 unit of capital this period, only  $1 - \delta$  is left for next period.
- Then, the *interest rate* faced by the household will be  $r(t) = R(t) - \delta$ .
- Why is it enough to keep track of the interest rate rather than other intertemporal prices?



# Firm Optimization I

- Only need to consider the problem of a *representative firm*:

$$\max_{L(t) \geq 0, K(t) \geq 0} F[K(t), L(t), A(t)] - w(t)L(t) - R(t)K(t).$$

- Since there are no irreversible investments or costs of adjustments, the production side can be represented as a static maximization problem.
- Equivalently, *cost minimization problem*.
- Features worth noting:
  - ① Problem is set up in terms of aggregate variables.
  - ② Nothing multiplying the  $F$  term, price of the final good has normalized to 1.
  - ③ Already imposes competitive factor markets: firm is taking as given  $w(t)$  and  $R(t)$ .
  - ④ Concave problem, since  $F$  is concave.

## Firm Optimization II

- Since  $F$  is differentiable, first-order necessary conditions imply:

$$w(t) = F_L[K(t), L(t), A(t)], \quad (2)$$

and

$$R(t) = F_K[K(t), L(t), A(t)]. \quad (3)$$

- Note also that in (2) and (3), we used  $K(t)$  and  $L(t)$ , the amount of capital and labor used by firms.
- In fact, solving for  $K(t)$  and  $L(t)$ , we can derive the capital and labor demands of firms in this economy at rental prices  $R(t)$  and  $w(t)$ .
- Thus we could have used  $K^d(t)$  instead of  $K(t)$ , but this additional notation is not necessary.

## Firm Optimization III

**Proposition** Suppose Assumption 1 holds. Then in the equilibrium of the Solow growth model, firms make no profits, and in particular,

$$Y(t) = w(t)L(t) + R(t)K(t).$$

- **Proof:** Follows immediately from Euler Theorem for the case of  $m = 1$ , i.e., constant returns to scale.
- Thus firms make no profits, so ownership of firms does not need to be specified.

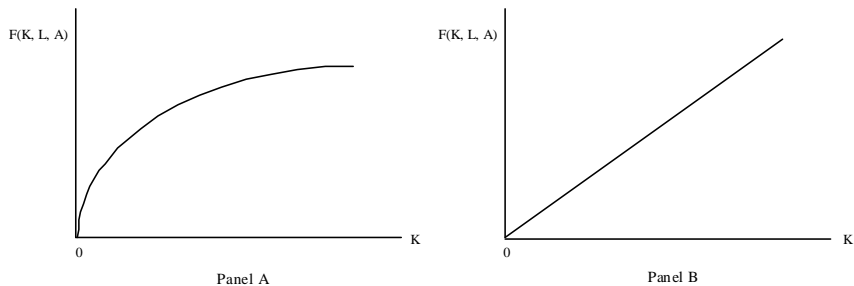
## Second Key Assumption

**Assumption 2 (Inada conditions)**  $F$  satisfies the Inada conditions

$$\begin{aligned}\lim_{K \rightarrow 0} F_K(\cdot) &= \infty \text{ and } \lim_{K \rightarrow \infty} F_K(\cdot) = 0 \text{ for all } L > 0 \text{ all } A \\ \lim_{L \rightarrow 0} F_L(\cdot) &= \infty \text{ and } \lim_{L \rightarrow \infty} F_L(\cdot) = 0 \text{ for all } K > 0 \text{ all } A.\end{aligned}$$

- Important in ensuring the existence of *interior equilibria*.
- It can be relaxed quite a bit, though useful to get us started.

# Production Functions



**Figure:** Production functions and the marginal product of capital. The example in Panel A satisfies the Inada conditions in Assumption 2, while the example in Panel B does not.

# Fundamental Law of Motion of the Solow Model I

- Recall that  $K$  depreciates exponentially at the rate  $\delta$ , so

$$K(t+1) = (1 - \delta) K(t) + I(t), \quad (4)$$

where  $I(t)$  is investment at time  $t$ .

- From national income accounting for a closed economy,

$$Y(t) = C(t) + I(t), \quad (5)$$

- Behavioral rule* of the constant saving rate simplifies the structure of equilibrium considerably.
- Note not derived from the maximization of utility function: welfare comparisons have to be taken with a grain of salt.

## Fundamental Law of Motion of the Solow Model II

- Since the economy is closed (and there is no government spending),

$$S(t) = I(t) = Y(t) - C(t).$$

- Individuals are assumed to save a constant fraction  $s$  of their income,

$$S(t) = sY(t), \quad (6)$$

$$C(t) = (1 - s)Y(t) \quad (7)$$

- Implies that the supply of capital resulting from households' behavior can be expressed as

$$K^s(t) = (1 - \delta)K(t) + S(t) = (1 - \delta)K(t) + sY(t).$$

# Fundamental Law of Motion of the Solow Model III

- Setting supply and demand equal to each other, this implies  $K^s(t) = K(t)$ .
- We also have  $L(t) = \bar{L}(t)$ .
- Combining these market clearing conditions with (1) and (4), we obtain *the fundamental law of motion* the Solow growth model:

$$K(t+1) = sF[K(t), L(t), A(t)] + (1 - \delta)K(t). \quad (8)$$

- Nonlinear *difference equation*.
- Equilibrium of the Solow growth model is described by this equation together with laws of motion for  $L(t)$  (or  $\bar{L}(t)$ ) and  $A(t)$ .



# Definition of Equilibrium I

- Solow model is a mixture of an old-style Keynesian model and a modern dynamic macroeconomic model.
- Households do not optimize, but firms still maximize and factor markets clear.

**Definition** In the basic Solow model for a given sequence of  $\{L(t), A(t)\}_{t=0}^{\infty}$  and an initial capital stock  $K(0)$ , an equilibrium path is a sequence of capital stocks, output levels, consumption levels, wages and rental rates  $\{K(t), Y(t), C(t), w(t), R(t)\}_{t=0}^{\infty}$  such that  $K(t)$  satisfies (8),  $Y(t)$  is given by (1),  $C(t)$  is given by (7), and  $w(t)$  and  $R(t)$  are given by (2) and (3).

- Note an equilibrium is defined as an entire path of allocations and prices: *not* a static object.

# Equilibrium Without Population Growth and Technological Progress I

- Make some further assumptions, which will be relaxed later:
  - 1 There is no population growth; total population is constant at some level  $L > 0$ . Since individuals supply labor inelastically,  $L(t) = L$ .
  - 2 No technological progress, so that  $A(t) = A$ .
- Define the capital-labor ratio of the economy as

$$k(t) \equiv \frac{K(t)}{L}, \quad (9)$$

- Using the constant returns to scale assumption, we can express output (income) per capita,  $y(t) \equiv Y(t) / L$ , as

$$\begin{aligned} y(t) &= F\left[\frac{K(t)}{L}, 1, A\right] \\ &\equiv f(k(t)). \end{aligned} \quad (10)$$

# Equilibrium Without Population Growth and Technological Progress II

- Note that  $f(k)$  here depends on  $A$ , so I could have written  $f(k, A)$ ; but  $A$  is constant and can be normalized to  $A = 1$ .
- From Euler Theorem,

$$\begin{aligned}R(t) &= f'(k(t)) > 0 \text{ and} \\w(t) &= f(k(t)) - k(t)f'(k(t)) > 0.\end{aligned}\tag{11}$$

- Both are positive from Assumption 1.

# Equilibrium Without Population Growth and Technological Progress I

- The per capita representation of the aggregate production function enables us to divide both sides of (8) by  $L$  to obtain:

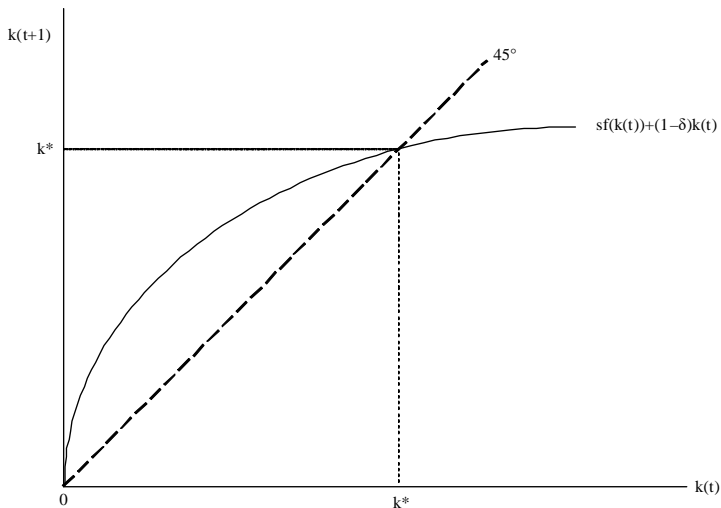
$$k(t+1) = sf(k(t)) + (1 - \delta)k(t). \quad (12)$$

- Since it is derived from (8), it also can be referred to as the *equilibrium difference equation* of the Solow model
- The other equilibrium quantities can be obtained from the capital-labor ratio  $k(t)$ .

**Definition** A steady-state equilibrium without technological progress and population growth is an equilibrium path in which  $k(t) = k^*$  for all  $t$ .

- The economy will tend to this steady state equilibrium over time (but never reach it in finite time).

# Steady-State Capital-Labor Ratio



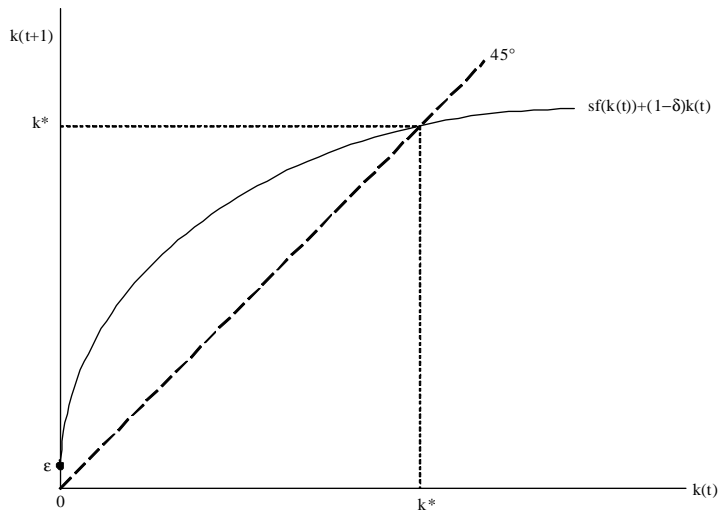
# Equilibrium Without Population Growth and Technological Progress II

- Thick curve represents (12) and the dashed line corresponds to the  $45^\circ$  line.
- Their (positive) intersection gives the steady-state value of the capital-labor ratio  $k^*$ ,

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}. \quad (13)$$

- There is another intersection at  $k = 0$ , because the figure assumes that  $f(0) = 0$ .
- Will ignore this intersection throughout:
  - 1 If capital is not essential,  $f(0)$  will be positive and  $k = 0$  will cease to be a steady state equilibrium
  - 2 This intersection, even when it exists, is an *unstable point*
  - 3 It has no economic interest for us.

# Equilibrium Without Population Growth and Technological Progress III

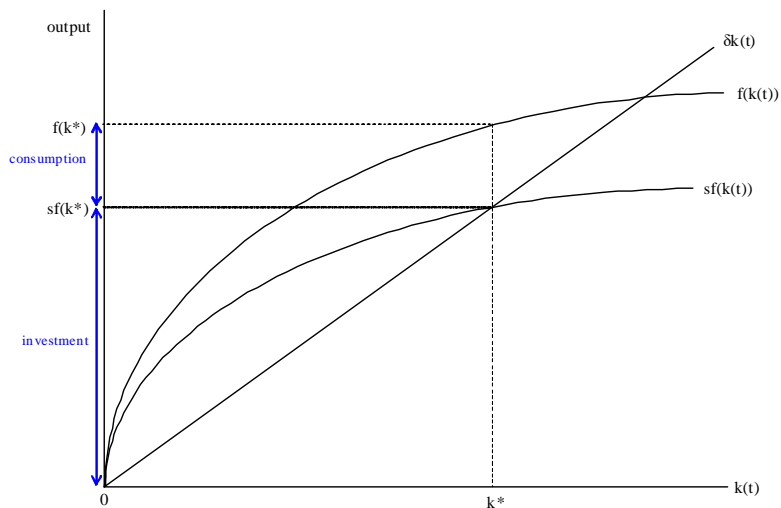


# Equilibrium Without Population Growth and Technological Progress IV

- Alternative visual representation of the steady state: intersection between  $\delta k$  and the function  $sf(k)$ . Useful because:
  - 1 Depicts the levels of consumption and investment in a single figure.
  - 2 Emphasizes the steady-state equilibrium sets investment,  $sf(k)$ , equal to the amount of capital that needs to be “replenished”,  $\delta k$ .



# Consumption and Investment in Steady State



# Equilibrium Without Population Growth and Technological Progress V

**Proposition** Consider the basic Solow growth model and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio  $k^* \in (0, \infty)$  is given by (13), per capita output is given by

$$y^* = f(k^*) \quad (14)$$

and per capita consumption is given by

$$c^* = (1 - s) f(k^*) . \quad (15)$$

# Proof

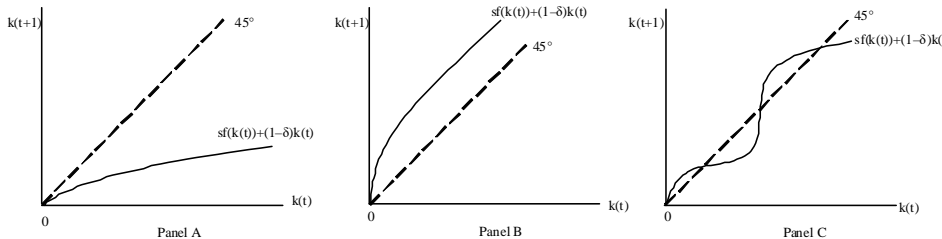
- The preceding argument establishes that any  $k^*$  that satisfies (13) is a steady state.
- To establish existence, note that from Assumption 2 (and from L'Hospital's rule),  $\lim_{k \rightarrow 0} f(k) / k = \infty$  and  $\lim_{k \rightarrow \infty} f(k) / k = 0$ .
- Moreover,  $f(k) / k$  is continuous from Assumption 1, so by the Intermediate Value Theorem there exists  $k^*$  such that (13) is satisfied.
- To see uniqueness, differentiate  $f(k) / k$  with respect to  $k$ , which gives

$$\frac{\partial [f(k) / k]}{\partial k} = \frac{f'(k) k - f(k)}{k^2} = -\frac{w}{k^2} < 0, \quad (16)$$

where the last equality uses (11).

- Since  $f(k) / k$  is everywhere (strictly) decreasing, there can only exist a unique value  $k^*$  that satisfies (13).
- Equations (14) and (15) then follow by definition.

# Non-Existence and Non-Uniqueness



**Figure:** Examples of nonexistence and nonuniqueness of interior steady states when Assumptions 1 and 2 are not satisfied.

# Equilibrium Without Population Growth and Technological Progress VI

- Comparative statics with respect to  $s$ ,  $a$  and  $\delta$  straightforward for  $k^*$  and  $y^*$ .
- But  $c^*$  will not be monotone in the saving rate (think, for example, of  $s = 1$ ).
- In fact, there will exist a specific level of the saving rate,  $s_{gold}$ , referred to as the “golden rule” saving rate, which maximizes  $c^*$ .
- But cannot say whether the golden rule saving rate is “better” than some other saving rate.
- Write the steady state relationship between  $c^*$  and  $s$  and suppress the other parameters:

$$\begin{aligned}c^*(s) &= (1 - s) f(k^*(s)), \\ &= f(k^*(s)) - \delta k^*(s),\end{aligned}$$

- The second equality exploits that in steady state  $sf(k) = \delta k$ .

# Equilibrium Without Population Growth and Technological Progress X

- Differentiating with respect to  $s$ ,

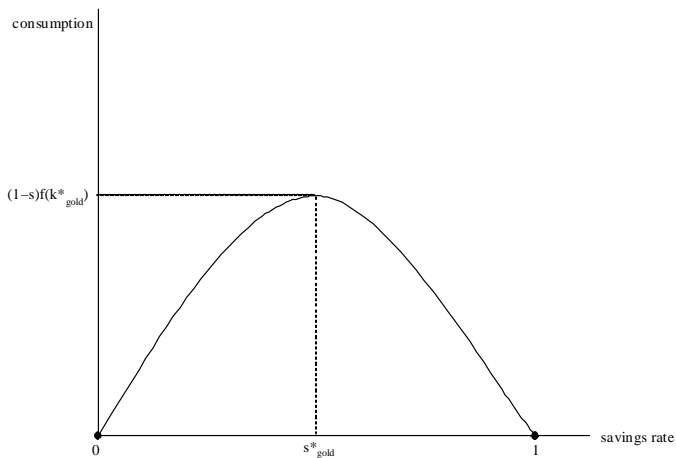
$$\frac{\partial c^*(s)}{\partial s} = [f'(k^*(s)) - \delta] \frac{\partial k^*}{\partial s}. \quad (17)$$

- $s_{gold}$  is such that  $\partial c^*(s_{gold}) / \partial s = 0$ . The corresponding steady-state golden rule capital stock is defined as  $k_{gold}^*$ .

**Proposition** In the basic Solow growth model, the highest level of steady-state consumption is reached for  $s_{gold}$ , with the corresponding steady state capital level  $k_{gold}^*$  such that

$$f'(k_{gold}^*) = \delta. \quad (18)$$

# The Golden Rule



**Figure:** The “golden rule” level of savings rate, which maximizes steady-state consumption.

# Dynamic Inefficiency

- When the economy is below  $k_{gold}^*$ , higher saving will increase consumption; when it is above  $k_{gold}^*$ , steady-state consumption can be increased by saving less.
- In the latter case, capital-labor ratio is too high so that individuals are investing too much and not consuming enough (*dynamic inefficiency*).
- But no utility function, so statements about “inefficiency” have to be considered with caution.
- Such dynamic inefficiency will not arise once we endogenize consumption-saving decisions.



# Discrete-Time Solow Model Redux

- Per capita capital stock evolves according to

$$k(t+1) = sf(k(t)) + (1 - \delta)k(t).$$

- The steady-state value of the capital-labor ratio  $k^*$  is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}.$$

- Consumption is given by

$$C(t) = (1 - s)Y(t)$$

- And factor prices are given by

$$R(t) = f'(k(t)) > 0 \text{ and}$$

$$w(t) = f(k(t)) - k(t)f'(k(t)) > 0.$$

# Transitional Dynamics

- *Equilibrium path*: not simply steady state, but entire path of capital stock, output, consumption and factor prices.
  - In engineering and physical sciences, equilibrium is point of rest of dynamical system, thus *the steady state equilibrium*.
  - In economics, non-steady-state behavior also governed by optimizing behavior of households and firms and market clearing.
- Need to study the “transitional dynamics” of the equilibrium difference equation (12) starting from an arbitrary initial capital-labor ratio  $k(0) > 0$ .
- Key question: whether economy will tend to steady state and how it will behave along the transition path.

# Transitional Dynamics: Review I

- Consider the nonlinear system of autonomous difference equations,

$$\mathbf{x}(t+1) = \mathbf{G}(\mathbf{x}(t)), \quad (19)$$

- $\mathbf{x}(t) \in \mathbb{R}^n$  and  $\mathbf{G} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ .
- Let  $\mathbf{x}^*$  be a fixed point of the mapping  $\mathbf{G}(\cdot)$ , i.e.,

$$\mathbf{x}^* = \mathbf{G}(\mathbf{x}^*).$$

- $\mathbf{x}^*$  is sometimes referred to as “an equilibrium point” of (19).
- We will refer to  $\mathbf{x}^*$  as a stationary point or a *steady state* of (19).

**Definition** A steady state  $\mathbf{x}^*$  is (locally) *asymptotically stable* if there exists an open set  $B(\mathbf{x}^*) \ni \mathbf{x}^*$  such that for any solution  $\{\mathbf{x}(t)\}_{t=0}^{\infty}$  to (19) with  $\mathbf{x}(0) \in B(\mathbf{x}^*)$ , we have  $\mathbf{x}(t) \rightarrow \mathbf{x}^*$ . Moreover,  $\mathbf{x}^*$  is *globally asymptotically stable* if for all  $\mathbf{x}(0) \in \mathbb{R}^n$ , for any solution  $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ , we have  $\mathbf{x}(t) \rightarrow \mathbf{x}^*$ .

# Transitional Dynamics: Review II

## Simple Result About Stability

- Let  $x(t)$ ,  $a, b \in \mathbb{R}$ , then the unique steady state of the linear difference equation  $x(t+1) = ax(t) + b$  is globally asymptotically stable (in the sense that  $x(t) \rightarrow x^* = b/(1-a)$ ) if  $|a| < 1$ .
- Suppose that  $g: \mathbb{R} \rightarrow \mathbb{R}$  is differentiable at the steady state  $x^*$ , defined by  $g(x^*) = x^*$ . Then, the steady state of the nonlinear difference equation  $x(t+1) = g(x(t))$ ,  $x^*$ , is locally asymptotically stable if  $|g'(x^*)| < 1$ . Moreover, if  $|g'(x)| < 1$  for all  $x \in \mathbb{R}$ , then  $x^*$  is globally asymptotically stable.

# Transitional Dynamics in the Discrete Time Solow Model

**Proposition** Suppose that Assumptions 1 and 2 hold, then the steady-state equilibrium of the Solow growth model described by the difference equation (12) is globally asymptotically stable, and starting from any  $k(0) > 0$ ,  $k(t)$  monotonically converges to  $k^*$ .

# Proof of Proposition: Transitional Dynamics I

- Let  $g(k) \equiv sf(k) + (1 - \delta)k$ . First observe that  $g'(k) > 0$  for all  $k$ .
- Next, from (12),

$$k(t+1) = g(k(t)), \quad (20)$$

with a unique steady state at  $k^*$ .

- From (13), the steady-state capital  $k^*$  satisfies  $\delta k^* = sf(k^*)$ , or

$$k^* = g(k^*). \quad (21)$$

- Recall that  $f(\cdot)$  is concave and differentiable from Assumption 1 and satisfies  $f(0) \geq 0$  from Assumption 2.

## Proof of Proposition: Transitional Dynamics II

- For any strictly concave differentiable function,

$$f(k) > f(0) + kf'(k) \geq kf'(k), \quad (22)$$

- The second inequality uses the fact that  $f(0) \geq 0$ .
- Since (22) implies that  $\delta = sf(k^*)/k^* > sf'(k^*)$ , we have  $g'(k^*) = sf'(k^*) + 1 - \delta < 1$ . Therefore,

$$g'(k^*) \in (0, 1).$$

- The Simple Result then establishes local asymptotic stability.

## Proof of Proposition: Transitional Dynamics III

- To prove global stability, note that for all  $k(t) \in (0, k^*)$ ,

$$\begin{aligned}k(t+1) - k^* &= g(k(t)) - g(k^*) \\ &= - \int_{k(t)}^{k^*} g'(k) dk, \\ &< 0\end{aligned}$$

- First line follows by subtracting (21) from (20), second line uses the fundamental theorem of calculus, and third line follows from the observation that  $g'(k) > 0$  for all  $k$ .



## Proof of Proposition: Transitional Dynamics IV

- Next, (12) also implies

$$\begin{aligned} \frac{k(t+1) - k(t)}{k(t)} &= s \frac{f(k(t))}{k(t)} - \delta \\ &> s \frac{f(k^*)}{k^*} - \delta \\ &= 0. \end{aligned}$$

Moreover, for any  $k(t) \in (0, k^* - \varepsilon)$ , this is uniformly so.

- Second line uses the fact that  $f(k)/k$  is decreasing in  $k$  (from (22) above) and last line uses the definition of  $k^*$ .
- These two arguments together establish that for all  $k(t) \in (0, k^*)$ ,  $k(t+1) \in (k(t), k^*)$ .
- An identical argument implies that for all  $k(t) > k^*$ ,  $k(t+1) \in (k^*, k(t))$ .
- Therefore,  $\{k(t)\}_{t=0}^{\infty}$  monotonically converges to  $k^*$  and is globally stable.

# Transitional Dynamics III

- Stability result can be seen diagrammatically in the Figure:
  - Starting from initial capital stock  $k(0) < k^*$ , economy grows towards  $k^*$ , *capital deepening* and growth of per capita income.
  - If economy were to start with  $k'(0) > k^*$ , reach the steady state by decumulating capital and contracting.
- As a consequence:

**Proposition** Suppose that Assumptions 1 and 2 hold, and  $k(0) < k^*$ , then  $\{w(t)\}_{t=0}^{\infty}$  is an increasing sequence and  $\{R(t)\}_{t=0}^{\infty}$  is a decreasing sequence. If  $k(0) > k^*$ , the opposite results apply.

- Thus far Solow growth model has a number of nice properties, but no growth, except when the economy starts with  $k(0) < k^*$ .

# Transitional Dynamics in Figure

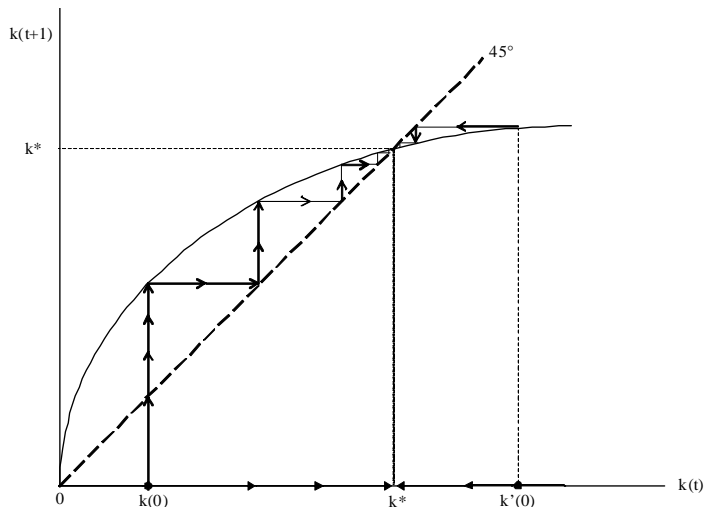


Figure: Transitional dynamics in the basic Solow model.

# From Difference to Differential Equations I

- Start with a simple difference equation

$$x(t+1) - x(t) = g(x(t)). \quad (23)$$

- Now consider the following approximation for any  $\Delta t \in [0, 1]$ ,

$$x(t + \Delta t) - x(t) \simeq \Delta t \cdot g(x(t)),$$

- When  $\Delta t = 0$ , this equation is just an identity. When  $\Delta t = 1$ , it gives (23).
- In-between it is a linear approximation, not too bad if  $g(x) \simeq g(x(t))$  for all  $x \in [x(t), x(t+1)]$

# From Difference to Differential Equations II

- Divide both sides of this equation by  $\Delta t$ , and take limits

$$\lim_{\Delta t \rightarrow 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \dot{x}(t) \simeq g(x(t)), \quad (24)$$

where

$$\dot{x}(t) \equiv \frac{dx(t)}{dt}$$

- Equation (24) is a differential equation representing (23) for the case in which  $t$  and  $t + 1$  is “small”.

# The Fundamental Equation of the Solow Model in Continuous Time I

- Nothing has changed on the production side, so (11) still give the factor prices, now interpreted as instantaneous wage and rental rates.
- Savings are again

$$S(t) = sY(t),$$

- Consumption is given by (7) above.
- Introduce population growth,

$$L(t) = \exp(nt) L(0). \quad (25)$$

- Recall

$$k(t) \equiv \frac{K(t)}{L(t)},$$

# The Fundamental Equation of the Solow Model in Continuous Time II

- Implies

$$\begin{aligned}\frac{\dot{k}(t)}{k(t)} &= \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}, \\ &= \frac{\dot{K}(t)}{K(t)} - n.\end{aligned}$$

- From the limiting argument leading to equation (24),

$$\dot{K}(t) = sF[K(t), L(t), A(t)] - \delta K(t).$$

- Using the definition of  $k(t)$  and the constant returns to scale properties of the production function,

$$\frac{\dot{k}(t)}{k(t)} = s \frac{f(k(t))}{k(t)} - (n + \delta), \quad (26)$$

# The Fundamental Equation of the Solow Model in Continuous Time III

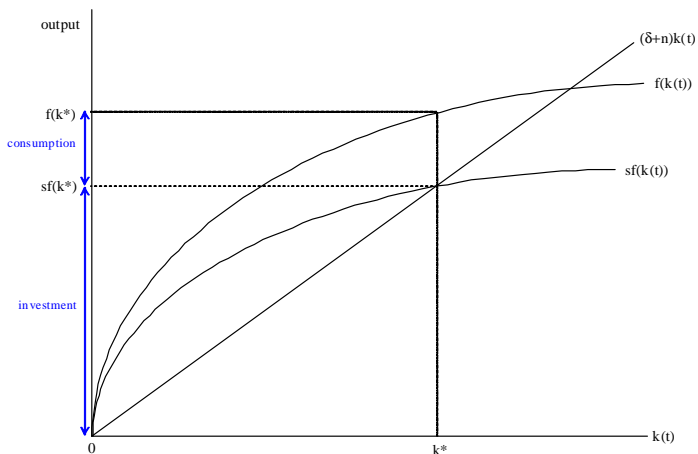
**Definition** In the basic Solow model in continuous time with population growth at the rate  $n$ , no technological progress and an initial capital stock  $K(0)$ , an equilibrium path is a sequence of capital stocks, labor, output levels, consumption levels, wages and rental rates

$[K(t), L(t), Y(t), C(t), w(t), R(t)]_{t=0}^{\infty}$  such that  $L(t)$  satisfies (25),  $k(t) \equiv K(t) / L(t)$  satisfies (26),  $Y(t)$  is given by the aggregate production function,  $C(t)$  is given by (7), and  $w(t)$  and  $R(t)$  are given by (11).

- As before, *steady-state* equilibrium involves  $k(t)$  remaining constant at some level  $k^*$ .



# Steady State With Population Growth



**Figure:** Investment and consumption in the steady-state equilibrium with population growth.

# Steady State of the Solow Model in Continuous Time

- Equilibrium path (26) has a unique *steady state* at  $k^*$ , which is given by a slight modification of (13) above:

$$\frac{f(k^*)}{k^*} = \frac{n + \delta}{s}. \quad (27)$$

**Proposition** Consider the basic Solow growth model in continuous time and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio is equal to  $k^* \in (0, \infty)$  and is given by (27), per capita output is given by

$$y^* = f(k^*)$$

and per capita consumption is given by

$$c^* = (1 - s) f(k^*).$$

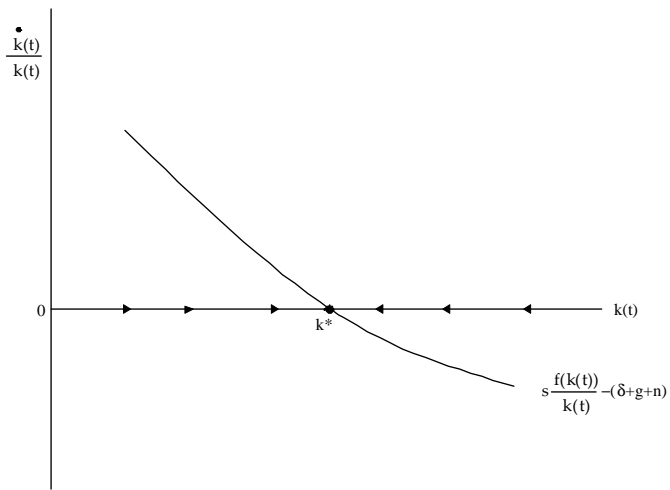
- Similar comparative statics to the discrete time model.

# Transitional Dynamics in the Continuous Time Solow Model I

## Simple Result about Stability In Continuous Time Model

- Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable function and suppose that there exists a unique  $x^*$  such that  $g(x^*) = 0$ . Moreover, suppose  $g(x) < 0$  for all  $x > x^*$  and  $g(x) > 0$  for all  $x < x^*$ . Then the steady state of the nonlinear differential equation  $\dot{x}(t) = g(x(t))$ ,  $x^*$ , is globally asymptotically stable, i.e., starting with any  $x(0)$ ,  $x(t) \rightarrow x^*$ .

# Simple Result in Figure



# Transitional Dynamics in the Continuous Time Solow Model II

**Proposition** Suppose that Assumptions 1 and 2 hold, then the basic Solow growth model in continuous time with constant population growth and no technological change is globally asymptotically stable, and starting from any  $k(0) > 0$ ,  $k(t) \rightarrow k^*$ .

- **Proof:** Follows immediately from the Theorem above by noting whenever  $k < k^*$ ,  $sf(k) - (n + \delta)k > 0$  and whenever  $k > k^*$ ,  $sf(k) - (n + \delta)k < 0$ .
- Figure: plots the right-hand side of (26) and makes it clear that whenever  $k < k^*$ ,  $\dot{k} > 0$  and whenever  $k > k^*$ ,  $\dot{k} < 0$ , so  $k$  monotonically converges to  $k^*$ .

# A First Look at Sustained Growth I

- Cobb-Douglas already showed that when  $\alpha$  is close to 1, adjustment to steady-state level can be very slow.
- Simplest model of sustained growth essentially takes  $\alpha = 1$  in terms of the Cobb-Douglas production function above.
- Relax Assumptions 1 and 2 and suppose

$$F [K (t) , L (t) , A (t)] = AK (t) , \quad (28)$$

where  $A > 0$  is a constant.

- So-called “AK” model, and in its simplest form output does not even depend on labor.
- Results we would like to highlight apply with more general constant returns to scale production functions,

$$F [K (t) , L (t) , A (t)] = AK (t) + BL (t) , \quad (29)$$

# A First Look at Sustained Growth II

- Assume population grows at  $n$  as before (cfr. equation (25)).
- Combining with the production function (28),

$$\frac{\dot{k}(t)}{k(t)} = sA - \delta - n.$$

- Therefore, if  $sA - \delta - n > 0$ , there will be sustained growth in the capital-labor ratio.
- From (28), this implies that there will be sustained growth in output per capita as well.

## A First Look at Sustained Growth III

**Proposition** Consider the Solow growth model with the production function (28) and suppose that  $sA - \delta - n > 0$ . Then in equilibrium, there is sustained growth of output per capita at the rate  $sA - \delta - n$ . In particular, starting with a capital-labor ratio  $k(0) > 0$ , the economy has

$$k(t) = \exp((sA - \delta - n)t) k(0)$$

and

$$y(t) = \exp((sA - \delta - n)t) A k(0).$$

- Note no transitional dynamics.



# Sustained Growth in Figure

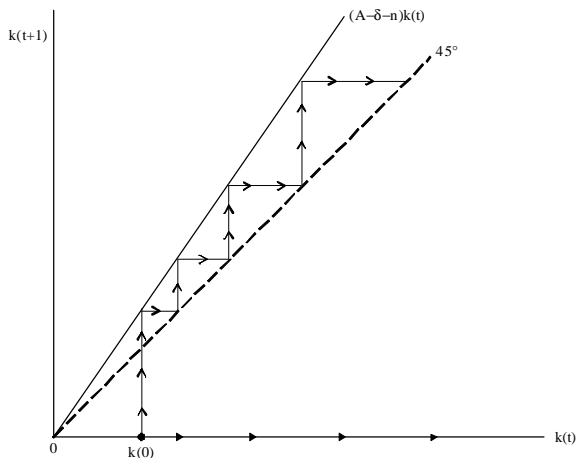


Figure: Sustained growth with the linear  $AK$  technology with  $sA - \delta - n > 0$ .

# A First Look at Sustained Growth IV

- Unattractive features:
  - ① Knife-edge case, requires the production function to be ultimately linear in the capital stock.
  - ② Implies that as time goes by the share of national income accruing to capital will increase towards 1.
  - ③ Technological progress seems to be a major (perhaps the most major) factor in understanding the process of economic growth.

# Balanced Growth I

- Production function  $F [K (t) , L (t) , A (t)]$  is too general.
- May not have *balanced growth*, i.e. a path of the economy consistent with the *Kaldor facts* (Kaldor, 1963).
- Kaldor facts:
  - while output per capita increases, the capital-output ratio, the interest rate, and the distribution of income between capital and labor remain roughly constant.
- We know that the capital share of national income is not really constant, and has been increasing over the last 30 years or so. Nevertheless, its “relative constancy” for almost a century might be an argument for sticking to Kaldor facts.
- More importantly, balanced growth is a very simple starting point.

## Balanced Growth II

- Note capital share in national income is about  $1/3$ , while the labor share is about  $2/3$ .
- Ignoring land, not a major factor of production.
- But in poor countries land is a major factor of production.
- This pattern often makes economists choose  $AK^{1/3}L^{2/3}$ .
- Main advantage from our point of view is that balanced growth is the same as a steady-state in transformed variables
  - i.e., we will again have  $\dot{k} = 0$ , but the definition of  $k$  will change.
- But important to bear in mind that growth has many non-balanced features.
  - e.g., the share of different sectors changes systematically.

# Types of Neutral Technological Progress I

- For some constant returns to scale function  $\tilde{F}$ :

- *Hicks-neutral* technological progress:

$$\tilde{F}[K(t), L(t), A(t)] = A(t) F[K(t), L(t)],$$

- Relabeling of the isoquants (without any change in their shape) of the function  $\tilde{F}[K(t), L(t), A(t)]$  in the  $L$ - $K$  space.
- *Solow-neutral* technological progress,

$$\tilde{F}[K(t), L(t), A(t)] = F[A(t)K(t), L(t)].$$

- Capital-augmenting progress: isoquants shifting with technological progress in a way that they have constant slope at a given labor-output ratio.
- *Harrod-neutral* technological progress,

$$\tilde{F}[K(t), L(t), A(t)] = F[K(t), A(t)L(t)].$$

- Increases output as if the economy had more labor: slope of the isoquants are constant along rays with constant capital-output ratio.

# Isoquants with Neutral Technological Progress

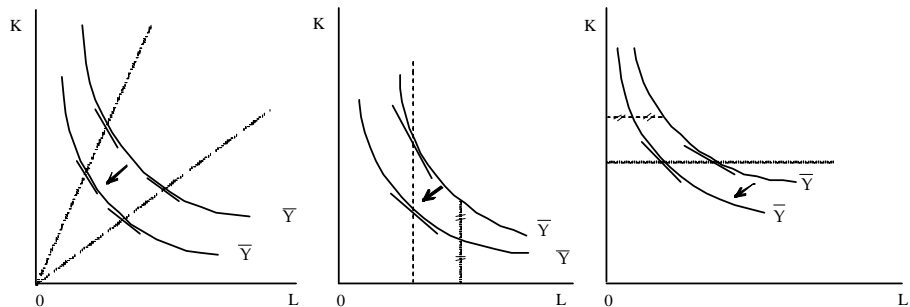


Figure: Hicks-neutral, Solow-neutral and Harrod-neutral shifts in isoquants.

## Types of Neutral Technological Progress II

- Could also have a vector valued index of technology  $\mathbf{A}(t) = (A_H(t), A_K(t), A_L(t))$  and a production function

$$\tilde{F}[K(t), L(t), \mathbf{A}(t)] = A_H(t) F[A_K(t) K(t), A_L(t) L(t)],$$

- Nests the constant elasticity of substitution production function introduced in the Example above.
- But even this is a restriction on the form of technological progress,  $A(t)$  could modify the entire production function.
- Balanced growth necessitates that all technological progress be labor augmenting or Harrod-neutral.

# Preliminaries

- Focus on continuous time models.
- Key elements of balanced growth: constancy of factor shares and of the capital-output ratio,  $K(t) / Y(t)$ .

- By factor shares, we mean

$$\alpha_L(t) \equiv \frac{w(t) L(t)}{Y(t)} \quad \text{and} \quad \alpha_K(t) \equiv \frac{R(t) K(t)}{Y(t)}.$$

- By Assumption 1 and Euler Theorem  $\alpha_L(t) + \alpha_K(t) = 1$ .



# Uzawa's Theorem

## Theorem

**(Uzawa I)** Suppose  $L(t) = \exp(nt) L(0)$ ,

$$Y(t) = \tilde{F}(K(t), L(t), \tilde{A}(t)),$$

$\dot{K}(t) = Y(t) - C(t) - \delta K(t)$ , and  $\tilde{F}$  is CRS in  $K$  and  $L$ .

Suppose for  $\tau < \infty$ ,  $\dot{Y}(t)/Y(t) = g_Y > 0$ ,  $\dot{K}(t)/K(t) = g_K > 0$  and  $\dot{C}(t)/C(t) = g_C > 0$ . Then,

- 1  $g_Y = g_K = g_C$ ; and
- 2 for any  $t \geq \tau$ ,  $\tilde{F}$  can be represented as

$$Y(t) = F(K(t), A(t)L(t)),$$

where  $A(t) \in \mathbb{R}_+$ ,  $F: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  is homogeneous of degree 1, and

$$\dot{A}(t)/A(t) = g = g_Y - n.$$

## Corollary

**Corollary** Under the assumptions of Uzawa Theorem, after time  $\tau$  technological progress can be represented as Harrod neutral (purely labor augmenting).

- Remarkable feature: stated and proved without any reference to equilibrium behavior or market clearing.
- Also, contrary to Uzawa's original theorem, not stated for a balanced growth path but only for an asymptotic path with constant rates of output, capital and consumption growth.
- **But**, not as general as it seems;
  - the theorem gives only one representation.

## Implications for Factor Shares

- Suppose the labor-augmenting representation of the aggregate production function applies.
- Then note that with competitive factor markets, as  $t \geq \tau$ ,

$$\begin{aligned}
 \alpha_K(t) &\equiv \frac{R(t) K(t)}{Y(t)} \\
 &= \frac{K(t)}{Y(t)} \frac{\partial F[K(t), A(t)L(t)]}{\partial K(t)} \\
 &= \alpha_K^*,
 \end{aligned}$$

- Second line uses the definition of the rental rate of capital in a competitive market
- Third line uses that  $g_Y = g_K$  and  $g_K = g + n$  from Uzawa Theorem and that  $F$  exhibits constant returns to scale so its derivative is homogeneous of degree 0.

## Intuition for the Uzawa's Theorems

- We assumed the economy features capital accumulation in the sense that  $g_K > 0$ .
- From the aggregate resource constraint, this is only possible if output and capital grow at the same rate.
- Either this growth rate is equal to  $n$  and there is no technological change (i.e., proposition applies with  $g = 0$ ), or the economy exhibits growth of per capita income and capital-labor ratio.
- The latter case creates an asymmetry between capital and labor: capital is accumulating faster than labor.
- Constancy of growth requires technological change to make up for this asymmetry
- But this intuition does not provide a reason for why technology should take labor-augmenting (Harrod-neutral) form.
- But if technology did not take this form, an asymptotic path with constant growth rates would not be possible.

# Interpretation

- Distressing result:
  - Balanced growth is only possible under a very stringent assumption.
  - Provides no reason why technological change should take this form.
- But when technology is endogenous, intuition above also works to make technology endogenously more labor-augmenting than capital augmenting.
- Not only requires labor augmenting asymptotically, i.e., along the balanced growth path.
- This is the pattern that certain classes of endogenous-technology models will generate.

## Implications for Modeling of Growth

- Does not require  $Y(t) = F[K(t), A(t)L(t)]$ , but only that it has a representation of the form  $Y(t) = F[K(t), A(t)L(t)]$ .
- Allows one important exception. If,

$$Y(t) = [A_K(t)K(t)]^\alpha [A_L(t)L(t)]^{1-\alpha},$$

then both  $A_K(t)$  and  $A_L(t)$  could grow asymptotically, while maintaining balanced growth.

- Because we can define  $A(t) = [A_K(t)]^{\alpha/(1-\alpha)} A_L(t)$  and the production function can be represented as

$$Y(t) = [K(t)]^\alpha [A(t)L(t)]^{1-\alpha}.$$

- Differences between labor-augmenting and capital-augmenting (and other forms) of technological progress matter when the elasticity of substitution between capital and labor is not equal to 1.

## Further Intuition

- Suppose the production function takes the special form  $F [A_K (t) K (t) , A_L (t) L (t)]$ .
- The stronger theorem implies that factor shares will be constant.
- Given constant returns to scale, this can only be the case when  $A_K (t) K (t)$  and  $A_L (t) L (t)$  grow at the same rate.
- The fact that the capital-output ratio is constant in steady state (or the fact that capital accumulates) implies that  $K (t)$  must grow at the same rate as  $A_L (t) L (t)$ .
- Thus balanced growth can only be possible if  $A_K (t)$  is asymptotically constant.

# The Solow Growth Model with Technological Progress: Continuous Time I

- From Uzawa Theorem, production function must admit representation of the form

$$Y(t) = F[K(t), A(t)L(t)],$$

- Moreover, suppose

$$\frac{\dot{A}(t)}{A(t)} = g, \quad (30)$$

$$\frac{\dot{L}(t)}{L(t)} = n.$$

- Again using the constant saving rate

$$\dot{K}(t) = sF[K(t), A(t)L(t)] - \delta K(t). \quad (31)$$



# The Solow Growth Model with Technological Progress: Continuous Time II

- Now define  $k(t)$  as the *effective capital-labor* ratio, i.e.,

$$k(t) \equiv \frac{K(t)}{A(t)L(t)}. \quad (32)$$

- Slight but useful abuse of notation.
- Differentiating this expression with respect to time,

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - g - n. \quad (33)$$

- Output per unit of effective labor can be written as

$$\begin{aligned} \hat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)} = F \left[ \frac{K(t)}{A(t)L(t)}, 1 \right] \\ &\equiv f(k(t)). \end{aligned}$$

# The Solow Growth Model with Technological Progress: Continuous Time III

- Income per capita is  $y(t) \equiv Y(t) / L(t)$ , i.e.,

$$\begin{aligned}y(t) &= A(t) \hat{y}(t) \\ &= A(t) f(k(t)).\end{aligned}\tag{34}$$

- Clearly if  $\hat{y}(t)$  is constant, income per capita,  $y(t)$ , will grow over time, since  $A(t)$  is growing.
- Thus should not look for “steady states” where income per capita is constant, but for *balanced growth paths*, where income per capita grows at a constant rate.
- Some transformed variables such as  $\hat{y}(t)$  or  $k(t)$  in (33) remain constant.
- Thus balanced growth paths can be thought of as steady states of a transformed model.

# The Solow Growth Model with Technological Progress: Continuous Time IV

- Hence use the terms “steady state” and balanced growth path interchangeably.
- Substituting for  $\dot{K}(t)$  from (31) into (33):

$$\frac{\dot{k}(t)}{k(t)} = \frac{sF[K(t), A(t)L(t)]}{K(t)} - (\delta + g + n).$$

- Now using (32),

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (\delta + g + n), \quad (35)$$

- Only difference is the presence of  $g$ :  $k$  is no longer the capital-labor ratio but the *effective* capital-labor ratio.

# The Solow Growth Model with Technological Progress: Continuous Time V

**Proposition** Consider the basic Solow growth model in continuous time, with Harrod-neutral technological progress at the rate  $g$  and population growth at the rate  $n$ . Suppose that Assumptions 1 and 2 hold, and define the effective capital-labor ratio as in (32). Then there exists a unique steady state (balanced growth path) equilibrium where the effective capital-labor ratio is equal to  $k^* \in (0, \infty)$  and is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta + g + n}{s}. \quad (36)$$

Per capita output and consumption grow at the rate  $g$ .

# The Solow Growth Model with Technological Progress: Continuous Time VI

- Equation (36), emphasizes that now total savings,  $sf(k)$ , are used for replenishing the capital stock for three distinct reasons:
  - 1 depreciation at the rate  $\delta$ .
  - 2 population growth at the rate  $n$ , which reduces capital per worker.
  - 3 Harrod-neutral technological progress at the rate  $g$ .
- Now replenishment of effective capital-labor ratio requires investments to be equal to  $(\delta + g + n)k$ .

# The Solow Growth Model with Technological Progress: Continuous Time VII

**Proposition** Suppose that Assumptions 1 and 2 hold, then the Solow growth model with Harrod-neutral technological progress and population growth in continuous time is asymptotically stable, i.e., starting from any  $k(0) > 0$ , the effective capital-labor ratio converges to a steady-state value  $k^*$  ( $k(t) \rightarrow k^*$ ).

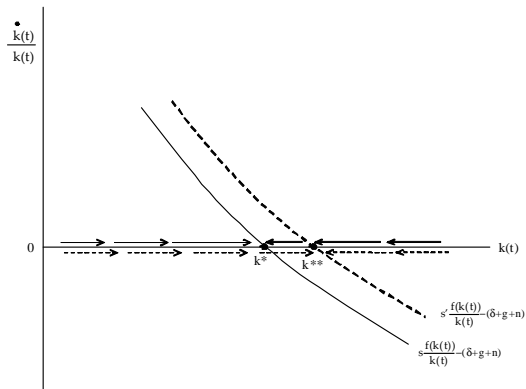
- Now model generates growth in output per capita, but entirely *exogenously*.

# Comparative Dynamics I

- Comparative dynamics: dynamic response of an economy to a change in its parameters or to shocks.
- Different from comparative statics in Propositions above in that we are interested in the entire path of adjustment of the economy following the shock or changing parameter.
- For brevity we will focus on the continuous time economy.
- Recall

$$\dot{k}(t) / k(t) = sf(k(t)) / k(t) - (\delta + g + n)$$

# Comparative Dynamics in Figure



**Figure:** Dynamics following an increase in the savings rate from  $s$  to  $s'$ . The solid arrows show the dynamics for the initial steady state, while the dashed arrows show the dynamics for the new steady state.



# Comparative Dynamics II

- One-time, unanticipated, permanent increase in the saving rate from  $s$  to  $s'$ .
  - Shifts curve to the right as shown by the dotted line, with a new intersection with the horizontal axis,  $k^{**}$ .
  - Arrows on the horizontal axis show how the effective capital-labor ratio adjusts gradually to  $k^{**}$ .
  - Immediately, the capital stock remains unchanged (since it is a *state* variable).
  - After this point, it follows the dashed arrows on the horizontal axis.
- $s$  changes in unanticipated manner at  $t = t'$ , but will be reversed back to its original value at some known future date  $t = t'' > t'$ .
  - Starting at  $t'$ , the economy follows the rightwards arrows until  $t'$ .
  - After  $t''$ , the original steady state of the differential equation applies and leftwards arrows become effective.
  - From  $t''$  onwards, economy gradually returns back to its original balanced growth equilibrium,  $k^*$ .

# Conclusions

- Simple and tractable framework, which allows us to discuss capital accumulation and the implications of technological progress.
- Solow model shows us that if there is no technological progress, and as long as we are not in the *AK* world, there will be no sustained growth.
- Generate per capita output growth, but only exogenously: technological progress is a blackbox.
- Capital accumulation: determined by the saving rate, the depreciation rate and the rate of population growth. All are exogenous.
- Need to dig deeper and understand what lies in these black boxes.