Network Markets and Coordination Games

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Introduction

Several oligopolistic industries that play a crucial role in modern economies are "network industries": industries where the benefit that an individual consumer derives from consuming a good increases with the number of other people consuming the same good. Network markets can be one-sided or two-sided. Examples of the first class of markets are software markets: the utility an individual derives from a given software is increasing in the total number of users because the more people own the software, the more people they can interact with, exchanging files and know-how. The second class of markets includes auction websites, credit card networks, directory services and all those markets where two groups of individuals or firms need a common platform to interact and one or more firms own platforms and sell access to them. In this case, the utility derived from accessing a platform is increasing in the number of potential counterparts who join the same platform. Also, network products can be divided into pure network goods, such as telecommunication networks, and non-pure network goods. The first class of products includes all those goods for which the network externality is the only source of utility a consumer derives from the product, while the second class includes al those products that also provide an intrinsic utility.

A classic methodological issue related to network markets is that modeling the demand function is particularly challenging because the problem faced by consumers choosing which network to join, for given prices, constitutes a coordination game and typically coordination games have multiple equilibria.

My dissertation explores three different approaches to this problem.

In chapter 1, I analyze one-sided markets for non-pure network goods where consumers can choose between two alternative products that are both vertically and horizontally differentiated. Modeling consumer choice as a global game with private values, I show that either a large amount of horizontal differentiation or a high positive correlation of consumers' private values of the goods are necessary and sufficient for the demand function to be well-defined. Using this result, I then address the issue of efficiency on these markets. I derive the equilibrium allocation of consumers to the networks for both the case of sponsored and unsponsored networks as well as the allocation that would maximize social welfare. The three allocations share two important features: in all of them both networks are active, due to the presence of sufficiently strong horizontal differentiation, and the high quality network attracts more than one half of the consumers. Nonetheless, two inefficiencies arise. First, since consumers fail to internalize network externalities, the equilibrium allocation with unsponsored networks is too balanced. Second, if access to the networks is priced by strategic firms, then the firm with a higher expected quality charges a price higher than the competitor's and this further reduces the asymmetry between market shares and therefore social welfare.

In chapter 2, which is coauthored with Attila Ambrus, we analyze two-sided markets for pure network goods. We address the issue of multiple equilibria of the coordination game played by consumers, assuming that the latter can coordinate their decisions to their advantage, if their interests coincide and if coordination can be achieved without communication. Using this methodology, we show that multiple asymmetric networks can coexist in equilibrium if consumers have heterogeneous reservation values. If the market is a monopoly, the network provider might choose to operate multiple networks to price differentiate consumers with different reservation values on both sides of the market. If the market is a duopoly, the two competing network providers might price their products in such a way that one of them attracts high reservation value consumers on one side of the market and low reservation value consumers on the other side, and vice versa. In these asymmetric equilibria, access to intrinsically identical networks can be sold at different prices because network externalities determine endogenous product differentiation: the larger the set of consumers from one side who join a network, the more attractive the network becomes for consumers on the other side.

In chapter 3, which is coauthored with Itzhak Gilboa, we address the issue of belief formation in coordination games. We take the view that players form their beliefs about other players' behavior by looking at history, that is, at the outcomes of similar coordination games played in the past, possibly by other players. A simple model is analyzed, in which a large population has to make a simultaneous decision regarding participation in a coup attempt. A dynamic process faces different populations with such games for randomly selected values of a parameter. We show that history serves as a coordination device, and determines for which values of the parameter a revolution would succeed. We also show that, for intermediate values of the parameter in question, the limit behavior depends on the way history unfolds, and cannot be determined from a priori considerations.

Chapter 1

Differentiated Networks: Equilibrium and Efficiency

1.1 Introduction

Many economic decisions, such as the purchase of a good by a population of consumers or the adoption of a standard of production by a set of firms, exhibit a specific form of strategic complementarity known as network externality. The set of consumers buying a specific product and the set of firms producing according to a given standard constitute virtual networks: the externality arises when the payoff of each player who belongs to a network is increasing in the total size of the network itself.

In most cases where two alternative network goods are available, they are strongly differentiated. Therefore, characterizing the social optimum is a non-trivial problem: the aggregate surplus from the network effect is maximized if all players join the same network, while the surplus from the intrinsic utility from consumption of the good is maximized if every player buys the product he likes the most.

This trade-off is exemplified by the case of PC and Macintosh. If there was only one standard, every computer owner could run all the existing software and easily exchange files and know-how with anybody else. On the other hand, one of the main reasons why the two standards do coexist is that the two types of computers have different strengths that appeal to the needs of different consumers.[1]

Another example is the case of matrix programming languages. Matlab seems to be best suited to analyze time-series data, while Gauss seems to perform better when analyzing panel data.[2] At the same time, all programmers would benefit from the existence of a unique common language because they could exchange suggestions and pieces of code with a larger group of people.

In this chapter, I investigate this trade-off between maximizing the aggregated network effect and letting consumers use their favorite product, and show that the market outcome is affected by two types of inefficiency.

I consider two cases, one where the networks are "sponsored" and one where the networks are "unsponsored". In the network literature, this distinction refers to the cost an individual bears to join a network: a network is sponsored if the access to it is priced by a strategic player. While this is the case for most telecommunication networks, there are many cases of unsponsored networks as well. For example, by learning a new language, an individual can join a "communication network" that clearly exhibits network externalities. In this case,

[1] This differentiation is both vertical and horizontal. On its website, Apple lists the top ten reasons to switch to a Mac. Among them, "reason number 2" is that a Mac doesn't crash (clearly a claim of higher vertical quality) while "reason number 3" is the fact that Mac offers the best technology to store and play digital music (which is a feature that different consumers might value differently, depending on the specific use they make of their computers).

[2] See Rust (1993).

there is no centralized entity that owns the language and strategically sets a price for the access to the network. The "price" is simply the time spent studying the language.

In my model, I assume that two new, alternative network products are introduced and that consumers simultaneously choose which one to join. If the networks are sponsored, access to each network is priced by a strategic firm. If they are unsponsored, access is available at marginal cost. The goods are both vertically and horizontally differentiated but neither firms nor consumers can perfectly observe the vertical dimension of quality. More precisely, they observe a noisy public signal about the vertical quality of each good, and one of the two networks has a higher expected quality. Each consumer privately and perfectly observes his private value for each good but he cannot distinguish the common component (the objective quality of the good) from his idiosyncratic taste component. When choosing which network to join, each individual takes into account three elements: his private valuation for each good, the expected size of each network, and the cost he has to bear to join it.

Consider, for example, the introduction of two new, non-interconnected telecommunication networks, such as two softwares for videoconference. The assumption of noisy information about the vertical quality of each good captures the fact that the overall performance of such products critically depends on how well they interact with the complementary hardware and software, and this interaction cannot be perfectly tested before the product is introduced. The assumption of horizontal differentiation captures the idiosyncratic taste consumers might have for the more "recreational" features of these products, such as the graphic interface. Finally, the assumption that each consumer perfectly observes her private value for each good before making a purchase captures the fact that software developers typically make trial versions of their products available for free.

The main question addressed in this chapter is how the equilibrium allocation of consumers across networks compares to the social optimum. I find that the optimal allocation of consumers across networks is asymmetric, with both networks having positive market shares and more than one half of the population joining the high-quality network. I then look at the allocation implemented by the market, distinguishing between the two cases of sponsored and unsponsored networks.

In the case of unsponsored networks, I identify a first source of inefficiency. Since consumers fail to internalize network effects, the equilibrium allocation is too balanced from a social point of view: the market share of the high-quality firm, though larger than one half, is still smaller than would be socially optimal. Looking at the case of sponsored networks, I find that strategic pricing makes the inefficiency even worse: the high-quality firm has an intrinsic advantage that is reflected in an equilibrium price higher than the one charged by the competitor, and this in turn reduces her market share even more.

The analysis in this chapter also brings a methodological contribution to the literature on network markets. There is a problem with modelling the demand function for network goods. The game played by consumers choosing which network to join, for given prices, constitutes a coordination game and typically these games have multiple equilibria. Therefore, in a basic model of Bertrand competition between identical networks, the demand for each good is not a well-defined function of prices. I enrich the basic model by allowing for both horizontal and vertical differentiation. Consumers' choices are then a global game with correlated private values. This allows me to derive necessary and sufficient conditions on the information structure of the game for the demand function to be well-defined.

A number of papers analyze price competition between networks. De Palma and Leruth (1993) allow only for horizontal differentiation. Their emphasis is on the conditions for a unique equilibrium of the coordination game played by consumers, rather than the efficiency issues emphasized here. They prove that a large degree of differentiation is necessary and sufficient for uniqueness of the equilibrium in that context. This condition is similar to the condition I derive in the limit case of my model where all the players perfectly observe the vertical dimension of quality and the latter is identical for the two goods. On the other hand, Farrell and Katz (1998) assume only vertical differentiation, therefore their model allows for multiple sets of self-fulfilling expectations in the coordination game played by consumers. My discussion of the optimal allocation of consumers between networks is also related to Farrell and Saloner (1986), who address the issue of whether complete standardization is efficient if consumers have heterogeneous preferences. My work is also connected to the literature on duopolistic price competition with differentiated goods. For a discussion of this literature, see Tirole (1988).

In this chapter, I model consumers' choice as a global game with correlated private values. Global games, first analyzed by Carlsson and van Damme (1993), have been typically applied to economic situations where players have common values, such as currency crises and regime switches. Morris and Shin (2004) analyze a private-value global game where two players choose between two alternative actions and each player privately observes his own payoff type which is the sum of a common component and an idiosyncratic component.

They derive a condition on the information structure that is necessary and sufficient for a unique equilibrium in that context. The solution of the model is identical in the case of a continuum of players choosing between two actions (Morris an Shin (2003)). In my model, I apply the Morris-Shin condition for uniqueness, and relate it to the degree of horizontal differentiation and the precision of the available information about the vertical dimension of quality.

The chapter is organized as follows. In Section 1.2 I introduce the formal model. In Section 1.3 I characterize the socially optimal allocation of consumers to networks. In Section 1.4 I characterize the equilibrium allocation for both the case of unsponsored and sponsored networks and compare it to the efficient one. Section 1.5 concludes. All the proofs are relegated to Appendix A.

1.2 The Model

I assume that two indivisible network goods, a and b, become available to a population of consumers represented by a continuum of mass 1. The two goods are produced at the same, constant marginal cost c. If the networks are sponsored, each good is produced and sold by a firm who strategically chooses her price to maximize her profits:

$$\pi_j = \left(p^j - c\right) n^j$$

where p^j is the price charged by firm j (with $j = a, b$) and n^j is the number of units she sells. If the networks are unsponsored, there are no strategic firms and each good can be purchased at marginal cost.

Preferences exhibit network externalities: the utility that any consumer i derives from joining network j is increasing in n^j. More precisely, I assume:

$$U_i^j = x_i^j + n^j - p^j$$

where x_i^j represents the intrinsic value of good j for consumer i.

I further assume that the two goods are both vertically and horizontally differentiated. The intrinsic value of a good for a consumer is the sum of a common value component $\widetilde{\theta}^j$, representing the vertical dimension of quality of brand j, and an idiosyncratic component $\widetilde{\varepsilon}_i^j$, representing individual taste:

$$\widetilde{x}_i^j \equiv \widetilde{\theta}^j + \widetilde{\varepsilon}_i^j.$$

I assume that each $\widetilde{\theta}^j$ is a normal random variable:

$$\widetilde{\theta}^j \sim N\left(y^j, \frac{2}{\alpha}\right).$$

The expected value of $\widetilde{\theta}^j$, given by y^j, can be interpreted as a noisy public signal about $\widetilde{\theta}^j$.

I model consumer heterogeneity assuming that each individual's idiosyncratic taste component is normally distributed around zero:

$$\widetilde{\varepsilon}_i^j \sim N\left(0, \frac{2}{\beta}\right).$$

Finally, I assume that $\widetilde{\theta}^a$, $\widetilde{\theta}^b$ and each $\widetilde{\varepsilon}_i^j$ are independently distributed. All the above distributional assumptions are common knowledge.

Each consumer privately observes the vector

$$\mathbf{x}_i = \left(x_i^a, x_i^b\right) \in \mathbb{R}^2$$

which constitutes his type. I will denote a type profile for all consumers as

$$\mathbf{x} = \times_{i\in[0,1]} \mathbf{x}_i \in \left(\mathbb{R}^2\right)^{[0,1]}.$$

Notice that consumers have correlated private values: types are correlated, due to the presence of a common component, but each consumer's payoff does not depend directly on other consumers' types. I emphasize that a consumer can only observe his type but he cannot observe θ^j and ε_i^j separately. More precisely, neither firms nor consumers can perfectly observe the vertical quality of each good.

I can now formally define actions, strategies and payoff functions for the players. If the networks are sponsored, then the two firms and the population of consumers play a game of incomplete information in two stages. In the first stage, firms announce prices $\left(p^a, p^b\right)$ simultaneously and noncooperatively. An action for firm j is a price $p^j \in \mathbb{R}$. The strategy space for the firms coincides with their action space and a strategy profile for the firms is a vector $\mathbf{p} = \left(p^a, p^b\right) \in \mathbb{R}^2$.

In the second stage of the game, consumers learn their types, observe prices and simultaneously choose a network. I assume complete market coverage and exclusivity: each consumer buys exactly one unit of one good, thus joining either network a or network b. Formally, an action for consumer i is $r_i \in \{a, b\}$, and an action profile for all the consumers is $\mathbf{r} \in \{a, b\}^{[0,1]}$. A pure strategy for a consumer is

$$s_i : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \{a, b\}$$

and a strategy profile for all consumers is $\mathbf{s} \equiv \times_{i \in [0,1]} s_i$. The size of network j when consumers play strategy profile $\mathbf{s}$ and firms play strategy profile $\mathbf{p}$ is $n^j(\mathbf{s}(\mathbf{x}, \mathbf{p}))$.

In the first stage, each firm maximizes her expected profits solving

$$\max_{p^j \in \mathbb{R}} \mathbb{E}_{\mathbf{x}} \left[\pi_j \left(n^j (\mathbf{s}(\mathbf{x}, \mathbf{p})), \mathbf{p} \right) \right] = \max_{p^j \in \mathbb{R}} (p^j - c) \mathbb{E}_{\mathbf{x}} \left[n^j (\mathbf{s}(\mathbf{x}, \mathbf{p})) \right].$$

In the second stage, each consumer maximizes his expected net surplus solving

$$\max_{j \in \{a,b\}} \mathbb{E}_{\mathbf{x}_{-i}} \left[U_i^j (\mathbf{x}_i, (\mathbf{s}(\mathbf{x}, \mathbf{p})), \mathbf{p}) | \mathbf{x}_i \right] =$$

$$= \max_{j \in \{a,b\}} \left(x_i^j + \mathbb{E}_{\mathbf{x}_{-i}} \left[n^j (\mathbf{s}(\mathbf{x}, \mathbf{p})) | \mathbf{x}_i \right] - p^j \right).$$

If networks are unsponsored, then both goods are priced at marginal cost and consumers play a static game of incomplete information equivalent to the second stage of the above game for the case $p^a = p^b = c$.

For notational convenience, I also define the following differences:

$$
\begin{aligned}
y &\equiv \frac{y^a - y^b}{2} \\
\widetilde{\theta} &\equiv \frac{\widetilde{\theta}^a - \widetilde{\theta}^b}{2} \sim N\left(y, \frac{1}{\alpha}\right) \\
\widetilde{\varepsilon}_i &\equiv \frac{\widetilde{\varepsilon}_i^a - \widetilde{\varepsilon}_i^b}{2} \sim N\left(0, \frac{1}{\beta}\right) \\
\widetilde{x}_i &\equiv \frac{\widetilde{x}_i^a - \widetilde{x}_i^b}{2} \sim N\left(y, \frac{\alpha+\beta}{\alpha\beta}\right).
\end{aligned}
$$

The above variables can be easily interpreted. The random variable $\widetilde{\theta}$ is an index of vertical differentiation. For positive realizations of $\widetilde{\theta}$, good a has a higher objective quality than good b, and the opposite is true for negative values. The parameter y is a public signal about the difference in quality between the two goods. Without loss of generality, I will assume that $y \geqslant 0$ throughout the chapter (i.e. I will assume that good a has a weakly higher expected quality than good b). The parameter α represents the precision of the public information about the difference in quality between the two goods. The random variable $\widetilde{\varepsilon}_i$ captures the idiosyncratic differences in taste among consumers and β, the precision of its distribution, is an index of the amount of heterogeneity among consumers.[3] More precisely, the smaller is β, the larger is the amount of horizontal differentiation between the two goods. The random variable $\widetilde{x}_i$ is a measure consumer i's preference for good a, the good with the highest expected quality. For positive realizations of $\widetilde{x}_i$, he prefers a to b. For negative realizations, his idiosyncratic preference for b is so strong that he prefers b to a. Notice that I use the notation $\mathbf{x}_i$ for the vector $\left(x_i^a, x_i^b\right)$ and the notation x_i for the difference $\frac{x_i^a - x_i^b}{2}$.

[3] Throughout the chapter, I will assume that the law of large number holds for a continuum of independent variables, i.e. we will assume that the distribution of the idiosyncratic component in the population is the same as the distribution of any individual ε_i.

Finally, I define

$$p\left(p^a, p^b\right) \equiv \frac{p^a - p^b}{2}.$$

In what follows, I will use the simplified notation p to denote $p\left(p^a, p^b\right)$.

1.3 Efficient Allocation

In this section, I derive the ex-ante efficient allocation of consumers across networks. I choose to address the issue of efficiency from an ex-ante point of view because I believe that the right benchmark to compare to the allocation implemented by the market is the allocation that maximizes welfare given all, and only, the public information available on the market.[4]

First, I need to define the welfare criterion. Given my assumption of quasi-linear utility functions, aggregate welfare equals consumer gross surplus minus total cost, since prices are a transfer from consumers to firms. Also, since I assumed that the two firms have the same marginal cost and that the total number of units sold in the market is constant, I can ignore costs as I solve the welfare maximization problem. Finally, the only choice variable in the welfare maximization problem is the allocation of consumers between the two networks.

[4] For the concept of ex-ante efficiency, see Holmström and Myerson (1983).

Therefore, the welfare maximization problem is equivalent to the following problem[5]:

$$\begin{aligned} \max_{\mathcal{A},\mathcal{B}} \mathbb{E}_{\mathbf{x}}\left[W\left(\mathcal{A},\mathcal{B}\right)\right] &= \max_{\mathcal{A},\mathcal{B}} \mathbb{E}_{\mathbf{x}}\left[\int_{i\in\mathcal{A}}\left(x_i^a+n^a\right)di+\int_{i\in\mathcal{B}}\left(x_i^b+n^b\right)di\right] \qquad (*) \\ s.t.\ \mathcal{A}\cup\mathcal{B} &= [0,1] \\ \mathcal{A}\cap\mathcal{B} &= \varnothing. \end{aligned}$$

Next, I show that the optimal choice of the sets $(\mathcal{A},\mathcal{B})$ can be characterized as a threshold allocation. By this I mean that a benevolent social planner with access to all the public information available on this market, and the power to assign each consumer to a network, would maximize social welfare by choosing a threshold t and allocating to network a those consumers with a preference for a larger than t and to network b those with a preference for a smaller than that. Formally, I define a threshold allocation as follows:

Definition 1.1.

*A **Threshold Allocation** is a couple* $(\mathcal{A}(t),\mathcal{B}(t))$ *such that* $\exists t\in\mathbb{R}$ *such that* $\mathcal{A}(t)=\{i\in[0,1]:x_i>t\}$ *and* $\mathcal{B}(t)=\{i\in[0,1]:x_i\leq t\}$.

Lemma 1.1.

The welfare maximizing allocation is a threshold allocation.

In what follows, I will denote by $(\mathcal{A}^*,\mathcal{B}^*)$ the solution to problem $(*)$ and by t^* the corresponding threshold. The intuition for the result presented in Lemma 1.1 is the following. Consider an arbitrary allocation of consumers to the networks. If this is not a threshold

[5] In the definition of problem $(*)$ I am giving a heuristic description of the welfare criterion ignoring measurability issues. In the rest of the chapter, all the allocations where I will evaluate welfare will be threshold allocations (see Definition 1.1). This fact, together with the assumption that the law of large number holds for a continuum of independent variables, will guarantee that welfare is well-defined for those allocations.

allocation, then there are at least two groups of consumers such that the first is allocated to network b, the second to network a, and those in the first group have a larger preference for a than those in the second group. Suppose I move a positive measure of consumers of the first group to network a and a set of consumers of the second group, with the same measure, to network b. Market shares stay constant, the gross surplus of the consumers I moved increases and that of any other consumer is unaffected. Therefore, total welfare increases and this proves that the initial allocation was not welfare maximizing.

In what follows, I will denote the welfare associated to a given threshold allocation as

$$\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right] \equiv \mathbb{E}_{\mathbf{x}}\left[W\left(\mathcal{A}\left(t\right),\mathcal{B}\left(t\right)\right)\right].$$

Given that the optimal allocation is associated to a threshold, the final step to solve the welfare maximization problem is to identify this optimal threshold. First, notice that the welfare function can be rewritten as the sum of three components:

$$\begin{aligned}\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right] = \mathbb{E}_{\mathbf{x}}\Bigg[&\left(\int_{i\in\mathcal{A}(t)}\theta^{a}di + \int_{i\in\mathcal{B}(t)}\theta^{b}di\right) + \\ &+\left(\int_{i\in\mathcal{A}(t)}\varepsilon_{i}^{a}di + \int_{i\in\mathcal{B}(t)}\varepsilon_{i}^{b}di\right) + \\ &+\left(\int_{i\in\mathcal{A}(t)}n^{a}di + \int_{i\in\mathcal{B}(t)}n^{b}di\right)\Bigg].\end{aligned} \tag{1.1}$$

The first component measures the aggregate surplus consumers derive from the vertical quality of the goods, the second measures the aggregate surplus derived from idiosyncratic taste and finally the third component measures the aggregate surplus derived from network effects. By evaluating expression 1.1 for a given realization of $\left(\widetilde{\theta}^{a},\widetilde{\theta}^{b}\right)$ and then taking the expectation, the welfare function can be finally be re-written as

$$\begin{aligned}\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right] &= \mathbb{E}_{\theta^a,\theta^b}\Big\{\theta^a - 2\theta\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right) + \\ &\quad + \sqrt{\frac{2}{\pi\beta}}e^{-\frac{(t-\theta)^2\beta}{2}} + \\ &\quad + 2\left[\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\right]^2 - 2\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right) + 1\Big\}\end{aligned} \tag{1.2}$$

where $\Phi(\cdot)$ denotes the cdf of a standard normal random variable.[6]

Before I characterize the optimal threshold, it is worth to analyze in larger detail the three components of the welfare function.

For a given realization of $\left(\widetilde{\theta}^a, \widetilde{\theta}^b\right)$, the first component,

$$\theta^a - 2\theta\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right), \tag{1.3}$$

measures the aggregate surplus derived from the vertical quality of the goods. Expression 1.3 can be easily interpreted. If $t = -\infty$ it reduces to θ^a : all consumers are assigned to network a and a measure 1 of individuals enjoy quality θ^a. For a finite t, if I denote by $F(\cdot)$ the cdf of x_i, then only $1 - F(t)$ consumers are assigned to a, while the remaining $F(t)$ consumers are assigned to b and enjoy quality θ^b. Substituting to $F(t)$ its expression and using the variable θ defined in section 1.2, I get 1.3. If the realized quality difference θ is positive, it easy to see that 1.3 is maximized by $t = -\infty$, i.e. by an allocation where all consumers join the high quality network. (See Figure 1.1[7]).

[6] The derivation of 1.2 is included in the proof of Proposition 1.1.

[7] The main purpose of Figure 1.3 and Figure 1.2 is to describe the qualitative features of the components of consumer surplus associated with vertical and horizontal quality for a given positive θ. For this illustrative purpose, I abstract from the absolute values each of these functions takes for specific values of the parameters

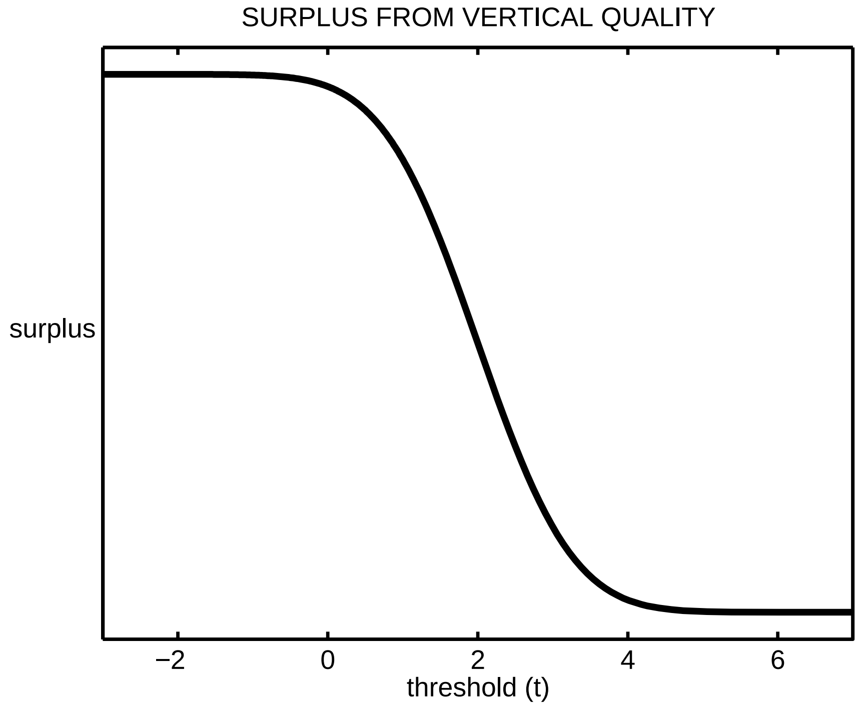

Figure 1.1: Aggregate surplus derived from the vertical quality of the goods for the case $\theta = 2$.

(α, β) and specific realizations of $\left(\theta^a, \theta^b\right)$. Therefore, I do not report the scale of the vertical axis.

The second welfare component for a given realization of $\left(\widetilde{\theta}^a, \widetilde{\theta}^b\right)$,

$$\sqrt{\frac{2}{\pi\beta}}e^{-\frac{(t-\theta)^2\beta}{2}}, \tag{1.4}$$

measures the portion of aggregate surplus due to consumers' idiosyncratic taste for the network to which they are allocated. Expression 1.4 is maximized at $t = \theta$. (See Figure. 1.2). Intuitively, this idiosyncratic surplus component is maximized when each consumer is assigned to the network for which he has an idiosyncratic preference. If θ were perfectly observable, this could be achieved by choosing $t = \theta$, so that every consumer with a positive ε_i would join a and the others would join b.

Finally, for given $\left(\widetilde{\theta}^a, \widetilde{\theta}^b\right)$, the last welfare component,

$$2\left[\Phi\left((t-\theta)\sqrt{\beta}\right)\right]^2 - 2\Phi\left((t-\theta)\sqrt{\beta}\right) + 1 \tag{1.5}$$

measures the aggregated network effect. Expression 1.5 can be easily interpreted. The total network effect is given by the sum of the squared market shares (all consumers in a network receive a positive effect measured by the size of the network itself). Denoting again by $F(\cdot)$ the cdf of x_i, for a given threshold t market shares are $n^a = 1 - F(t)$ and $n^b = F(t)$. Substituting to $F(t)$ its expression I get 1.5.

This component of consumer surplus is maximized by $t \in \{-\infty, +\infty\}$ since for any of those values all consumers join the same network and therefore each of them enjoys the largest possible network effect. (See Figure 1.3).

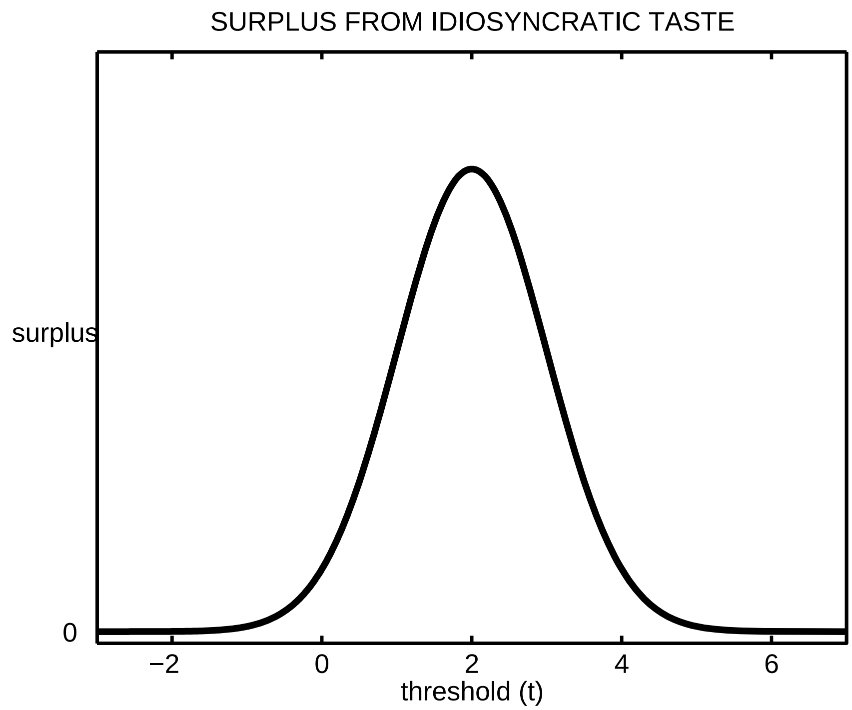

Figure 1.2: Aggregate surplus derived from idiosyncratic taste for the case $\theta = 2$.

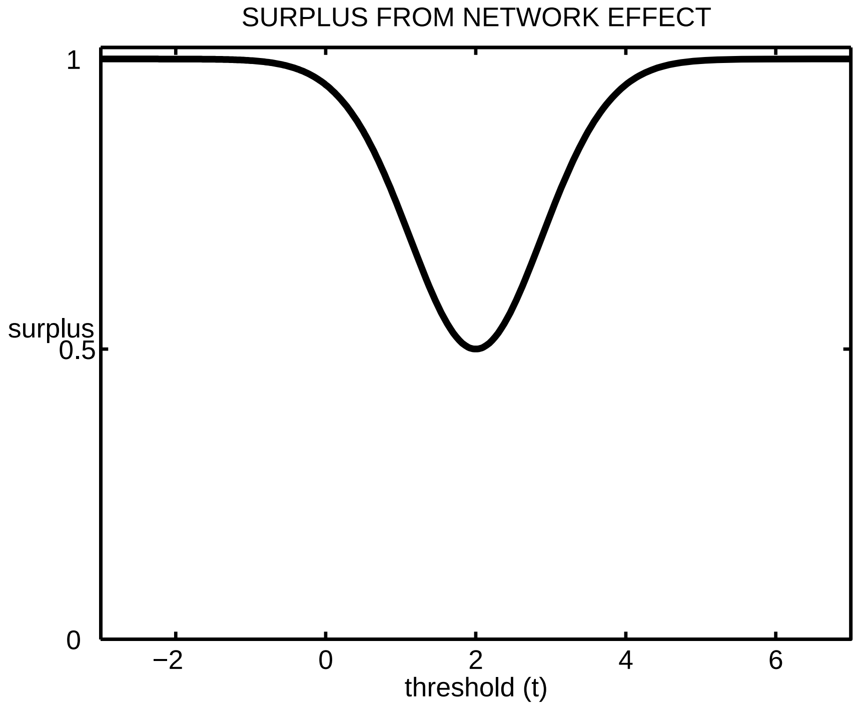

Figure 1.3: Aggregate surplus derived from the network effect for the case $\theta = 2$.

Given this decomposition, it appears that there is a trade-off between maximizing the first and the third component of welfare by assigning all consumers to the same network and maximizing the second component by splitting consumers equally. Proposition 1.1 characterizes the socially optimal allocation.

Proposition 1.1 (Ex-Ante Efficient Allocation).

Social welfare is maximized by an asymmetric threshold allocation such that

$$-1 < t^* < 0 < y.$$

As one would intuitively expect, $t^* < y$: the optimal allocation is such that the network with the largest expected quality has the largest market share.

Nonetheless, the tension between vertical differentiation and network effect on the one hand, and heterogeneity in consumers' taste on the other hand, results in an interior optimum. All the consumers with a positive x_i, who therefore have a private preference for a and who constitute more than one half of the population, and some of those with a small preference for b, should join the large network a. The minority of consumers with a strong preference for b should form another, much smaller network.

1.4 The Equilibrium Allocation

In this section, I characterize the allocations that prevail in equilibrium in the market, for the case of unsponsored and sponsored networks, in order to compare them to the efficient one that I characterized in Section 1.3. First, I will solve the coordination game consumers play when they choose which network to join, for given prices. Then, I will consider separately the case where prices are simply equal to the marginal cost and the case where they are set by strategic firms.

1.4.1 Consumer Coordination

The next result characterizes the equilibrium of the coordination game played by consumers for given price difference p and given expected quality difference y. This coordination game constitutes a global game with correlated private values, such as the one analyzed by Morris and Shin (2004). I can apply their methodology to characterize the equilibrium and identify the necessary and sufficient condition for its uniqueness.

Suppose consumers observe a quadruple $\left\{p^a, p^b, y^a, y^b\right\}$. Consider a type $\widehat{\mathbf{x}}_i$ of consumer i with the following property: if a consumer has type $\widehat{\mathbf{x}}_i$, and he believes that every consumer with a preference for a larger than his own will join a, and every consumer with a preference for a smaller than his own will choose b, then he is indifferent between the two networks. Formally, $\widehat{\mathbf{x}}_i(p, y)$ is a solution to:

$$\begin{aligned} & \left(x_i^a + \Pr\left(x_{i'} > x_i\right) | \, \mathbf{x_i} - p^a\right) = \qquad (1.6) \\ = \quad & \left(x_i^b + \Pr\left(x_{i'} \leq x_i\right) | \, \mathbf{x_i} - p^b\right). \end{aligned}$$

Let $t(p, y)$ be the intrinsic preference for a of a consumer of type $\widehat{\mathbf{x}}_i(p, y)$. That is:

$$t(p, y) \equiv \frac{\widehat{x}_i^a(p, y) - \widehat{x}_i^b(p, y)}{2}.$$

By rearranging 1.6, it can be shown that $t(p, y)$ is implicitly defined by the following equation:

$$t - \Phi\left[(t-y)\,z\right] + \frac{1}{2} - p = 0 \tag{1.7}$$

where

$$z = z(\alpha,\beta) \equiv \sqrt{\frac{\alpha^2\beta}{(\alpha+\beta)(\alpha+2\beta)}} \quad . \tag{1.8}$$

The following proposition holds:

Proposition 1.2 (Equilibrium of the Coordination Game).

If $z \leqslant \sqrt{2\pi}$, the coordination game played by consumers after any price announcement $\left(p^a, p^b\right)$ has the following unique Nash-equilibrium:

$$s_i^*(\mathbf{x},\mathbf{p}) = \begin{cases} a \text{ if } x_i > t(p,y), \\ \\ b \text{ if } x_i \leq t(p,y), \end{cases} \qquad \forall i \in [0,1]\,.$$

where $t(p,y)$ satisfies

$$t(p,y) - \Phi\left[(t(p,y)-y)\,z\right] + \frac{1}{2} - p = 0.$$

The threshold $t(p,y)$ is strictly increasing in p and strictly decreasing in y.

The equilibrium described by Proposition 1.2 is a symmetric equilibrium in switching strategies around the threshold $t(p,y)$: consumers with a preference for a larger than $t(p,y)$ join a and the others join b. Therefore, the equilibrium allocation is a threshold allocation. This will prove particularly useful as I compare the market allocation to the efficient one.

The two available pieces of public information, namely the public signal y about the relative quality of the two goods and the difference p between the two prices, affect the equilibrium threshold in a very intuitive way. For a given quality signal, if a becomes more expensive with respect to b, the threshold moves to the right, i.e. a loses some consumers to b.

For given prices, instead, if a's expected quality advantage y increases, the threshold moves to the left, i.e. a gains some consumers.

To capture the intuition behind the uniqueness condition mentioned in the statement of Proposition 1.2, first notice that the above equilibrium in switching strategies is unique if there exists only one value of the threshold t that solves 1.7. In turn, 1.7 has a unique solution $t(p, y)$ if the second term is (almost) invariant with respect to t. That term represents the beliefs of a player who observes a private signal exactly equal to the threshold t. More precisely, it represents his expectation of the proportion of consumers with a private value x_i smaller than his own, i.e. smaller than t. If this expectation is very sensitive to changes in t, then the strategic uncertainty faced by a consumer is very sensitive to his type. As shown by Morris and Shin (2004), uniqueness of equilibrium in global games with correlated private values is guaranteed by a low degree of sensitivity of strategic uncertainty to a player's type. In particular, this low level of sensitivity can be achieved by one of two conditions, both summarized by the inequality $z \leqslant \sqrt{2\pi}$: either the private signals have to be very correlated or consumers preferences have to be very heterogeneous. In the first case, no matter what specific realization of $\widetilde{x}_i$ he observed, any consumer i knows $\widetilde{x}_{i'}$ and $\widetilde{x}_i$ are highly correlated and therefore thinks that it is about as likely for any consumer i' to observe $x_{i'} > x_i$ or $x_{i'} < x_i$. In the second case, the high variance of the heterogeneous idiosyncratic component makes the private signals almost independent, therefore x_i is not very informative on $x_{i'}$ and so, once again, whatever the realization of $\widetilde{x}_i$ he observed consumer i thinks that it is about as likely for any consumer i' to observe $x_{i'} > x_i$ or $x_{i'} < x_i$.

Throughout the rest of the chapter I will assume that the condition for uniqueness, $z \leqslant \sqrt{2\pi}$, is satisfied, focusing my attention on those network markets where the demand function is naturally well-defined. In other words, I will focus on markets where either there is a sufficient amount of horizontal differentiation or the information on the vertical quality of the goods is sufficiently noisy.

Before I consider the two cases of sponsored and unsponsored networks in more detail, I present one more result about the issue of consumer coordination.

Proposition 1.3 (Impact of Public Information).

Let $z \leqslant \sqrt{2\pi}$. For any couple (p, y) and for a given realization of θ, in equilibrium the networks realized market shares are

$$n^a = 1 - n^b = \Pr\left[x_i > t(p,y)\right] = 1 - \Phi\left[\left(t\left(p,y\right) - \theta\right)\sqrt{\beta}\right]. \tag{1.9}$$

Each firm's market share is strictly increasing in her expected quality.

This result has a natural interpretation. For a given θ, the market share of each network is given by the proportion of consumers with a value of x_i above or below the equilibrium threshold. After θ has been drawn, the distribution of x_i is uniquely determined, regardless of what its expected value y was. Therefore, if a network's market share is increasing in her *expected* quality, it has to be the case that the effect comes from a change in the threshold $t\left(p,y\right)$. The fact that $t\left(p,y\right)$ is actually affected by y might appear counterintuitive because this is a model with private values, where each consumer perfectly (and privately) observes the quality he would derive from the use of each good. Therefore, one might expect that after observing his own x_i consumer i would discard the noisy information contained in its expected value y. Still, consumer i cares not only about his own x_i but also about the distribution of the private values of every other consumer in the market. This is true because of the presence of network effects: the signals observed by other consumers will affect their choices and i needs to take such choices into account because of the strategic complementarity. This is where y becomes relevant. The prior distribution of each $x_{i'}$ is centered around y. When consumer i observes x_i, and updates his belief about the distribution of $x_{i'}$, whatever the value of x_i he observed, the posterior expectation of $x_{i'}$ is increasing in y. Since in equilibrium consumers with a high $x_{i'}$ choose a, the higher y, the higher $a's$ expected market share from the point of view of consumer i, the higher his own convenience in choosing a.

1.4.2 Unsponsored Networks

I now consider the case where the two networks are unsponsored, in order to highlight the possible coordination failure arising on these markets when we abstract from the issue of strategic pricing. Without sponsors, consumers can join any network by paying a price equal to the marginal cost:

$$p^a = p^b = c, \qquad \text{and} \qquad p = 0.$$

The next proposition describes the equilibrium allocation and compares it to the efficient one.

Proposition 1.4 (Inefficiency with Unsponsored Networks).

With unsponsored networks, the equilibrium allocation is a threshold allocation with threshold

$$t^u \equiv t(0, y) \in (-0.5, 0) .$$

Moreover,

$$t^u > t^* \quad and \quad \mathbb{E}_{\mathbf{x}} \left[W \left(t^u \right) \right] < \mathbb{E}_{\mathbf{x}} \left[W \left(t^* \right) \right] .$$

The allocation implemented by the market if the networks are unsponsored shares some qualitative features with the efficient allocation: both are threshold allocations and in both cases, since the threshold is smaller than y, the network with higher expected quality has a market share larger than one half. Moreover, in both cases network a will be composed not only by those consumers with a positive value of x_i, but also by some consumers with a moderate private preference for b.

Nonetheless, the market does not implement the welfare maximizing allocation. In particular, the market allocation is more balanced than the efficient one. The source of this market failure is the presence of network effects that consumers fail to internalize. To get an intuition for this result, it is useful to revise the decomposition of the welfare function that I presented in Section 1.3. For any realized level of quality of the two goods, social welfare is the sum of the gross surplus derived from the objective quality of the goods, gross surplus derived from idiosyncratic taste and, finally, gross surplus derived from network effects. The efficient threshold t^* is therefore the result of the compromise among different forces: from a social point of view, horizontal differentiation makes a symmetric allocation more desirable while vertical differentiation and the presence of network externalities make asymmetry more desirable. Since individual consumers do not internalize the network externality, society implements an allocation that is more symmetric than the efficient one.

1.4.3 Sponsored Networks

After identifying a source of inefficiency arising on network markets, I now ask the question of whether strategic pricing mitigates or aggravates this inefficiency. More precisely, I will now assume that the two networks are sponsored and that therefore access to each of them is priced by a strategic, profit-maximizing firm. Then, I will derive the equilibrium of the two-stage price competition game and compare the allocation of consumers induced in equilibrium to both the ex-ante efficient one and the one implemented by the market in the absence of strategic pricing.

Let

$$v \equiv v(\alpha, \beta) = \sqrt{\frac{\alpha\beta}{\alpha + \beta}}$$

denote the inverse of the standard error of $\widetilde{x}_i$.

The next Lemma describes the demand functions for the two goods.

Lemma 1.2 (Expected Demand Functions).

Let $z \leqslant \sqrt{2\pi}$. The expected demand function for each network is well defined for any price couple $\left(p^a, p^b\right)$ and it is given by

$$\mathbb{E}_{\mathbf{x}}\left[n^a\right] = 1 - \mathbb{E}_{\mathbf{x}}\left[n^b\right] = 1 - \Phi\left[\left(t\left(p, y\right) - y\right) v\right]$$

Moreover, for given prices each firm's expected market share is strictly increasing in the expected quality of her product.

The assumption $z \leqslant \sqrt{2\pi}$ guarantees that there is a unique equilibrium of the coordination game played by consumers in the second stage, therefore the demand function is well defined.

It can be interesting to compare the comparative statics result contained in Lemma 1.2 with that of Proposition 1.3. Proposition 1.3 looks at the market shares for a given realization of θ, and the only reason why they might be a function of y is through a change in $t(p,y)$. Lemma 1.2 instead, considers the expected market shares given the ex-ante information that is available to the firms when they choose prices: in this case, there are two mechanisms through which a network's expected size is increasing in its expected quality: one is the change in $t(p,y)$, the other is the shift in the distribution of x_i (which is centered around y).

I now have the tools to characterize the pure strategy subgame perfect equilibrium of the price competition game. Substituting the expected demand functions into the profit functions I can write the two firms' optimization problem at time 1 as

$$\begin{aligned}
\max_{p^a \in \mathbb{R}} \mathbb{E}_{\mathbf{x}}\left[\pi_a\right] &= \left(p^a - c\right)\left[1 - \Phi\left[\left(t\left(p,y\right) - y\right)\right] v\right] \\
\max_{p^b \in \mathbb{R}} \mathbb{E}_{\mathbf{x}}\left[\pi_b\right] &= \left(p^b - c\right)\Phi\left[\left(t\left(p,y\right) - y\right) v\right].
\end{aligned}$$

Let p^s denote the value of p associated to the firms' equilibrium strategies and t^s denote $t(p^s, y)$. The following proposition holds:

Proposition 1.5 (Strategic Pricing Inefficiency).

If a pure strategy SPNE of the price competition game exists, then

$$p^s \in (0, y) \quad \textit{and} \quad t^s \in (t^u, y)\,.$$

Moreover,

$$t^* < t^u < t^s$$

and

$$\mathbb{E}_{\mathbf{x}}\left[W\left(t^s\right)\right] < \mathbb{E}_{\mathbf{x}}\left[W\left(t^u\right)\right] < \mathbb{E}_{\mathbf{x}}\left[W\left(t^*\right)\right].$$

As in many models of price competition with differentiated products, it is hard to prove the existence of an equilibrium in pure strategies for generic values of the parameters. A general sufficient condition for existence was derived by Caplin and Nalebuff (1991). My model satisfies that condition for some limit cases. I have solved the model for a grid of values of the parameters (α, β) that satisfy the condition $z \leqslant \sqrt{2\pi}$ and in all those cases a pure strategy equilibrium exists and is unique.

The main qualitative features of the equilibrium of the price competition game are consistent with the standard results in duopolistic models of price competition with vertical differentiation[8]: in equilibrium, the firm selling the best product charges the highest price ($p^s > 0$) but the difference in quality more than compensates the difference in prices ($y > p^s$), so that she also attracts a fraction of consumers larger than one half.[9]

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[8]See, for example, Shaked and Sutton (1982).

[9]Note that in this model, the firm that has an advantage is not literally the firm "selling the best product" but is the firm that is expected to sell the best product.

What is relevant for efficiency, though, is *by how much* this market share exceeds one half. I have already shown that in the absence of strategic pricing the market share of firm a is too small, from a social point of view. According to Proposition 1.5, a's market share is even smaller if the two networks are sponsored and this equilibrium provides even less welfare than the equilibrium with unsponsored networks. The reason why strategic pricing reduces welfare is quite straightforward: with vertical differentiation, the firm selling the best product has a natural advantage that is reflected in a higher equilibrium price. In turn, the fact that a is more expensive than b, ceteris paribus, shifts some consumers from a to b.

I have therefore identified two separate sources of inefficiency on network markets: the first is the presence of network externalities that consumers do not internalize, which determines a market allocation that is too balanced from a social point of view. The second, which arises only if networks are sponsored, is the presence of strategic pricing, that further reduces the asymmetry between network sizes.

1.5 Conclusions

I described the optimal allocation of consumers between two vertically and horizontally differentiated networks. I found that social welfare is maximized by an asymmetric allocation such that the network with the highest expected quality has the largest market share. I then compared this allocation with the ones implemented by the market if the networks are unsponsored or sponsored, respectively. Two sources of inefficiency emerged from the analysis: the failure to internalize network externalities and the presence of sponsors that strategically set prices for the products. Both these facts induce the market to implement an allocation of consumers that is too symmetric.

I analyzed efficiency from an ex-ante point of view. One possible alternative is to analyze efficiency from an ex-post point of view, assuming that a benevolent social planner can observe the realized distribution of private values in the population. It can be shown that also in that case the efficient allocation is a threshold allocation such that both networks are active and one of them has a larger market share than the other. The main difference with the ex-ante case is that the ex-post optimal threshold depends on the realization of θ. Also, in the ex-post efficient allocation the network with the largest market share is the one with the highest *realized* quality, while in the ex-ante case it is the one with the highest *expected* quality.

Comparing the equilibrium allocation to the ex-post efficient allocation, the main result is that strategic pricing can increase welfare if the firm with the highest expected quality has the lowest realized quality and decrease welfare in the opposite case. The intuition for this result is the following. If the public signal is in favor of, say, firm a, in equilibrium she will charge a price higher than the competitor and ex-post this will negatively affect her market share (everything else equal). Therefore, if ex-post a is the best firm, strategic pricing shifts some consumers towards the worst firm, while if a is the worst firm, then strategic pricing shifts some consumers towards the best firm.

In my model, I assumed that consumer heterogeneity is unbounded while the utility derived from the network effect is bounded. The result that the ex-ante efficient threshold is an interior optimum holds under more general assumptions. For symmetric, bounded network effects, if the distribution of x_i is continuous and symmetric around the mean, then it is sufficient that the upper bound of the support of x_i is weakly larger than the individual surplus from being in a network of size one.

I also assumed complete market coverage. This assumption allowed us to abstract from the possible deadweight loss associated to strategic pricing and to focus only on the inefficiency arising from the difference in equilibrium prices.

Finally, I assumed that firms have access only to the public information available about the vertical quality of the products. A natural extension of the model would be to allow each firm to observe a private signal about the vertical quality of her product. This would add the issue of how firms use prices to signal their quality.

I leave for future research an extension of this model to a dynamic environment, where consumers choose sequentially and firms can adjust prices over time.

Chapter 2

Network Markets and Consumer Coordination[1]

2.1 Introduction

A market has network externalities if consumers' utility from purchasing a product depends on which other consumers buy the same product. A highlighted special case of this is two-sided markets with network externalities. In these markets consumers are divided into two distinct subgroups. A consumer's utility on one side increases in the total number of consumers on the other side of the market who buy the same product (and possibly decreases in the number of consumers on the same side of the market). This applies to various situations in which two groups of agents need a common platform to interact and one or more firms own platforms and sell access to them. The higher the number of agents on one side who join a platform, the higher the utility of an agent on the other side of the platform because she has a higher number of potential partners with whom to trade

[1] Attila Ambrus and Rosa Argenziano

or interact. One example of this is the market for on-line matchmaking services where the two sides are women and men. In other cases the two sides are buyers and sellers and the platforms are auction websites, directory services, classified advertisers and credit card networks (the sellers are merchants who accept the credit card and the buyers are credit card holders).

This chapter investigates the decisions of firms regarding how many networks to operate and how to price them, and the network choices of consumers in two-sided markets with network externalities. We consider an extensive form game in which in the first stage firms establish networks, in the second stage they announce registration fees for these networks, and in the third stage consumers simultaneously choose networks or decide to stay out of the market.

The first contribution of this chapter is methodological. Equilibrium analysis on network markets is an involved task. The coordination problems that arise among consumers result in a severe multiplicity of equilibria both in games in which there is a monopolist network provider and in games in which there are multiple providers. Consumers can have various self-fulfilling expectations regarding which networks other consumers join and whether they join any network at all. To address the issue of multiplicity of equilibria we use the concept of coalitional rationalizability, proposed by Ambrus (2002 and 2003) to select equilibria and therefore derive qualitative predictions. Informally, this method corresponds to assuming that consumers can coordinate their network choices as long as it is in their joint interest and this coordination does not require explicit communication.

The second contribution of this chapter is that this methodology allows us to analyze pricing games in which consumers in the market are heterogeneous. This makes it possible to ask a new set of questions. These include whether price discrimination among different type of consumers is possible without product differentiation, whether there can be multiple active networks operated by competing firms, attracting different types of consumers, and whether it can be in the interest of a monopolistic network provider to operate multiple networks, aimed at different sets of consumers.

Coordination failures and the resulting inefficiencies are relevant phenomena in network markets, since typically there are many anonymous consumers who cannot communicate with each other. Nevertheless, in some cases it is reasonable to expect consumers to be able to coordinate on a particular network. The simplest example is if there are two networks and one is cheaper on both sides of the market. Choosing this network is then a natural focal point on which consumers can coordinate. The central assumption of the model presented in this chapter is that consumers can coordinate their decisions to their advantage if their interests coincide and if coordination can be achieved without communication, as in the above case. In contrast, if there is no unique candidate network that consumers would agree to join, we do not assume successful coordination even if it is in the common interest of consumers. Our motivation for this is that if there are lot of small consumers on the market then it is practically impossible for them to get together and make explicit agreements on network choices.

Coalitional rationalizability allows us to incorporate the above assumption into the analysis. This noncooperative solution concept assumes that players can coordinate to restrict their play to a subset of the original strategy set if it is in the interest of every participant to do so. This defines a set of implicit agreements, which puts restrictions on beliefs that players can have at different stages of the game. These agreements are based on public information, the description of the game, and therefore do not require explicit communication. Furthermore, they are self-fulfilling in the sense that if participants expect each other to choose networks according to an agreement, then it is in their best interest to act according to the agreement. In the games that we analyze there can be groups of consumers who can coordinate their network choices this way after certain price announcements.

We investigate the subgame perfect equilibria of the above market games that are compatible with the additional assumption of coalitional rationalizability. We call these equilibria coalition perfect.

We analyze participation rates of consumers, prices charged by the firms, network sizes and firms' profits in coalition perfect equilibria. The analysis is carried out for the cases of both one and two firms operating on the market. In the former case we distinguish between the case that the firm can operate only one network and the case in which it can decide to operate multiple networks.

We show that if consumers are homogeneous on the same side of the market then a monopolist network provider only establishes one network in coalition perfect equilibrium. On the other hand, if there is enough heterogeneity among consumers, a monopolist might want to establish two networks, which in equilibrium are joined by different type of consumers. The intuition is that, if there are high reservation value consumers on both sides of the market, then the monopolist wants to extract surplus from both of these groups. However, if there are relatively few of these consumers and the monopolist operates only one network, then he can charge a high price on at most one side. In order to charge a high price on one side, there have to be enough consumers on the other side of the network, which is only possible if the price charged on that side is low. On the other hand, if the monopolist establishes two networks such that one of them is cheap on one side of the market and the other network is cheap on the other side, then all consumers are willing to join some network and consumers with high reservation values are willing to join the network which is more expensive for them. The way price discrimination is achieved in these equilibria is through endogenous product differentiation. Networks are physically the same, but if one side of a network attracts a lot of consumers, then the other side of the network becomes more valuable for consumers.

In the case of two network providers competing for consumers we show that homogeneity of consumers on the same side of the market ensures that both firms' profits are zero in coalition perfect equilibrium, reestablishing the classic Bertrand result. This holds despite the fact that equilibrium prices do not have to be equal to the marginal cost (consumers on one side of the market can be subsidized). Homogeneity of the consumers also implies that there cannot be multiple asymmetric networks in coalition perfect equilibrium. On the other hand, we show that if consumers are heterogeneous, then there can be equilibria in which there are two networks that attract different type of consumers and in which firms earn positive profits. The intuition is that although firms can steal each other's consumers by undercutting their rival's prices on both sides of the market, this move is not necessarily profitable. In particular, undercutting might be unprofitable if it increases the number of consumers to be subsidized more than the number of consumers who pay a positive price.

In both the monopoly and the duopoly case the coalition perfect equilibria with two asymmetric networks have the feature that one network is larger and cheaper on one side of the market, while the other one is larger and cheaper on the other side. One example of this configuration is when a town has both a freely and universally distributed newspaper with classified ads and one that is not freely distributed. The first newspaper is cheaper and larger on the buyers' side. In order to compete with the freely distributed newspaper, typically the other newspaper has to have more ads posted on it, and therefore can only charge a smaller fee for posting ads. Therefore this newspaper is typically cheaper and larger on the sellers' side. Another example is on-line job search, where the two main platforms are Careerbuilder.com and Monster.com. Monster has a database of 25 million resumes versus Careerbuilder's 9 million, therefore larger on the job seekers' side. On the other hand, Careerbuilder has 45.2% of the job postings of the on-line job search market in the US, while Monster has only 37.5%.[2] Therefore Careerbuilder is larger on this side. And to post a job on Careerbuilder, a firm pays \$269, while to post a job on Monster a firm pays \$335.[3]

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[2] 2004 February figures. The information about the size of the two databases is taken form www.careerbuilder.com and www.monster.com. The information about the number of job postings is obtained from Corzen.

[3] The base cost of posting a resume is zero on both sites, but job seekers pay extra fees for preferential treatment of their resumes (for example if they want them to come up at the top of search result lists obtained by firms). We do not have information on how many job seekers pay these extra fees, therefore we cannot make a correct price comparison on this side of the market.

2.2 Related Literature

Recently a number of papers investigated the issue of optimal pricing and price competition in markets with two-sided network externalities. For a more extensive literature review, see for example Armstrong (2002).

Rochet and Tirole (2003) and Armstrong (2002) study monopolistic pricing and price competition between two firms on markets where the firms are platforms that try to attract two groups of agents. The models in these papers abstract away from coordination problems among consumers. They assume differentiable demand functions for the networks, implicitly assuming differentiated networks (that consumers have heterogeneous inherent preferences between networks). Also, they focus on a particular symmetric equilibrium. Moreover, Rochet and Tirole emphasize the case in which the networks' primary pricing instrument are transaction fees.

Jullien (2001) constructs a duopoly model that allows for more than two subgroups of consumers and for both inter-groups and intra-group network externalities. The setup of this paper differs from ours in that the intrinsic value of the good sold by each firm is assumed to be high compared to the network effect and also that one of the firms is highlighted in the sense that consumers always coordinate on the equilibrium which is the most favorable for this firm.

Ellison, Fudenberg and Mobius (2002) study competition between two auction sites. In their model, like in ours, multiple asymmetric platforms can coexist in equilibrium, despite no product differentiation. In addition to this, they assume heterogeneous agents on both sides of the market. On the other hand, in their model consumers choose platforms ex ante, while in ours they do it after learning about their types. Furthermore, the reason that multiple active networks can coexist in equilibrium is completely different in their model. They consider a finite number of buyers and sellers, therefore one of them switching from one platform to another adversely affects the market price on the latter platform.[4] In our model there is a continuum of consumers on both sides of the market, therefore this market-impact effect is absent.

The model in Damiano and Li (2003a and 2003b) is similar to ours in that consumers on a two-sided market are heterogeneous and that registration fees serve the role to separate different types of consumers. The main difference between our setup and theirs is that in the latter there is no network externality. Consumers care about the average quality of consumers on the other side of the network and not their number. On the other hand, in our model consumers are symmetric with respect to the external effect they generate on consumers on the other side.

The most similar model to ours is presented by Caillaud and Jullien (2001 and 2003). They analyze markets where firms are intermediaries offering matchmaking services to two groups of agents. The above papers assume that consumers on each side are homogeneous and their utility is linear in the number of consumers on the other side of the network.[5] The assumption of homogeneity implies that these papers do not address most of the issues we investigate in this chapter. Also, Caillaud and Jullien select among equilibria by imposing monotonicity on the demand function of consumers and by assuming full market coverage

[4] See Ellison and Fudenberg (2003) for a detailed analysis of this point.

[5] For a comparison between our results in this context and the ones of Caillaud and Jullien (2003), see Subsection 2.7.1.

in equilibrium. As opposed to making an assumption on the aggregate demand function our model imposes restrictions directly on the expectations of individual players.

In our model different types of consumers might select to join different networks. In this aspect, our analysis is connected to the literature on price discrimination (for an overview see Varian (1987)) multiproduct pricing (see Baumol et al. (1982)) and the theory of screening (for an overview see Salanie (1997)).

2.3 The Model

We consider a standard model of price competition in two-sided markets with network externalities. It is a sequential move game in which first firms announce prices, then consumers observe the announcements, and finally consumers choose which network to join, if any. We examine the cases of one or two firms operating in the market. The added features of our model are the following. First, in the monopoly case we allow the firm to choose the number of networks to be established. Second, consumers are not assumed to be homogeneous in that how much they value the network good. Third, we do not make any restriction on the utility functions of consumers besides quasilinearity in money.

Formally, the set of players in the model is (F, C^1, C^2), where F denotes the firms, while C^1 and C^2 the consumers of the corresponding sides. We assume there is a continuum of consumers on both sides of the market, indexed by the interval $[0, 1]$. Let C_i^k denote consumer i on side k. As far as firms are concerned, we restrict attention to the cases $F = \{A\}$ (only one firm, A is present) and $F = \{A, B\}$ (two firms, A and B are present).

We consider a three-stage game with observable actions (after every stage all players observe all action choices made in that stage).

In the first stage the firms simultaneously choose how many networks to establish. We will restrict attention to cases when the maximum number of networks a firm can operate is either one (in which case decisions at this stage are trivial) or two. Let n_k denote the first stage action of firm k. Furthermore, let $(\tau_k^m)_{m=1,\dots,n_k}$ denote the networks that firm k establishes.

In the second stage the firms simultaneously set prices (registration fees) on the established networks. Firms can charge different registration fees on different networks or on different sides of the same network. Furthermore, they can charge negative prices on either side of the networks (subsidizing consumers on that side). Let $p_{k,n}^j$ denote the price that firm k sets on side j of τ_k^n. If every firm can only have one network, then we use the simpler notation p_k^j.

In the third stage the consumers simultaneously choose which network to join, if any. We assume that a consumer can join at most one network (exclusivity of networks).[6]

Firms maximize profits. Let π_k denote the profit of firm k. We assume that firms are symmetric and that the cost of operating a network is zero, independently of the number of consumers joining the network.[7] Then the payoff of the firm is the sum of the revenues collected from the firm's networks, where the revenue collected from a network is sum of the revenue collected on side 1 and the revenue collected on side 2. Let $N_{k,n}^j$ denote the number of consumers on side j who join network τ_k^n.[8] If the maximum number of networks

[6] See Section 2.8 for a discussion on relaxing this assumption.

[7] See Section 2.8 for a discussion on how the results affected by assuming a positive marginal cost.

[8] More precisely, the Lebesgue measure of consumers on that side joining the network. We leave payoffs

k can operate is 1 then we use the simpler notation N_k^j. Then $\pi_k = \sum_{n=1,\ldots,n_k} \sum_{j=1,2} p_{k,n}^j N_{k,n}^j$.

Consumer i on side j maximizes the individual-specific utility function U_i^j. Let $U_i^j = 0$ if she does not join any network. Let $U_i^j = g_i^j(N_{k,n}^{-j}) - p_{c_i^j}^j$ if she joins network τ_k^n. Assume $g_i^j(0) = 0$ and that g_i^j is increasing for every consumer C_i^j.[9] A consumer's utility if joining a network is quasilinear in money and increases in the number of people joining the network from the other side of the market.[10] Implicit in the construction is that consumers do not have any inherent preference for joining one network or another, they only care about the number of people joining the networks and the price they have to pay.

Let $u_i^j = g_i^j(1)$. We call u_i^j the reservation value of consumer i on side j.

If $g_i^j = g_{i'}^j \ \forall \ i, i' \in [0,1]$ and $j \in \{1,2\}$, then we say consumers are homogeneous. A special case of the above specification, that received highlighted attention in the existing literature, is when for every $j = 1,2$ and $i \in [0,1]$ it holds that $g_i^j(N_{k,n}^{-j}) = u^j N_{k,n}^{-j}$ (consumers on the same side have the same linear utility function).

2.4 Coalitional Rationalizability and Coalition Perfect Equilibrium

The central assumption of our model is that at every stage of the game players can coordinate their actions whenever it is in their joint interest and it does not require communica-

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undefined for cases where the latter set is not measurable.

[9] We assume that network participation is a pure network good for analytical convenience. Most results of the paper could be generalized to the case of $g_i^j(0) \geq 0$.

[10] The above specification makes the simplifying assumption that a consumer's utility is independent of how many consumers join the network on her side, implicitly assuming that interacting with consumers on the other side is a nonrival activity. Section 2.8 discusses implications of partially relaxing this assumption.

tion. The formal concept we use is coalitional rationalizability. It is a solution concept that builds on the idea that whenever it is in the mutual interest of some group of players (a coalition) to restrict their play to a certain subset of the strategy space (to implicitly agree upon not playing some strategies) then these players indeed expect each other to make this restriction. These restrictions are called supported. The set of coalitionally rationalizable strategies are the outcomes that are consistent with common certainty of the assumption that play is consistent with every supported restriction by every coalition. For the formal construction see Ambrus (2002).

In the forthcoming analysis we restrict attention to pure strategy subgame perfect equilibria in which players play coalitionally rationalizable strategies in every subgame. We call these outcomes coalition perfect equilibria.[11]

Definition 2.1. *A strategy profile is a coalition perfect equilibrium if it is a subgame perfect Nash equilibrium and in every subgame every player plays some coalitionally rationalizable strategy.*

Intuitively, coalition perfect equilibrium requires that supported restrictions can be made not only at the beginning of the game, but after any publicly observed history, and that players at any stage of the game foresee restrictions that are made at later stages. In the games we consider coalitional rationalizability puts restrictions in certain subgames on consumers' beliefs concerning other consumers' choices. To provide some intuition on this,

[11] It is possible to show that in our context the requirement that players play coalitionally rationalizable strategies in every subgame is outcome equivalent to the concept of extensive form coalitional rationalizability (see Ambrus (2003)).

we proceed with an informal definition of supported restriction in consumer subgames, and three examples of such restrictions.

Consider a third stage (consumer) subgame. Assume that players are certain that play is in some subset S' of the set of strategies. Then a coalition of consumers supports restricting play to a subset S'' of S' if for any player in the coalition the following condition holds. No matter what beliefs she has concerning the choices of consumers outside the coalition, her expected payoff is always strictly higher if the restriction is made (if every player in the coalition plays inside S'') than if the restriction is not made and she plays some strategy outside S''.

For the first example suppose $F = \{A, B\}$ and that for every $j = 1, 2$ and $i \in [0, 1]$ it holds that $g_i^j(N^{-j}) = uN^{-j}$ for some $u > 0$. Consider the consumer subgame that follows price announcements $p_k^j = 0 \; \forall \; j = 1, 2$ and $k = A, B$. Then the restriction to join either A's network or B's network (or agreeing upon not to stay out of the market) is a supported restriction for the coalition of all consumers, because if the restriction is made, then any possible conjecture that is compatible with it is such that a best response to it yields expected payoff of at least $u/2$ (the conjecture should allocate an expected size of at least $1/2$ to one of the two networks), while staying out of the market yields zero payoff. Because prices charged by the two networks are the same, no more strategies are eliminated by coalitional rationalizability in this subgame. Both joining A's network and joining B's network are coalitionally rationalizable for every consumer and therefore this subgame has three coalitionally rationalizable Nash equilibria. Either every consumer joins A or every consumer joins B, or one half of the consumers on both sides joins each network.

Next, consider the subgame in the same game that follows price announcements $p_A^j = 0$ and $p_B^j = u/4 \; \forall \; j = 1, 2$. In this subgame A is a supported restriction for the coalition of all players, since it yields payoff u to all consumers, while joining B's network can yield a payoff of at most $3u/4$ and staying out yields 0. Coalitional rationalizability pins down a unique strategy profile in this subgame.

If consumers are heterogeneous then the set of coalitionally rationalizable outcomes in a subgame might only be reached after multiple rounds of agreements. Suppose $F = \{A, B\}$, consumers have linear utility functions and $u_i^1 = 1 \; i \in [0, 1/2]$, $u_i^1 = 1/2 \; i \in (1/2, 1]$ and $u_i^2 = 1 \; i \in [0, 1]$. In words, consumers on side 2 are homogeneous, while half of the consumers on side 1 have relatively low reservation values. Consider the subgame following price announcements $p_A^1 = .4, p_A^2 = .8$ and $p_B^1 = .8, p_B^2 = .4$. Initially, there is no supported restriction for the coalition of all consumers. Consumers would prefer to coordinate their network choices, but coordinating on A is better for side 1 consumers, while coordinating on B is better for side 2 consumers. However, note that joining B is not rationalizable for any consumer C_i^1 for $i \in (1/2, 1]$ and therefore not joining B is a supported restriction for these consumers. Once it is established that players C_i^1 for $i \in (1/2, 1]$ only consider strategies $\emptyset$ or A, it is a supported restriction for the coalition of all consumers to join A. Therefore coalitional rationalizability pins down a unique outcome in this subgame as well.

The sequential structure of the game implies that there are no supported restrictions in the game that involve both firms and consumers. By the time consumers move, firms already made their choices and those choices were observed by the consumers. And since consumers cannot commit themselves to make choices that are not in their interest, there are no credible implicit agreements between firms and consumers. Showing this formally is straightforward and therefore omitted from here. It is less straightforward to establish that there are no supported restrictions between two firms, and we could not establish a result like that for the general specification. However, we did not find any example in which there is a supported restriction by firms. Firms in our model are competitors and they only move once, therefore their possibilities to make credible and mutually advantageous restrictions are very limited.

We note that the solution concept we use is not equivalent to coalition-proof Nash equilibrium (Bernheim, Peleg and Whinston (1987)) or Pareto efficiency in the consumer subgames. In consumer subgames in which there is a unique Pareto efficient outcome coalition perfect equilibrium implies that the efficient outcome is played. But in subgames with multiple Pareto efficient outcomes it is consistent with coalition perfect equilibrium that a Pareto inefficient outcome is played. Also if consumers are heterogeneous, then not every Pareto efficient outcome is consistent with coalition perfect equilibrium. In general, none of the results presented in this chapter for heterogeneous consumers would hold if instead of coalitional rationalizability we assumed Pareto efficiency in consumer subgames.[12] As far as coalition-proof Nash equilibrium is concerned, it can be shown that in our model it is a stronger concept than coalitionally rationalizable Nash equilibrium, implying that it assumes a more effective form of coordination than what we impose.[13] It can be established

[12] One intuition for this is that if in every equilibrium of a consumer subgame there are some consumers who do not join any network, then the requirement of Pareto efficiency is very weak (consumers who stay out get utility zero, no matter how many other consumers stay out or how many consumers join one network versus the other).

[13] In particular, it assumes successful coordination even when there are multiple outcomes in a subgame that cannot be ruled out by the logic of coalitional agreements. In our view in these cases coordination can

that both the range of network choices and equilibrium prices can be larger in coalition perfect equilibrium than in coalition-proof Nash equilibrium.[14] The above implies that those results of this chapter that establish properties of all coalition perfect equilibria are valid for (the strictly smaller set of) perfect coalition-proof equilibria too.

We do not investigate the issue of existence of pure strategy coalition perfect equilibrium here. It is possible to show that under some technical conditions every game with one firm has a pure strategy coalition perfect equilibrium, while every game with two firms has a coalition perfect equilibrium in which consumers play pure strategies and only firms use mixed strategies.[15] We restrict attention to pure strategies to keep the analysis tractable. In all examples we provide in this chapter there exists a pure strategy coalition perfect equilibrium.

2.5 Monopolist with One Network

In this subsection we assume $F = \{A\}$ and that firm A can only establish one network.

In a market without network externalities, subgame perfection, which is implicitly assumed when the demand function of consumers is derived, guarantees that a monopolist can achieve the maximum profit compatible with Nash equilibrium.[16] In markets with network

only be achieved through explicit agreements, which is not possible in the settings we consider.

[14] See the earlier version of our paper for an example.

[15] The conditions are that utility functions are differentiable and there is a uniform bound on their derivatives.

[16] To give a simple example, if all consumer have the same reservation value $u > 0$ for some indivisible good and the firm charges a price strictly below u, then subgame perfection implies that all consumers buy the good. Then in any subgame perfect Nash equilibrium the firm gets a profit of u times the number of consumers.

externalities the above result does not hold. Typically there are many different equilibria of the pricing game with one network provider. The consumers face a coordination problem when deciding on whether to join a network or not, which results in a wide range of subgame perfect equilibrium prices, consumer participation measures and profit levels.

In sharp contrast, Theorem 2.1 below establishes that in every coalition perfect equilibrium the monopolist gets the maximum profit compatible with Nash equilibrium. The intuition is the following. In any Nash equilibrium consumers who join the network get nonnegative utility. Then coalitional rationalizability guarantees that all these consumers join the network if the price is smaller than the previous equilibrium price. But then the firm is guaranteed to get a profit that is arbitrarily close to the above equilibrium profit by charging slightly smaller prices (note that if the equilibrium price on one side is negative, then all consumers on that side join the network in equilibrium, so a price decrease cannot result in having to subsidize more consumers).

The proofs of all theorems that are stated in this chapter are in Appendix B.

Theorem 2.1. π_A *is uniquely determined in coalition perfect equilibrium and it is equal to the maximum possible profit of* A *in Nash equilibrium.*

If consumers are homogeneous and the reservation values of consumers are u^1 on side 1 and u^2 on side 2, then Theorem 2.1 implies that in any coalition perfect equilibrium that $p_A^k = u^k$ and $N_A^k = 1 \; \forall \; k = 1, 2$. The assumption that consumers can implicitly coordinate their choices in this case ultimately hurts them because the firm can extract all the potential consumer surplus on the market.

A corollary of the previous observation is that if consumers are homogeneous, then it is never in the interest of the firm to establish more than one network. By providing one network the firm can extract the maximum possible gross consumer surplus on this market. If there are two active networks then gross consumer surplus is smaller than in the above case, and since no consumer can get negative utility in any Nash equilibrium, the profit of the firm is strictly smaller than in the one network case.

2.6 Monopolist Who Can Operate Multiple Networks

This section investigates a monopolist network provider's decision on how many networks to operate and how to price them. Since the analysis of the model with more than two networks and heterogeneous consumers is involved, we restrict attention to the case when the monopolist can operate at most two networks.

First we show in a simple setting that it can indeed be better for the monopolist to operate two networks, despite the fact that it would be socially optimal to have all consumers on the same network. The example also demonstrates that in network markets the monopolist can effectively price discriminate among consumers through registration fees, even without product differentiation. We compare social welfare in the case the monopolist can operate only one network versus if it is allowed to operate multiple networks. For general distribution of types we provide a necessary condition for two networks being established in equilibrium.

We note that the results of this section carry over to the case when consumers can join multiple networks at the same time.[17] Furthermore, the latter framework is formally

[17] See Subsection 2.8.2 for a discussion.

equivalent to one in which the monopolist operates only one network but can sell restricted access to the network (can control which transactions can occur on the network). Therefore in that setting if there is enough heterogeneity among consumers then the monopolist might want to sell both limited access at a lower price, and full access at a higher price, on both sides of the network.

2.6.1 Two Types of Consumers on Each Side

Here we restrict attention to a context in which the two sides of the market are symmetric, there are only two types of consumers on each side, and consumers have linear utility functions.

Assume that $g_i^j(N) = u_i^j \cdot N \; \forall \; j = 1, 2$ and $i \in [0, 1]$. Also assume that for every $u_i^1 = u_i^2 = h$ for $i \in [0, a]$ and $u_i^1 = u_i^2 = l$ for $i \in (a, 1]$, where $l < h$ and $a \in [0, 1]$.

In each side a fraction a of the consumers have a reservation value h, which is higher than l, the reservation value of the rest of the consumers. We refer to consumers with reservation value h as high types, and consumers with reservation value l as low types[18].

It is possible to show that for all parameter values a, l and h there exists a coalition perfect equilibrium, no matter what is the maximum number of networks that A can establish.[19]

As a first step we characterize the set of coalition perfect equilibria when the firm can only establish one network. The coalition perfect equilibrium is almost always unique, but

[18] Note that high and low type only refers to the reservation value of consumers and not to their quality in terms of how desirable a consumer's presence is on the network for consumers on the other side. In our model all consumers are ex ante identical in terms of this external effect.

[19] In a previous version of this model we establish a more general result.

depends on the values of parameters a, h and l. If l is relatively low and a is high, then the monopolist targets only the high type consumers and charges a high price on both sides. If l is relatively high and a is small, then the monopolist targets all consumers and charges a low price on both sides. In cases in between the monopolist might target all consumers on one side and only the high types on the other side, by charging a low and a high price. These results are both intuitive and in accordance with classic results from the literature on multi product pricing with heterogeneous consumers.

Define the following cutoff points:

$$t_1 \equiv 2a - 1 \tag{2.1}$$

$$t_2 \equiv \frac{a}{2-a} \tag{2.2}$$

Notice that if $a \geq \frac{1}{2}$ then $0 \leq t_1 \leq t_2 \leq \frac{1}{2}$, while if $a \leq \frac{1}{2}$ then $t_1 \leq 0 \leq t_2 \leq \frac{1}{2}$. Also, notice that both t_1 and t_2 are strictly increasing in a.

Theorem 2.2. *For every coalition perfect equilibrium the following hold:*

1. *If* $\frac{l}{h} < \max\{0, t_1\}$, *then only the high types on both sides join the network and* $p_A^1 = p_A^2 = ah$

2. *If* $\frac{l}{h} \in (\max\{0, t_1\}, t_2)$ *then there is* $j \in \{1, 2\}$ *such that on side* j *all consumers join the network and* $p_A^j = al$ *and on the other side only high types join the network and* $p_A^{-j} = h$.

3. *If* $\frac{l}{h} \in (t_2, 1)$ *then all consumers on both sides join the network and* $p_A^1 = p_A^2 = l$.

Finally, if $\frac{l}{h} = t_1$ *there are coalition perfect equilibria of both type 1 and type 2 above. Similarly if* $\frac{l}{h} = t_2$ *then there are coalition perfect equilibria of both type 2 and type 3 above.*

Note that if $a < 1/2$, then there is no coalition perfect equilibrium in which the monopolist charges a high price on both sides of the market, targeting only high types. Charging a high price on one side of the market has to be accompanied by charging a low price on the other side. The reason is that there have to be enough consumers on the other side on the network for high types on the first side to be willing to pay the high price. The monopolist therefore cannot extract a high level of consumer surplus from both sides of the market simultaneously.

Assume now that the maximum number of networks A can establish is 2. The next theorem shows that for a range of parameter values in every coalition perfect equilibrium the monopolist chooses to operate two networks and high and low type consumers on the same side of the market choose different networks.

Define the following cutoff points:

$$z_1 \equiv 4a - 1 \tag{2.3}$$

$$z_2 \equiv \frac{a(1-2a)}{1-a} \tag{2.4}$$

Notice that if $a \in \left[0, 1 - \frac{\sqrt{2}}{2}\right]$ then $t_1 \leq z_1 \leq t_2 \leq z_2$. and both z_1 and z_2 are strictly increasing in a.

Theorem 2.3. *If* $a \in \left(0, 1 - \frac{\sqrt{2}}{2}\right)$ *and* $\frac{l}{h} \in (\max\{0, z_1\}, z_2)$ *then the following hold for every coalition perfect equilibrium:*

(1) $n_A = 2$

(2) *There is* $n \in \{1, 2\}$ *such that all high types on side 1 and all low types on side 2 join* τ_k^n*, while all low types on side 1 and all high types on side 2 join* τ_k^{-n}*. The prices are* $p^1_{A,n} = p^2_{A,-n} = h(1-2a) + al$ *and* $p^2_{A,n} = p^1_{A,-n} = al$*.*

Note that this range of parameter values cuts into the region in which a monopolist with one network would target all consumers on both sides and into the region in which it targets only high type consumers on one side and all consumers on the other.

By establishing two networks and pricing them differently the monopolist implements a form of second degree price discrimination. In particular, if the proportion of high types is sufficiently low, then the monopolist can separate the low types and the high types on each side even if reservation values are unobservable, by charging a high price on side 1 and a low price on side 2 in one network, and doing the opposite on the other network[20]. An appropriate choice of prices results in low type consumers choosing networks that are relatively cheap for them, while high type consumers choosing the ones that are relatively expensive for them. In equilibrium the two networks, despite being physically equivalent, end up being of different quality. In our framework the quality of a network for a consumer is determined by how many consumers join the network on the other side of the market. If the majority of consumers on each side of the market are low types, then when all low type consumers on side 1 join one network, that network becomes higher quality for side 2 consumers. Similarly when all low type consumers on side 2 join one network, that

[20] Note that coalitional rationalizability implies that in order to have two active networks in equilibrium they either have to have exactly the same prices, or one network has to be relatively cheaper on one side, while the other one on the other side. Therefore there cannot be coalition perfect equilibria with two networks with one being large and expensive on both sides, while the other small and cheap on both sides.

network becomes higher quality for side 1 consumers. Since in the above equilibria the low type consumers join different networks, one network ends up being high quality for side 1 consumers, while the other one for side 2 consumers. High type consumers have a higher willingness to pay for quality and therefore are willing to join the networks that are more expensive for them.

The result that in equilibrium the monopolist separates consumers on the same side by offering them two products that have different prices and qualities is standard in the adverse selection literature[21]. What is special to this model is that the two networks are ex-ante identical and product differentiation is endogenous. The quality of a network is determined in equilibrium by the network choices of the consumers, which are driven by the prices of the networks.

The reason the monopolist might be better off by the price discrimination is that it can extract a large consumer surplus from high type consumers simultaneously on both sides of the market, something that it cannot achieve by operating only one network (see Figure 2.1 for an illustration).

Notice that in the above equilibrium the firm sacrifices some gross consumer surplus (it is socially efficient if all participating consumers are on the same network) in order to be able to extract a high share of the surplus from consumers with high reservation values on both sides of the market.

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[21] See Mussa-Rosen (1978) and Maskin-Riley (1984).

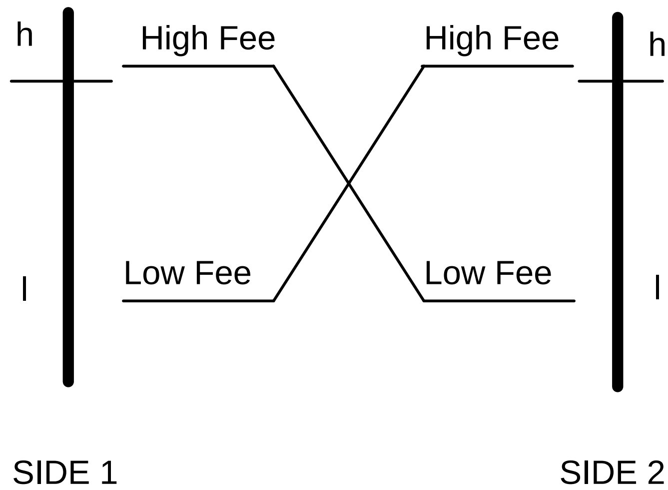

Figure 2.1: Optimal Price Structure for a Monopolist with Two Networks

Despite this, the aggregate social welfare in the situation in which the monopolist is not allowed to operate multiple networks can be both higher or lower than in the situation in which it can only operate one. If $\frac{l}{h} \in (t_2, z_2)$ then a monopolist operating only one network charges prices (l, l) and all consumers join the network. This generates a higher aggregate surplus than if the monopolist can operate two networks, because the same set of consumers participate in the market both cases, but more surplus is generated if they all join same network. As far as consumer surplus is concerned, high types are better off if the monopolist can only run one network and low types are indifferent (they get zero utility in both cases). On the other hand if $\frac{l}{h} \in (z_1, t_2)$, then being restricted to operate one network the monopolist sets a price of h on one side and la on the other. Only high types join the network on the first side and all consumers on the other side. In this case high type consumers are better off if the monopolist can operate two networks and low types are again indifferent. Furthermore, it is straightforward to establish that aggregate social surplus is higher in the case of two networks. In case of two networks aggregate surplus is $2(ah\,(1-2a)+al)+2a^2\,(h-l)$, while in case of one network it is $ah+al+a^2\,(h-l)$. The difference of the two surpluses is $ah+al-3a^2h-a^2l$, which is positive given that $\frac{l}{h}<1$ and $\frac{l}{h} \geq z_1 = 4a-1$ implies $a<\frac{1}{4}$.

Equilibrium prices and quantities have to satisfy the "incentive compatibility constraints" that a high type consumer should prefer the more expensive network, while a low type consumer should prefer the cheaper network. Furthermore, since staying out of the market is an option to every consumer, consumers have to get nonnegative utility in equilibrium - a "participation constraint". One feature of the above result, which is consistent with the literature on adverse selection, is that the incentive compatibility constraints for the high types and the participation constraints for the low types are binding in equilibrium.

2.6.2 General Specification

The question of when it is in the interest for the monopolist to operate multiple networks is difficult to answer in general.[22] One result in the restricted setting of the previous subsection is that the reservation values of high and low type consumers have to be sufficiently different for the monopolist wanting to establish two networks. Below we show that this can be extended to a more general setting. In particular we provide a lower bound on the ratio of the highest and lowest reservation value which is consistent with two networks being operated by the monopolist, provided that consumers' utility functions are convex (including the linear specification).

Let $F = \{A\}$ and let the maximum number of networks A can establish be 2. Assume that g_i^j *is weakly convex for every* $j = 1, 2$ *and* $i \in [0, 1]$.

Theorem 2.4. *If* $u_i^j / u_{i'}^j < 3 + 2\sqrt{2}$ *for every* $j = 1, 2$ *and* $i, i' \in [0, 1]$ *then in every coalition perfect equilibrium either* $n_A = 1$, *or* $\exists\ n \in \{1, 2\}$ *such that* $N_{A,n}^k = 0$ *for some* $k \in \{1, 2\}$.

The central point of the proof is that the above assumptions guarantee that establishing one network and charging prices equal to the lowest reservation values is always more prof-

[22] One complication is that as opposed to the case of two types of consumers on each side, in general the profit of a monopolist might not be uniquely determined in coalition perfect equilibrium. The previous version of this work provides an example in which there are coalition perfect equilibia in which only one network is established, but there are also coalition perfect equilibria in which two networks are established, yielding different profit levels. This is possible because coalitional rationalizability does not uniqely determine choices in every consumer subgame.

itable than establishing two active networks. Recall that establishing two active networks implies sacrificing some consumer surplus, which limits the price that the monopolist can charge on the networks. Therefore it can only be profitable if there are some consumers on both sides of the market with sufficiently high reservation values relative to the rest of the consumers.

2.7 Duopoly

In this section we consider two firms operating in the market, each allowed to establish only one network. Just like in the case of one firm operating in the market, typically there are many different types of subgame perfect Nash equilibria, including one in which no consumers participate in the market. There are also equilibria in which firms get positive profits.[23]

We investigate whether the assumption of coalitional rationalizability reestablishes the result that firms' profits are zero in Bertrand competition for markets with network externalities. Furthermore we ask whether there exist coalition perfect equilibria in which asymmetric networks coexist in the market, with different types of consumers choosing different networks. First we address these questions in the special case that all consumers on the same side have the same reservation values, where we can characterize the set of coalition perfect equilibria. Then we investigate the case of heterogeneous consumers, where

[23] The presence of so called "divide and conquer" strategies (introduced by Innes and Sexton (1993), and then analyzed in the context of network markets by Jullien (2001) and Caillaud and Jullien (2001)) restricts the set of SPNE. If a firm charges a sufficiently low (negative) price compared to its rival on one side of the market, it can make its network a dominant choice for consumers on that side. Then it can charge a high price on the other side of the market and still make sure that consumers join its network on that side. Despite this there is typically a severe multiplicity of equilibria.

we provide an example to show that there might be coalition proof equilibria with novel features, and obtain a partial characterization result.

2.7.1 Homogeneous Consumers

In this subsection we assume that consumers on the same side have the same reservation value, u^1 on side 1 and u^2 on side 2. It can be shown that in all these games there exists coalition perfect equilibria.

Theorem 2.5 establishes that in this case coalition perfect equilibria of the duopoly game have similar properties to subgame perfect equilibria of duopoly pricing games with no network externalities. Namely in every coalition perfect equilibrium both firms' profits are zero, and if both of them are active then they charge the same prices and have the same size. The difference is that in this two-sided market environment prices do not have to be zero, despite the assumption that the marginal cost is zero. It is compatible with coalition perfect equilibrium that consumers on one side of the market pay a positive price for joining a network, while consumers on the other side are subsidized to join. In fact Theorem 2.6 below shows that if the two sides are asymmetric ($u^1 \neq u^2$), then in every coalition perfect equilibrium the side with the smaller reservation value gets strictly subsidized, and consumers on the other side pay a strictly positive price.

Theorem 2.5. *There can be two types of coalition perfect equilibria:*

1. $\exists\ k \in \{A, B\}$ *such that* $N_k^j = 1\ \forall\ j = 1, 2$ *and* $i \in [0, 1]$, $p_k^1 = -p_k^2$ *and* $p_k^j \leq u^j\ \forall$ $j = 1, 2$

2. $N_A^1 = N_A^2 = N_B^1 = N_B^2 = 1/2$, $p_A^1 = p_B^1 = -p_A^2 = -p_B^2$ *and* $p_A^j \leq u^j\ \forall\ j = 1, 2$

Either all consumers join the same network, or the two networks charge the same prices and split the consumers equally. In all equilibria no consumer stays out of the market and all consumers on the same side of the market pay the same price. This market price on one side of the market is just the negative of the price on the other side. Therefore either both prices are zero, or consumers on one side pay a positive price while consumers on the other side receive an equivalent subsidy. In all equilibria both firms' profits are zero.

The intuition behind the zero profit result is that coalitional rationalizability implies that slightly undercutting the competitor's price on both sides of the market results in stealing the whole market. Then if in some profile at least one firm's profit is positive, then at least one firm could profitably deviate by slightly undercutting the other firm's prices. Furthermore, even in profiles in which firms get zero profit but active firms do not charge the same prices at least one firm could profitably deviate by undercutting.

The result of full consumer participation comes from the fact that in all the above equilibria either the market price (the price charged by the active firm(s)) is negative on one side, or the market price is zero on both sides. The first case implies that consumers on the side with the negative market price do not stay out of the market, and then the zero profit result can be used to show that there has to be full consumer participation on the other side as well. The intuition behind the result that every consumer joins some network is that if prices are zero then it is a supported restriction for the coalition of all consumers to agree upon joining some network. This is because even the most pessimistic expectation compatible with the agreement (namely that other consumers are equally dispersed between the two networks) yields a positive expected payoff, while staying out of the market gives a zero payoff.

Theorem 2.6. *Assume $u^j < u^{-j}$ for $j \in \{1, 2\}$. Then in every coalition perfect equilibrium $N^1_k + N^2_k > 0$ for some $k \in \{A, B\}$ implies that $p^j_k \in [-u^{-j}, u^j - u^{-j}]$*

Theorem 2.6 states that if consumers on one side have a higher reservation value than consumers on the other side, then in a coalition perfect equilibrium the price charged on the side which has the smaller reservation value has to be in an interval that is strictly below zero. The minimal amount of subsidy that consumers on this side get is the difference between the reservation values. This result is not a consequence of coalitional rationalizability, but comes from the restrictions that "divide and conquer" strategies put on any subgame perfect Nash equilibrium[24]. If the market price is positive on the side with the low reservation value, then it is relatively cheap to steal consumers on that side, and then a higher price can be charged on the side with the high reservation value.

The following claim concerning the context of linear utility functions is straightforward to establish using Theorems 2.5 and 2.6.

Claim 2.1. *Let $g^j_i(N^{-j}) = N^{-j}u^j \; \forall \; j = 1, 2$ and $i \in [0, 1]$. Then the following hold:*

- *If $u^1 = u^2 \equiv u$, then two types of coalition perfect equilibria exist:*

1. (monopoly equilibria with zero profits) $\exists \; k \in \{A, B\}$ such that all consumers join network τ_k, $p^1_k = -p^2_f$ and $p^j_k \leq u \; \forall \; j = 1, 2$

2. (symmetric equilibria with zero profits) $N^1_A = N^2_A = N^1_B = N^2_B = 1/2$, $p^1_A = p^1_B = -p^2_A = -p^2_B$ and $p^j_A \leq u/2 \; \forall \; j = 1, 2$

[24] this is why Caillaud and Jullien (2001) obtain a similar result for equilibria in which both firms are active

- *If* $u^j < u^{-j}$ *for some* $j \in \{1, 2\}$ *and* $u^{-j} \leq 2u^j$*, then two types of coalition perfect equilibria exist:*

1. (monopoly equilibria with zero profits) $\exists\ k \in \{A, B\}$ *such that all consumers join network* τ_k*,* $p_k^1 = -p_k^2$ *and* $p_k^j \in [-u^{-j}, u^j - u^{-j}]$

2. (symmetric equilibria with zero profits) $N_A^1 = N_A^2 = N_B^1 = N_B^2 = 1/2$*,* $p_A^1 = p_B^1 = -p_A^2 = -p_B^2$ *and* $p_k^j \in [\frac{-u^{-j}}{2}, u^j - u^{-j}]$

- *If* $u^j < u^{-j}$ *for some* $j \in \{1, 2\}$ *and* $u^{-j} > 2u^j$*, then only one type of coalition perfect equilibrium exists:*

(monopoly equilibria with zero profits) $\exists\ k \in \{A, B\}$ *such that all consumers join network* τ_k*,* $p = -p_k^2$ *and* $p_k^j \in [-u^{-j}, u^j - u^{-j}]$.

The above linear specification gives an opportunity for a direct comparison with the predictions of Caillaud and Jullien (2001) in the case of asymmetric sides, since that is exactly the context of their investigation. By assuming monotonicity of the demand function, they obtain the same set of equilibria with two active firms. On the other hand, their refinement selects a larger set of equilibria with one active firm, including equilibria in which the active firm gets positive profits. Furthermore, full participation is an extra assumption in their model, while it is a result in our chapter.

2.7.2 The General Case

The next example shows that, as opposed to the case of homogeneous consumers, price competition does not necessarily drive profits to zero if consumers are heterogeneous. The example also points out that consumers with different reservation values might join different networks and pay different prices in equilibrium, even in the absence of product differentiation.

Claim 2.2. *Assume* $g_i^j(N^{-j}) = N^{-j} \cdot u_i^j \ \forall \ j = 1, 2$ *and* $i \in [0, 1]$. *Assume that on both sides of the market, a mass of consumers with measure* 0.4 *have reservation value* 2.55 *('I' types), a mass of consumers with measure* 0.15 *have reservation value* 0.51 *('II' types), a mass of consumers with measure* 0.1 *have reservation value* 0.46 *('III' types), while a mass of consumers with measure* 0.35 *have reservation value* 0.15 *('IV' types).*
Then there exists a coalition perfect equilibrium in which one firm charges a price of 0.31 *on side 1 and* -0.2 *on side 2, while the other firm charges* -0.2 *on side 1 and* 0.31 *on side 2. All type 'I' consumers on side 1 and type 'II'-'IV' consumers on side 2 join the first firm, while all type 'I' consumers on side 2 and type 'II'-'IV' consumers on side 1 join the second firm.*

Notice that in this profile both firms get a profit of $0.31 \times 0.4 - 0.2 \times 0.6 = 0.04$, which is strictly positive. Furthermore, in the described equilibrium the firms charge different prices, and consumers on the same side of the market with different reservation values end up paying different prices for the market good, despite the fact that reservation prices are private information of the consumers.

Every consumer on both sides of the market joins some network. Type 'I' consumers on both sides of the market pay a registration fee of 0.31 for joining a network, and in equilibrium they face a measure of 0.6 consumers from the other side of the market. All other consumers on both sides of the market are subsidized, they pay a registration fee of -0.2. In the equilibrium they face only a measure of 0.4 consumers from the other side of the market.

This equilibrium structure is similar to the equilibria in the previous section, in which the monopolist achieved price discrimination by operating two networks. In particular, one network is cheaper on one side of the market, while the other one is cheaper on the other side. A larger fraction of consumers, those having relatively low reservation values, join the cheap network sides, which makes it worthwhile for the remaining, high reservation value, consumers to join the expensive network sides. These similarities are consequences of assuming that after every price announcement consumers play some coalitionally rationalizable Nash equilibrium. In the monopoly case it is never in the interest of the firm to establish two networks that are priced equally, since that would just split consumers into two networks, generating less consumer surplus and therefore less profit. In the duopoly case there cannot be a coalition perfect equilibrium with positive profits and equally priced networks, because of the usual Bertrand competition undercutting argument. Therefore in both cases, for different reasons, the two networks have to be priced differently.

The intuition why competition does not drive profits down to zero in the above example is that with heterogeneous consumers deviation strategies based on undercutting, which are always effective due to the assumption of coalitional rationalizability, are not necessarily profitable. For example if B announces slightly smaller prices than the equilibrium prices of firm A then coalitional rationalizability implies that all type 'I'-'III' from side 1, and all consumers from side 2 join its network. But the highest profit B can achieve this way is strictly smaller than the equilibrium profit of B. The reason is that the proposed undercutting increases the number of consumers joining the network by a larger amount on the side where the price is negative.

The same intuition applies to "divide and conquer" type strategies. A firm can lower its price so that it makes it a dominant choice for some type of consumers to join its network and then it can charge a high price on the other side of the market and still make sure that some consumers join its network on that side as well. But if consumers are heterogeneous, then the proportion of consumers who are willing to pay the increased price on the latter side might be too low to compensate for the costs associated with lowering the price (increasing the subsidy) on the first side.

Theorem 2.7 establishes that the basic features of the above example hold for any coalition perfect equilibrium in which some firm's profit is positive, for any game with two firms.

Theorem 2.7. *Suppose that in a coalition perfect equilibrium $\pi_k > 0$ for some $k \in \{A, B\}$. Then (1) $N_k^j > 0 \ \forall \ j \in \{1, 2\}$ and $k \in \{A, B\}$; (2) $\exists \ j \in \{1, 2\}$ such that $p_A^j > p_B^j$, $p_A^{-j} \leq p_B^{-j}$, $N_A^j \leq N_B^j$ and $N_A^{-j} > N_B^{-j}$.*

In all these equilibria both firms have to be active and firm A's network has to be (weakly) more expensive and smaller on one side of the market and (weakly) cheaper and larger on the other than firm B's network. Furthermore, the two networks have to be asymmetric in the sense that on at least one side the networks charge different prices and on at least one side a different fraction of consumers choose A's network than B's.

We conclude the section by establishing that just like in the case of a monopolist network provider, enough heterogeneity among the consumers is needed for the existence equilibria with two asymmetric networks (and therefore for the existence of equilibria with nonzero profits). The next theorem is an extension of Theorem 2.5 to the case of nearly homogeneous consumers, provided that consumers' utility functions are weakly convex.

Theorem 2.8. *Assume that g_i^j is weakly convex for every $j = 1, 2$ and $i \in [0, 1]$. If $u_i^j / u_{i'}^j \leq 4/3$ for every $j = 1, 2$ and $i, i' \in [0, 1]$ then $\pi_A = \pi_B = 0$. Furthermore, there can be two types of coalition perfect equilibria:*

1. $\exists\ k \in \{A, B\}$ *such that* $N_k^j = 0\ \forall\ j = 1, 2$ *and* $i \in [0, 1]$
2. $N_A^j = N_B^j$ and $p_A^j = p_B^j\ \forall\ j \in \{1, 2\}$

2.8 Discussion

In this section, we discuss how the results presented in this chapter would be affected by changing different assumptions we made in the model.

2.8.1 Positive Marginal Cost

If firms face a positive constant marginal cost, all the qualitative results still hold, with one exception. If the marginal cost is higher than a certain threshold, then even if every consumer's reservation value is still higher than the marginal cost, coalitional rationalizability does not exclude the possibility that in price competition between two firms consumers do not join any network in equilibrium. The reason is that if marginal costs are high, then the average price charged on consumers is high and ex-ante coordination to join some network becomes harder.

2.8.2 Multi-Homing

In some two-sided markets network choices are naturally mutually exclusive, at least over a given period of time. For example people looking for a date can only be in one entertainment facility at a time. In other contexts consumers can join multiple platforms, which is called multi-homing in the literature. The qualitative conclusions of the chapter remain valid if we allow for multi-homing. In particular if there are multiple active networks in coalition perfect equilibrium that are not equally priced then it has to be that one network is cheaper and larger on one side of the market and the other network is cheaper and larger on the other side. If consumers are homogeneous, then it still holds that in every coalition perfect equilibrium both firms get zero profit. On the other hand firms can have positive profits in price competition if consumers are heterogeneous, the intuition being the same as in Section 2.7. Finally, multi-homing does not change the result that a monopolist might find it more profitable to operate two networks rather than one. In fact, if multi-homing is allowed, then the high reservation value consumers join multiple networks, increasing the monopolist's revenue. This makes operating two networks more attractive and therefore there is a larger set of games in which the monopolist runs two networks in coalition perfect equilibrium.

2.8.3 Conflict of Interest among Consumers on the Same Side

The assumption that a consumer's utility is not affected by the number of consumers from the same side of the market who join the same platform as she does can be restrictive in a variety of contexts. If the networks are matchmaking services or auction sites, then people on the same side of the market might compete for the same transactions. In other contexts transactions are non-rival goods, validating our assumption of no conflict of interest on the same side.

If consumers are homogeneous, it is possible to partially relax the assumption of no conflict of interest on the same side. In particular, it is enough to assume no conflict of interest on the same side only if every consumer from the other side is present at the same network, for the main qualitative conclusions of the chapter to remain valid.[25] In all other cases consumers' utilities might depend negatively on the number of consumers from the same side joining the same network. One scenario that validates this assumption is if there exist ideal matches in the market, in which case two consumers from the same side are competitors only if their ideal partners from the other side of the market are not present at their network (which cannot happen if all consumers from the other side are present).

2.8.4 More than Two Firms

With more than two firms operating on the market the analysis of the price competition game becomes complicated and it therefore omitted. However, we note that there can exist coalition perfect equilibria with multiple asymmetric networks in the case of more than two firms too. In fact, in general there is a wider range of coalition perfect equilibria. The intuition behind this is that coordination among consumers is more difficult if there are more than two firms. Formally, coalitional rationalizability puts less restriction on what can happen in consumer subgames. We plan to investigate the effect of the number of firms in the market on the range of equilibria in a future project.

[25] See the previous version of our work for a formal investigation of this.

2.9 Conclusions and Possible Extensions

This chapter analyzes pricing decisions of firms and platform choices of consumers on two-sided markets with network externalities, assuming that groups of consumers can coordinate their choices if coordination is focal. A key feature of the analysis is that consumers are allowed to be heterogeneous with respect to their willingness to pay for the network good. To keep the analysis tractable, several simplifying assumptions are made in other dimensions though. Section 2.8 discusses the consequences of relaxing some of these, but there are several other extensions that would make the model more realistic in important settings. Besides registration fees, in several contexts firms might be able to charge usage fees, fees for successful matches, or more complicated pricing instruments like contingent offers. In other two-sided markets, for example the market for health insurance, adverse selection is a central problem. Our model assumes that serving every consumer induces the same marginal cost to the firm and therefore abstracts away from this issue. Finally, there are contexts in which consumers are not ex ante symmetric with respect to the network externality they generate (their "attractiveness" to consumers on the other side). Some of these directions are addressed in the existing literature, others are left for future research.

Chapter 3

History as a Coordination Device[1]

3.1 Introduction

Consider a population of identical individuals who have to make a simultaneous decision regarding participation in a coup attempt. The probability of success increases with the proportion of individuals who decide to participate. The nature of the problem could be explained in the context of a two-player stag-hunt game. Assume that each player has to decide whether to join the rebellion (R) or to opt out (O).

	O	R
O	$7,7$	$6,0$
R	$0,6$	$9,9$

In this two-person example, the rebellion will succeed if and only if both players choose R. In this case, both players will be better off than in the status quo ($9 > 7$). That is, the equilibrium (R, R) Pareto dominates the equilibrium (O, O). Yet, strategy O guarantees a

[1]Rosa Argenziano and Itzhak Gilboa

higher minimal payoff than does R. Indeed, the equilibrium (O, O) risk dominates (R, R) (Harsanyi and Selten (1988)), and may be a reasonable prediction of the outcome of the game even though (R, R) is a Pareto dominant equilibrium.

In this chapter we analyze a large population version of this revolution game. In this game, a continuum of players have to make a simultaneous decision regarding their participation in a coup attempt. The probability that the revolution succeeds depends on the proportion of players who decided to join the attempt. As in the game above, everyone will be better off if the revolution succeeds, but in case of failure the attempting rebels will be punished. We will also assume that, should the revolution succeed, every individual has an incentive to be among the rebels rather than to remain obedient.

Consider the decision of a single player in this game. Imagine that rumors have been spreading that the revolution would start tonight. She can ignore the rumors and go to sleep, or take to the streets. For simplicity, assume that this is a one-shot, binary decision. The potential rebel sits at home and attempts to assess the probability that the revolution would succeed. How would she do that?

We maintain that the assessment of this probability would and should be based on the results of past coup attempts in similar games. These games may have been played by the same population or by others. They may have been more or less similar. Both the nature of the game and the identity of the population playing it should be taken into account in the evaluation of the similarity of past games to the present one. But ignoring these past games would hardly seem a rational way of generating beliefs.[2]

[2]The belief formation process may be embedded in a meta-game, which will also have a flavor of a coordination game. We assume, however, that people have a fundamental tendency to expect the future to be similar to the past. To quote Hume (1748), "From similar causes we expect similar effects."

In this chapter we are interested in a dynamic process, according to which large populations are called upon to play a simple "revolution" game, where the games differ from each other by one parameter at most. This parameter designates the status quo, and the lower it is, the more do the people have to gain from a successful revolution. We assume that players generate beliefs regarding the success of a revolution based on the similarity-weighted relative frequency of successes in the past: they calculate relative frequencies, but each past case is assigned a weight that is proportional to its similarity to the game at hand.

Our dynamic process may explain a "Domino" effect. Consider, for example, the revolutions in the Soviet block in the late '80s and early '90s. A successful revolution in one country renders the success of a revolution in another country more likely, and vice versa. This process will also exhibit path-dependence. Assume, first, that the first attempted revolution occurs in a country in which a revolution is almost inevitable, because the present conditions leave people with nothing to lose. A successful attempt in this country would make it more likely that a revolution would succeed in a similar country, even if the conditions in the latter are not as dire. Continuing in this way, one may generate a sequence of successful revolutions.

If, on the other hand, the first revolutionary attempt occurs in a country in which most people have little to gain from a revolution, this attempt might fail. If it is then followed by a revolution attempt in a similar country, where conditions are worse yet similar, a revolution in the latter may also fail, and, continuing in this way one may generate a sequence of failed revolutions.

We conclude that history serves as a coordination device. It informs the belief formation process of all individuals, and, being commonly known, it coordinates among them. However, beliefs that are history-dependent may lead to different behavior, depending on the way history unfolds.

The rest of this chapter is organized as follows. We first discuss related literature. Section 3.3 describes the stage game. We devote Section 3.4 to modeling the way players generate beliefs given history. Finally, Section 3.5 describes the dynamic process and provides the main result.

3.2 Related Literature

The game theoretic literature has witnessed many attempts to select equilibria based on the parameters of the game. The equilibrium selection literature includes many notions that are defined by the game itself (see van Damme (1983)), such as the risk-dominance criterion mentioned above. Other types of considerations attempted to embed the game in a dynamic process (Young (1993), Kandori, Mailath, and Rob (1993), Burdzi, Frankel and Pauzner (2001)) or in incomplete information set-up (Carlsson and van Damme (1994)).

It is noteworthy that risk dominance has emerged as the preferred selection criterion based on quite different types of considerations. On the other hand, the literature on network externalities tends to favor Pareto dominant equilibria over risk dominant ones (see Katz and Shapiro (1986)). This suggests a more agnostic view, according to which the parameters of the game cannot, in general, predict equilibrium selection. It appears that game theoretic considerations could be used to impose certain restrictions on the possible outcomes, but the actual selection of an equilibrium is often left to history, chance, institutional details, or other unmodeled factors.

The conceptualization of a revolution as a coordination game dates back to Schelling (1960) at the latest. There exist alternative conceptualizations in the political science literature, such as Muller and Opp (1986), who emphasize the public good aspect of a revolution. Yet, the coordination game model of a revolution has been the subject of many studies. Lohmann (1994) studied the weekly demonstrations in Leipzig and the evolution of beliefs along the process. More recently, Edmond (2003) studied the effect of mass media on revolutions, whereas Angeletos, Hellwig, and Pavan (2004) focus on a learning process by which individuals form beliefs. As in Lohmann (1994) and Angeletos, Hellwig, and Pavan (2004), we study the evolution of beliefs in a game that is played repeatedly. However, as opposed to these papers, our game is played by a new population at every stage. Thus, our focus is on the generation of prior beliefs (over other players' actions), based on similar games, rather than on the update of already existing prior beliefs by Bayes's law.

3.3 The Stage Game

We describe a symmetric two-stage extensive form game G_x depending on a parameter $x \in [0,1]$. There is a continuum of players $[0,1]$. In stage 1 all players move simultaneously. The set of moves for each player i is $S_i = \{0,1\}$, where 1 stands for participation, and 0 – for opting out.

In stage 2, after each player determined her move in $\{0,1\}$, nature chooses a move in $\{F,S\}$, which stand for *Failure* and for *Success* of the revolution, respectively. Nature's move depends on the set of players choosing 1 in stage 1, $A \subset [0,1]$. Specifically, if A

is Lebesgue-measurable, we assume that nature chooses S with probability $\lambda(A)$, where λ stands for Lebesgue's measure. If A is non-measurable, the probability of nature choosing S can be defined arbitrarily (say, by the inner measure of A). At equilibrium, the set A will be measurable.

After each player determined her choice of participation (0 or 1) and nature determined the success of the revolution (by the probability $\lambda(A)$), the game is over. The payoff of each player depends only on her own choice of participation, and on nature's move (i.e., on the success of the revolution). The payoff function $u = u_i$ for every $i \in [0, 1]$ is given by the following matrix:

$$\begin{array}{ccc} & S(uccess) & F(ailure) \\ 1(YES) & 1 & 0 \\ 0(NO) & \frac{x+1}{2} & x \end{array} \tag{3.1}$$

where $x \in [0, 1]$ is the parameter of the game.

The interpretation of this matrix is as follows. The worst thing that can happen to an individual in this game is to participate in a failed coup. The result is likely to involve imprisonment, exile, decapitation, and the like. We normalize this worst outcome to 0. The best thing that can happen to an individual is that she participates in a revolution that succeeds. In this case she is a part of a (presumably) better and more just society. We normalize this payoff to 1.

An individual who decides to participate in the revolution therefore decides to bet on its success with the extreme payoff of 0 and 1. Between these extreme payoffs lie the payoffs for an individual who decides to opt out, foregoing the chance of being part of the revolution. The payoff of such an individual still depends on the outcome of the revolutionary attempt.

Should this attempt fail, such an individual would get x, which is a measure of the well-being of the people in the status quo. We implicitly assume that such an individual, who did not participate in a failed coup d'etat, will be unaffected by the attempted coup. If, however, the revolution succeeds, even the individuals who were passive will benefit from the new regime. However, not being part of the revolutionary forces, they would not reap the benefits of revolution in its entirety. We choose to set their payoff to the arithmetic average between the full benefit, 1, and the status quo, x.

Observe that if $x = 1$, there is nothing to be gained from a revolution. In this case the well-being of the people in the status quo is just as good as it could possibly be in the case of a successful revolution, and joining a revolution is a dominated strategy. This is what we would expect the situation to be in a democracy.

If, on the other hand, $x = 0$, the well-being of the people in the status quo is comparable to the well-being of an individual who participated in a failed revolution. This describes a situation in which the people has nothing to lose, as in a situation of starvation. In this case, indeed, joining the revolution is a dominant strategy.

In between, when $0 < x < 1$, lie the cases that are strategically more interesting: in these cases, each individual will be better off if the revolution succeed, but she prefers to be passive if the revolution is doomed to fail. In these situations there is no dominant strategy, and each individual player has to determine her choice depending on her beliefs about the other players's choices.

Assume, then, that an individual i attempts to estimate the expected utility of a player playing 1 (participating in the revolution) versus 0 (opting out). Realizing the nature of the game, the individual knows that the probability of a revolution succeeding is independent of her own choice. Suppose that individual i's belief over the measure of other individuals who choose 1 is given by a measure μ_i over (the Lebesgue σ-algebra on) $[0,1]$. That is, for every Lebesgue-measurable set $B \subset [0,1]$, individual i assigns probability $\mu_i(B)$ to the event that the measure of individuals who eventually choose 1 (with or without herself) lies in B. Individual i's subjective probability that the revolution would succeed is, therefore,

$$\widehat{p}_i = \int_{[0,1]} p d\mu_i(p)$$

That is, individual i is assumed to calculate the overall probability of a successful revolution by Bayes's formula, taking into account the assumption that the probability of success equals the measure of the set of individuals who participate in the coup.

Let $U(k)$ denote the expected payoff for an individual i who chooses strategy $k \in \{0,1\}$, given beliefs $\widehat{p}_i$. That is, $U(1) = \widehat{p}_i$ and $U(0) = \frac{x+1}{2}\widehat{p}_i + x\left[1 - \widehat{p}_i\right]$.

It is useful to calculate the *critical belief* $p^*(x)$, that is the minimal value of $\widehat{p}_i$ that is necessary for an individual to participate in the revolutionary attempt. If we set $p^*(x) = \frac{2x}{1+x}$, it is easily observed that $U(1) \geq U(0)$ iff $\widehat{p}_i \geq p^*(x)$.

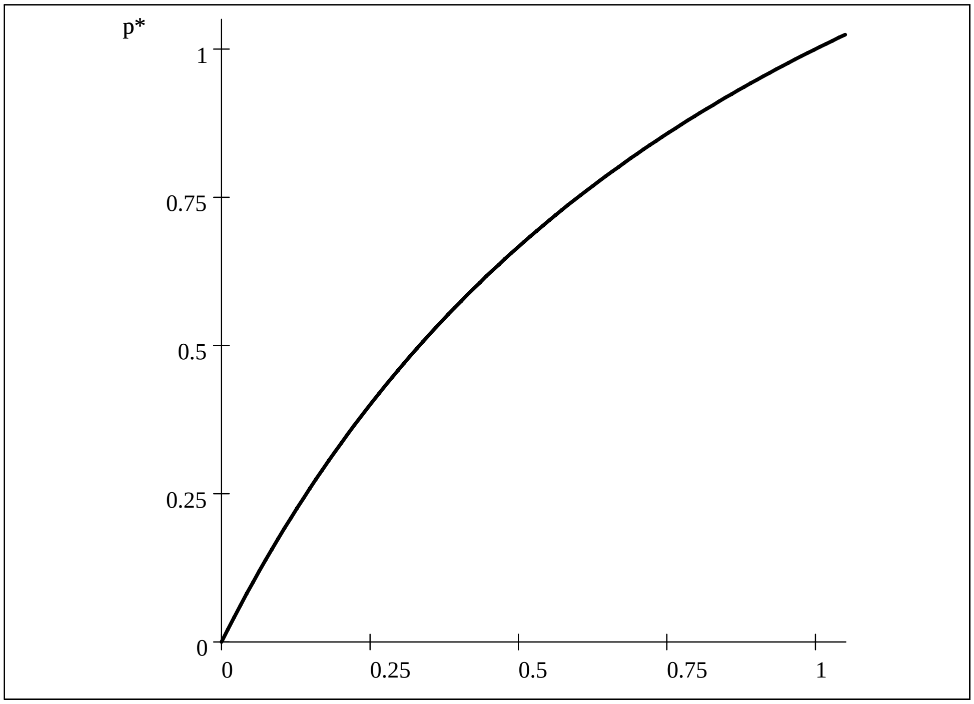

Figure 3.1: Threshold $p^*(x)$

At equilibrium, $\widehat{p}_i$ is independent of i, and it coincides with the actual probability of a successful revolution. Thus, for every $x \in [0,1]$ the game has three symmetric Nash equilibria (where $\sigma_i \in \Delta(\{0,1\})$ denotes player i's mixed strategy):[3]

1. $\sigma_i^* = \{1,0\}\ \forall i \in [0,1]$
2. $\sigma_i^* = \{0,1\}\ \forall i \in [0,1]$
3. $\sigma_i^* = (\frac{2x}{1+x}, 1 - \frac{2x}{1+x})\ \forall i \in [0,1]$

These equilibria are easy to describe. Revolution will succeed in equilibrium if $\widehat{p}_i \geq p^*(x)$, that is, if the subjective belief of each agent about the proportion of individuals who would participate in the coup, $\widehat{p}_i$, exceeds the critical belief $p^*(x)$. The revolution would fail otherwise. Simply put, if everyone believes that the revolution is likely to succeed, it will, and if everyone believes that the revolution is unlikely to succeed, it won't. This begs the question, however, of how do individuals form their beliefs.

3.4 Where Do Expectations Come from?

Our approach to the belief formation question is history- and context-dependent. Specifically, we assume that games of the type G_x above are being played over and over again, by different populations $[0,1]$, in different countries, at different times, and for different values of x. Yet, the history of similar games played in the past, which is assumed to be common knowledge, determines the beliefs $\widehat{p}_i$ of the individuals in question.

[3] As usual, the mixed equilibrium is rather arbitrary and dynamically unstable. Yet, at this point we do not rule out mixed or asymmetric equilibria.

More concretely, we assume that time is discrete and that the game G_x is played in every period by a new generation of players. We further assume that at the beginning of each period t nature selects a value for x_t in an i.i.d. manner, according to a known discrete distribution. For simplicity, we assume that the possible values for x are only $\{0, \frac{1}{2}, 1\}$. Thus the process is determined by a probability vector $(p_0, p_{\frac{1}{2}}, p_1)$ where $\Pr(x_t = \alpha) = p_\alpha$ for $\alpha \in \{0, \frac{1}{2}, 1\}$.

In each period, all the players of the current generation have a common memory of all the games that have been played in the past. Before playing, they observe the current state of the world and form an expectation on the probability of a success that is based on the similarity between the current state of the world and the state of the world in previous games that ended, respectively, with a success or a failure.

In particular, denoting by S_t the set of (indices of) past games that ended with a success and with F_t the set of (indices of) past games that ended in a failure, we assume that for every i and and every $t \geq 0$:

$$\widehat{p}_{it} = \widehat{p}_t = \frac{\sum_{k \in S_t} s^+(x_k, x_t)}{\sum_{k \in S_t} s^+(x_k, x_t) + \sum_{k \in F_t} s^-(x_k, x_t)} \tag{3.2}$$

where $s^+(x_k, x_t)$ is a function that measures the degree of support that a success in a game where the state of the world was x_k brings to the possibility that there is a success in the game being currently played and $s^-(x_k, x_t)$ measures the degree of support that a failure in a game where the state of the world was x_k leads to the possibility that there is a failure in the game being currently played. (We normalize the "degree of support" in such a way that a past success only lends support to a prediction of a future success, and a past failure

– to a future failure.)

The formula (3.2) is not well-defined for the first period, $t = 1$. Also, it allows $\widehat{p}_t$ to be 0 or 1, if history consists of failures alone, or of successes alone, respectively. We find such extreme beliefs unwarranted. Hence we use Equation (3.2) only when history contains both successes and failures. Formally, we assume that $t \geq 3$, and that history contains at least one success and at least one failure, so that $\widehat{p}_t \in (0, 1)$.

Possible functional forms for $s^+(x_k, x_t)$ and for $s^-(x_k, x_t)$ are

$$s^+(x_k, x_t) = 1 + (x_k - x_t) \tag{3.3}$$

$$s^-(x_k, x_t) = 1 - (x_k - x_t) \tag{3.4}$$

Observe that the denominator of $\widehat{p}_t(x_t)$ doesn't vanish for any $x_t \in [0, 1]$ because $\sum_{k \in S_t} s^+(x_k, x_t)$ and $\sum_{k \in F_t} s^-(x_k, x_t)$ are both nonnegative and cannot be simultaneously equal to zero because the first one can be zero only for $x_t = 1$ while the second one can be zero only for $x_t = 0$.

The interpretation of these formulae is as follows. If $x_t = x_k$, the current game t is practically identical to the game that was played in period k in the past. In this case, the degree of support is normalized to 1: whatever happened in period k (success or failure) lends empirical support to the belief that it is going to occur again in period t.

Assume, now, that $x_t > x_k$. In this case the game played in period k is similar, but not identical to, the game played in period t. The similarity is smaller than in the case $x_t = x_k$, because now (period t) people have a lower incentive to rebel than in the past (period k), since the status quo is more agreeable. Suppose that in period k the revolution succeeded. An individual is expected to reason as follows, "Well, in period k, when people were hungry and had little to lose, they rebelled. But it is still possible that today, when things are better, they won't. Hence, the success of the revolution in period k does lend *some* support to the assumption that people would rebel today, but this support is lower than it would be if the situations were identical." This is captured by the function $s^{+}(x_k, x_t) = 1 + (x_k - x_t) = 1 - (x_t - x_k)$. The bigger is the difference $x_t - x_k > 0$, the lower is the support that one gets for a successful revolution in period t from a successful revolution in period k. In the extreme case in which $x_t = 1$ and $x_k = 0$, a revolution in period k (when people had nothing to lose) lends no support to the prediction of a successful revolution in period t (when people have nothing to gain).

Next suppose that the revolution in period k failed (retaining the assumption $x_t > x_k$). In this case there is no support for a prediction of a success, but there is support for a prediction of a failure. Past failures make future failures more likely. But to what degree? Here the function $s^{-}(x_k, x_t) = 1 - (x_k - x_t) = 1 + (x_t - x_k)$ is larger than 1, that is, larger than in the case in which the game of period k were identical to the game in period t. This assumption is supposed to reflect the following reasoning, "The revolution in period t is unlikely to succeed. Even in period k, when people were hungrier, the revolution failed. Why would it succeed now, when the status quo is better?"

Finally, if $x_t < x_k$ the logic is reversed: a success in period k lends support greater than 1 to a success in period t, because in period t, a-priori, there is a stronger motivation to rebel, and a failure in period k lends support lower than 1 for a failure in period t for the same reason.

Observe that if the same game is repeatedly played over and over again, that is, if $x_k = x_t$ for every k, then the expected probability of a success is simply the observed relative frequency of success in the past. Hence, our belief formation process can be viewed as generalizing empirical frequencies (as in fictitious games, Robinson (1951)) to the case in which the game that is played is not identical to past games, but only similar to them.

The following notation may prove useful. Let N_t^S be the cardinality of S_t, and N_t^F – the cardinality of F_t. Equation (3.2) can be written as

$$\widehat{p}_t = \widehat{p}_t(x_t) = \frac{N^S - N^S x_t + \sum_{k\in S_t} x_k}{N^S + N^F + (N^F - N^S)\, x_t + \sum_{k\in S_t} x_k - \sum_{k\in F_t} x_k}$$

It seems natural that the expected probability of a successful revolution, $\widehat{p}_t$, be a monotonically decreasing function of the well-being at the status quo, x_t. Indeed,

$$\frac{d\widehat{p}_t}{dx_t} = \frac{+N^S\left(\sum_{k\in F_t} x_k - 2N^F\right) - N^F \sum_{k\in S_t} x_k}{\left[N^S + N^F + (N^F - N^S)\, x_t + \sum_{k\in S_t} x_k - \sum_{k\in F_t} x_k\right]^2} < 0 \tag{3.5}$$

(The last inequality holds because the numerator is the sum of two negative terms).

3.5 The Dynamic Process

We now wish to study the dynamic process in which at every stage $t \geq 1$ x_t is drawn from $\{0, \frac{1}{2}, 1\}$ according to probabilities $(p_0, p_{\frac{1}{2}}, p_1)$, beliefs are formed in accordance with equation (3.2), and an equilibrium in G_{x_t} is chosen by the beliefs $\widehat{p}_t(x_t)$.

Under our assumptions, for almost every history we can predict the outcome of the game by looking at a graph representing the two curves $p^*(x_t)$ (increasing in x_t) and $\widehat{p}_t(x_t)$ (decreasing in x_t). Note that the two curves intersect exactly once in $[0,1]$ because their difference $\widehat{p}_t(x_t) - p^*(x_t)$ is continuous, strictly decreasing, takes a nonnegative value at zero and a non-positive value at one. Let α_t denote the value such that $p^*(\alpha_t) = \widehat{p}_t(\alpha_t)$.

If the current value x_t is to the left of the intersection point, that is, $x_t < \alpha_t$, all players' expectation $\widehat{p}_t(x_t)$ will be above the critical belief $p^*(x_t)$, and they will therefore all play 1, resulting in a successful revolution. If, however, the current x_t is to the right of the intersection point, that is, $x_t > \alpha_t$, then all players will play 0 and there will be a failure with probability 1. If $\widehat{p}_t(x_t) = p^*(x_t)$, namely, $x_t = \alpha_t$, then all individuals are indifferent between playing 0 and playing 1. For simplicity we assume that they break ties in favor of the status quo and choose 0. Observe that this is the only case in which we *assume* that the equilibrium is symmetric. For other values of x_t symmetry is a result of optimization.[4]

Next, we observe that a state of the process is fully summarized by a matrix of relative frequencies

$F_t =$

	$x=0$	$x=\frac{1}{2}$	$x=1$
1	$f_{t,10}$	$f_{t,1\frac{1}{2}}$	$f_{t,11}$
0	$f_{t,00}$	$f_{t,0\frac{1}{2}}$	$f_{t,01}$

where $f_{t,ij}$ is the relative frequency, up to time t, of periods in which the game was G_j and all players played i. We are interested in the limit of F_t as $t \to \infty$.

Given our assumption that $t \geq 3$, and that history contains at least one success and at least one failure, $\widehat{p}_t \in (0,1)$, which in turn implies that $f_{t,00} = f_{t,11} = 0$ for all t. Observe

[4] Other assumptions are possible, allowing players to select different strategies, and/or to play mixed strategies. Note that in the latter case one has to assume that the law of large numbers holds (see Judd (1985)). The main point of this paper does not depend on these assumptions.

that the relative frequencies of the columns are governed only by the selection of x, and are independent of the players' behavior. Hence the only candidates for limit frequencies can be the following matrices:

$L_1 =$

	$x=0$	$x=\frac{1}{2}$	$x=1$
1	p_0	$p_{\frac{1}{2}}$	0
0	0	0	p_1

$L_2 =$

	$x=0$	$x=\frac{1}{2}$	$x=1$
1	p_0	0	0
0	0	$p_{\frac{1}{2}}$	p_1

$L_3 =$

	$x=0$	$x=\frac{1}{2}$	$x=1$
1	p_0	$wp_{\frac{1}{2}}$	0
0	0	$(1-w)\,p_{\frac{1}{2}}$	p_1

for $w \in (0,1)$.

We can finally present our main result.

Theorem 3.1. *For every $(p_0, p_{\frac{1}{2}}, p_1)$, F_t converges to L_1 or to L_2 with probability 1. For an open and convex set of vectors $(p_0, p_{\frac{1}{2}}, p_1)$, containing $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, there is a positive probability that F_t converges to L_1, and also a positive probability that F_t converges to L_2.*

The proof is relegated to Appendix C.

The theorem states that games $G_{\frac{1}{2}}$, which are partly similar to games G_0 but also to games G_1, will eventually be played either like G_0 or like G_1 with probability 1. That is, history will determine the outcome of the non-trivial games: either they all result, in the limit, in successful revolutions (if they end up being played like G_0), or they all result in failed ones (if they end up being played like G_1). The main point of the theorem is its second part, which states that the limit distribution of F_t cannot be computed a-priori. That is, the fundamentals of the game and of the dynamic process do not suffice for a unique determination of the limit behavior in the non-trivial games. The random process generating the sequence x_t will determine the results of the games $G_{\frac{1}{2}}$. Specifically, if the process starts with a large proportion of $x_t = 0$, then for $x_t = \frac{1}{2}$ most similar games will be found to have resulted in a successful revolution, and therefore the game at stage t will also end in success. This will establish an equilibrium at which revolutions succeed in games $G_{\frac{1}{2}}$. Conversely, if the process starts with many draws of $x_t = 1$, every new game $G_{\frac{1}{2}}$ will be considered similar to games in which revolutions failed, and will therefore also result in a failed revolution.

Appendices

Appendix A

This Appendix collects the proofs of all the results presented in Chapter 1.

Proof of Lemma 1.1 The welfare function can be rewritten as

$$\mathbb{E}_{\mathbf{x}}\left[\int_{i\in\mathcal{A}}(x_i^a+n^a)\,di+\int_{i\in\mathcal{B}}\left(x_i^b+n^b\right)di\right]=$$

$$=\mathbb{E}_{\mathbf{x}}\left[\int_{i\in\mathcal{A}}n^a di+\int_{i\in\mathcal{A}}x_i^a di+\int_{i\in\mathcal{B}}n^b di+\int_{i\in\mathcal{B}}x_i^b di\right]=$$

$$=\mathbb{E}_{\mathbf{x}}\left[[n^a]^2+\left[n^b\right]^2+\int_{i\in\mathcal{A}}x_i^a di+\int_{i\in\mathcal{B}}x_i^b di\right].$$

We will prove by contradiction that any allocation $(\mathcal{A}',\mathcal{B}')$ that is not a threshold allocation cannot maximize the welfare function. By construction, there exists no $X\in\mathbf{R}$ such that $\mathcal{B}'=\{i\in[0,1]:x_i\leq X\}$ and $\mathcal{A}'$=$\{i\in[0,1]:x_i>X\}$. Therefore, there exists at least a couple $(\mathcal{S},\mathcal{T})$ such that $\mathcal{S}\subset\mathcal{A}$, $\mathcal{T}\subset\mathcal{B}$, $\int_{i\in\mathcal{S}}di=\int_{i\in\mathcal{T}}di$ and $x_i<x_{i'}$ for every (i,i') such that $i\in\mathcal{S}$ and $i'\in\mathcal{T}$. Consider now a different allocation $(\mathcal{A}'',\mathcal{B}'')$ such that $\mathcal{A}''=(\mathcal{A}'/\mathcal{S})\cup\mathcal{T}$ and $\mathcal{B}''=(\mathcal{B}'/\mathcal{T})\cup\mathcal{S}$. We will show that this allocation yields a larger welfare than $(\mathcal{A}',\mathcal{B}')$. Market shares are the same under both allocations, therefore the utility of every consumer in $\mathcal{A}'/\mathcal{S}$ and in $\mathcal{B}'/\mathcal{T}$ is the same. The utility of each individual i

$\in \mathcal{S}$ is equal to $n^a + x_i^a$ under allocation $(\mathcal{A}', \mathcal{B}')$ and to $n^b + x_i^b$ under allocation $(\mathcal{A}'', \mathcal{B}'')$. Analogously, the utility of each consumer $i \in \mathcal{T}$ is $n^b + x_i^b$ under allocation $(\mathcal{A}', \mathcal{B}')$ and $n^a + \varepsilon_i^a$ under allocation $(\mathcal{A}'', \mathcal{B}'')$. Therefore, the total change in welfare when moving from allocation $(\mathcal{A}', \mathcal{B}')$ to allocation $(\mathcal{A}'', \mathcal{B}'')$ is

$$\mathbb{E}_{\mathbf{x}}\left[\int_{i\in\mathcal{S}}\left(n^b + x_i^b - n^a - x_i^a\right)di + \int_{i\in\mathcal{T}}\left(n^a + x_i^a - n^b - x_i^b\right)di\right] =$$

$$= \mathbb{E}_{\mathbf{x}}\left[\left(n^b - n^a\right)\left(\int_{i\in\mathcal{S}} di - \int_{i\in\mathcal{T}} di\right) - \int_{i\in\mathcal{S}} 2x_i di + \int_{i\in\mathcal{T}} 2x_i di\right] =$$

$$= 0 - \mathbb{E}_{\mathbf{x}}\left[\int_{i\in\mathcal{S}} 2x_i di + \int_{i\in\mathcal{T}} 2x_i di\right] > 0$$

where the last inequality holds because $x_i < x_{i'}$ for every (i, i') such that $i \in \mathcal{S}$ and $i' \in \mathcal{T}$.

■

Proof of Proposition 1.1 The proof of Proposition 1.1 is divided into 4 claims. In Claim A.1 we derive expression 1.2 for the welfare function. In Claim A.2 we prove that the welfare function has a global maximum and the maximum is attained at a point $t \in (-1, 1)$. In Claim A.3 we prove that the optimum cannot be in the interval $t \in [0, y]$. For the case $y \geq 1$, this completes the proof. Claim A.4 considers the case where $y < 1$ and shows that it cannot be the case that $t^* \in (y, 1)$. This concludes the proof.

Claim A.1. *Expression 1.2 represents the ex-ante welfare function associated to a generic threshold allocation.*

For a given realization of $\left(\widetilde{\theta}^a, \widetilde{\theta}^b\right)$, take a generic threshold allocation $\left(\mathcal{A}(t), \mathcal{B}(t)\right)$. Let $I(\cdot)$ denote the indicator function. The level of welfare associated to $\left(\mathcal{A}(t), \mathcal{B}(t)\right)$ is

$$\mathbb{E}_{\mathbf{x}}\left[W(t)\middle|\theta^a, \theta^b\right] =$$

$$\begin{aligned}
= & \int_{-\infty}^{t-\theta} \theta^b f(\varepsilon_i)\, d\varepsilon_i + \int_{t-\theta}^{+\infty} \theta^a f(\varepsilon_i)\, d\varepsilon_i + \\
& + \int_{-\infty}^{t-\theta} (1-n^a) f(\varepsilon_i)\, d\varepsilon_i + \int_{t-\theta}^{+\infty} n^a f(\varepsilon_i)\, d\varepsilon_i + \\
& + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varepsilon_i^a I\left(\frac{\varepsilon_i^a - \varepsilon_i^b}{2} \geqslant t-\theta\right) f(\varepsilon_i^a) f\left(\varepsilon_i^b\right) d\varepsilon_i^a d\varepsilon_i^b + \\
& + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varepsilon_i^b I\left(\frac{\varepsilon_i^a - \varepsilon_i^b}{2} \leqslant t-\theta\right) f(\varepsilon_i^a) f\left(\varepsilon_i^b\right) d\varepsilon_i^a d\varepsilon_i^b =
\end{aligned}$$

$$\begin{aligned}
= & \ \theta^b \int_{-\infty}^{t-\theta} f(\varepsilon_i)\, d\varepsilon_i + \theta^a \int_{t-\theta}^{+\infty} f(\varepsilon_i)\, d\varepsilon_i + \\
& + (1-n^a) \int_{-\infty}^{t-\theta} f(\varepsilon_i)\, d\varepsilon_i + n^a \int_{t-\theta}^{+\infty} f(\varepsilon_i)\, d\varepsilon_i + \\
& + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varepsilon_i^a I\left(\frac{\varepsilon_i^a - \varepsilon_i^b}{2} \geqslant t-\theta\right) f(\varepsilon_i^a) f\left(\varepsilon_i^b\right) d\varepsilon_i^a d\varepsilon_i^b + \\
& + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \varepsilon_i^b I\left(\frac{\varepsilon_i^a - \varepsilon_i^b}{2} \leqslant t-\theta\right) f(\varepsilon_i^a) f\left(\varepsilon_i^b\right) d\varepsilon_i^a d\varepsilon_i^b =
\end{aligned}$$

$$\begin{aligned}
= & \ \theta^b \Phi\left((t-\theta)\sqrt{\beta}\right) + \theta^a \left[1 - \Phi\left((t-\theta)\sqrt{\beta}\right)\right] + \\
& (1-n^a)\Phi\left((t-\theta)\sqrt{\beta}\right) + n^a \left[1 - \Phi\left((t-\theta)\sqrt{\beta}\right)\right] + \\
& + \int_{-\infty}^{+\infty}\left[\int_{\varepsilon_i^b + 2(t-\theta)}^{+\infty} \varepsilon_i^a f(\varepsilon_i^a)\, d\varepsilon_i^a\right] f\left(\varepsilon_i^b\right) d\varepsilon_i^b + \\
& + \int_{-\infty}^{+\infty}\left[\int_{\varepsilon_i^a - 2(t-\theta)}^{+\infty} \varepsilon_i^b f\left(\varepsilon_i^b\right) d\varepsilon_i^b\right] f(\varepsilon_i^a)\, d\varepsilon_i^a =
\end{aligned}$$

$$
\begin{aligned}
= \quad & \theta^b \Phi\left(\left(t-\theta\right)\sqrt{\beta}\right) + \theta^a\left[1-\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\right] + \\
& + \left[\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\right]^2 + \left[1-\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\right]^2 + \\
& + \int_{-\infty}^{+\infty}\left[\sqrt{\frac{2}{\beta}}\phi\left(\left(\varepsilon_i^b + 2\left(t-\theta\right)\right)\sqrt{\frac{\beta}{2}}\right)\right] f\left(\varepsilon_i^b\right) d\varepsilon_i^b + \\
& + \int_{-\infty}^{+\infty}\left[\sqrt{\frac{2}{\beta}}\phi\left(\left(\varepsilon_i^a - 2\left(t-\theta\right)\right)\sqrt{\frac{\beta}{2}}\right)\right] f\left(\varepsilon_i^a\right) d\varepsilon_i^a =
\end{aligned}
$$

$$
= \Phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\left[\theta^b - \theta^a\right] + \theta^a +
$$

$$
+2\left[\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\right]^2 - 2\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right) + 1 +
$$

$$
+\sqrt{\frac{2}{\pi\beta}} e^{-\frac{(t-\theta)^2\beta}{2}}.
$$

Therefore, the expected welfare as a function of a generic threshold t is:

$$
\begin{aligned}
\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right] \quad = \quad & \mathbb{E}_{\theta^a,\theta^b}\left\{\theta^a - 2\theta\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right) + \right. \\
& + \sqrt{\frac{2}{\pi\beta}} e^{-\frac{(t-\theta)^2\beta}{2}} + \\
& \left. +2\left[\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\right]^2 - 2\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right) + 1\right\}.
\end{aligned}
$$

Claim A.2. *The welfare function has a global maximum and the maximum is attained at a point* $t \in (-1, 1)$.

Given Claim A.1, the optimal threshold t^* solves the following problem:

$$\max_t \mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right] = \max_t \mathbb{E}_{\theta^a,\theta^b}\left\{\left[\theta^a - 2\theta\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\right] + \right.$$
$$+\left[\sqrt{\frac{2}{\pi\beta}}e^{-\frac{(t-\theta)^2\beta}{2}}\right] +$$
$$\left. + \left[2\left[\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\right]^2 - 2\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right)+1\right]\right\}.$$

The welfare function is continuous and differentiable in t, and by Leibnitz' rule, its first derivative can be written as:

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right] =$$

$$= \mathbb{E}_{\theta^a,\theta^b}\left[\left(2\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right) - 1 - \theta - t + \theta\right)\left(2\sqrt{\beta}\phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\right)\right] =$$

$$= \mathbb{E}_{\theta^a,\theta^b}\left[\left(2\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right) - 1 - t\right)\left(2\sqrt{\beta}\phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\right)\right].$$

Notice that

$$2\sqrt{\beta}\phi\left(\left(t-\theta\right)\sqrt{\beta}\right) > 0 \qquad \forall\theta.$$

This implies that, since for every $t \leq -1$

$$2\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right) - 1 - t > 0 \qquad \forall\theta$$

then it is true that for every $t \leq -1$

$$\mathbb{E}_{\theta^a,\theta^b}\left[\left(2\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right) - 1 - t\right)\left(2\sqrt{\beta}\phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\right)\right] > 0.$$

So the derivative is always positive for any $t \leq -1$.

Analogously, since for every $t \geq 1$ it is true that

$$2\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right) - 1 - t < 0 \qquad \forall\theta,$$

then for every $t \geq 1$ it is also true that

$$\mathbb{E}_{\theta^a,\theta^b}\left[\left(2\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right) - 1 - t\right)\left(2\sqrt{\beta}\phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\right)\right] < 0,$$

i.e. the derivative is always negative for any $t \geq 1$.

If we restrict the domain of the welfare function to any set $[t_1, t_2]$ such that $(-1,1) \subset [t_1, t_2]$, the function has a global maximum $t^* \in [t_1, t_2]$ by Weierstrass theorem. If then we extend the domain to $\mathbb{R}$, it is true that t^* is still the global maximum because $\mathbb{E}_{\mathbf{x}}[W(t)] < \mathbb{E}_{\mathbf{x}}[W(-1)]$ $\forall t < t_1$ and $\mathbb{E}_{\mathbf{x}}[W(t)] < \mathbb{E}_{\mathbf{x}}[W(1)]$ $\forall t > t_2$. By a similar argument, it has to be the case that $t^* \in (-1,1) \subset [t_1, t_2]$ because

$\mathbb{E}_{\mathbf{x}}[W(t)] < \mathbb{E}_{\mathbf{x}}[W(-1)]$ $\forall t$ such that $t_1 < t \leq -1$ and $\mathbb{E}_{\mathbf{x}}[W(t)] < \mathbb{E}_{\mathbf{x}}[W(1)]$ $\forall t$ such that $1 \leq t < t_2$.

Claim A.3. *The optimum cannot be in the interval* $t \in [0, y)$.

By the definition of a derivative,

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbf{x}}[W(t)] = \lim_{\delta\to 0}\frac{\mathbb{E}_{\mathbf{x}}[W(t+\delta)] - \mathbb{E}_{\mathbf{x}}[W(t)]}{\delta} =$$

$$= \lim_{\delta\to 0^+}\frac{\mathbb{E}_{\mathbf{x}}[W(t+\delta)] - \mathbb{E}_{\mathbf{x}}[W(t)]}{\delta}.$$

Let $F(\cdot)$ denote the cdf of the random variable x_i.

Notice that

$$\mathbb{E}_{\mathbf{x}}\left[W\left(t+\delta\right)\right]-\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right]=F\left(t\right)\left[F\left(t+\delta\right)-F\left(t\right)\right]-\left[1-F\left(t+\delta\right)\right]\left[F\left(t+\delta\right)-F\left(t\right)\right]+$$

$$+\left[F\left(t+\delta\right)-F\left(t\right)\right]\left[F\left(t+\delta\right)-1+F\left(t\right)\right]+\int_{t}^{t+\delta}-2x_i f\left(x_i\right)dx_i.$$

Let

$$H\left(\delta\right)\equiv F\left(t\right)\left[F\left(t+\delta\right)-F\left(t\right)\right]-\left[1-F\left(t+\delta\right)\right]\left[F\left(t+\delta\right)-F\left(t\right)\right]+$$

$$+\left[F\left(t+\delta\right)-F\left(t\right)\right]\left[F\left(t+\delta\right)-1+F\left(t\right)\right]-2t\left[F\left(t+\delta\right)-F\left(t\right)\right].$$

Since

$$-\int_{t}^{t+\delta}2x_i f\left(x_i\right)dx_i<-2t\int_{t}^{t+\delta}{}_i f\left(x_i\right)dx_i=-2t\left[F\left(t+\delta\right)-F\left(t\right)\right]\qquad\forall\delta>0,$$

then

$$\mathbb{E}_{\mathbf{x}}\left[W\left(t+\delta\right)\right]-\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right]<H\left(\delta\right)\qquad\forall\delta>0,$$

which implies

$$\frac{\mathbb{E}_{\mathbf{x}}\left[W\left(t+\delta\right)\right]-\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right]}{\delta}<\frac{H\left(\delta\right)}{\delta}\qquad\forall\delta>0.$$

Taking the limit:

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right]=\lim_{\delta\to0^+}\frac{\mathbb{E}_{\mathbf{x}}\left[W\left(t+\delta\right)\right]-\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right]}{\delta}\leq\lim_{\delta\to0^+}\frac{H\left(\delta\right)}{\delta}=$$

$$=\lim_{\delta\to0^+}\frac{\left\{F\left(t\right)\left[F\left(t+\delta\right)-F\left(t\right)\right]-\left[1-F\left(t+\delta\right)\right]\left[F\left(t+\delta\right)-F\left(t\right)\right]\right\}}{\delta}+$$

$$+\lim_{\delta\to0^+}\frac{\left\{\left[F\left(t+\delta\right)-F\left(t\right)\right]\left[F\left(t+\delta\right)-1+F\left(t\right)\right]-2t\left[F\left(t+\delta\right)-F\left(t\right)\right]\right\}}{\delta}=$$

$$= \lim_{\delta \to 0^+} \frac{2 \left\{ \left[F(t+\delta) - F(t) \right] \left[F(t) - 1 + F(t+\delta) \right] - t \left[F(t+\delta) - F(t) \right] \right\}}{\delta}. \tag{A.1}$$

By l'Hôpital's rule, A.1 is equal to:

$$2 \lim_{\delta \to 0^+} \left\{ f(t+\delta) \left[F(t) - 1 + F(t+\delta) \right] + \left[F(t+\delta) - F(t) \right] f(t+\delta) - t f(t+\delta) \right\} =$$

$$= 2 f(t) \left\{ 2F(t) - 1 - t \right\} < 0$$

The last inequality holds because $y > 0$ and $t \in [0, y]$.

Claim A.4. *It cannot be the case that* $t^* \in (y, 1)$.

We will prove Claim A.4 by proving that for any $t > y$ there exists a $t' < y$ such that $\mathbb{E}_{\mathbf{x}} [W(t')] > \mathbb{E}_{\mathbf{x}} [W(t)]$.

Let $t \in (y, 1)$ and take $t' < y$ such that

$$\left| y - t' \right| = |y - t| \equiv \Delta.$$

It holds that

$$\mathbb{E}_{\mathbf{x}} \left[W(t') \right] - \mathbb{E}_{\mathbf{x}} \left[W(t) \right] =$$

$$= \mathbb{E}_{\theta^a, \theta^b} \left[\theta^a - 2\theta \Phi \left(\left(t' - \theta \right) \sqrt{\beta} \right) \right] - \mathbb{E}_{\theta^a, \theta^b} \left[\theta^a - 2\theta \Phi \left((t - \theta) \sqrt{\beta} \right) \right] +$$

$$+ \mathbb{E}_{\theta^a, \theta^b} \left[\sqrt{\frac{2}{\pi \beta}} e^{-\frac{\left(t' - \theta\right)^2 \beta}{2}} \right] - \mathbb{E}_{\theta^a, \theta^b} \left[\sqrt{\frac{2}{\pi \beta}} e^{-\frac{(t - \theta)^2 \beta}{2}} \right] +$$

$$+ \mathbb{E}_{\theta^a, \theta^b} \left[\left[2 \left[\Phi \left(\left(t' - \theta \right) \sqrt{\beta} \right) \right]^2 - 2\Phi \left(\left(t' - \theta \right) \sqrt{\beta} \right) + 1 \right] \right] +$$

$$- \mathbb{E}_{\theta^a, \theta^b} \left[\left[2 \left[\Phi \left((t - \theta) \sqrt{\beta} \right) \right]^2 - 2\Phi \left((t - \theta) \sqrt{\beta} \right) + 1 \right] \right] =$$

$$= \mathbb{E}_{\theta} \left[-2\theta \Phi \left(\left(t' - \theta \right) \sqrt{\beta} \right) \right] - \mathbb{E}_{\theta} \left[-2\theta \Phi \left((t - \theta) \sqrt{\beta} \right) \right].$$

The last equality holds because both

$$\mathbb{E}_{\theta^a,\theta^b}\left[\theta^a - 2\theta\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\right]$$

and

$$\mathbb{E}_{\theta^a,\theta^b}\left[\sqrt{\frac{2}{\pi\beta}}e^{-\frac{(t-\theta)^2\beta}{2}}\right]$$

are symmetric with respect to y.

To establish the sign of

$$= \mathbb{E}_{\theta}\left[-2\theta\Phi\left(\left(t'-\theta\right)\sqrt{\beta}\right)\right] - \mathbb{E}_{\theta}\left[-2\theta\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\right]$$

rewrite the expression as

$$\int_{-\infty}^{+\infty} 2\theta\left[\Phi\left(\left(y+\Delta-\theta\right)\sqrt{\beta}\right) - \Phi\left(\left(y-\Delta-\theta\right)\sqrt{\beta}\right)\right] g\left(\theta\right) d\theta$$

where $g(\theta)$ denotes the pdf of the random variable θ.

Notice that

$$\int_{-\infty}^{+\infty} 2\theta\left[\Phi\left(\left(y+\Delta-\theta\right)\sqrt{\beta}\right) - \Phi\left(\left(y-\Delta-\theta\right)\sqrt{\beta}\right)\right] g\left(\theta\right) d\theta =$$

$$= \int_{-\infty}^{0} 2\theta\left[\Phi\left(\left(y+\Delta-\theta\right)\sqrt{\beta}\right) - \Phi\left(\left(y-\Delta-\theta\right)\sqrt{\beta}\right)\right] g\left(\theta\right) d\theta +$$

$$+ \int_{0}^{2y} 2\theta\left[\Phi\left(\left(y+\Delta-\theta\right)\sqrt{\beta}\right) - \Phi\left(\left(y-\Delta-\theta\right)\sqrt{\beta}\right)\right] g\left(\theta\right) d\theta +$$

$$+ \int_{2y}^{+\infty} 2\theta\left[\Phi\left(\left(y+\Delta-\theta\right)\sqrt{\beta}\right) - \Phi\left(\left(y-\Delta-\theta\right)\sqrt{\beta}\right)\right] g\left(\theta\right) d\theta.$$

It holds that

$$\int_0^{2y} 2\theta \left[\Phi\left((y+\Delta-\theta)\sqrt{\beta}\right) - \Phi\left((y-\Delta-\theta)\sqrt{\beta}\right)\right] g(\theta)\, d\theta > 0$$

since the integrand is nonnegative $\forall \theta \in [0, 2y]$ and strictly positive $\forall \theta \in (0, 2y]$.

Moreover,

$$\int_{-\infty}^{0} 2\theta \left[\Phi\left((y+\Delta-\theta)\sqrt{\beta}\right) - \Phi\left((y-\Delta-\theta)\sqrt{\beta}\right)\right] g(\theta)\, d\theta \leq 0$$

because the integrand is nonpositive $\forall \theta \in (-\infty, 0]$ and

$$\int_0^{2y} 2\theta \left[\Phi\left((y+\Delta-\theta)\sqrt{\beta}\right) - \Phi\left((y-\Delta-\theta)\sqrt{\beta}\right)\right] g(\theta)\, d\theta \geq 0$$

because the integrand is nonnegative $\forall \theta \in [2y, +\infty)$.

Next, we show that

$$\int_{-\infty}^{0} 2\theta \left[\Phi\left((y+\Delta-\theta)\sqrt{\beta}\right) - \Phi\left((y-\Delta-\theta)\sqrt{\beta}\right)\right] g(\theta)\, d\theta +$$

$$+\int_0^{2y} 2\theta \left[\Phi\left((y+\Delta-\theta)\sqrt{\beta}\right) - \Phi\left((y-\Delta-\theta)\sqrt{\beta}\right)\right] g(\theta)\, d\theta > 0.$$

Take any $\theta \in [2y, +\infty)$ and denote $\theta - 2y = d > 0$.

For any such θ, there exists $\theta' \in (-\infty, 0]$ such that $\theta' = -d$. By symmetry of the distribution of θ, $g(\theta') = g(\theta)$.

Moreover,

$$2\theta' \left[\Phi\left((y+\Delta-\theta')\sqrt{\beta}\right) - \Phi\left((y-\Delta-\theta')\sqrt{\beta}\right)\right] +$$

$$+2\theta\left[\Phi\left(\left(y+\Delta-\theta\right)\sqrt{\beta}\right)-\Phi\left(\left(y-\Delta-\theta\right)\sqrt{\beta}\right)\right]=$$

$$=-2d\left[\Phi\left(\left(y+d+\Delta\right)\sqrt{\beta}\right)-\Phi\left(\left(y+d-\Delta\right)\sqrt{\beta}\right)\right]+$$

$$+\left(4y+2d\right)\left[\Phi\left(\left(-y-d+\Delta\right)\sqrt{\beta}\right)-\Phi\left(\left(-y-d-\Delta\right)\sqrt{\beta}\right)\right]=$$

$$=4y\left[\Phi\left(\left(-y-d+\Delta\right)\sqrt{\beta}\right)-\Phi\left(\left(-y-d-\Delta\right)\sqrt{\beta}\right)\right]>0.$$

Since $\forall\theta\in[2y,+\infty)$ there exists $\theta'\in(-\infty,0]$ such that

$$2\theta\left[\Phi\left(\left(y+\Delta-\theta\right)\sqrt{\beta}\right)-\Phi\left(\left(y-\Delta-\theta\right)\sqrt{\beta}\right)\right]g\left(\theta\right)+$$

$$+2\theta'\left[\Phi\left(\left(y+\Delta-\theta'\right)\sqrt{\beta}\right)-\Phi\left(\left(y-\Delta-\theta'\right)\sqrt{\beta}\right)\right]g\left(\theta'\right)>0,$$

then

$$\int_{-\infty}^{0}2\theta\left[\Phi\left(\left(y+\Delta-\theta\right)\sqrt{\beta}\right)-\Phi\left(\left(y-\Delta-\theta\right)\sqrt{\beta}\right)\right]g\left(\theta\right)d\theta+$$

$$+\int_{0}^{2y}2\theta\left[\Phi\left(\left(y+\Delta-\theta\right)\sqrt{\beta}\right)-\Phi\left(\left(y-\Delta-\theta\right)\sqrt{\beta}\right)\right]g\left(\theta\right)d\theta\geq 0$$

and combining this inequality with

$$+\int_{0}^{2y}2\theta\left[\Phi\left(\left(y+\Delta-\theta\right)\sqrt{\beta}\right)-\Phi\left(\left(y-\Delta-\theta\right)\sqrt{\beta}\right)\right]g\left(\theta\right)d\theta>0$$

we get

$$\mathbb{E}_{\theta}\left[-2\theta\Phi\left(\left(t'-\theta\right)\sqrt{\beta}\right)\right]-\mathbb{E}_{\theta}\left[-2\theta\Phi\left(\left(t-\theta\right)\sqrt{\beta}\right)\right]>0.$$

This implies that if $0<y\leq 1$ it cannot be the case that the welfare function attains its global maximum for $t\in\left(y,1\right]$.

We conclude that both if $y \in (0,1]$ and if $y > 1$ it has to be the case that the global maximizer of the welfare function, t^*, satisfies

$$-1 < t^* < 0 < y.$$

■

Proof of Proposition 1.2 The characterization of the equilibrium and derivation of the uniqueness condition follow directly from the Appendix of Morris and Shin (2004).

The evaluate the sign of $\frac{\partial t(p,y)}{\partial p}$,and $\frac{\partial t(p,y)}{\partial y}$, notice that since $t(p,y)$ has to satisfy expression 1.7, then:

$$\begin{aligned}
\frac{\partial t(p,y)}{\partial p} &= -\frac{\frac{\partial\left[t-\Phi[(t-y)z]+\frac{1}{2}-p\right]}{\partial p}}{\frac{\partial\left[t-\Phi[(t-y)z]+\frac{1}{2}-p\right]}{\partial t}} = -\frac{-1}{1-\phi\left[(t-y)z\right]z} \\
\frac{\partial t(p,y)}{\partial y} &= -\frac{\frac{\partial\left[t-\Phi[(t-y)z]+\frac{1}{2}-p\right]}{\partial y}}{\frac{\partial\left[t-\Phi[(t-y)z]+\frac{1}{2}-p\right]}{\partial t}} = -\frac{\phi\left[(t-y)z\right]z}{1-\phi\left[(t-y)z\right]z}.
\end{aligned}$$

When the uniqueness condition is satisfied, the denominators of the two expressions above are strictly positive, therefore

$$\begin{aligned}
\frac{\partial t(p,y)}{\partial p} &> 0 \\
\frac{\partial t(p,y)}{\partial y} &< 0.
\end{aligned}$$

■

Proof of Proposition 1.3 If consumers play the equilibrium strategy profile, then all the consumers who observe $x_i > t(p,y)$ and who therefore have an idiosyncratic taste component $\varepsilon_i > t(p,y) - \theta$, buy a and everyone else buys b.

For the second part of the claim, note that

$$\begin{aligned}\frac{\partial n^a(\cdot)}{\partial y^a} &= \frac{\partial n^a}{\partial y}\frac{1}{2} = \left(\frac{1}{2}\right)\left\{-\phi\left[\left(t\left(p,y\right)-\theta\right)\sqrt{\beta}\right]\sqrt{\beta}\left(\frac{\partial t}{\partial y}\right)\right\} = \\ &= \left(\frac{1}{2}\right)\left\{-\phi\left[\left(t(p,y)-\theta\right)\sqrt{\beta}\right]\sqrt{\beta}\left(-\frac{\phi\left[\left(t-y\right)z\right]z}{1-\phi\left[\left(t-y\right)z\right]z}\right)\right\} > 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial n^b(\cdot)}{\partial y^b} &= \frac{\partial n^b}{\partial y}\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)\left\{\phi\left[\left(t(p,y)-\theta\right)\sqrt{\beta}\right]\sqrt{\beta}\left(\frac{\partial t}{\partial y}\right)\right\} = \\ &= \left(-\frac{1}{2}\right)\left\{\phi\left[\left(t(p,y)-\theta\right)\sqrt{\beta}\right]\sqrt{\beta}\left(-\frac{\phi\left[\left(t-y\right)z\right]z}{1-\phi\left[\left(t-y\right)z\right]z}\right)\right\} > 0.\end{aligned}$$

■

Proof of Proposition 1.4 For the first statement, notice that the equilibrium threshold for the case of unsponsored networks, $t^u \equiv t\left(0,y\right)$, satisfies

$$t^u - \Phi\left[\left(t^u - y\right)z\right] + \frac{1}{2} = 0. \tag{A.2}$$

Under the uniqueness condition, the lhs of A.2 is monotonically increasing in t and t^u is well-defined. Since $\Phi\left[\cdot\right] \in [0,1]$, for any $t \geq 0$ it holds that the lhs of A.2 is strictly negative if $t \leq -\frac{1}{2}$. Therefore, it has to be the case that $t^u > \frac{1}{2}$. Moreover, for $y = 0$ the solution to A.2 is $t^u = 0$, and since $t(p,y)$ is strictly decreasing in y, it has to be the case that $t^u \in \left(-\frac{1}{2},0\right)$.

For the second and the third statement, we know from Proposition 1.1 that $t^* \in (-1,0)$. Next, we will prove that the welfare function is strictly decreasing in the interval $t \in [t^u,0]$, which in turn implies that $t^* \in (-1, t^u)$ and that t^u cannot be a point where the function attains its global maximum.

Let $t = t^u$. From the proof of Proposition 1.1, we know:

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbf{x}}[W(t)] = \lim_{\delta \to 0} \frac{\mathbb{E}_{\mathbf{x}}[W(t+\delta)] - \mathbb{E}_{\mathbf{x}}[W(t)]}{\delta} =$$

$$= \lim_{\delta \to 0^+} \frac{\mathbb{E}_{\mathbf{x}}[W(t+\delta)] - \mathbb{E}_{\mathbf{x}}[W(t)]}{\delta} =$$

$$= 2f(t^u)\{2F(t^u) - 1 - t^u\} \tag{A.3}$$

We will show that

$$2f(t^u)\{2F(t^u) - 1 - t^u\} < 0$$

and that therefore

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbf{x}}[W(t)] < 0.$$

First, notice that A.3 has the same sign as $2F(t^u) - 1 - t^u$, since $f(t^u) > 0$.

From the definition of t^u

$$t^u = \Phi[(t^u - y)z] - \frac{1}{2}.$$

By assumption, $F(t^u) = \Phi[(t^u - y)v]$ where

$$v \equiv v(\alpha, \beta) = \sqrt{\frac{\alpha\beta}{\alpha+\beta}}.$$

Therefore, we need to check the sign of

$$2\Phi\left[\left(t^{u}-y\right)v\right]-1-\Phi\left[\left(t^{u}-y\right)z\right]+\frac{1}{2}=$$

$$=\Phi\left[\left(t^{u}-y\right)v\right]-\Phi\left[\left(t^{u}-y\right)z\right]+\Phi\left[\left(t^{u}-y\right)v\right]-\frac{1}{2}.$$

Since $v>z$ and $t^{u}-y<0$, $\Phi\left[\left(t^{u}-y\right)v\right]-\Phi\left[\left(t^{u}-y\right)z\right]<0$ and $\Phi\left[\left(t^{u}-y\right)v\right]<\frac{1}{2}$, therefore A.3 is strictly negative, and so is the derivative of the welfare function evaluated at t^{u}.

Finally, let $t\in\left(t^{u},0\right]$. We will prove that in this interval $\frac{\partial}{\partial t}\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right]<0$ as well. As shown above, it holds that

$$\frac{\partial}{\partial t}\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right]<0=\lim_{\delta\rightarrow0^{+}}\frac{\mathbb{E}_{\mathbf{x}}\left[W\left(t+\delta\right)\right]-\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right]}{\delta}\leq\lim_{\delta\rightarrow0^{+}}\frac{H\left(\delta\right)}{\delta}=$$

$$=2f\left(t\right)\left\{2F\left(t\right)-1-t\right\}$$

and that the last expression has the same sign as $2F\left(t\right)-1-t$. The function $2F\left(t\right)-1-t$ is the difference of the two strictly increasing functions $r\left(t\right)=2F\left(t\right)$ and $s\left(t\right)=1+t$. This difference is positive for $t<-1$, negative for $t=t^{u}$ and negative in $t=0$ (since $F\left(\cdot\right)$ is the cdf of x_{i} which is distributed normally around $y>0$).

Since the slope of $s(t)$ is constant and the slope of $r(t)$ is increasing in the interval $t\in\left(-1,0\right]$, it cannot be the case that their difference is nonnegative in any point $t\in\left(t^{u},0\right)$. Therefore, it holds that $2F\left(t\right)-1-t<0$ in the interval $t\in\left(t^{u},0\right]$, which in turn implies that the welfare function is strictly decreasing in the interval $t\in\left[t^{u},0\right]$.

We can conclude that $t^* \in (-1, t^u)$ and that t^u cannot be a point where the function attains its global maximum. ■

Proof of Lemma 1.2 To calculate the expected market shares, it is sufficient to take the expectation over θ of the market shares expressed in Proposition 1.3. For the second part of the claim, note that

$$\begin{aligned}\frac{\partial \mathbb{E}_{\mathbf{x}}[n^a]}{\partial y^a} &= \frac{\partial \mathbb{E}_{\mathbf{x}}[n^a]}{\partial y}\frac{1}{2} = \left(\frac{1}{2}\right)\left\{-\phi\left[\left(t\left(p,y\right)-y\right)v\right]v\left(\frac{\partial t}{\partial y}-1\right)\right\} = \\ &= \left(\frac{1}{2}\right)\left\{-\phi\left[\left(t\left(p,y\right)-y\right)v\right]v\left(-\frac{\phi\left[\left(t-y\right)z\right]z}{1-\phi\left[\left(t-y\right)z\right]z}-1\right)\right\} > 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \mathbb{E}_{\mathbf{x}}\left[n^b\right]}{\partial y^b} &= \frac{\partial \mathbb{E}_{\mathbf{x}}\left[n^b\right]}{\partial y}\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)\left\{\phi\left[\left(t\left(p,y\right)-y\right)v\right]v\left(\frac{\partial t}{\partial y}-1\right)\right\} = \\ &= \left(-\frac{1}{2}\right)\left\{\phi\left[\left(t\left(p,y\right)-y\right)v\right]v\left(-\frac{\phi\left[\left(t-y\right)z\right]z}{1-\phi\left[\left(t-y\right)z\right]z}-1\right)\right\} > 0.\end{aligned}$$

■

Proof of Proposition 1.5 First, we prove that if a pure strategy SPNE exists, then it has to be the case that $p^s \in (0, y)$.

The first order conditions of the firms optimization problem are

$$\begin{aligned}\frac{\partial \mathbb{E}_{\mathbf{x}}[\pi_a]}{\partial p^a} &= 1-\Phi\left[\left(t\left(p,y\right)-y\right)v\right]-\left(p^a-c\right)\phi\left[\left(t\left(p,y\right)-y\right)v\right]\frac{\partial t}{\partial p}\frac{v}{2} = 0 \\ \frac{\partial \mathbb{E}_{\mathbf{x}}[\pi_b]}{\partial p^b} &= \Phi\left[\left(t\left(p,y\right)-y\right)v\right]-\left(p^b-c\right)\phi\left[\left(t\left(p,y\right)-y\right)v\right]\frac{\partial t}{\partial p}\frac{v}{2} = 0.\end{aligned}$$

Both must be satisfied in equilibrium. Therefore, it has to be the case that $\left(p^a, p^b\right)$ and $p^s = \frac{p^a - p^b}{2}$ satisfy:

$$1 - 2\Phi\left[\left(t\left(p, y\right) - y\right)v\right] = \left(p^a - p^b\right)\phi\left[\left(t\left(p, y\right) - y\right)v\right]\frac{\partial t}{\partial p}\frac{v}{2}. \tag{A.4}$$

First, note that A.4 can't be satisfied if $p^s < 0$ because that would imply that the lhs is positive and the rhs is negative. Also, it cannot be the case that $p^s = 0$ in equilibrium because in that case the lhs would be positive and the rhs would be equal to zero. Moreover, it cannot be that $p^s = y$ in equilibrium because in that case the lhs would be equal to zero and the right hand side would be positive. Finally, it cannot be that $p^s > y$ in equilibrium because in that case the lhs would be negative and the rhs would be positive. Therefore, if a pure strategy SPNE exists it has to be such that $p^s \in (0, y)$. Notice that both the lhs and the rhs of A.4 are continuous functions of p, therefore $G\left(p^s\right) \equiv lhs - rhs$ is continuous as well. Moreover, $G(p)$ is positive at $p = 0$ and is negative at $p = y$, therefore it must have at least one zero in the interval $p \in (0, y)$.

Next, we prove that $t^s \in (t^u, y)$.

From Proposition 1.2, $t\left(p, y\right)$ is strictly increasing in p and since $p^s > 0$ then it has to be the case that $t^u \equiv t\left(0, y\right) < t\left(p^s, y\right) \equiv t^s$.

From the definition of $t\left(p, y\right)$, it follows that $t\left(y, y\right) = y$. Since $t\left(p, y\right)$ is strictly increasing in p and $p^s < y$, then it has to be the case that $t^s < y$.

Finally, we need to show that $\mathbb{E}_{\mathbf{x}}\left[W\left(t^u\right)\right] < \mathbb{E}_{\mathbf{x}}\left[W\left(t^s\right)\right]$. This result holds because, as we have shown in the proof of Proposition 4, $\mathbb{E}_{\mathbf{x}}\left[W\left(t\right)\right]$ is strictly decreasing in the interval $t \in [t^u, y)$ and $t^s \in (t^u, y)$. ■

Appendix B

This Appendix collects the proofs of all the results presented in Chapter 2

Some extra notation for this Appendix.

Let $C = C^1 \cup C^2$.

For every $D \subset C$ such that $\{i : C^i \in D\}$ is measurable with respect to the Lebesgue measure, let $N(D) = \int\limits_{C^i \in D} di$.

Let S_A denote the set of strategies of player A, S_B the set of strategies of player B and S_i^j the set of strategies of C_i^j for $j = 1, 2$ and $i \in [0, 1]$. Note that strategies in S_i^j specify actions for C_i^j after any history of length 2. Let $S^c = \underset{i \in [0,1]}{\times} S_i^1 \underset{i \in [0,1]}{\times} S_i^2$

For any game, let Γ^c be the set of all subgames that start in the third stage. We refer to elements of Γ^cas consumer subgames.

Moreover, let c_i^j denote the action choice of consumer i on side j.

Finally, for every $s \in S$, $k \in F$, $j \in \{1, 2\}$, $n = 1, ..., n_k$ and $i \in [0, 1]$ let $n_k(s)$, $\pi_k(s)$, $p_{k,n}^j(s)$, $N_{k,n}^j(s)$ and $c_i^j(s)$ denote the realized n_k, π_k, $p_{k,n}^j$, $N_{k,n}^j$ and c_i^j if s is played

Lemma B.1. *Let $G^c = (C, S^c, u^c) \in \Gamma^c(G)$. Then $\exists\ s \in S^c$ such that s is a coalitionally rationalizable Nash equilibrium of G^c.*

Proof of Lemma B.1

For any $A \subset S^c$ such that $A = \underset{j=1,2}{\times} \underset{i\in[0,1]}{\times} A_i^j \neq \emptyset$ let $\mathcal{F}(A)$ be the collection of supported restrictions in S^c given A. Let $A^0, A^1, ...$ be such that $A^0 = S^c$ and $A^k = \underset{B\in\mathcal{F}(A^{k-1})}{\cap} B$ $\forall\ k \geq 1$. Note that $A^* = \lim_{k\to\infty} A^k$ is the set of coalitionally rationalizable strategies in G^c. Let $k \geq 0$ and assume $A^k \neq \emptyset$. For every $i \in [0,1]$ and $j = 1,2$ let $\overline{u}_i^j(k) = \sup_{\omega_{-i,j}\in\Omega_{-i,j}(A^k), a_{i,j}\in A_{i,j}^k} u_{i,j}^c(a_{i,j}, \omega_{-i,j})$. Let $(a_{i,j}^m, \omega_{-i,j}^m)_{m=1,2,...}$ be such that $a_{i,j}^m \in A_{i,j}^k$ and $\omega_{-i,j}^m \in \Omega_{-i,j}(A^k)\ \forall\ m \geq 1$, and $u_{i,j}^c(a_{i,j}^m, \omega_{-i,j}^m) \to \overline{u}_i^j(k)$ as $m \to \infty$. Since $A_{i,j}^k$ is finite, $(a_{i,j}^m)$ has a subsequence $(a_{i,j}^{m_n})$ such that $\lim_{m_n\to\infty} a_{i,j}^{m_n} = a \in A_{i,j}^k$. Then by the definition of supported restriction $a \in B_{i,j}$ for every $B \subset A^k$ such that B is a supported restriction in G^c given A^k. Since i and j were arbitrary, this implies $A^{k+1} \neq \emptyset$. Then by induction $A^0 = S^c \neq \emptyset$ implies $A^* \neq \emptyset$.

Note that $(S^c)_i^j = (S^c)_{i'}^{j'} \equiv \overline{S}^c\ \forall\ i, i' \in [0,1]$ and $j, j' \in \{1,2\}$. Order pure strategies in $\overline{S}^c$ any way, such that $\overline{S}^c = \{a_1, ..., a_n\}$. Define $x(s) : \overline{S}^c \to R^{2n}$ such that $x_k(s) = N(C_i^1 \in C^1 : c_i^1(s) = a_k)\ \forall\ i \in 1, ..., n$ and $x_k(s) = N(C_i^2 \in C^2 : c_i^2(s) = a_{k-n})\ \forall\ i \in n+1, ..., 2n$.

Let $\Theta = \{\theta \in R^{2n} : \exists\ s \in A^*$ s.t. $x(s) = \theta\}$. It is easy to establish that Θ is a nonempty, compact and convex subset of R^{2n}. For every $\theta \in \Theta$, $i \in [0,1]$ and $j = 1,2$ let $BR_i^j(\theta) = \{a \in \overline{S}^c : \exists\ s \in S$ s.t. $x(s) = \theta$ and $a \in BR_i^j(s_{-j,i})\}$. Now define the correspondence $F : \Theta \to R^{2n}$ such that $F(\theta) = \{y : \exists\ s \in S^c$ s.t. $s_i^j \in BR_i^j(\theta)$, and $y = x(s)\}$. It is straightforward to establish that $BR_i^j(s_{-j,i}) \subset (A^*)_i^j\ \forall\ i \in [0,1]$ and $j = 1,2$. This implies that $F(\theta) \subset \Theta\ \forall\ \theta \in \Theta$, therefore F is a correspondence from Θ to itself. Note that F is nonempty valued, since $\overline{S}^c$ is finite, so the best response correspondence is nonempty valued. Furthermore, if $s, t \in \overline{S}^c$ are such that $s_i^j, t_i^j \in BR_i^j(\theta)$ for some $\theta \in \Theta$, then $z_i^j \in BR_i^j(\theta)$ for every $z \in \overline{S}^c$ for which it holds that $\forall\ i \in [0,1]$ and $j = 1,2$ either $z_i^j = s_i^j$ or $z_i^j = t_i^j$. This implies that F is convex valued. Finally, since for every $i \in [0,1]$ and $j = 1,2$ g_i^j is continuous, it holds that for every $i \in [0,1]$ and $j = 1,2$ $BR_i^j(\theta)$ is upper hemicontinuous, which implies that F is upper hemicontinuous.

The above imply that all the conditions for Kakutani's fixed point theorem hold for F , therefore it has a fixed point θ^*. That implies $\exists\ s^* \in A^*$ such that $x(s^*) = \theta^*$ and $(s^*)_i^j \in BR_i^j(\theta^*)$ $\forall\ \theta \in \Theta$, $i \in [0,1]$ and $j = 1,2$. This implies $(s^*)_i^j \in BR_i^j(s^*_{-j,i})$ $\forall$ $\theta \in \Theta$, $i \in [0,1]$ and $j = 1,2$, which establishes that s^* is a coalitionally rationalizable Nash equilibrium. ■.

Proof of Theorem 2.1

Let $\pi_A(s) = \pi$. If $\pi = 0$ the claim is trivial, since in every Nash equilibrium and therefore in every coalition perfect Nash equilibrium the firm has to get nonnegative profit (since announcing prices above 0 guarantees that).

Suppose now that $\pi > 0$. That implies that at least on one side of the market the monopolist charges a strictly positive profit and has a strictly positive market share.

Let $\widehat{C^j}(s) = \{C_i^j : c_i^j(s) = A\}$ for $j = 1,2$. Then for $C_i^j \in \widehat{C^j}(s)$ and $N^j \in [0,1]$ it holds that $g_i^j(N^{-j}) - p_A^j(s) \geqslant 0$ (note that $g_i^j(N_A^{-j}(s))$ is constant in N^j). Then for every $\varepsilon > 0$ and $C_i^j \in \widehat{C^j}(s)$ it holds that $g_i^j(N_A^{-j}(s)) - p_A^j(s) + \varepsilon > 0$, which implies that after a price announcement of $\left(p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon\right)$ joining the network is a supported restriction for $\widehat{C^1}(s)$ and $\widehat{C^2}(s)$.

If $p_A^j(s) > 0$ for $j = 1,2$, then it has to be the case that $N_A^j(s) \geq 0$ and $Max\left\{N_A^j(s), N_A^{-j}(s) > 0\right\}$.
Then the firm can guarantee a profit arbitrarily close to π by charging prices $\left(p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon\right)$.

If $p_A^j(s) > 0$ and $p_A^{-j}(s) = 0$ then it has to be the case that $N_A^j(s) > 0$ and $N_A^{-j}(s) \geq 0$. In this case, by charging $\left(p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon\right)$ the monopolist gets market shares $N_A^j \geq N_A^j(s)$ and $N_A^{-j} = 1$ and profits arbitrarily close to π. Finally, if $p_A^j(s) > 0$ and $p_A^{-j}(s) = 0$ it has to be the case that $N_A^j(s)$ and $N_A^{-j}(s) = 1$. Then, by charging $\left(p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon\right)$ the monopolist gets market shares $N_A^j \geq N_A^j(s)$ and $N_A^{-j} = 1$ and again profits arbitrarily close to π.

That means that if consumers play only coalitionally rationalizable strategies, then the firm can guarantee a profit arbitrarily close to π by charging prices $\left(p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon\right)$ where $\varepsilon > 0$ is small enough. This implies that it cannot be that $\pi(s') < \pi$. ■

Proof of Theorem 2.2

Let $s \in S$ be a coalition perfect equilibrium. If $p^1(s), p^2(s) < l$ then joining the network is a supported restriction for the coalition of all consumers. The supremum of the profit the firm in this price range is $2l$ and the firm can get a profit arbitrarily close to it by charging $\left(p^1(s), p^2(s)\right) = (l - \varepsilon, l - \varepsilon)$ for small enough $\varepsilon > 0$. If $p^1(s), p^2(s) < ah$ then joining the network is a supported restriction for the coalition of consumers that involve the high types from both sides of the market. Therefore the monopolist can guarantee a profit arbitrarily close to $2a^2h$ by charging prices $\left(p^1(s), p^2(s)\right) = (ah - \varepsilon, ah - \varepsilon)$ for small enough $\varepsilon > 0$. If $p^j(s) < h$ and $p^{-j}(s) < al$ for some $j \in \{1, 2\}$ then joining the network is a supported restriction for $C^{-j} \cup \{C_i^j : i \in [0, a]\}$. Therefore the monopolist can guarantee a profit arbitrarily close to $a(h + l)$ by charging prices $\left(p^1(s), p^2(s)\right) = (h - \varepsilon, al - \varepsilon)$ for small enough $\varepsilon > 0$.

The above establish that $\pi_A(s) \geq \max(2l, 2a^2h, a(h + l))$.

If $p^j(s) > h$ for some $j \in \{1,2\}$, then $N^j(s) = 0$. Then $N^{-j}(s) > 0$ only if $p^{-j}(s) \leq 0$. In any case $\pi_A(s) < \max(2l, 2a^2h, a(h+l))$. Therefore $p^j(s) \leq h$ for $j = 1, 2$.

It cannot be that $p^j(s) \leq l \; \forall \; j \in \{1,2\}$ and $p^j(s) < l$ for some $j \in \{1,2\}$, since then $\pi_A(s) < 2l$.

If $p^j(s) > l \; \forall \; j \in \{1,2\}$ then $c_i^j(s) = \emptyset \; \forall \; j = 1, 2$ and $i \in (a, 1]$. Then it cannot be that $p^j(s) > ah$ for some $j \in \{1,2\}$, otherwise $c_i^j(s) = \emptyset \; \forall \; j = 1, 2$ and $i \in [0, 1]$ and therefore $\pi_A(s) = 0$. Furthermore, it cannot be that $p^j(s) < ah$ for some $j \in \{1,2\}$, otherwise $\pi_A(s) < 2a^2h$.

Suppose now that $p^j(s) > l$ and $p^{-j}(s) \leq l$ for some $j \in \{1,2\}$. It cannot be that $p^{-j}(s) < al$ since then $\pi_A(s) < a(l+h)$. If $p^{-j}(s) > al$, then $p^j(s) > ah$ or $p^{-j}(s) > ah$ implies $\pi_A(s) = 0$. Then $p^j(s) < ah$ or $p^{-j}(s) < ah$ implies $\pi_A(s) < 2a^2h$. Finally, $p^j(s) = ah$ and $p^{-j}(s) = ah$ contradict that $p^j(s) > l$ and $p^{-j}(s) \leq l$. This concludes that $p^{-j}(s) = al$. Then $p^j(s) = h$, otherwise $\pi_A(s) < a(l+h)$.

Consider first the case that $ah \geqslant l$. If $p^j(s) \geq ah$ then only high types can join the network in equilibrium. Furthermore, consumers on side j only join in equilibrium if at least some low types join the network from the other side. For that to be possible in equilibrium, it has to be the case that $p^{-j}(s) \leq al$. The above imply that if ${}^j(s) > h$ for $j = 1$ or $j = 2$ then $\pi(s) \leq ah + al$ But note that the firm can get a profit arbitrarily close to this amount by charging $(h - \varepsilon, al - \varepsilon)$ for small enough $\varepsilon > 0$, since then joining the network is a supported restriction for the coalition of consumers involving all high types on side 1 and all consumers on side 2.

Consider now the case that $l > ah$. If ${}^j(s) > l$ for $j = 1$ or $j = 2$ the same arguments as above establish that $\pi(s) \leq ah + al$, but the monopolist can get a profit arbitrarily close to $ah + al$ by charging $(h - \varepsilon, al - \varepsilon)$ for small enough $\varepsilon > 0$.

This concludes that $\pi(s) \leq \max(2l, 2a^2h, ah + al)$, but if consumers play only coalitionally rationalizable strategies then the monopolist can always guarantee a profit arbitrarily close to $\max(2l, 2a^2h, ah + al)$. But then it has to be the case that in every coalition perfect equilibrium $\pi(s) = \max(2l, 2a^2h, ah + al)$. This establishes that the prices charged by the monopolist are either (l, l) or (ah, ah) or (h, al) or (al, h) in any coalition perfect equilibrium, and in the first case $c_i^1(s) = c_i^2(s) = A\ \forall i \in [0, 1]$, in the second case $c_i^1(s) = c_i^2(s) = A\ \forall\ i \in [0, a]$ and $c_i^1(s) = c_i^2(s) = \emptyset\ \forall\ i \in (a, 1]$ and in the third case either $c_i^1 = A\ \forall\ i \in [0, a]$, $c_i^1 = A\ \forall\ i \in (a, 1]$ and $c_i^2 = A\ \forall\ i \in [0, 1]$ or $p^1(s) = al$, $p^2(s) = h$, $c_i^1 = A\ \forall\ i \in [0, 1]$, $c_i^1 = A\ \forall\ i \in [0, a]$ and $c_i^1 = \emptyset\ \forall\ i \in (a, 1]$.

The above imply that if $2l > \max(2a^2h, ah + al)$ then $p^j(s) = l$ and $N^j(s) = 1\ \forall\ j \in \{1, 2\}$. If $2a^2h > \max(2l, ah + al)$ then $p^j(s) = ah$ and $N^j(s) = a\ \forall\ j \in \{1, 2\}$. And if $ah + al > \max(2l, 2a^2h)$ then $p^j(s) = h$, $p^{-j}(s) = al$, $N^j(s) = a$ and $N^{-j}(s) = 1$ for some $j \in \{1, 2\}$.

Suppose $2a^2h > al + ah$. Since $a > 0$, this is equivalent to $(2a - 1)h > l$. The latter implies $a^2h > l$ since $a^2h - (2a - 1)h = (a - 1)^2h > 0$. Therefore $(2a - 1)h > l$ implies that $2a^2h > \max(2l, ah + al)$.

Suppose now that $ah + al < 2l$. It is equivalent to $l > \frac{a}{2-a}h$. The latter implies $l > a^2h$ since $\frac{a}{2-a}h - a^2h = ah\frac{1-2a+a^2}{2-a} > 0$. Therefore $2a^2h < 2l$. This establishes that if $l > \frac{a}{2-a}h$ then $2l > \max(2a^2h, ah + al)$.

Note that $(2a-1)h < \frac{a}{2-a}h$. If $l \in ((2a-1)h, \frac{a}{2-a}h)$ then $2a^2h < al + ah$ and $ah + al > 2l$

and therefore $ah + al > \max(2l, 2a^2h)$.

Let $G^1 = (I, S^1, u^1) \in \Gamma^c$ be the subgame following price announcements (ah, ah) and let $s^1 \in S^1$ be such that $(s^1)^1_i = (s^1)^2_i = A \ \forall\, i \in [0, a]$ and $(s^1)^1_i = (s^1)^2_i = \emptyset \ \forall\, i \in (a, 1]$. Let $G^2 = (I, S^2, u^2) \in \Gamma^c$ be the subgame following price announcements (h, al) and let $s^2 \in S^2$ be such that $(s^1)^1_i = A \ \forall\, i \in [0, a]$, $(s^1)^1_i = \emptyset \ \forall\, i \in (a, 1]$ and $(s^1)^2_i = A \ \forall\, i \in [0, 1]$. Let $G^3 = (I, S^3, u^3) \in \Gamma^c$ be the subgame following price announcements (l, l) and let $s^3 \in S^3$ be such that $(s^3)^1_i = (s^3)^2_i = A \ \forall\, i \in [0, 1]$. It is straightforward to establish that for $k = 1, 2, 3$ s^k is a coalitionally rationalizable Nash equilibrium in G^k. Let now $s_{-A} \in S_{-A}$ be any profile that specifies s^k in G^k for every $k = 1, 2, 3$ and specifies some arbitrary coalitionally rationalizable Nash equilibrium in every other $G^c \in \Gamma^c$. By Lemma B.1 there exists a profile like that. Let s_A be such that $p_A(s_A) = (ah, ah)$. Let s'_A be such that $p_A(s'_A) = (h, al)$. And let s''_A be such that $p_A(s''_A) = (l, l)$. If $\frac{l}{h} = 2a - 1$ then $2a^2h = ah + al = \max(2l, 2a^2h, ah + al)$. The above then establish that both (s_A, s_{-A}) and (s'_A, s_{-A}) are coalition perfect equilibria. If $\frac{l}{h} = \frac{a}{2-a}$ then $2l = ah + al = \max(2l, 2a^2h, ah + al)$. The above then establish that both (s'_A, s_{-A}) and (s''_A, s_{-A}) are coalition perfect equilibria. ■

Proof of Theorem 2.3

If $s_{-A} \in S_{-A}$ is such that consumers play a coalitionally rationalizable Nash equilibrium in every consumer subgame, then in the subgame following $n_A(s_A, s_{-A}) = 2$ and $p^1_1 = p^2_2 = la - \varepsilon$, $p^2_1 = p^1_2 = la + (1 - a)h - 2\varepsilon$ $(\varepsilon > 0)$, it has to hold that $n^1_i = 2$, $n^2_i = 1 \ \forall\, i \in [0, a]$ and $n^1_i = 1$, $n^2_i = 2 \ \forall\, i \in (a, 1]$. To see this, define $A \subset S$ such that $A = \underset{i \in [0,1], j=1,2}{\times} A^j_i$ and $A^j_i \equiv \{\emptyset, 1, 2\} \ \forall\, i \in [0, a]$, $j = 1, 2$, $A^1_i \equiv \{\emptyset, 1\} \ \forall\, i \in (a, 1]$ and $A^2_i \equiv \{\emptyset, 2\} \ \forall\, i \in (a, 1]$. Also define $B \subset S$ such that $B = \underset{i \in [0,1], j=1,2}{\times} B^j_i$ and $B^1_i \equiv \{2\} \ \forall\, i \in [0, a]$, $B^2_i \equiv \{1\} \ \forall\, i \in [0, a]$,

$j = 1, 2$, $B_i^1 \equiv \{1\}\ \forall\ i \in (a, 1]$ and $A_i^2 \equiv \{2\}\ \forall\ i \in (a, 1]$. First note that $A_i^j \times S_{-1,i}$ is a supported restriction by C_i^j given $S\ \forall\ i \in [0, 1]$ and $j = 1, 2$ (since strategies in S_i^j / A_i^j are never best responses for C_i^j). Next, B is a supported restriction given A by $C^1 \cup C^2$ since it gives the best possible payoff to every consumer in this subgame, given A. Therefore $n_i^1 = 2$, $n_i^2 = 1\ \forall\ i \in [0, a]$ and $n_i^1 = 1$, $n_i^2 = 2\ \forall\ i \in (a, 1]$ is the only coalitionally rationalizable strategy in the above subgame.

Since ε can be arbitrarily small positive, the above establishes that if $s \in S$ is a coalition perfect equilibrium, then $\pi_A(s) \geq 2(la + (1-a)ah) \equiv \pi^*$.

Suppose $s_{-A} \in S_{-A}$ is such that consumers play a coalitionally rationalizable Nash equilibrium in every consumer subgame and $\exists\ \widehat{s}_A \in S_A$ such that $\pi_A(\widehat{s}_A, s_{-A}) \geq \pi^*$ and it is not the case that $n_A(\widehat{s}_A, s_{-A}) = 2$, and $p_j^1(\widehat{s}_A, s_{-A}) = p_{-j}^2(\widehat{s}_A, s_{-A}) = la$, $p_j^2(\widehat{s}_A, s_{-A}) = p_{-j}^1(\widehat{s}_A, s_{-A}) = la + (1-a)h$ for some $j = 1, 2$.

Let $s = (\widehat{s}_A, s_{-A})$.

First suppose $n_A(s) = 1$. If $l \in \left((4a-1)h, \frac{a}{2-a}h\right)$, then by Theorem 2.4 $\pi_A(s) \leq (l+h)a$. But for $l > (4a-1)h$ it holds that $(l+h)a < 2(la + (1-a)ah) \equiv \pi^*$, a contradiction. If $l \in \left((\frac{a}{2-a}h, \frac{a(1-2a)}{1-a}h\right)$, then by Theorem 2.4 $\pi_A(s) \leq 2l$. But $l < \frac{a(1-2a)}{1-a}h$ implies $2l < 2(la + (1-a)ah) \equiv \pi^*$, a contradiction.

Therefore $n_A(s) = 2$.

It cannot be that $N_k^j = 0$ for some $j = 1, 2$ and $k = 1, 2$ since then either $N_k^{-j}(s) = 0$ or $p_k^{-j}(s) \leq 0$ (otherwise consumers choosing network k in s would get negative utility, contradicting the assumption on s_{-A}). In either case $\pi_A(s)$ is smaller or equal to the supremum of profits attainable by a strategy in which A operates only one network. Then, as established above, $\pi_A(s) < \pi^*$. Therefore $N_k^j > 0 \; \forall \; j = 1,2$ and $k = 1,2$.

Let $H^j = \{C_i^j : i \in [0,a]\}$ and $L^j = \{C_i^j : i \in (a,1]\} \; \forall \; j = 1,2$.

Let $X_k^j = \{C_i^j : n_i^j(s) = k\} \; \forall \; j = 1,2$ and $k = 1,2$.

First we establish that it cannot be that for some $j = 1,2$ both $X_1^j \cap L^j = \emptyset$ and $X_2^j \cap L^j = \emptyset$. If $X_1^j \cap L^j = \emptyset$ and $X_2^j \cap L^j = \emptyset \; \forall \; j = 1,2$ then $\pi_A(s) < 2a^2h < \pi^*$. Otherwise, w.l.o.g. assume $X_1^2 \cap L^2 = \emptyset$ and $X_2^2 \cap L^2 = \emptyset$ and $X_2^1 \cap L^1 \neq \emptyset$. Then consider a deviation s' by the firm such that $n_A(s') = 1$, $p^1(s') = \max(0, p_1^1(s)\frac{N_1^2+N_2^2}{N_2^2} - \varepsilon)$ and $p^2(s') = \max(0, p_2^2(s) - \varepsilon)$. In the subgame following the above prices it is a supported restriction for $X_1^1 \cup X_2^1 \cup X_1^2 \cup X_2^2$ (note that $X_2^1 \cap L^1 \neq \emptyset$ and therefore $p_1^1(s) \leq lN_2^2$) to choose 1, which guarantees a profit of at least $\pi' \equiv p^1(s')(N_1^1 + N_2^1) + p^2(s')(N_1^2 + N_2^2)$. Consider now deviation s'' by the firm such that $n_A(s'') = 1$, $p^1(s'') = \max(0, p_1^2(s)\frac{N_1^2+N_2^2}{N_1^2} - \varepsilon)$ and $p^2(s'') = \max(0, p_1^2(s) - \varepsilon)$. In the subgame following the above prices it is a supported restriction for $X_2^1 \cup X_1^2 \cup X_2^2$ to choose 1, which guarantees a profit of at least $\pi'' \equiv p^1(s'')N_2^1 + p^2(s'')(N_1^2 + N_2^2)$. It is straightforward to verify that both $\pi' \leq \pi_A(s)$ and $\pi'' \leq \pi_A(s)$, and therefore at least one of the above deviations yields higher profit than $\pi_A(s)$. And since $n_A(s') = 1$ and $n_A(s'') = 1$, it holds that $\pi' < \pi^*$ and $\pi'' < \pi^*$.

Next we establish that it cannot be that for some $j = 1,2$ both $X_1^j \cap L^j \neq \emptyset$ and $X_2^j \cap L^j \neq \emptyset$. If $X_1^j \cap L^j \neq \emptyset$ and $X_2^j \cap L^j \neq \emptyset \; \forall \; j = 1,2$ then $p_k^j(s) \leq lN_k^{-j}(s) \; \forall \; k = 1,2$ and $j = 1,2$. Then $\pi_A(s) < 2l < \pi^*$. Otherwise w.l.o.g. assume $X_1^1 \cap L^1 \neq \emptyset$, $X_2^1 \cap L^1 \neq \emptyset$ and $X_1^2 \cap L^2 \neq \emptyset$. Then $\pi_A(s) < (h+l)N_1^1(s)N_2^2(s) + 2lN_1^2(s)N_2^1(s)$, since $p_1^1(s) \leq lN_2^2(s)$, $p_2^1(s) \leq lN_2^1(s)$, $p_1^2(s) \leq lN_1^1(s)$ and $p_2^2(s) < hN_2^1(s)$. Note that $(h+l)N_2^2(s) < (h+l)a < \pi^*$ and therefore $(h+l)N_2^2(s) < (h+l)N_1^1(s)N_2^2(s) + 2lN_1^2(s)N_2^1(s)$. This implies $(h+l)N_2^2(s) < 2lN_1^2(s) < 2l$.

Furthermore, $2l < \pi^*$ and therefore $2l < (h+l)N_1^1(s)N_2^2(s) + 2lN_1^2(s)N_2^1(s)$. This implies $2l(1-N_1^2(s)N_2^1(s)) < (h+l)N_1^1(s)N_2^2(s)$ which implies $2lN_1^1(s) < (h+l)N_1^1(s)N_2^2(s)$ which implies $2l < (h+l)N_2^2(s)$, a contradiction.

Next we establish that it cannot be that for some $k = 1,2$ both $X_k^1 \cap L^1 \neq \emptyset$ and $X_k^2 \cap L^2 \neq \emptyset$. Suppose otherwise. Then, as established above, $X_{-k}^1 \cap L^1 = \emptyset$ and $X_{-k}^2 \cap L^2 = \emptyset$, but $X_{-k}^1 \neq \emptyset$ and $X_{-k}^2 \neq \emptyset$. This implies $N_k^{-j}(s)l - p_k^j(s) \geq N_{-k}^{-j}(s)l - p_{-k}^j(s)\ \forall\ j = 1,2$ and $N_k^{-j}(s)h - p_k^j(s) \leq N_{-k}^{-j}(s)h - p_{-k}^j(s)\ \forall\ j = 1,2$, which implies $p_k^j(s) \leq p_{-k}^j(s)\ \forall\ j = 1,2$. Then by Lemma B.1 it has to be that $p_k^j(s) = p_{-k}^j(s)\ \forall\ j = 1,2$. But since $p_k^j(s) \leq N_k^{-j}(s)l < l\ \forall\ j = 1,2$, this implies $\pi_A(s) < 2l < \pi^*$.

Therefore $\exists\ k \in \{1,2\}$ such that $X_k^1 \cap H^1 = X_k^1$ and $X_{-k}^2 \cap H^2 = X_k^2$. W.l.o.g. let $k = 1$.

Note that $p_1^2(s) \leq lN_1^1$ and $p_2^1(s) \leq lN_2^2$ since $X_1^2 \cap L^2 \neq \emptyset$, $X_2^1 \cap L^1 \neq \emptyset$ and by definition no consumer can get negative utility in any subgame if s is played. Then $hN_1^2 - p_1^1(s) \geq hN_2^2 - p_2^1(s)$ implies $p_1^1(s) \leq lN_2^2 + h(1-N_2^2)$ and $hN_2^1 - p_2^2(s) \geq hN_1^1 - p_1^2(s)$ implies $p_2^2(s) \leq lN_1^1 + h(1-N_1^1)$. This establishes that $\pi_A(s) \leq l(N_1^1N_1^2 + N_2^1N_2^2) + (lN_2^2 + h(1-N_2^2))N_1^1 + (lN_1^1 + h(1-N_1^1))N_2^2 \leq l(N_1^1 + N_2^2) + h(N_1^1 + N_2^2 - 2N_1^1N_2^2)$.

Note that $\frac{\partial(l(N_1^1+N_2^2)+h(N_1^1+N_2^2-2N_1^1N_2^2))}{\partial N_1^1} = h + l - 2hN_2^2 \geq h + l - 2ha > 0$ (since the starting assumptions imply $a < 1/2$). Similarly it holds that $\frac{\partial(l(N_1^1+N_2^2)+h(N_1^1+N_2^2-2N_1^1N_2^2))}{\partial N_2^2} = h + l - 2hN_1^1 \geq h + l - 2ha > 0$. Therefore $\pi_A(s) < l2a + h(2a - 2a^2) = \pi^*$ unless $p_1^2(s) = p_2^1(s) = al$, $N_1^2(s) = N_2^1(s) = 1-a$, $p_1^1(s) = p_2^2(s) = la + h(1-a)$ and $N_1^1(s) = N_2^2(s) = a$.

This concludes the claim. ■

Proof of Theorem 2.4 Suppose $\exists\ s \in S$ such that s is a coalition perfect equilibrium,

$n_A(s) = 2$ and $N_k^j(s) > 0\ \forall\ j \in \{1,2\}$ and $k \in \{A,B\}$.

By Lemma B.2 (see below) $\exists\, k \in \{1,2\}$ such that $p^1_k(s) \leq p^1_{3-k}(s)$ and $p^2_k(s) \geq p^2_{3-k}(s)$, otherwise $N^1_k(s) = N^2_k(s) = 0$ for some $k \in \{1,2\}$. Let $l^1 = (\inf\limits_{i\in[0,1]:c^j_i(s)=k} u^1_i)(N^2_k(s) + N^2_{3-k}(s))$, $h^1 = (\inf\limits_{i\in[0,1]:c^j_i(s)=3-k} u^1_i)(N^2_k(s) + N^2_{3-k}(s))$, $l^2 = (\inf\limits_{i\in[0,1]:c^j_i(s)=3-k} u^2_i)(N^1_k(s) + N^1_{3-k}(s))$ and $h^2 = (\inf\limits_{i\in[0,1]:c^j_i(s)=k} u^2_i)\ (N^1_k(s) + N^1_{3-k}(s))$. Then Nash equilibrium implies $p^1_k(s) \leq N^2_k(s)l^1/(N^2_k(s) + N^2_{3-k}(s))$, $p^2_{3-k}(s) \leq N^1_k(s)l^2/(N^1_k(s) + N^1_{3-k}(s))$, $p^1_{3-k}(s) \leq (N^2_k(s) - N^2_k(s))\ h^1/(N^2_k(s) + N^2_{3-k}(s)) + p^1_k(s)$ and $p^2_k(s) \leq (N^1_{3-k}(s) - N^1_k(s))h^2/(N^1_k(s) + N^1_{3-k}(s)) + p^2_{3-k}(s)$. Let $x^1 = N^1_k(s)/(N^1_k(s) + N^1_{3-k}(s))$ and $x^2 = N^2_{3-k}(s)/\ (N^2_{3-k}(s) + N^2_k(s))$. Then $\pi_A(s) \leq x^2l^1(1-x^1) + x^1((1-2x^2)h^1 + x^2l^1) + x^1l^2(1-x^2) + x^2((1-2x^1)h^2 + x^1h^2) = h^1x^1 + h^2x^2 + l^1x^2 + l^2x^1 - 2h^1x^1x^2 - 2h^2x^1x^2$. Taking first order conditions it is easy to verify that the latter expression is maximized at $x_1 = \frac{h^2+l^1}{2h^1+2h^2}$, $x_2 = \frac{h^1+l^1}{2h^1+2h^2}$. Substituting these values into the expression yields $\pi_A(s) \leq \frac{\left(h^1+l^2\right)\left(h^2+l^1\right)}{2(h^1+h^2)}$.

Let $\widehat{C}(s) = \{C^j_i \in C : c^j_i(s) \neq \emptyset\}$. Notice that if $s'_A \in S_A$ is such that $n_A(s'_A) = 1$ and $p^j_A(s'_A) = l^j/(N^{3-j}_k(s) + N^{3-j}_{3-k}(s)) - \varepsilon\ \forall\ j \in \{1,2\}$, where $\varepsilon > 0$, then $\pi_A(s'_A, s_A) \geq l^1 + l^2 - \varepsilon(\sum\limits_{j=1,2}\sum\limits_{k=1,2} N^j_k(s))$, since the assumptions that there is no conflict of interest among consumers on the same side and that s is a Nash equilibrium together guarantee that in the consumer subgame following the above price announcements joining the network is a supported restriction for all players in $\widehat{C}(s)$. Since s is a Nash equilibrium, this implies $\pi_A(s) \geq l^1 + l^2$. Therefore $\frac{\left(h^1+l^2\right)\left(h^2+l^1\right)}{2(h^1+h^2)} \geq l^1 + l^2$.

It is straightforward to verify that for any $h_1 + h_2 = h > 0$ and $l_1 + l_2 = l > 0$ the expression $\frac{(h_1+l_2)(h_2+l_1)}{2(h_1+h_2)} - l_1 - l_2$ is maximized at $h_1 = h_2 = h/2$, $l_1 = l_2 = l/2$. In that case $\frac{(h_1+l_2)(h_2+l_1)}{2(h_1+h_2)} - l_1 - l_2 = h^2 - 6hl + l^2$. Then s being a Nash equilibrium implies $h^2 - 6hl + l^2 \geq 0$. Since $h > l$, this implies $h \geq (3 + 2\sqrt{2})l$. Therefore if $(\max\limits_{i\in[0,1]} u^j_i)/(\min\limits_{i\in[0,1]} u^j_i) < 3 + 2\sqrt{2}\ \forall$

$j \in \{1, 2\}$, then s cannot be a Nash equilibrium, a contradiction. ■

Lemma B.2. *Let* $G^c = (C, S^c, u^c)$ *be the subgame following price announcements* $(p_A^1, p_A^2, p_B^1, p_B^2)$. *If* $p_A^j < p_B^j \ \forall\ j = 1, 2$ *then* B *is not coalitionally rationalizable in* G^c *for any* $C_i^j \in C$. *If also* $p_A^j < u^j \ \forall\ j \in \{1, 2\}$ *then* A *is the unique coalitionally rationalizable strategy in* G^c *for every* $C_i^j \in C$. *Similarly if* $p_B^j < p_A^j \ \forall\ j = 1, 2$ *then* A *is not coalitionally rationalizable in* G^c *for any* $C_i^j \in C$. *If also* $p_B^j < u^j \ \forall\ j \in \{1, 2\}$ *then* B *is the unique coalitionally rationalizable strategy in* G^c *for every* $C_i^j \in C$.

Proof of Lemma B.2 Consider $p_A^j < p_B^j \ \forall\ j = 1, 2$. If $p_B^j > u^j$ for some $j \in \{1, 2\}$ then in G^c choosing B is not rationalizable for any $C_i^j \in C$. Observe that $p_A^j \leq u^j \ \forall\ j \in \{1, 2\}$ implies $p_A^j < u^j \ \forall\ j \in \{1, 2\}$. For every $C_i^j \in C$ the maximum utility C_i^j can expect when joining B is $u^j - p_B^j$. The minimum utility that C_i^j can expect if every consumer in C joins A is $u^j - p_A^j(s) > u^j - p_B^j$. If $p_A^j(s) < u^j \ \forall\ j \in \{1, 2\}$ then this implies that joining A is a supported restriction for consumers in C, which implies the claim.

The other case is perfectly symmetric. ■

Lemma B.3. *For every* $k = 1, 2$ *and* $i \in [0, 1]$ *it holds that* $\emptyset$ *is not coalitionally rationalizable for* C_i^k *in the subgame following price announcements* $(0, 0, 0, 0)$.

Proof of Lemma B.3

Let $\widehat{S}_i^k = \{A, B\} \ \forall\ k = 1, 2$ and $i \in [0, 1]$. Consider the restriction $\widehat{S} \equiv \underset{i \in [0,1]}{\times} \widehat{S}_i^1 \underset{i \in [0,1]}{\times} \widehat{S}_i^2$ given S^c by C in the subgame following price announcements $(0, 0, 0, 0)$. Let $k \in \{1, 2\}$, $i \in [0, 1]$ and $\omega_{-k,i} \in \Omega_{-k,i}(\widehat{S})$. Let $n_A^{-k}(\omega_{-k,i}) = \int_{t_{-i} \in S_{-i}} g_j(C_j^{-k} \in C^{-k}/C_i^{-k} : t_j^k(0, 0, 0, 0) =$

$A)d\omega_{-k,i}$ and let $n_B^{-k}(\omega_{-k,i}) = \int\limits_{t_{-i}\in S_{-i}} g_j(C_j^{-k} \in C^{-k}/C_i^{-k} : t_j^k(0,0,0,0) = B)d\omega_{-k,i}$. Since $\omega_{-k,i} \in \Omega_{-k,i}(\widehat{S})$, $\min(n_A^{-k}(\omega_{-i}), n_B^{-k}(\omega_{-i})) > 0$. Then playing a best response strategy to $\omega_{-k,i}$ yields a positive expected payoff to C_i^k. Since $U_i^k(\emptyset) = 0\ \forall\ i \in [0,1]$, this implies that the above restriction is supported for C and therefore $\emptyset$ is not a coalitionally rationalizable strategy for any $k = 1,2$ and $i \in [0,1]$. ■

Lemma B.4. *Let s be a coalition perfect equilibrium. If $N_A^k(s) > 0$ for some $k = 1,2$ and $N_B^1(s) = N_B^2(s) = 0$ then (i) $p_A^1(s) = -p_A^2(s)$, (ii) $p_A^k(s) \leq u^k\ \forall\ k = 1,2$ and (iii) $N_A^1(s) = N_A^2(s) = 1$. Similarly if $N_B^k(s) > 0$ for some $k = 1,2$ and $N_A^1(s) = N_A^2(s) = 0$ then (i) $p_B^1(s) = -p_B^2(s)$, (ii) $p_B^k(s) \leq u^k\ \forall\ k = 1,2$ and (iii) $N_B^1(s) = N_B^2(s) = 1$.*

Proof of Lemma B.4

Note that $N_B^1(s) = N_B^2(s) = 0$ implies $\pi_B(s) = 0$. Suppose $p_A^k(s) > u^k$ for some $k = 1,2$. Then $N_A^k(s) = 0$ since consumers cannot get negative utility in s. Then $N_A^{-k}(s) > 0$ implies that $p_A^{-k}(s) \leq 0$, again because consumers cannot get negative utility in s. Since A cannot have negative profit in s, this implies $p_A^{-k}(s) = 0$. Consider the deviation $(u - \varepsilon, -\varepsilon)$ by B, where $\varepsilon > 0$. In the subgame following this deviation it is a supported restriction for $C^1 \cup C^2$ to play B, because that profile yields the highest possible payoff in this subgame for every $C_i^k \in C$, and choosing A or $\emptyset$ yields a strictly smaller payoff than this maximum no matter what strategies other consumers play. Therefore $s_i^k(p_A^1(s), p_A^2(s), u - \varepsilon, -\varepsilon) = B$ $\forall\ k = 1,2$ and $i \in [0,1]$. Then B's profit after this deviation is $u - 2\varepsilon$, which is positive for small enough profits, a contradiction. This concludes that $p_A^k(s) \leq u^k\ \forall\ k = 1,2$.

Suppose now that $p_A^1(s) + p_A^2(s) > 0$. Consider the deviation $(p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon)$ by B, where $\varepsilon > 0$. By lemma B.2 $s_i^k(p_A^1(s), p_A^2(s), p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon) = B\ \forall\ k = 1,2$ and

$i \in [0,1]$. Then B's profit after this deviation is $p_A^1(s) + p_A^2(s) - 2\varepsilon$, which is positive for small enough ε, a contradiction. This concludes that $p_A^1(s) + p_A^2(s) \leq 0$.

Suppose now that $p_A^1(s) + p_A^2(s) < 0$. This implies $p_A^k(s) < 0$ for some $k = 1, 2$. Then $N_B^k(s) = 0$ implies $N_A^k(s) = 1$, since A strictly dominates $\emptyset$ for side 1 consumers. But then $p_A^1(s) + p_A^2(s)$ implies $\pi_A(s) = p_A^1(s)N_A^1(s) + p_A^2(s)N_A^2(s) < 0$, a contradiction. This concludes that $p_A^1(s) + p_A^2(s) \leq 0$.

If $p_A^k(s) < 0$ for some $k = 1, 2$, then $N_B^k(s) = 0$ implies $N_A^k(s) = 1$. Then $\pi_A(s) \geq 0$ implies that $N_A^k(s) = 1 \; \forall \; k = 1, 2$.

Consider now $p_A^1(s) = p_A^2(s) = 0$. Then $\pi_A(s) = 0$. If $p_B^k(s) < 0$ for some $k = 1, 2$, then $\emptyset$ is a strictly dominated strategy for side k consumers, and therefore $N_B^k(s) = 0$ implies $N_A^k(s) = 1$. Then choosing A yields utility $u_{-k} > 0$ for side $-k$ consumers, and therefore $N_B^{-k}(s) = 0$ implies $N_A^{-k}(s) = 1$. Suppose now that $p_B^1(s) > 0$ and $p_B^2(s) = 0$. Then by lemma B.2 a deviation $\min(u - \varepsilon, p_B^1(s) - \varepsilon), -\varepsilon$ by A for $\varepsilon > 0$ guarantees that all consumers join A, which for small enough ε yields positive profit for A, contradicting that s is an equilibrium. A symmetric argument rules out that $p_B^1(s) = 0$ and $p_B^2(s) > 0$. If $p_B^1(s) = p_B^2(s) = 0$, then lemma B.3 implies that $N_A^k(s) + N_B^k(s) = 1 \; \forall \; k = 1, 2$, and then $N_B^1(s) = N_B^2(s) = 0$ implies $N_A^1(s) = N_A^2(s) = 1$. ■

Lemma B.5. *Let s be a coalition perfect equilibrium such that $N_A^k(s) > 0$ for some $k = 1, 2$ and $N_B^k(s) > 0$ for some $k = 1, 2$. Then $p_A^1(s) = p_B^1(s) = -p_A^2(s) = -p_B^2(s)$ and $p_A^k(s) \leq u^k$ $\forall \; k = 1, 2$. Moreover, $N_A^1(s) = N_A^2(s) = N_B^1(s) = N_B^2(s) = 1/2$.*

Proof of Lemma B.5

Suppose $p_f^k(s) > u^k$ for some $k = 1, 2$ and $f \in \{A, B\}$. W.l.o.g. assume $p_A^1(s) > u^1$.

Then $N_A^1(s) = 0$ and therefore $N_A^2(s) > 0$. This is only compatible with consumers choosing A in s playing a best response and A not getting negative profits if $p_A^2(s) = 0$. Then by lemma B.2 a price announcement $(u^1 - \varepsilon, -\varepsilon)$ by B for $\varepsilon > 0$ guarantees that all consumers join B, which for small enough ε yields positive profit for B. Therefore $\pi_A(s) = 0$ and $\pi_B(s) > 0$. The latter can only be if both $N_B^1(s) > 0$ and $N_B^2(s) > 0$, which imply that $p_B^k(s) \leq u^k \; \forall \; k = 1, 2$. Then by Lemma B.2 a deviation $p_B^1(s) - \varepsilon, p_B^2(s) - \varepsilon$ by A for $\varepsilon > 0$ guarantees that all consumers join A. For small enough ε this deviation profit is close to $p_B^1(s) + p_B^2(s)$. If $p_B^k(s) \geq 0 \; \forall \; k = 1, 2$, then $\pi_B(s) > 0$ implies $p_B^1(s) + p_B^2(s) > 0$, which implies that the above deviation is profitable for small enough ε. If $p_B^2(s) \leq 0$, then $\pi_B(s) > 0$ implies $N_B^1(s) > 0$, but then $N_A^2(s) > 0$ contradicts that every consumer plays a best response in s. Therefore $p_B^2(s) > 0$. If $p_B^1(s) < 0$ and $p_B^2(s) > 0$, then $N_B^1(s) = 1$ since B is the unique best response in s after the equilibrium price announcements for side 1 consumers, and therefore $p_B^1(s) + p_B^2(s) \geq p_B^1(s) N_B^1(s) + p_B^2(s) N_B^2(s) = \pi_B(s) > 0$. This again implies that the above deviation for A is profitable for small enough ε, contradicting that s is a Nash equilibrium. This concludes that $p_f^k(s) \leq u^k \; \forall \; k = 1, 2$ and $f \in \{A, B\}$. Suppose $p_A^k(s) \neq p_B^k(s)$ for some $k = 1, 2$. W.l.o.g. assume $p_A^1(s) > p_B^1(s)$. Then $p_A^2(s) \leq p_B^2(s)$, otherwise lemma B.2 implies $N_A^1(s) = N_A^2(s) = 0$. Suppose first that $N_A^1(s) = N_B^1(s) = 0$. Then $N_A^2(s) > 0$ and $N_B^2(s) > 0$. This is only compatible with consumers being in equilibrium and firms not getting negative profit if $p_A^2(s) = p_B^2(s) = 0$. Then $\pi_B(s) = 0$. Then by lemma B.2 a deviation $\min(u - \varepsilon, p_A^1(s) - \varepsilon), -\varepsilon$ by B for $\varepsilon > 0$ guarantees that all consumers join A, which for small enough ε yields positive profit for B, contradicting that s is an equilibrium. Suppose next that $N_A^2(s) = N_B^2(s) = 0$. Then $N_A^1(s) > 0$ and

$N_B^1(s) > 0$, which contradicts that s is a Nash equilibrium, since $N_A^2(s) = N_B^2(s) = 0$ and $p_A^1(s) > p_B^1(s)$ implies that given $s_{-1,i}$ B is a better response than A in the subgame following the equilibrium price announcements for every $C_i^1 \in C^1$. This concludes that $N_k^1(s) > 0$ for some $k = A, B$ and $N_k^1(s) > 0$ for some $k = A, B$. But then $N_A^1(s) \leq N_B^1(s)$ and $N_A^2(s) > N_B^2(s)$, otherwise $p_A^1(s) > p_B^1(s)$ and $p_A^2(s) \leq p_B^2(s)$ imply that some consumers are not playing a best response in s. Consider now following two deviations. The first is $(p_B^1(s) - \varepsilon, p_B^2(s) - \varepsilon)$ by A, and the second is $(p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon)$ by B. Since $p_f^k(s) \leq u^k$ $\forall$ $k = 1, 2$ and $f \in \{A, B\}$, lemma B.2 implies that $s_i^k(p_B^1(s) - \varepsilon, p_B^2(s) - \varepsilon, p_B^1(s), p_B^2(s)) = A$ and $s_i^k(p_A^1(s), p_A^2(s), p_B^1(s) - \varepsilon, p_B^2(s) - \varepsilon) = B$ $\forall$ $k = 1, 2$ and $i \in [0, 1]$. Then the first deviation yields a profit $p_B^1(s) + p_B^2(s) - 2\varepsilon$ to A, while the second yields $p_A^1(s) + p_A^2(s) - 2\varepsilon$ to B. The sum of these deviation profits is $p_A^1(s) + p_A^2(s) + p_B^1(s) + p_B^2(s) - 4\varepsilon$. The sum of the two firms' equilibrium profits is $N_A^1(s)p_A^1(s) + N_A^2(s)p_A^2(s) + N_B^1(s)p_B^1(s) + N_B^2(s)p_B^2(s) \equiv \pi^*$. Note that $p_B^1(s) < 0$ implies that $N_A^1(s) + N_A^2(s) = 1$, since then $\emptyset$ is never a best response for any $C_i^1 \in C^1$. Similarly, $p_A^2(s) < 0$ implies that $N_B^1(s) + N_B^2(s) = 1$. Then by $N_A^1(s) \leq N_B^1(s)$, $N_A^2(s) > N_B^2(s)$, $p_A^1(s) > p_B^1(s)$ and $p_A^2(s) \leq p_B^2(s)$ it has to hold that $N_A^1(s)p_A^1(s) + N_A^2(s)p_A^2(s) + N_B^1(s)p_B^1(s) + N_B^2(s)p_B^2(s) < \frac{1}{2}(p_A^1(s) + p_A^2(s) + p_B^1(s) + p_B^2(s))$. The left hand side of this inequality is nonnegative (it is the sum of equilibrium profits), therefore the right hand side is positive, which implies that also $N_A^1(s)p_A^1(s) + N_A^2(s)p_A^2(s) + N_B^1(s)p_B^1(s) + N_B^2(s)p_B^2(s) < p_A^1(s) + p_A^2(s) + p_B^1(s) + p_B^2(s)$. But that implies that for small enough ε the sum of the two deviation profits above is larger than the sum of the two equilibrium profits, implying that at least one of the deviations is profitable, a contradiction. This concludes that $p_A^k(s) = p_B^k(s)$ $\forall$ $k = 1, 2$. Suppose that $\pi_A(s) + \pi_B(s) > 0$. W.l.o.g.

assume $\pi_A(s) \geq \pi_B(s)$. Then $\pi_B(s) < p_A^1(s) + p_A^2(s) \leq \pi_A(s) + \pi_B(s)$ (note that $p_A^k(s) = p_B^k(s) \; \forall \; k = 1,2$, and that $p_A^k(s) < 0$ implies that $C_i^k(s) \neq \emptyset \; \forall \; C_i^k \in C^k$). By lemma B.2 a deviation $p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon$ by B for $\varepsilon > 0$ guarantees that all consumers join B, which yields a profit of $p_A^1(s) + p_A^2(s) - 2\varepsilon$ to B. This implies that the above deviation is profitable for small enough ε, a contradiction. Then $\pi_A(s) + \pi_B(s) \leq 0$ and since equilibrium profits have to be nonnegative, $\pi_A(s) = \pi_B(s) = 0$. Suppose $p_A^1(s) + p_A^2(s) > 0$. By lemma B.2 a deviation $p_A^1(s) - \varepsilon, p_A^2(s) - \varepsilon$ by B for $\varepsilon > 0$ guarantees that all consumers join B, which yields a profit of $p_A^1(s) + p_A^2(s) - 2\varepsilon$ to B. But for small enough ε this profit is positive, which contradicts that $\pi_B(s) = 0$ and that s is an equilibrium. This concludes that $p_A^1(s) + p_A^2(s) \leq 0$. Suppose $p_A^1(s) + p_A^2(s) < 0$. Then $p_A^k(s) = p_B^k(s) < 0$ for some $k = 1,2$. Then $N_A^k(s) + N_B^k(s) = 1$, since $\emptyset$ is never a best response for any $C_i^k \in C^k$ in the subgame after the equilibrium price announcements. But then $\min(\pi_A(s), \pi_B(s)) < 0$, contradicting that s is a Nash equilibrium. This concludes that $p_A^1(s) + p_A^2(s) = 0$. If $p_A^k(s) = p_B^k(s) < 0$ for some $k = 1,2$, then $N_A^k(s) + N_B^k(s) = 1$. Then nonnegativity of equilibrium profits implies that also $N_A^{-k}(s) + N_B^{-k}(s) = 1$ and that $N_A^1(s) = N_A^2(s)$, $N_B^1(s) = N_B^2(s)$. If $p_A^1(s) = p_B^1(s) = p_A^2(s) = p_B^2(s) = 0$, then by lemma B.3 $N_A^k(s) + N_B^k(s) = 1 \; \forall \; k = 1,2$. As shown above, $N_A^k(s) + N_B^k(s) > 0 \; \forall \; k = 1,2$. Then $p_A^k(s) = p_B^k(s) \; \forall \; k = 1,2$ implies $N_A^k(s) = N_B^k(s) \; \forall \; k = 1,2$. This implies $\pi_A(s) = \pi_B(s)$. If $p_A^k(s) = p_B^k(s) < 0$ for some $k = 1,2$, then the above implies that $N_A^k(s) = N_B^k(s) = 1/2$. Then $p_A^1(s) + p_A^2(s) = 0$ and nonnegativity of equilibrium profits together imply that also $N_A^{-k}(s) = N_B^{-k}(s) = 1/2$. If $p_A^1(s) = p_B^1(s) = p_A^2(s) = p_B^2(s) = 0$, then $N_A^j(s) + N_B^j(s) = 1 \; \forall \; j = 1,2$ and the fact that s is a Nash equilibrium imply that $N_A^j(s) = N_B^j(s) = 1/2 \; \forall \; j = 1,2$. ■

Proof of Theorem 2.5

Lemma B.4 and lemma B.5 establish that there are no other coalition perfect equilibria with one or two active firms than those stated in the claim. All that remains to be shown is that there is no coalition perfect equilibrium with no active firm.

Suppose $N_A^k(s) + N_B^k(s) = 0 \; \forall \; k = 1, 2$. Then $\pi_A(s) = \pi_B(s) = 0$. If $p_f^k(s) < 0$ for some $k = 1, 2$ and $f = A, B$, then $N_A^k(s) + N_B^k(s) = 1$, since $\emptyset$ is a never best response strategy for any $C_i^k \in C^i$, a contradiction. Suppose now that $\exists \; k \in \{A, B\}$ such that $p_k^j(s) \geq 0 \; \forall \; j = 1, 2$ and $p_j^l(s) > 0$ for some $l \in \{1, 2\}$. W.l.o.g. assume $p_A^1(s) > 0$ (and $p_A^2(s) \geq 0$). By lemma B.2 the deviation $\min(u^1 - \varepsilon, p_A^1(s) - \varepsilon)$, $\min(u^2 - \varepsilon, p_A^2(s) - \varepsilon)$ by B for $\varepsilon > 0$ guarantees that every consumer joins B, and it yields strictly positive profit for small enough ε, a contradiction. If $p_k^j(s) = 0 \; \forall \; j = 1, 2$ and $k = A, B$, then $N_A^j(s) + N_B^j(s) = 1 \; \forall \; j = 1, 2$ by lemma B.4 . This concludes that if s is a coalition perfect equilibrium, then it cannot be that $N_A^j(s) + N_B^j(s) = 0 \; \forall \; j = 1, 2$. ■

Proof of Theorem 2.6

W.l.o.g. assume that $k = 1$ (the other case is perfectly symmetric), so $u^1 < u^2$.

By Theorem 2.5 if $N_k^1(s) + N_k^2(s) > 0$ for some $k \in \{A, B\}$, then $p_k^1(s) = -p_k^2(s)$ and $p_k^l(s) \leq u^l \; \forall \; l = 1, 2$. Furthermore, $\pi_A(s) = \pi_B(s) = 0$.

Assume $N_A^1(s) + N_A^2(s) > 0$ and suppose $p_A^1(s) > u^1 - u^2$. Consider the deviation $p_A^1(s) - u^1 - \varepsilon, u^2 - \varepsilon$ by B for $\varepsilon > 0$. In the subgame following the deviation B is a strictly dominant strategy for every $C_i^1 \in C^1$, therefore it is the only rationalizable strategy. But then B is the only rationalizable strategy in the subgame for every $C_i^2 \in C^2$ too. Therefore after the above deviation B's profit in s is $p_A^1(s) - u^1 + u^2 - 2\varepsilon$. Since $p_A^1(s) > u^1 - u^2$, this profit is strictly positive for small enough ε, contradicting that s is an equilibrium. A perfectly symmetric argument shows that it cannot be that $N_B^1(s) + N_B^2(s) > 0$ and $p_B^1(s) > u^1 - u^2$.■

Proof of Theorem 2.7

Let s be a coalition perfect equilibrium.

Suppose first that $N_k^j(s) = 0$ for some $j \in \{1, 2\}$ and $k \in \{A, B\}$. W.l.o.g. assume $k = A$ and $j = 1$. Then either $N_A^1(s) = N_A^2(s) = 0$ or $N_A^2(s) > 0$ and $p_A^1(s) = 0$. In either case $\pi_A = 0$ and then by the starting assumption $\pi_B(s) > 0$. Let $\widehat{C} = \{C_i^j : c_i^j(s) = B\}$. Note that $p_B^j(s) < 0$ for some $j \in \{1, 2\}$ implies that $C^j \subset \widehat{C}$. Consider now deviation $(p_B^1(s) - \varepsilon, p_B^2(s) - \varepsilon)$ for $\varepsilon > 0$ by A. Similar arguments as in lemma B.2 establish that in the subgame after this deviation A is the unique coalitionally rationalizable strategy for every $C_i^j \in \widehat{C}$. But then for small enough ε the deviation is profitable, a contradiction. Therefore $N_k^j(s) > 0 \; \forall \; j \in \{1, 2\}$ and $k \in \{A, B\}$.

If $p_A^j(s) > p_B^j(s) \; \forall \; j = 1, 2$ then similar arguments as in B.2 establish that $N_A^1(s) + N_A^2(s) = 0$, contradicting the above result. Similarly it cannot be that $p_A^j(s) < p_B^j(s) \; \forall \; j = 1, 2$.

Consider now $p_A^j(s) = p_B^j(s) \; \forall \; j = 1, 2$. Let $\widehat{C} = \{C_i^j : c_i^j(s) \neq \emptyset\}$. There exists $k \in \{A, B\}$ such that $\pi_k(s) \leq (\pi_1(s) + \pi_2(s))/2 > 0$. W.l.o.g. assume $k = A$. Consider deviation $(p_B^1(s) - \varepsilon, p_B^2(s) - \varepsilon)$ by A. Similar arguments as in Lemma B.2 establish that in the subgame after this deviation A is the unique coalitionally rationalizable strategy for every $C_i^j \in \widehat{C}$. Therefore if ε is small enough then after this deviation A's profit is larger than $(\pi_1(s) + \pi_2(s))/2$ (note that $p_B^j(s) < 0$ for some $j = 1, 2$ implies that $C^j \subset \widehat{C}$), a contradiction.

Finally, notice that if $p_A^j(s) \leq p_B^j(s)$ for some $j \in \{1,2\}$, then $N_B^j(s) > 0$ and the assumption that s is a Nash equilibrium imply that $N_B^{3-j}(s) \geq N_A^{3-j}(s)$. If $p_A^j(s) < p_B^j(s)$ for some $j \in \{1,2\}$, then $N_B^j(s) > 0$ and the assumption that s is a Nash equilibrium imply that $N_B^{3-j}(s) > N_A^{3-j}(s)$. Similarly if $p_B^j(s) \leq p_A^j(s)$ (correspondingly $p_B^j(s) < p_A^j(s)$) for some $j \in \{1,2\}$, then $N_B^{3-j}(s) \leq N_A^{3-j}(s)$ (correspondingly $N_B^{3-j}(s) < N_A^{3-j}(s)$). ■

Lemma B.6. *Let $G^c = (C, S^c, u^c)$ be the subgame following price announcements $(p_A^1, p_A^2, p_B^1, p_B^2)$. If $p_A^j < p_B^j \ \forall\ j = 1,2$ then B is not coalitionally rationalizable in G^c for any $C_i^j \in C$. Furthermore, if A is rationalizable for some $C_i^j \in C$ in a subgame following price announcements $(q_A^1, q_A^2, p_B^1, p_B^2)$ where $q_A^1 > p_A^1$ and $q_A^2 > p_A^2$ then A is the unique coalitionally rationalizable strategy in G^c for C_i^j. Similarly if $p_B^j < p_A^j \ \forall\ j = 1,2$ then A is not coalitionally rationalizable in G^c for any $C_i^j \in C$. Furthermore, if B is rationalizable for some $C_i^j \in C$ in a subgame following price announcements $(p_A^1, p_A^2, q_B^1, q_B^2)$ where $q_B^1 > p_B^1$ and $q_B^2 > p_B^2$ then B is the unique coalitionally rationalizable strategy in G^c for C_i^j.*

Proof of Lemma B.6

Analogous to the proof of Lemma B.2, therefore omitted.

Proof of Theorem 2.8

Let $\Delta = \min(\frac{\sup_k u_k^1}{\inf_{j,k} u_k^1}, \frac{\sup_k u_k^2}{\inf_{j,k} u_k^2})$ and let s be a coalition perfect equilibrium. By the starting assumption $\Delta \leq 4/3$. Assume the theorem does not hold for s. Then by Theorem 2.7 $\exists\ j \in \{1,2\}$ such that $p_A^j(s) < p_B^j(s)$, $N_A^j(s) \geq N_B^j(s)$ and $p_A^{-j}(s) \geq p_B^{-j}(s)$, $N_A^{-j}(s) < N_B^{-j}(s)$. For every $j \in \{1,2\}$ let $\inf_k u_k^j \equiv l^j$. If $p_k^j(s) \leq l^j\ \forall\ k \in \{A,B\}$ and $j \in \{1,2\}$ then an analogous proof to the proof of lemma B.5 establishes that there is a profitable deviation for at least one firm, contradicting that s is a coalition perfect equilibrium. The same holds if $p_k^j(s) \geq 0 \; \forall \; k \in \{A, B\}$ and $j \in \{1, 2\}$. It is straightforward to show that it cannot be that for some $k \in \{A, B\}$ it holds that $p_k^j(s) > l^j \; \forall \; j \in \{1, 2\}$. Below we consider the remaining possibilities. Consider first the case that for some $k \in \{A, B\}$ and $j \in \{1, 2\}$ it holds that $p_k^j(s) > l^j$, $p_k^{-j}(s) < 0$ and $0 \leq p_{-k}^1(s), p_{-k}^2(s)$. Since $\sup_k u_k^j \leq \Delta l^j \leq \frac{4}{3} l^j$ and $p_k^j(s) > l^j$, for $N_k^j(s) > 0$ it has to be that $N_k^{-j}(s) > \frac{3}{4}$. Furthermore, $p_k^j(s) > p_{-k}^j(s)$ implies $N_k^j(s) \leq \frac{1}{2}$, and $\sup_k u_k^j \leq \Delta l^j \leq \frac{4}{3} l^j$ implies $p_k^j(s) < \frac{4}{3} l^j$. Therefore $\pi_k(s) \leq \frac{2}{3} l^j + \frac{3}{4} p_k^{-j}(s)$. Then $\pi_k(s) \geq 0$ implies $l^j + p_k^{-j}(s) > \frac{2}{3} l^j + \frac{3}{4} p_k^{-j}(s)$ and therefore $\pi_k(s) < l^j + p_k^{-j}(s)$. But note that $-k$ can get a profit arbitrarily close to $l^j + p_k^{-j}(s)$ by deviating to price announcements $(l^j - \varepsilon, p_k^{-j}(s) - \varepsilon)$ for small enough $\varepsilon > 0$ (since by lemma B.6 after that price announcement all consumers join B). So if $\pi_{-k}(s) \leq \pi_k(s)$ then $-k$ has a profitable deviation from s. On the other hand note that k can get a profit arbitrarily close to $l^j + p_k^{-j}(s)$ by deviating to price announcements $(p_{-k}^1(s) - \varepsilon, p_{-k}^2(s) - \varepsilon)$ for small enough $\varepsilon > 0$. So if $\pi_{-k}(s) > \pi_k(s)$ then k has a profitable deviation from s. This concludes that s cannot be a coalition perfect equilibrium, a contradiction. Consider now the case that for some $k \in \{A, B\}$ and $j, h \in \{1, 2\}$ it holds that $p_k^j(s) > l^j$ and $0 > p_{-k}^h(s)$. Just like in the previous case, it has to be that $N_k^{-j}(s) > \frac{3}{4}$ and therefore $N_{-k}^{-j}(s) < \frac{1}{4}$. Then $\sup_k u_k^j \leq \Delta l^j \leq \frac{4}{3} l^j$ implies $p_{-k}^j(s) < \frac{1}{3} l^j$. Then $0 > p_{-k}^h(s)$ and $\frac{4}{3} l^j > p_{-k}^{-h}(s)$ imply that $\pi_{-k}(s) < \frac{1}{3} l^j$. Since $N_k^j(s) \leq \frac{1}{2}$ and $p_k^j(s) \leq \frac{4}{3} l^j N_k^{-j}(s)$ and $\pi_k(s) \geq 0$, it holds that $\frac{2}{3} l^j N_k^{-j}(s) + p_k^{-j}(s) N_k^{-j}(s) \geq 0$, therefore $p_k^{-j}(s) \geq -\frac{2}{3} l^j$. Therefore $l^j + p_k^{-j}(s) \geq \frac{1}{3} l^j$. But note that $-k$ can get a profit arbirarily close to $l^j + p_k^{-j}(s)$ by deviating to price announcement $(l^j - \varepsilon, p_k^{-j}(s) - \varepsilon)$ for small enough

$\varepsilon > 0$. This concludes that s cannot be a coalition perfect equilibrium, a contradiction. ■

Appendix C

This Appendix contains the proof of the Theorem presented in Chapter 3

Proof of Theorem 3.1

We have observed that the behavior of the process depends on whether $\widehat{p}_t(\frac{1}{2})$ is above or below $p^*_t(\frac{1}{2}) = \frac{2}{3}$.

Recall that

$$\widehat{p}_t = \frac{\displaystyle\sum_{k \in S_t} s^+ (x_k, x_t)}{\displaystyle\sum_{k \in S_t} s^+ (x_k, x_t) + \sum_{k \in F_t} s^- (x_k, x_t)}$$

We simplify notation by defining

$$A_t = \sum_{k \in S_t} s^+ \left(x_k, \frac{1}{2} \right) \tag{C.1}$$

$$B_t = \sum_{k \in F_t} s^- \left(x_k, \frac{1}{2} \right) \tag{C.2}$$

$$c = p^*_t(\frac{1}{2}) = \frac{2}{3} \tag{C.3}$$

$$\widehat{p}_t \left(\frac{1}{2} \right) = \frac{A_t}{A_t + B_t} \tag{C.4}$$

$$z_t = A_t - cB_t - cA_t \tag{C.5}$$

so that

$$\widehat{p}_t\left(\frac{1}{2}\right) > c \Leftrightarrow \frac{A_t}{A_t+B_t} > c \Leftrightarrow \quad \text{(C.6)}$$

$$A_t - cB_t - cA_t > 0 \Leftrightarrow z_t > 0 \quad \text{(C.7)}$$

Suppose we start from $z_t > 0$.

Then,

- with probability p_0 the next draw will be 0 , there will be a success and z_t will change (increase) by $(1-c)(A_{t+1}-A_t) = (1-c)\left(1+\left(0-\frac{1}{2}\right)\right) = (1-c)\frac{1}{2} = +\frac{1}{6}$
- with probability $p_{\frac{1}{2}}$ the next draw will be $\frac{1}{2}$, there will be a success and z_t will change (increase) by $(1-c)(A_{t+1}-A_t) = (1-c)\,1 = +\frac{1}{3}$
- with probability p_1 the next draw will be 1 , there will be a failure and and z_t will change (decrease) by $-c(B_{t+1}-B_t) = -c\left(1-\left(1-\frac{1}{2}\right)\right) = -\frac{c}{2} = -\frac{1}{3}$.

For $z_t > 0$ (∗)

Prob.	$z_{t+1} - z_t$
p_0	$\frac{1}{6}$
$p_{\frac{1}{2}}$	$\frac{1}{3}$
p_1	$-\frac{1}{3}$

Similarly, if we start from $z_t \leq 0$ what will happen is that:

- with probability p_0 the next draw will be 0 , there will be a success and z_t will change (increase) by $(1-c)(A_{t+1}-A_t) = (1-c)\left(1+\left(0-\frac{1}{2}\right)\right) = (1-c)\frac{1}{2} = +\frac{1}{6}$
- with probability $p_{\frac{1}{2}}$ the next draw will be $\frac{1}{2}$, there will be a failure and z_t will change (increase) by $-c(B_{t+1}-B_t) = -c(1) = -\frac{2}{3}$

- with probability p_1 the next draw will be 1 , there will be a failure and and z_t will change (decrease) by $-c\left(B_{t+1}-B_t\right)=-c\left(1-\left(1-\frac{1}{2}\right)\right)=-\frac{c}{2}=-\frac{1}{3}$.

For $z_t \leq 0$ (**)

Prob.	$z_{t+1}-z_t$
p_0	$\frac{1}{6}$
$p_{\frac{1}{2}}$	$-\frac{2}{3}$
p_1	$-\frac{1}{3}$

We conclude that z_t is a Markov process, which, for non-negative values is governed by $(*)$, and for negative ones – by $(**)$.

To establish convergence, we wish to show that, with probability 1, z_t ends up being always positive or always negative from some point on. To see that this is the case, we note that for every $(p_0, p_{\frac{1}{2}}, p_1)$, (i) the process $(*)$ has a positive drift, or (ii) the process $(**)$ has a negative drift, or both. If (i) holds but not (ii), then z_t will be always positive form some point on with probability 1. Conversely, it will be always negative (from some point on with probability 1) if (ii) holds but not (i). We are left with the case in which both (i) and (ii) hold. In this case, starting with any positive value of z_t, there is a positive probability (independent of t) that z_t will never be non-positive, and vice versa for negative values of z_t. Hence the probability of switching infinitely many times between positive and negative values is zero.

To establish convergence, it remains to note that if, for some T, for all $t \geq T$ we have $z_t \geq 0$, then F_t converges to L_1, whereas if, for some T, for all $t \geq T$ we have $z_t < 0$, then F_t converges to L_2.

Finally, we observe that both (i) and (ii) hold when $p_0 + 2p_{\frac{1}{2}} > 2p_1$ and $p_0 - 4p_{\frac{1}{2}} < 2p_1$. This defines a convex and non-empty set of vectors $(p_0, p_{\frac{1}{2}}, p_1)$, of which $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a member. ■

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