Advanced Economic Growth: Lecture 22: Stochastic Growth

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November 21, 2007

Stochastic Growth Models I

- Brock and Mirman (1972): generalization of neoclassical growth and starting point of *Real Business Cycle* models
  - Baseline neoclassical growth: complete markets, households and firms can trade using any Arrow-Debreu commodity.
  - With uncertainty implies that full set of *contingent claims* is traded competitively.
  - Implies that individuals can fully insure against idiosyncratic risks.
  - Source of interesting uncertainty thus is aggregate shocks.
- Bewley (1970s and the 1980s): households cannot use contingent claims and can only trade in riskless bonds.
  - Explicitly prevent risk-sharing and thus "incomplete markets".
  - Stochastic stream of labor income: can only achieve smoothing via "self-insurance".
  - Does *not* admit a representative consumer.
    - ★ Trading in contingent claims not only sufficient, but also necessary for representative household assumption with uncertainty.
  - Key for study of questions related to risk, income fluctuations and policy.

Stochastic Growth Models II

- Stochastic overlapping generations models: process of takeoff from low growth and to sustained growth:
  - Takeoff: faster growth and also less variable.
  - Key: tradeoff between investment in risky activities and safer activities with lower returns.
  - Early stages:
    - ★ Not enough resources to invest in many activities to achieve diversification.
    - ★ To reduce this risk, invest in low-return safe activities
  - Equilibrium: lengthy period of slow or no growth, and high levels of variability.
  - Takeoff into sustained growth when risky investments are successful for a number of periods.
  - Then, better diversification and risk management, less risk and more investments in higher return activities.

The Brock-Mirman Model I

- With competitive and complete markets, the First and Second Welfare Theorems so equilibrium growth path is identical to the optimal growth path.
- But analysis is more involved and introduces new concepts.
- Economy as baseline neoclassical growth model, but production technology now given by

$$Y(t) = F(K(t), L(t), z(t)), \tag{1}$$

- $z(t)$=stochastic *aggregate* productivity term
- Suppose $z(t)$ follows a monotone Markov chain (as defined in Assumption 16.6) with values in the set $\mathcal{Z} \equiv \{z_1, ..., z_N\}$.
- Many applications assume aggregate production function takes the form $Y(t) = F(K(t), z(t) L(t))$.

The Brock-Mirman Model II

- Assume that the production function F satisfies usual assumptions and define

$$\begin{aligned} y(t) &\equiv \frac{Y(t)}{L(t)} \\ &\equiv f(k(t), z(t)), \end{aligned}$$

- Fraction δ of the existing capital stock depreciates at each date.
- Suppose $z_1, ..., z_N$ are arranged in ascending order and that $j > j'$ implies $f(k, z_j) > f(k, z_{j'})$ for all $k \in \mathbb{R}_+$.
- Thus higher values of the stochastic shock z correspond to greater productivity at *all* capital-labor ratios.
- Representative household with instantaneous utility function $u(c)$ that satisfies the standard assumptions.
- Supplies one unit of labor inelastically, so $K(t)$ and $k(t)$ can be used interchangeably (and no reason to distinguish $C(t)$ from $c(t)$).

The Brock-Mirman Model III

- Consumption and saving decisions at time t are made after observing $z(t)$.
- Sequence version of the expected utility maximization problem of a social planner:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c(t)) \tag{2}$$

subject to

$$k(t+1) = f(k(t), z(t)) + (1-\delta) k(t) - c(t) \text{ and } k(t) \geq 0, \tag{3}$$

with given $k(0) > 0$.

- To characterize the optimal growth path using the sequence problem: define feasible plans, mappings $\tilde{k}[z^t]$ and $\tilde{c}[z^t]$ with $z^t \equiv (z(0), ..., z(t))$.

The Brock-Mirman Model IV

- Instead look at the recursive version:

$$V(k,z) = \max_{k' \in [0, f(k,z)+(1-\delta)k]} \left\{ u\left(f(k,z) + (1-\delta)k - k'\right) + \beta \mathbb{E}\left[V\left(k',\right.\right.\right. \tag{4}$$

Proposition In the stochastic optimal growth problem described above, the value function $V(k,z)$ is uniquely defined, strictly increasing in both of its arguments, strictly concave in k and differentiable in $k > 0$. Moreover, there exists a uniquely defined policy function $\pi(k,z)$ such that the capital stock at date $t+1$ is given by $k(t+1) = \pi(k(t), z(t))$.

- Proof: verifying that Assumptions 16.1-16.6 from the previous chapter are satisfied and apply Theorems.
- To do this, first define $\bar{k}$ such that $\bar{k} = f(\bar{k}, z_N) + (1-\delta)\bar{k}$, and show that starting with $k(0) \in (0, \bar{k})$, the capital-labor ratio will always remain within the compact set $(0, \bar{k})$.

The Brock-Mirman Model V

Proposition In the stochastic optimal growth problem described above, the policy function for next period's capital stock, $\pi(k, z)$, is strictly increasing in both of its arguments.

- Proof:
  - By assumption u is differentiable and from the Proposition above V is differentiable in k.
  - By the same argument as before, $k \in (0, \bar{k})$; thus we are in the interior of the domain of the objective function.
  - Thus, the value function V is differentiable in its first argument and

$$u'\left(f(k,z) + (1-\delta)k - k'\right) - \beta \mathbb{E}\left[V'\left(k', z'\right) \mid z\right] = 0,$$

  - Proposition above: V is strictly concave in k. Thus this can hold when k or z increases only if k' also increases.
  - For example, an increase in k reduces the first-term (because u is strictly concave), hence an increase in k' is necessary to increase the first term and to reduce the second term (by the concavity of V).
  - Argument for increase in z is similar.

The Brock-Mirman Model VI

- Define the policy function for consumption as

$$\pi^c(k,z) \equiv f(k,z) + (1-\delta)k - \pi(k,z),$$

where $\pi(k,z)$ is the optimal policy function for next date's capital stock determined in Proposition above.

- Using this notation, the stochastic Euler equation can be written as

$$u'(\pi^c(k,z)) = \beta\mathbb{E}\left[\left(f'(\pi(k,z),z') + (1-\delta)\right)u'\left(\pi^c\left(\pi(k,z),z'\right)\right)\right] \tag{5}$$

- A different way of expressing this equation makes it both simpler and more intuitive:

$$u'(c(t)) = \beta\mathbb{E}_t\left[p(t+1)\,u'(c(t+1))\right], \tag{6}$$

- $p(t+1)$ is the stochastic marginal product of capital (including undepreciated capital) at date $t+1$.

The Brock-Mirman Model VII

- Also useful for comparison with the competitive equilibrium because $p(t+1)$ corresponds to the stochastic (date $t+1$) dividends paid out by one unit of capital invested at time t.
- Proposition above characterizes form of the value function and policy functions, but:.
  1. Not an analog of the "Turnpike Theorem": does not characterize the long-run behavior of the neoclassical growth model under uncertainty.
  2. Qualitative results about the value and the policy functions, but no comparative static results.
- Stochastic law of motion of the capital-labor ratio:

$$k(t+1) = \pi(k(t), z(t)), \tag{7}$$

The Brock-Mirman Model VIII

- Defines a general Markov process, since before the realization of $z(t)$, $k(t+1)$ is a random variable, with its law of motion governed by the last period's value of $k(t)$ and the realization of $z(t)$.
- If $z(t)$ has a non-degenerate distribution, $k(t)$ does not typically converge to a single value.
- But may hope that it will converge to an *invariant limiting distribution*.
- Markov process (7): starting with any $k(0)$, converges to a unique invariant limiting distribution.
- I.e., when we look at sufficiently faraway horizons, the distribution of k should be independent of $k(0)$.

The Brock-Mirman Model IX

- Moreover, the average value of $k(t)$ in invariant limiting distribution will be the same as the time average of $\{k(t)\}_{t=0}^{T}$ as $T \to \infty$ (stochastic process for the capital stock is "ergodic").
- A "steady-state" equilibrium now corresponds not to specific values but to invariant limiting distributions.
- If $z(t)$ takes values within a sufficiently small set, this limiting invariant distribution would hover around some particular values ("quasi-steady-state" values)
- But in general the range of the limiting distribution could be quite wide.

Example: Brock-Mirman with Closed-form Solution I

- Suppose $u(c) = \log c$, $F(K, L, z) = zK^{\alpha}L^{1-\alpha}$, and $\delta = 1$.
- Again z follows a Markov chain over the set $\mathcal{Z} \equiv \{z_1, ..., z_N\}$, with transition probabilities denoted by $q_{jj'}$.
- Let $k \equiv K/L$. The stochastic Euler equation (5):

$$\frac{1}{zk^{\alpha} - \pi(k, z)} = \beta \mathbb{E}\left[\frac{\alpha z' \pi(k, z)^{\alpha-1}}{z'\pi(k, z)^{\alpha} - \pi(\pi(k, z), z')} \middle| z\right], \tag{8}$$

- Relatively simple functional equation in a single function $\pi(\cdot, \cdot)$.
- Here "guessing and verifying" is handy. Conjecture that

$$\pi(k, z) = B_0 + B_1 zk^{\alpha}.$$

Example: Brock-Mirman with Closed-form Solution II

- Substituting this guess into (8):

$$\frac{1}{(1-B_1)\,zk^{\alpha}-B_0} = \beta\mathbb{E}\left[\frac{\alpha z'\left(B_0+B_1zk^{\alpha}\right)^{\alpha-1}}{z'\left(B_0+B_1zk^{\alpha}\right)^{\alpha}-B_0-B_1z'\left(B_0+B_1zk^{\alpha}\right)^{\alpha}}\middle| z\right]. \tag{9}$$

- This equation cannot be satisfied for any $B_0 \neq 0$.
- Thus imposing $B_0 = 0$ and writing out the expectation explicitly with $z = z_{j'}$, this expression becomes

$$\frac{1}{(1-B_1)\,z_{j'}k^{\alpha}} = \beta\sum_{j=1}^{N} q_{jj'}\frac{\alpha z_j\left(B_1z_{j'}k^{\alpha}\right)^{\alpha-1}}{z_j\left(B_1z_{j'}k^{\alpha}\right)^{\alpha}-B_1z_j\left(B_1z_{j'}k^{\alpha}\right)^{\alpha}}.$$

- Simplifying each term within the summation:

$$\frac{1}{(1-B_1)\,z_{j'}k^{\alpha}} = \beta\sum_{j=1}^{N} q_{jj'}\frac{\alpha}{B_1\left(1-B_1\right)z_{j'}k^{\alpha}}.$$

Example: Brock-Mirman with Closed-form Solution III

- Now taking $z_{j'}$ and k out of the summation and using the fact that, by definition, $\sum_{j=1}^{N} q_{jj'} = 1$, we can cancel the remaining terms and obtain
$$B_1 = \alpha\beta,$$
- Thus irrespective of the exact Markov chain for z, the optimal policy rule is
$$\pi(k, z) = \alpha\beta z k^{\alpha}.$$
- Identical to deterministic case, with z there corresponding to a non-stochastic productivity term.
- Thus stochastic elements have not changed the form of the optimal policy function.
- Same result applies when z follows a general Markov process rather than a Markov chain.

Example: Brock-Mirman with Closed-form Solution IV

- Here can fully analyze the stochastic behavior of the capital-labor ratio and output per capita.
- Stochastic behavior of the capital-labor ratio in this economy is identical to that of the overlapping generations model
- But just one of the few instances of the neoclassical growth model that admit closed-form solutions.
- In particular, if the depreciation rate of the capital stock δ is not equal to 1, the neoclassical growth model under uncertainty does not admit an explicit form characterization.

Equilibrium Growth under Uncertainty I

- Environment identical to that in the previous section, z an aggregate productivity shock affecting all production units and households.
- Arrow-Debreu commodities defined so that goods indexed by different realizations of the history z^t correspond to different commodities.
- Thus economy with a countable infinity of commodities.
- Second Welfare Theorem applies and implies that the optimal growth path characterized in the previous section can be decentralized as a competitive equilibrium
- Moreover, since we are focusing on an economy with a representative household, this allocation is a competitive equilibrium without any redistribution of endowments.
- Justifies the frequent focus on social planner's problems in analyses of stochastic growth models in the literature.

Equilibrium Growth under Uncertainty II

- But explicit characterization of competitive equilibria shows the equivalence, and introduces ideas related to pricing of contingent claims.
- Complete markets: in principle, any commodity, including any contingent claim, can be traded competitively.
- In practice no need to specify or trade all of these commodities; a subset sufficient to provide all necessary trading opportunities.
- Will also show what subsets are typically sufficient.
- Preferences and technology as in previous model: economy admits representative household and production side can be represented by a representative firm.
- Household maximize the objective function given by (2) subject to the lifetime budget constraint (written from the viewpoint of time $t = 0$).

Equilibrium Growth under Uncertainty III

- No loss of generality in considering the viewpoint of time $t = 0$ relative to formulating with sequential trading constraints.
- $\mathcal{Z}^t$=set of all possible histories of the stochastic variable z^t up to date t .
- $\mathcal{Z}^\infty$=set of infinite histories.
- $z^t \in \mathcal{Z}^\infty$=a possible history of length t.
- $p_0\left[z^t\right]$=price of the unique final good at time t in terms of the final good of date 0 following a history z^t,
- $c\left[z^t\right]$ and $w_0\left[z^t\right]$ similarly defined.
- Household's lifetime budget constraint:

$$\sum_{t=0}^{\infty}\sum_{z^t\in\mathcal{Z}^\infty} p_0\left[z^t\right] c\left[z^t\right] \leq \sum_{t=0}^{\infty}\sum_{z^t\in\mathcal{Z}^\infty} w_0\left[z^t\right] + k\left(0\right). \qquad (10)$$

Equilibrium Growth under Uncertainty IV

- No expectations:
  - Complete markets: all trades at $t = 0$ at price vector for all Arrow-Debreu commodities.
  - Household buys claims to different "contingent" consumption bundles; i.e. conditioned on z^t.
- Left-hand=total expenditure taking the prices of all possible claims as given.
- Right-hand side=labor earnings and value of initial capital stock per capita.
- Right-hand side of (10) could also include profits accruing to the individuals, but constant returns and competitive markets implies that equilibrium profits will be equal to 0.
- Objective function at time $t = 0$:

$$\sum_{t=0}^{\infty} \beta^t \sum_{z^t \in \mathcal{Z}^\infty} q\left[z^t \mid z^0\right] u\left(c\left[z^t\right]\right), \tag{11}$$

Equilibrium Growth under Uncertainty V

- $q\left[z^t \mid z^0\right]$=probability at time 0 that the history z^t will be realized at time t.
- Sequence problem of maximizing (11) subject to (10). Assuming interior solution, first-order conditions: is

$$\beta^t q\left[z^t \mid z^0\right] u'\left(c\left[z^t\right]\right) = \lambda p_0\left[z^t\right] \tag{12}$$

  for all t and all z^t.
- λ is the Lagrange multiplier on (10) corresponding to the marginal utility of income at date $t = 0$
- Combining two different date t histories z^t and $\hat{z}^t$:

$$\frac{u'\left(c\left[\hat{z}^t\right]\right)}{u'\left(c\left[z^t\right]\right)} = \frac{p_0\left[\hat{z}^t\right] / q\left[\hat{z}^t \mid z^0\right]}{p_0\left[z^t\right] / q\left[z^t \mid z^0\right]},$$

- Right-hand side=relative price of consumption claims conditional on histories z^t and $\hat{z}^t$

Equilibrium Growth under Uncertainty VI

- Combining for histories z^t and z^{t+1} such that $z^{t+1} = (z^t, z(t+1))$:

$$\frac{\beta u'\left(c\left[z^{t+1}\right]\right)}{u'\left(c\left[z^t\right]\right)} = \frac{p_0\left[z^{t+1}\right] / q\left[z^{t+1} \mid z^0\right]}{p_0\left[z^t\right] / q\left[z^t \mid z^0\right]},$$

- Right-hand side=contingent interest rate between date t and $t+1$ conditional on z^t (and contingent on the realization of z^{t+1}).
- To characterize equilibrium need prices $p_0\left[z^t\right]$, from the profit maximization problem of firm.
- $R_0\left[z^t\right]$=price of one unit of capital after the state z^t
- $K^e\left[z^t\right]$ and $L\left[z^t\right]$ =capital and labor employment levels of the representative firm after history z^t.
- Value of the firm:

$$\sum_{t=0}^{\infty} \beta^t \sum_{z^t \in \mathcal{Z}^\infty} \left\{ \begin{array}{c} p_0\left[z^t\right]\left(F\left(K^e\left[z^t\right], L\left[z^t\right], z(t)\right) + (1-\delta) K^e\left[z^t\right]\right) \\ -R_0\left[z^t\right] K^e\left[z^t\right] - w_0\left[z^t\right] L\left[z^t\right] \end{array} \right\}$$

Equilibrium Growth under Uncertainty VII

- Profit maximization implies:

$$p_0\left[z^t\right]\left(\frac{\partial F\left(K^e\left[z^t\right],L\left[z^t\right],z\left(t\right)\right)}{\partial K^e}+\left(1-\delta\right)\right) = R_0\left[z^t\right]$$
$$p_0\left[z^t\right]\frac{\partial F\left(K^e\left[z^t\right],L\left[z^t\right],z\left(t\right)\right)}{\partial L} = w_0\left[z^t\right].$$

- Using constant returns to scale and expressing everything in per capita terms:

$$p_0\left[z^t\right]\left(f'\left(k^e\left[z^t\right],z\left(t\right)\right)+\left(1-\delta\right)\right) = R_0\left[z^t\right] \quad (13)$$
$$p_0\left[z^t\right]\left(f\left(k^e\left[z^t\right],z\left(t\right)\right)-k^e\left[z^t\right]f'\left(k^e\left[z^t\right],z\left(t\right)\right)\right) = w_0\left[z^t\right]$$

- Relation between prices and marginal productivity of factors.
- But (13) also stating that $R_0\left[z^t\right]$ is equal to the value of dividends paid out by a unit of capital inclusive of undepreciated capital.

Equilibrium Growth under Uncertainty VIII

- Alternative, equivalent, way of formulating competitive equilibrium and writing (13) is to assume that capital goods are *rented*.
- Labor market clearing:

$$L\left[z^t\right] = 1 \text{ for all } z^t. \tag{14}$$

- Production after history z^t is $f\left(k^e\left[z^t\right], z\left(t\right)\right) + \left(1-\delta\right) k^e\left[z^t\right]$, divided between consumption $c\left[z^t\right]$ and savings $s\left[z^t\right]$.
- Capital used at time $t+1$ (after history z^{t+1}) must be equal to $s\left[z^t\right]$.
- Market clearing for capital implies that for any $z^{t+1} = \left(z^t, z\left(t+1\right)\right)$,

$$k^e\left[z^{t+1}\right] = s\left[z^t\right], \tag{15}$$

- Capital market clearing condition:

$$c\left[z^t\right] + s\left[z^t\right] \leq f\left(s\left[z^{t-1}\right], z\left(t\right)\right) + \left(1-\delta\right) s\left[z^{t-1}\right] \tag{16}$$

for any $z^{t+1} = \left(z^t, z\left(t+1\right)\right)$.

Equilibrium Growth under Uncertainty IX

- Capital market clearing condition also implies *no arbitrage* condition linking $R_0\left[z^{t+1}\right]$ to $p_0\left[z^t\right]$.
- Consider the following riskless arbitrage:
  - Buy one unit of the final good after z^t to be used as capital at time $t+1$ and simultaneously sell claims on capital goods for each $z^{t+1} = \left(z^t, z\left(t+1\right)\right)$.
  - No risk, since unit of final good bought after history z^t will cover the obligation to pay capital good after any $z^{t+1} = \left(z^t, z\left(t+1\right)\right)$.
- Implies the no arbitrage condition

$$p_0\left[z^t\right] = \sum_{z(t+1)\in\mathcal{Z}} R_0\left[\left(z^t, z\left(t+1\right)\right)\right]. \tag{17}$$

Equilibrium Growth under Uncertainty X

- Competitive equilibrium: $\left\{c\left[z^t\right], s\left[z^t\right], k^e\left[z^{t+1}\right]\right\}_{z^t \in \mathcal{Z}^t}$, and $\left\{p_0\left[z^t\right], R_0\left[z^t\right], w_0\left[z^t\right]\right\}_{z^t \in \mathcal{Z}^t}$, such that households maximize utility (i.e., satisfy (12)), firms maximize profits (i.e., satisfy (13) and (17)), and labor and capital markets clear (i.e., (14), (15), and (16) are satisfied).
- Substitute from (13) and (17) into (12) and rearrange:

$$u'\left(c\left[z^t\right]\right) = \sum_{z(t+1) \in \mathcal{Z}} \frac{\lambda p_0\left[z^{t+1}\right]}{\beta^t q\left[z^t \mid z^0\right]}\left(f'\left(k\left[z^{t+1}\right], z(t+1)\right) + (1-\delta)\right) \tag{18}$$

- Next using (12) for $t+1$:

$$\begin{aligned} \beta u'\left(c\left[z^{t+1}\right]\right) &= \frac{\lambda p_0\left[z^{t+1}\right]}{\beta^t q\left[z^{t+1} \mid z^0\right]} \\ &= \frac{\lambda p_0\left[z^{t+1}\right]}{\beta^t q\left[z^{t+1} \mid z^t\right] q\left[z^t \mid z^0\right]}, \end{aligned}$$

Equilibrium Growth under Uncertainty XI

- Second line uses the law of iterated expectations, $q\left[z^{t+1} \mid z^0\right] \equiv q\left[z^{t+1} \mid z^t\right] q\left[z^t \mid z^0\right]$.
- Substituting into (18), we obtain

$$\begin{aligned} & u'\left(c\left[z^t\right]\right) \\ = & \beta \sum_{z(t+1)\in\mathcal{Z}} q\left[z^{t+1} \mid z^t\right] u'\left(c\left[z^{t+1}\right]\right) \left(\begin{array}{c} f'\left(k\left[z^{t+1}\right], z(t+1)\right) \\ +(1-\delta) \end{array} \right) \\ = & \beta \mathbb{E}\left[u'\left(c\left[z^{t+1}\right]\right)\left(f'\left(k\left[z^{t+1}\right], z(t+1)\right)+(1-\delta)\right) \mid z^t\right], \end{aligned}$$

- Identical to (6).

Proposition In the above-described economy, optimal and competitive growth path coincide.

Equilibrium Growth under Uncertainty XII

- Equilibrium problem in its equivalent form with sequential trading rather than all trades taking place at the initial date $t = 0$.
- Write the budget constraint of the representative household somewhat differently.
- Normalize the price of the final good at each date to 1.
- $a\left[z^t\right]$s=Basic Arrow securities that pay out only in specific states on nature.
- $\left\{a\left[z^t\right]\right\}_{z^t \in \mathcal{Z}^t}$=set of contingent claims that the household has purchased that will pay $a\left[z^t\right]$ units of the final good at date t when history z^t is realized.
- Price of claim to one unit of $a\left[z^t\right]$ at time $t-1$ after history z^{t-1} denoted by $\bar{p}\left[z\left(t\right) \mid z^{t-1}\right]$, where $z^t = \left(z^{t-1}, z\left(t\right)\right)$.
- Amount of these claims purchased by the household is denoted by $a\left[\left(z^{t-1}, z\left(t\right)\right)\right]$.

Equilibrium Growth under Uncertainty XIII

- Thus flow budget constraint of the household:

$$c\left[z^t\right]+\sum_{z(t+1)\in\mathcal{Z}}\bar{p}\left[z\left(t+1\right)\mid z^t\right]a\left[\left(z^{t-1},z\left(t\right)\right)\right]\leq w\left[z^t\right]+a\left[z^t\right],$$

- $w\left[z^t\right]$=equilibrium wage rate after history z^t in terms of final goods dated t.
- Let a denote the current asset holdings of the household (realization of current assets after some z^t has been realized).
- Then flow budget constraint of the household can be written as

$$c+\sum_{z'\in\mathcal{Z}}\bar{p}\left[z'\mid z\right]a'\left[z'\mid z\right]\leq w+a,$$

- Function $\bar{p}\left[z'\mid z\right]$=prices of contingent claims (for next date's state z' given current state z).
- $a'\left[z'\mid z\right]$=corresponding asset holdings.

Equilibrium Growth under Uncertainty XIV

- $V(a,z)$=value function of the household.
- Choice variables: $a'[z' \mid z]$ and consumption today, $c[a,z]$.
- $q[z' \mid z]$ =probability that next period's stochastic variable will be equal to z' conditional on today's value being z.
- Then taking the sequence of equilibrium prices $\bar{p}$ as given, the value function of the representative household:

$$V(a,z) = \sup_{\{a'[z'|z]\}_{z' \in \mathcal{Z}}} \left\{ \begin{array}{c} u\left(a + w - \sum_{z' \in \mathcal{Z}} \bar{p}[z' \mid z]\, a'[z' \mid z]\right) \\ +\beta \sum_{z' \in \mathcal{Z}} q[z' \mid z]\, V\left(a'[z' \mid z], z'\right) \end{array} \right\}. \tag{19}$$

- All Theorems on the value function can again be applied to this value function.

Equilibrium Growth under Uncertainty XV

- First-order condition for current consumption:

$$\bar{p}\left[z' \mid z\right] u'\left(c\left[a, z\right]\right) = \beta q\left[z' \mid z\right] \frac{\partial V\left(a'\left[z' \mid z\right], z'\right)}{\partial a}$$

  for any $z' \in \mathcal{Z}$.

- Capital market clearing:

$$a'\left[z' \mid z\right] = a'\left[z\right],$$

- Thus in the aggregate the same amount of assets will be present in all states at the next date.
- Thus first-order condition for consumption can be alternatively written as

$$\bar{p}\left[z' \mid z\right] u'\left(c\left[a, z\right]\right) = \beta q\left[z' \mid z\right] \frac{\partial V\left(a'\left[z\right], z'\right)}{\partial a}. \tag{20}$$

Equilibrium Growth under Uncertainty XVI

- No arbitrage condition implies

$$\sum_{z' \in \mathcal{Z}} \bar{p}\left[z' \mid z\right] R\left[z' \mid z\right] = 1, \tag{21}$$

where $R\left[z' \mid z\right]$ is the price of capital goods when the current state is z' and last period's state was z.

- Intuition:
  - Cost of one unit of the final good now, 1, has to be equal to return of carrying it to the next period and selling it as a capital good then.
  - Summing over all possible states z' tomorrow must have total return of 1 to ensure no arbitrage
- Combine (20) with the envelope condition

$$\frac{\partial V(a,z)}{\partial a} = u'\left(c\left[a,z\right]\right),$$

Equilibrium Growth under Uncertainty XVII

- Multiply both sides of (20) by $R\left[z' \mid z\right]$ and sum over all $z' \in \mathcal{Z}$ to obtain the first-order condition of the household as

$$\begin{aligned} u'\left(c\left[a,z\right]\right) &= \beta \sum_{z' \in \mathcal{Z}} q\left[z' \mid z\right] R\left[z' \mid z\right] u'\left(c\left[a',z'\right]\right). \\ &= \beta \mathbb{E}\left[R\left[z' \mid z\right] u'\left(c\left[a',z'\right]\right) \mid z\right]. \end{aligned}$$

- Market clearing condition for capital, combined with the fact that the only asset in the economy is capital, implies:

$$a = k.$$

- Therefore first-order condition can be written as

$$u'\left(c\left[k,z\right]\right) = \beta \mathbb{E}\left[R\left[z' \mid z\right] u'\left(c\left[k',z'\right]\right) \mid z\right]$$

which is identical to (6).

Equilibrium Growth under Uncertainty XVIII

- Again shows the equivalence between the social planner's problem and the competitive equilibrium path.
- Social planner's problem (the optimal growth problem) is considerably simpler, characterizes the equilibrium path of all the real variables and various different prices are also straightforward to obtain from the Lagrange multiplier.

Application: Real Business Cycle Models I

- Real Business Cycle (RBC): one of the most active research areas in the 1990s and also one of the most controversial.
- Conceptual simplicity and relative success in matching certain moments of employment, consumption and investment fluctuations vs. the absence of monetary factors and demand shocks.
- But exposition of RBC model useful for two purposes:
  1. one of the most important applications of the neoclassical growth model under uncertainty
  2. new insights from introduction of labor supply choices into the neoclassical growth model under uncertainty generates.
- Only difference is instantaneous utility function of the representative household now takes the form

$$u(C, L),$$

Application: Real Business Cycle Models II

- u is jointly concave and continuously differentiable in both of its arguments and strictly increasing in C and strictly decreasing in L.
- Also assume that L has to lie in some convex compact set $[0, \bar{L}]$.
- Focus on the optimal growth formulation: maximization of

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t u\left(C\left(t\right), L\left(t\right)\right)$$

subject to the flow resource constraint

$$K\left(t+1\right) \leq F\left(K\left(t\right), L\left(t\right), z\left(t\right)\right) + \left(1-\delta\right) K\left(t\right) - C\left(t\right).$$

- $z\left(t\right)$ again represents an aggregate productivity shock following a monotone Markov chain.

Application: Real Business Cycle Models III

- Social planner's problem can be written recursively as

$$V(K,z) = \sup_{\substack{L\in[0,\bar{L}] \\ K'\in[0,F(K,L,z)+(1-\delta)K]}} \left\{ \begin{array}{c} u\left(F\left(K,L,z\right)+\left(1-\delta\right)K-K',L\right) \\ +\beta\mathbb{E}\left[V\left(K',z'\right)\mid z\right] \end{array} \right. \tag{22}$$

Proposition The value function $V(K,z)$ defined in (22) is continuous and strictly concave in K, strictly increasing in K and z, and differentiable in $K>0$. There exist uniquely defined policy functions $\pi^k(K,z)$ and $\pi^l(K,z)$ that determine the level of capital stock chosen for next period and the level of labor supply as a function of the current capital stock K and the stochastic variable z.

- Assuming an interior solution, relevant prices can be obtained from the appropriate multipliers and the standard first-order conditions characterize the form of the equilibrium.

Application: Real Business Cycle Models IV

- Define the policy function for consumption:

$$\pi^c(K,z) \equiv F\left(K, \pi^l(K,z), z\right) + (1-\delta)K - \pi^k(K,z),$$

- Key first order conditions (write π^J short for $\pi^J(K,z)$, $J = c, l, k$):

$$\begin{aligned} u_c\left(\pi^c, \pi^l\right) &= \beta \mathbb{E}\left[R\left(\pi^k, z'\right) u_c\left(\pi^c\left(\pi^k, z'\right), \pi^l\left(\pi^k, z'\right)\right.\right. \\ w(K,z)\, u_c\left(\pi^c, \pi^l\right) &= -u_l\left(\pi^c, \pi^l\right). \end{aligned}$$

where

$$\begin{aligned} R(K,z) &= F_k(K,z) + (1-\delta) \\ w(K,z) &= F_l(K,z) \end{aligned}$$

Application: Real Business Cycle Models V

- First condition in (23) is essentially identical to (5), whereas the second is a static condition determining the level of equilibrium (or optimal) labor supply.
- Second condition does not feature expectations: conditional on the current value K and the current z.
- Analysis of macroeconomic fluctuations: period in which z is low.
  - If no offsetting change in labor supply, "recession".
  - Under standard assumptions, $w(K, z)$ and labor supply decline: low employment and output.
  - If Markov process for z exhibits persistence, *persistent fluctuations*.
  - Provided $F(K, L, z)$ is such that low output is associated with low marginal product of capital, expectation of future low output will typically reduce savings and thus future levels of capital stock
    - ★ This effect depends also on form of utility function (consumption smoothing and income and substitution effects).

Application: Real Business Cycle Models VI

- Thus model may generate some of the major qualitative features of macroeconomic fluctuations.
- RBC literature argues it generates the *major* quantitative features such as correlations between output, investment, and employment.
- Debate on whether:
  1. the model did indeed match these moments in the data;
  2. these were the right empirical objects to look at; and
  3. focusing on exogenous changes in aggregate productivity sidestep why there are shocks.
- RBC debate is not as active today as it was in the 1990s, but not a complete agreement.

Example: RBC model with closed-form solution I

- $u(C, L) = \log C - \gamma L$, $F(K, L, z) = zK^{\alpha}L^{1-\alpha}$, and $\delta = 1$.
- z follows a monotone Markov chain over the set $\mathcal{Z} \equiv \{z_1, ..., z_N\}$, with transition probabilities denoted by $q_{jj'}$.
- Conjecture that
$$\pi^k(K, z) = BzK^{\alpha}L^{1-\alpha}.$$
- Then with these functional forms, the stochastic Euler equation for consumption (23) implies
$$\frac{1}{(1-B)\,zK^{\alpha}L^{1-\alpha}} = \beta\mathbb{E}\left[\frac{\alpha z'\left(BzK^{\alpha}L^{1-\alpha}\right)^{-(1-\alpha)}(L')^{1-\alpha}}{(1-B)\,z'\left(BzK^{\alpha}L^{1-\alpha}\right)^{\alpha}(L')^{1-\alpha}}\middle| z\right],$$
where L' denotes next period's labor supply.

Example: RBC model with closed-form solution II

- Canceling constants within the expectations and taking terms that do not involve z' out of the expectations:

$$\frac{1}{zK^{\alpha}L^{1-\alpha}} = \beta\mathbb{E}\left[\alpha\left(BzK^{\alpha}L^{1-\alpha}\right)^{-1}\middle| z\right],$$

which yields

$$B = \alpha\beta.$$

- Resulting policy function for the capital stock is therefore

$$\pi^{k}(K,z) = \alpha\beta zK^{\alpha}L^{1-\alpha},$$

which is identical to that in Example before.

- Next, considering the first-order condition for labor:

$$\frac{(1-\alpha)\,zK^{\alpha}L^{-\alpha}}{(1-B)\,zK^{\alpha}L^{1-\alpha}} = \gamma.$$

Example: RBC model with closed-form solution III

- The resulting policy function for labor as

$$\pi^{l}(K, z) = \frac{(1-\alpha)}{\gamma(1-\alpha\beta)},$$

- Labor supply is constant: with the preferences as specified here, the income and the substitution effects cancel out, increase in wages induced by a change in aggregate productivity has no effect on labor supply.
- Same result obtains whenever the utility function takes the form of $U(C, L) = \log C + h(L)$ for some decreasing and concave function h.
- Replicates the covariation in output and investment, but does not generate labor fluctuations.

Growth with Incomplete Markets: The Bewley Model I

- Economy is populated by a continuum 1 of households and the set of households is denoted by $\mathcal{H}$.
- Each household has preferences given by (2) and supplies labor inelastically.
- Suppose also that the second derivative of this utility function, $u''(\cdot)$, is increasing.
- Efficiency units that each household supplies vary over time.
- In particular, each household $h \in \mathcal{H}$ has a labor endowment of $z^h(t)$ at time t, where $z^h(t)$ is an independent draw from the set $\mathcal{Z} \equiv [z_{\min}, z_{\max}]$, where $0 < z_{\min} < z_{\max} < \infty$.
- Labor endowment of each household is identically and independently distributed with distribution function $G(z)$ defined over $[z_{\min}, z_{\max}]$.
- Production side is the same as in the canonical neoclassical growth model under certainty.

Growth with Incomplete Markets: The Bewley Model II

- Only difference is $L(t)$ is now the sum (integral) of the heterogeneous labor endowments of all the agents:

$$L(t) = \int_{h \in \mathcal{H}} z^h(t)\, dh.$$

- Appealing to a law of large numbers type argument, we assume that $L(t)$ is constant at each date and we normalize it to 1.
- Thus output per capita in the economy can be expressed as

$$y(t) = f(k(t)),$$

  with $k(t) = K(t)$.
- No longer any aggregate productivity shock; only uncertainty at the individual level (i.e., it is idiosyncratic).
- Individual households will experience fluctuations in their labor income and consumption, but can imagine a *stationary* equilibrium in which aggregates are constant over time.

Growth with Incomplete Markets: The Bewley Model III

- Focus on such a stationary equilibrium: wage rate w and the gross rate of return on capital R will be constant .
- First take these prices as given and look at the behavior of a typical household $h \in \mathcal{H}$
- Maximize (2) subject to the flow budget constraint

$$a^h(t+1) \leq Ra^h(t) + wz^h(t) - c^h(t)$$

  for all t, where $a^h(t)$ is the asset holding of household $h \in \mathcal{H}$ at time t.
- Consumption cannot be negative, so $c^h(t) \geq 0$.

Growth with Incomplete Markets: The Bewley Model IV

- Requirement that individual should satisfy its lifetime budget constraint in all histories imposes the endogenous borrowing constraint:

$$\begin{aligned} a^h(t) &\geq -\frac{z_{\min}}{R-1} \\ &\equiv -b, \end{aligned}$$

for all t.

- Maximization problem of household $h \in \mathcal{H}$ recursively:

$$V^h(a,z) = \sup_{a' \in [-b, Ra+wz]} \left\{ u\left(Ra + wz - a'\right) + \beta \mathbb{E}\left[V^h\left(a', z'\right) \mid z\right] \right\}. \tag{24}$$

Growth with Incomplete Markets: The Bewley Model V

Proposition The value function $V^h(a, z)$ defined in (24) is uniquely defined, continuous and strictly concave in a, strictly increasing in a and z, and differentiable in $a \in (-b, Ra + wz)$. Moreover, the policy function that determines next period's asset holding $\pi(a, z)$ is uniquely defined and continuous in a.

Proposition The policy function $\pi(a, z)$ derived in Proposition **??** is strictly increasing in a and z.

- Total amount of capital stock in the economy=asset holdings of all households in the economy, thus in a stationary equilibrium:

$$\begin{aligned} k(t+1) &= \int_{h \in \mathcal{H}} a^h(t)\, dh \\ &= \int_{h \in \mathcal{H}} \pi\left(a^h(t), z^h(t)\right) dh. \end{aligned}$$

Growth with Incomplete Markets: The Bewley Model VI

- Integrates over all households taking their asset holdings and the realization of their stochastic shock as given.
- Both the average of current asset holdings and also the average of tomorrow's asset holdings must be equal by the definition of a stationary equilibrium.
- Recall policy function $a' = \pi(a, z)$ defines a general Markov process: under fairly weak it will admit a unique invariant distribution.
- If not economy could have multiple stationary equilibria or even there might be problems of non-existence.
- Ignore this complication and assume the existence of a unique invariant distribution, $\Gamma(a)$, so stationary equilibrium capital-labor ratio is:

$$k^* = \int \int \pi(a, z)\, d\Gamma(a)\, dG(z),$$

which uses the fact that z is distributed identically and independently across households and over time.

Growth with Incomplete Markets: The Bewley Model VII

- Turning to the production side:

$$\begin{aligned} R &= f'(k^*) + (1-\delta) \\ w &= f(k^*) - k^* f'(k^*). \end{aligned}$$

- Recall neoclassical growth model with complete markets and no uncertainty implies unique steady state in which $\beta R = 1$, i.e.,

$$f'(k^{**}) = \beta^{-1} - (1-\delta), \tag{25}$$

  where k^{**} refers to the capital-labor ratio of the neoclassical growth model under certainty.
- In Bewley economy this is no longer true.

Growth with Incomplete Markets: The Bewley Model VIII

Proposition In any stationary equilibrium of the Bewley economy, we have that the stationary equilibrium capital-labor ratio k^* is such that

$$f'(k^*) < \beta^{-1} - (1 - \delta) \tag{26}$$

and

$$k^* > k^{**}, \tag{27}$$

where k^{**} is the capital-labor ratio of the neoclassical growth model under certainty.

- **Sketch of proof:**
  - Suppose $f'(k^*) \geq \beta^{-1} - (1 - \delta)$.
  - Then each household's expected consumption is strictly increasing.
  - This implies that average consumption in the population, which is deterministic, is strictly increasing and would tend to infinity.
  - This is not possible since aggregate resources must always be finite.
  - This establishes (26).
  - Given this result, (27) immediately follows from (25) and from the strict concavity of $f(\cdot)$.

Growth with Incomplete Markets: The Bewley Model IX

- Interest rate is "depressed" relative to the neoclassical growth model with certainty because each household has an additional self-insurance (or precautionary) incentive to save.
- These additional savings increase the capital-labor ratio and reduce the equilibrium interest rate.
- Two features, potential shortcomings, are worth noting:
  1. Inefficiency from overaccumulation of capital unlikely to be important for explaining income per capita differences across countries.
     - ★ model is not interesting because of this but as an illustration of stationary equilibrium in which aggregates are constant while individual households have uncertain and fluctuating consumption and income profiles.
  2. Incomplete markets assumption in this model may be extreme.

The Overlapping Generations Model with Uncertainty I

- Time is discrete and runs to infinity.
- Each individual lives for two periods.
- Suppose utility of a household in generation t is given by

$$U(t) = \log c_1(t) + \beta \log c_2(t+1). \tag{28}$$

- There is a constant rate of population growth equal to n, so that

$$L(t) = (1+n)^t L(0), \tag{29}$$

  where $L(0)$ is the size of the first generation.
- Total output at time t is given by

$$Y(t) = z(t) K(t)^{\alpha} L(t)^{1-\alpha}.$$

The Overlapping Generations Model with Uncertainty II

- Expressing this in per capita terms

$$y(t) = z(t)\, k(t)^{\alpha}.$$

- Capital depreciates fully, i.e., $\delta = 1$.
- Factor prices clearly only depend on the current values of z and the capital-labor ratio k, and can be expressed as

$$\begin{aligned} R(k,z) &= \alpha z k^{\alpha-1} \\ w(k,z) &= (1-\alpha)\, z k^{\alpha}. \end{aligned} \tag{30}$$

- The consumption Euler equation for an individual of generation t, then can be expressed as

$$\begin{aligned} \frac{c_2(t+1)}{c_1(t)} &= \beta R(t+1) \\ &= \beta R(k,z), \end{aligned}$$

with $R(k,z)$ given by (30).

The Overlapping Generations Model with Uncertainty III

- The total amount of savings at time t is then given by $s(t) = s(k(t), z(t))$ such that

$$s(k, z) = \frac{\beta}{1+\beta} w(k, z), \tag{31}$$

- As in the canonical overlapping generations model and baseline Solow growth model corresponds to a constant savings rate now equal to $\beta/(1+\beta)$.
- Combining (31) with (29) and the fact that $\delta = 1$:

$$\begin{aligned} k(t+1) &= \pi(k, z) \\ &= \frac{s(k, z)}{(1+n)} \\ &= \frac{\beta(1-\alpha) z k^{\alpha}}{(1+n)(1+\beta)}. \end{aligned} \tag{32}$$

The Overlapping Generations Model with Uncertainty IV

- Clearly, if $z = \bar{z}$, this equation would have a unique steady state with capital-labor ratio given by

$$k^* = \left[\frac{\beta (1-\alpha) \bar{z}}{(1+n)(1+\beta)} \right]^{\frac{1}{1-\alpha}}. \tag{33}$$

- However, when z has a non-degenerates distribution, (32) defines a stochastic first-order difference equation.
- As in the neoclassical growth model under uncertainty, the long-run equilibrium of this model will correspond to an invariant distribution of the capital stock.
- Since (32) is very tractable, we can obtain more insights about the behavior of the economy with a diagrammatic analysis.
- Suppose z is distributed between $[z_{\min}, z_{\max}]$: the behavior of the economy can be analyzed by plotting *the stochastic correspondence* associated with (32).

The Overlapping Generations Model with Uncertainty V

- Stochastic correspondence plots the entire range of possible values of $k(t+1)$ for a given value of $k(t)$.
- Upper thick curve corresponds to the realization of $z_{\max}$, and lower to the realization of $z_{\min}$.
- Dotted curve in the middle is for $z = \bar{z}$.
- Curves for both $z_{\min}$ and $z_{\max}$ start above the 45^0 line, a consequence of the Inada condition implied by the Cobb-Douglas technology
- Particular sample path of capital-labor ratio:
  - starting with $k(0)$, economy first receives a fairly favorable productivity shock moving to $k(1)$.
  - another moderately favorable productivity realization and the capital-labor ratio increases to $k(2)$.
  - in the following period realization of the stochastic variable is quite bad and the capital-labor ratio and thus output per capita decline sharply.

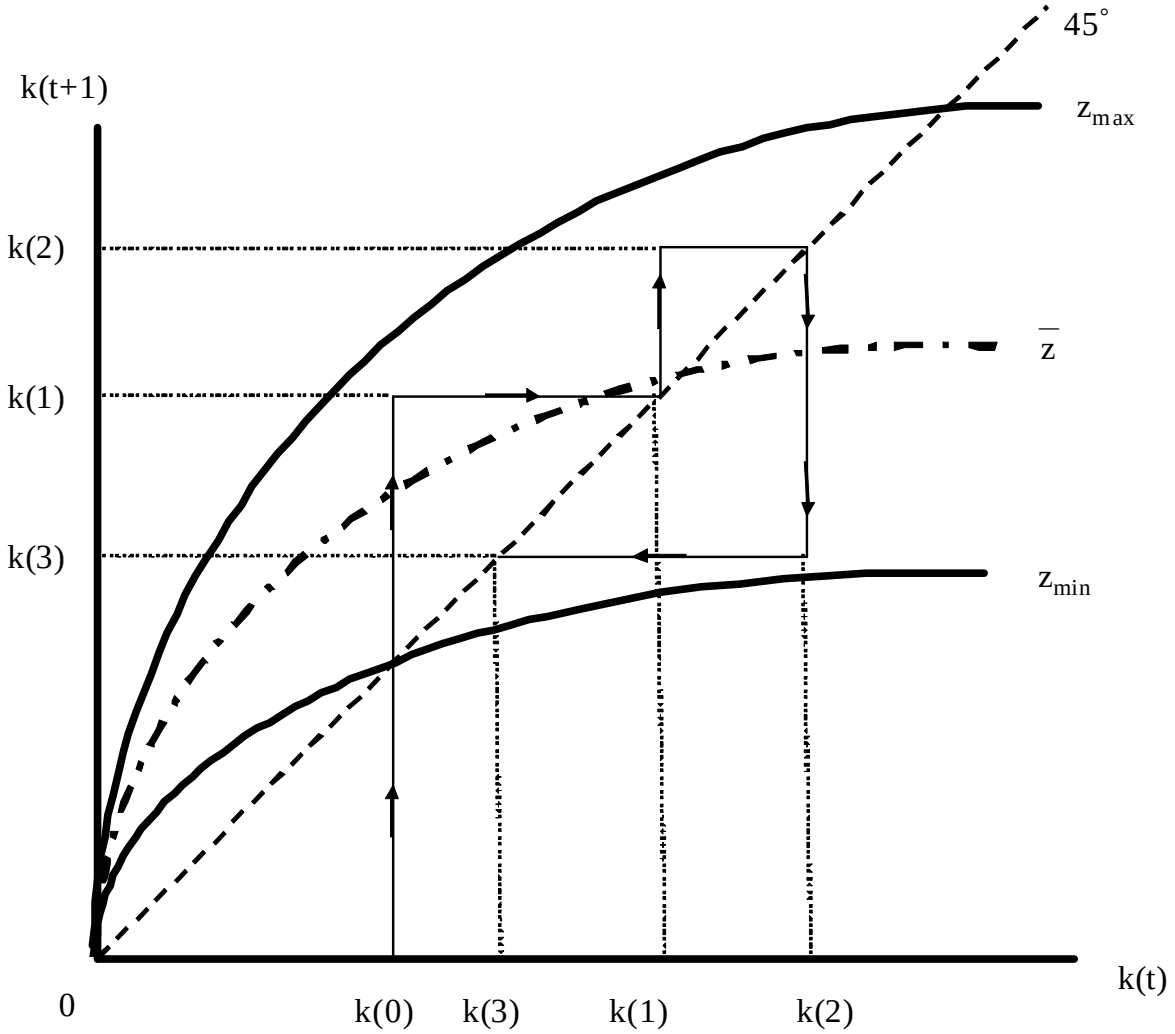

Figure: Stochastic correspondence of the overlapping generations model. Any value for next period's capital-labor ratio within curves $z_{\min}$ and $z_{\max}$ has positive probability. The path $k(0) \rightarrow k(1) \rightarrow k(2) \rightarrow k(3)$ illustrates a particular realization of the stochastic path.

Risk, Diversification and Growth I

- Economic growth over past few thousand years much less "orderly" than implied by models with balanced growth and well-behaved transitional dynamics.

  "The advance occurred very slowly over a long period and was broken by sharp recessions. The right road was reached and thereafter never abandoned, only during the eighteenth century, and then only by a few privileged countries. Thus, before 1750 or even 1800 the march of progress could still be affected by unexpected events, even disasters." F. Braudel (1973, p. xi).

Risk, Diversification and Growth II

- In the model here these patterns arise endogenously:
  - Extent to which can diversify risks by investing in imperfectly correlated activities is limited by the amount of capital.
  - Greater variability and risk at early stages of development; risks significantly reduced after economy "takes off."
  - Desire to avoid risk makes individuals invest in lower return less risky activities during early stages:
    - ★ growth rate endogenously limited
  - Economic development goes hand-in-hand with financial development: availability of capital enables better risk sharing through asset markets.
- Endogenously incomplete markets:
  - Price-taking behavior not sufficient to guarantee Pareto optimality.
  - Inefficiency form is interesting both on substantive and methodological grounds.

The Environment I

- Overlapping generations economy. Each generation lives for two periods.
- No population growth and the size of each generation is normalized to 1.
- Production sector consists of two sectors.
- First sector produces final goods with the Cobb-Douglas production function

$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha}, \tag{34}$$

- Capital depreciates fully after use (i.e., $\delta = 1$).
- Second sector transforms savings at time $t-1$ into capital to be used for production at time t.
- This sector consists of a continuum $[0,1]$ of intermediates, and stochastic elements only affect this sector.

The Environment II

- Represent possible states of nature also with the unit interval: intermediate sector $j \in [0, 1]$ pays a positive return only in state j and nothing in any other state.
- Thus investing in a sector is equivalent to buying a Basic Arrow Security that only pays in one state of nature.
- Since there is a continuum of sectors, the probability that a single sector will have positive payoff is 0.
- But if an individual invests in some subset J of $[0, 1]$, there will be positive returns with probability equal to the (Lebesgue) measure on the set J.
- Thus each intermediate sector is a risky activity but can diversify risks by investing in multiple sectors.
- If invest in all of the sectors, positive returns with probability 1.
- But economic interactions non-trivial because investing in all sectors not be possible at every date because of potential *nonconvexities*.

The Environment III

- Assume each sector has a *minimum size requirement*, $M(j)$: positive returns only if aggregate investment in sector exceeds $M(j)$.
- Let $I(j,t)$ be aggregate investment in intermediate sector j at time t.
- Assume this investment will generate date $t+1$ capital equal to $QI(j,t)$ if state j is realized *and* $I(j,t) \geq M(j)$, and nothing otherwise.
- Aggregate investment exceeding minimum size requirement is thus *necessary* for any positive returns.
- Assume also a safe intermediate sector which transforms one unit of savings at date t into q units of date $t+1$ capital.
- Key assumption is:

$$q < Q, \tag{35}$$

The Environment IV

- Adopt a simple distribution of minimum size requirements by intermediate sector:

$$M(j) = \max\left\{0, \frac{D}{(1-\gamma)}(j-\gamma)\right\}. \tag{36}$$

- Thus intermediate sectors $j \leq \gamma$ have no minimum size requirement.
- The figure shows the minimum size requirements with the thick line.

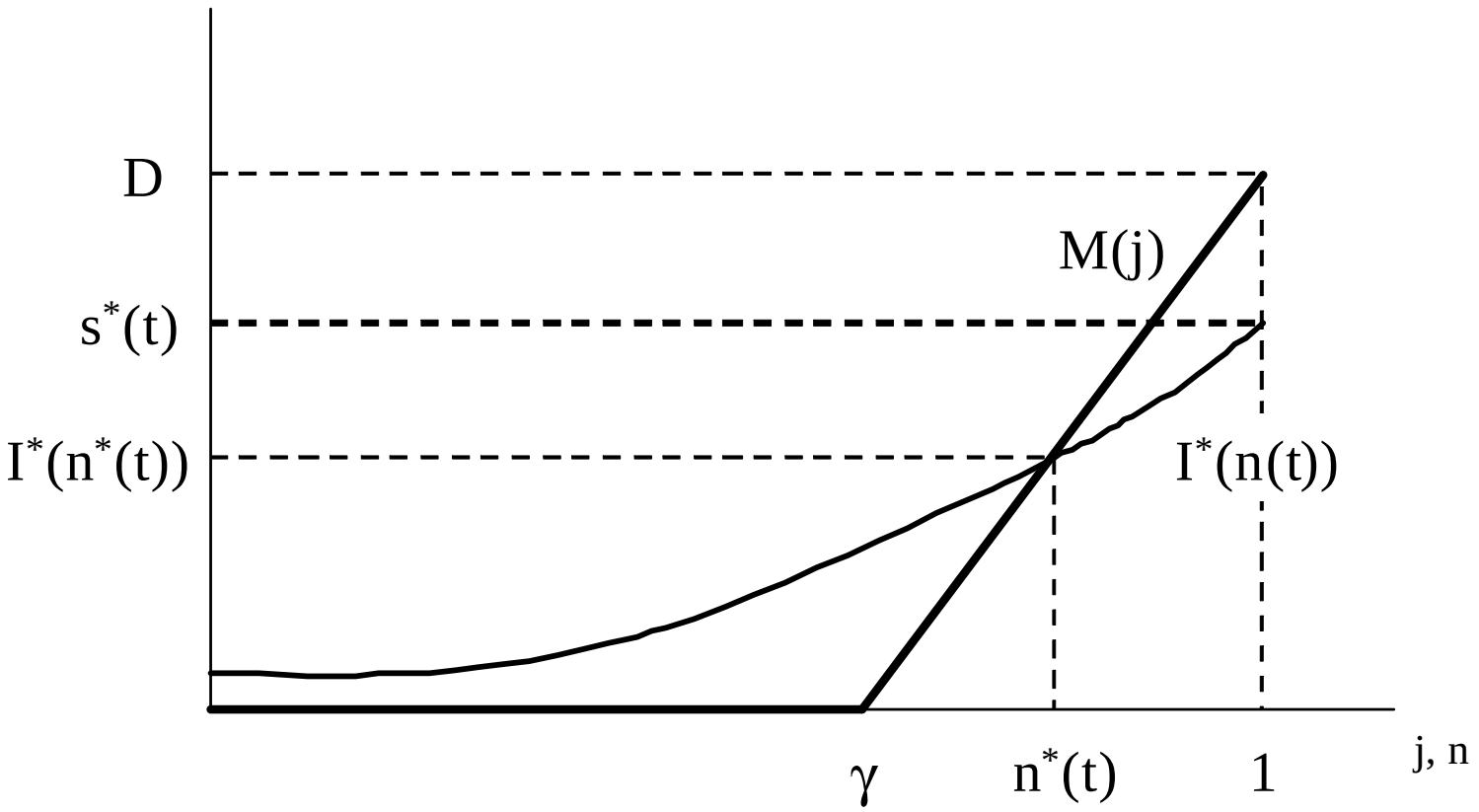

Figure: Minimum size requirements, $M(j)$, of different sectors and demand for assets, $I^*(n)$.

The Environment V

- Three important features introduced so far.
  1. Risky investments have a higher expected return than the safe investment, $Q > q$.
  2. Output of risky investments (intermediate sectors) are imperfectly correlated so there is safety in numbers.
  3. Mathematical formulation implies simple relationship between investments and returns.
     - ★ If individual holds portfolio consisting of equi-proportional investment I in all sectors $j \in \bar{J} \subseteq [0,1]$, and measure of the set $\bar{J}$ is p, portfolio pays return QI with probability p, and nothing with probability $1-p$.
- First two features imply that if aggregate production set of this economy had been convex, e.g. $D = 0$, all agents would invest an equal amount in all intermediate sectors and manage to diversify all risks without sacrificing any of the high returns.
- But nonconvexities, captured by the minimum size requirements: tradeoff between insurance and high productivity.

The Environment VI

- Preferences of households: each generation has size normalized to 1.
- Household from a generation born at time t:

$$\mathbb{E}_t U(c_1(t), c_2(t+1)) = \log(c_1(t)) + \beta \int_0^1 \log c_2(j, t+1)\, dj, \tag{37}$$

- Integral replaces the expectation using the fact that all states are equally likely.
- Each household has 1 unit of labor when young and no labor endowment when old.
- Thus total supply of labor in the economy is 1.
- Moreover, in the second period of their lives, each individual consumes the return from their savings.
- Set of young households at time t is denoted by $\mathcal{H}_t$ and the figuredepicts the life cycle and the various decisions of a typical household.
- Note uncertainty affects the return on their savings and thus the amount of capital they will have in the second period.

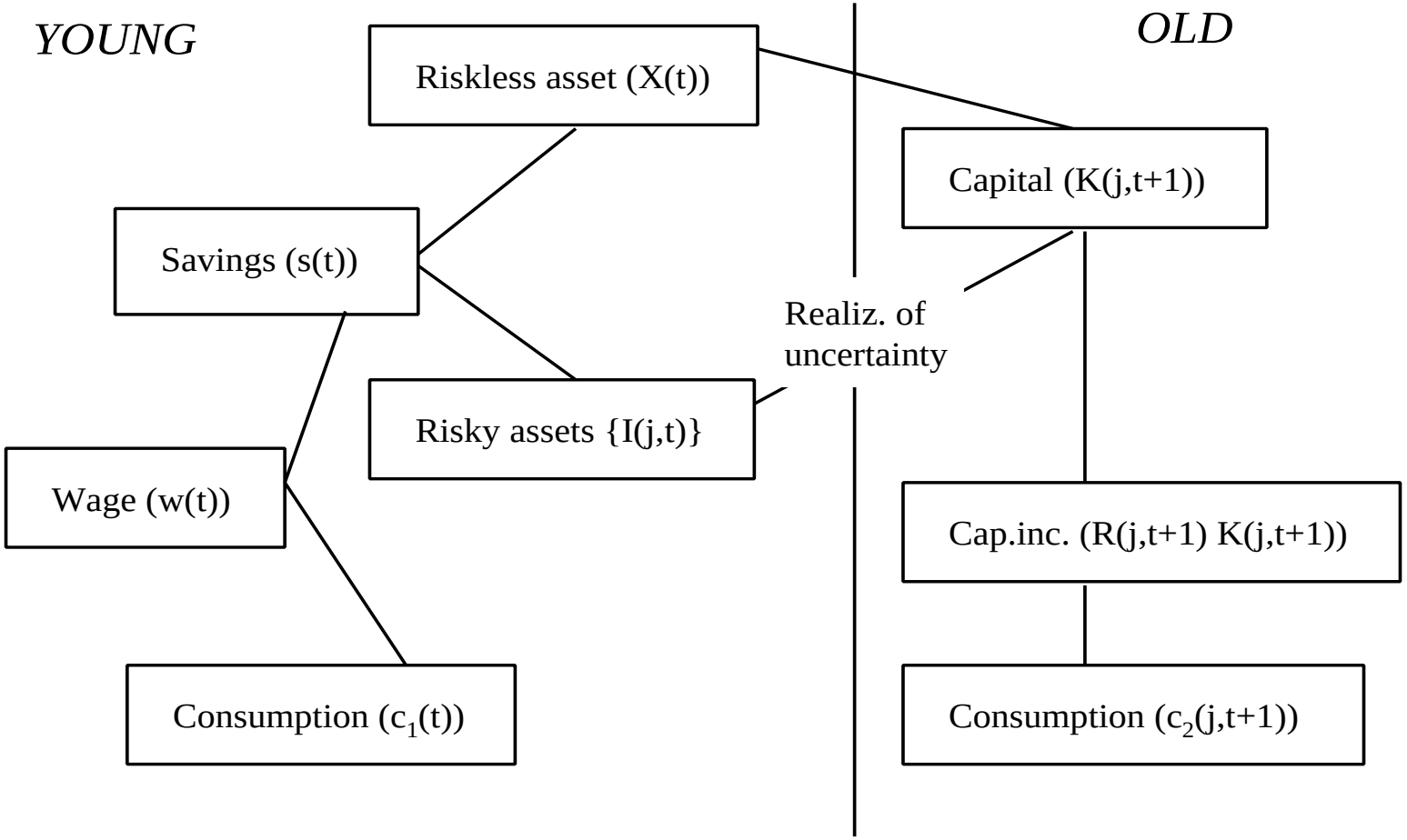

Figure: Life cycle of a typical household.

The Environment VII

- Capital stock at time $t+1$ depends on realization of the state of nature and composition of investment of young agents.
- In state j, the aggregate stock of capital is

$$K(j,t+1) = \int_{h \in \mathcal{H}_t} (qX^h(t) + QI^h(j,t))dh$$

- $I^h(j,t)$=amount of savings invested by (young) agent $h \in \mathcal{H}_t$ in sector j at time t, and
- $X^h(t)$=amount invested in the safe intermediate sector.
- Since capital stock is potentially random, so will be output and factor prices.

The Environment VIII

- Labor and capital assumed to be traded in competitive markets, so equilibrium factor prices will be given by their marginal products:

$$\begin{aligned} w(j,t+1) &= (1-\alpha)K(j,t+1)^{\alpha} \\ &= (1-\alpha)\left(\int_{h\in\mathcal{H}_t}(qX^h(t)+QI^h(j,t))dh\right)^{\alpha}. \end{aligned} \tag{38}$$

and

$$\begin{aligned} R(j,t+1) &= \alpha K(j,t+1)^{\alpha-1} \\ &= \alpha\left(\int_{h\in\mathcal{H}_t}(qX^h(t)+QI^h(j,t))dh\right)^{\alpha-1}. \end{aligned} \tag{39}$$

- Assume households make investments in different intermediate sectors through *financial intermediaries*.

The Environment IX

- Free entry into financial intermediation.
  - Any intermediary can form costlessly and mediate funds for a particular sector
- Important requirement is any financial intermediary should raise enough funds to cover the minimum size requirement.
- Assume each financial intermediary can operate only a single sector: ruling out formation of a grand financial intermediary
- Denote price charged for a security associated with intermediate sector j at time t by $P(j,t)$.
- $P(j,t) < 1$ is not possible: one unit of the security requires one unit of the final good, so $P(j,t) < 1$ would lose money.
- $P(j,t) > 1$ ruled out by free entry: if intermediary offers $P(j,t) > 1$ and raises $I(j,t) > M(j)$, some other can enter, offer a lower price for the security, and attract all the funds.
- Hence equilibrium behavior will force $P(j,t) = 1$ for all securities that are being supplied.

Equilibrium I

- Not all intermediate sectors will be *open* at each date: let set of intermediate sectors that are open at date t be $J(t)$.
- For any $j \in J(t)$ free entry implies that $P(j,t) = 1$.
- This allows to write problem of a representative household $h \in \mathcal{H}_t$ taking prices and the set of available securities at time t as given:

$$\max_{s(t),X(t),[I(j,t)]_{0\leq j\leq 1}} \log c(t) + \beta \int_0^1 \log c(j,t+1)\, dj, \tag{40}$$

subject to:

$$X(t) + \int_0^1 I(j,t)\, dj = s(t), \tag{41}$$

$$c(j,t+1) = R(j,t+1)\left(qX(t) + QI(j,t)\right), \tag{42}$$

$$I(j,t) = 0, \quad \forall j \notin J(t), \tag{43}$$

$$c(t) + s(t) \leq w(t), \tag{44}$$

Equilibrium II

- Suppressed the superscript h to simplify the notation.
- (41): investment in safe sector and sum of the investments in all other securities are equal to total savings of individual, $s(t)$.
- (42): consumption in state j at time $t+1$:
  - supply labor only when young and consume capital income when old: second period consumption equal to capital holdings times $R(j, t+1)$ given by (39).
  - $R(j, t+1)$ conditioned on the state j (at time $t+1$) since amount of capital and thus the marginal product of capital will differ across states.
  - amount of capital available to the household is equal to what it receives from the safe investment, $qX(t)$, plus the return from the Basic Arrow Security for state j, $QI(j, t)$.
- (43): household cannot invest in any security that is not being supplied in the market.
- (44): sum of consumption and savings less than or equal to the income of the individual, wage income from (38).

Equilibrium III

- *Static equilibrium*: equilibrium for time t, taking $K(t)$ and thus $w(t)$ as given.
  $\left\langle s^*(t), X^*(t), [I^*(j,t)]_{0\leq j\leq J^*(t)}, J^*(t), [P^*(j,t)]_{0\leq j\leq J^*(t)}, w^*(j,t), \right.$
  such that:
  - $s^*(t), X^*(t), [I^*(j,t)]_{0\leq j\leq 1}$ solve the maximization of (40) subject to (41)-(44) for given $[P^*(j,t)]_{0\leq j\leq J^*(t)}$, $J^*(t)$,$w^*(j,t)$ and $R^*(j,t)$;
  - $w^*(j,t)$ and $R^*(j,t)$ are given by (38) and by (39); and
  - $J^*(t)$ and $[P^*(j,t)]_{0\leq j\leq J^*(t)}$ are such that for all $j \in J^*(t)$, $P^*(j,t) = 1$ and the set $J^*(t)$ is determined by free entry in the sense that if some $j' \notin J^*(t)$ were offered for a price $P(j',t) \geq 1$, then the solution to the modified maximization problem (40) subject to (41)-(44) would involve $I(j',t) < M(j)$;
- Last condition: there is no more room for one more intermediate sector to open and attract sufficient funds to cover the minimum size requirement.

Equilibrium IV

- *Dynamic equilibrium*: sequence of static equilibria linked to each other through (38) given the realization of the state $j(t)$ at each $t = 1, 2, ...$
- Because preferences in (40) are logarithmic, saving rate of all households will be constant, regardless of the risk-return tradeoff:

$$s^*(t) \equiv s^*(w(t)) = \frac{\beta}{1+\beta} w(t). \tag{45}$$

- Thus problem can be broken in two: first, amount of savings, then optimal portfolio is chosen.
- Decomposition is useful because:
  1. For any $j, j' \in J(t)$, $I^*(j,t) = I^*(j',t)$: each individual is facing the same price for all traded *symmetric* Basic Arrow Securities, want to purchase a *balanced portfolio*.
  2. Set of open projects takes form $J^*(t) = [0, n^*(t)]$ for some $n(t) \in [0,1]$: intermediate sectors with small minimum size requirements will be opened before.

Equilibrium V

- Can divide the states of nature at time t into two sets:
  1. states in $[0, n(t)]$="good," society is lucky and risky investments have delivered positive returns, return to capital is $R^G(t+1)$,
  2. states in $(n(t), 1]$="bad," society is unlucky and risky investments have zero returns, return to capital is $R^B(t+1)$.
- Returns dated $t+1$, because they are paid out at time $t+1$.
- Representative household problem written in simpler form:

$$\max_{X(t),I(t)} \left\{ \begin{array}{c} n^*(t) \log\left[R^G(t+1)(qX(t)+QI(t))\right] \\ +(1-n^*(t)) \log\left[R^B(t+1)\, qX(t)\right] \end{array} \right\}, \tag{46}$$

subject to:

$$X(t) + n^*(t)\, I(t) \leq s^*(t) \tag{47}$$

where $n^*(t)$, $R^G(t+1)$ and $R^B(t+1)$ are taken as given by the representative household, and $s^*(t)$ is given by (45).

Equilibrium VI

- From (39)

$$R^B(t+1) = \alpha (qX(t))^{\alpha-1}$$

is the marginal product of capital in the "bad" state, when the realized state is $j > n^*(t)$ and no risky investment pays off,

- And

$$R^G(t) = \alpha (qX(t) + QI(t))^{\alpha-1}$$

applies in the "good state", i.e. when the realized state is $j\,[0, n^*(t)]$.

- Maximization of (46) subject to (47) yields unique solution:

$$X^*(t) = \frac{(1-n^*(t))Q}{Q-qn^*(t)} s^*(t), \tag{48}$$

and

$$I^*(j,t) = \begin{cases} I^*(n^*(t)) \equiv \frac{Q-q}{Q-qn(t)} s^*(t), & \text{for } j \leq n^*(t) \\ 0 & \text{for } j > n^*(t) \end{cases}. \tag{49}$$

Equilibrium VII

- (49): $I^*(n)$ is strictly increasing in n: demand for *each asset* (or investment in each intermediate sector) grows as the measure of open sectors increases.
- When more securities are available, risk-diversification opportunities improve, consumers become willing to reduce investments in safe asset and increase in risky projects.
- What holds back investment in the higher productivity sectors in this economy is the fact that they are riskier.
- But "safety in numbers" implies first-order benefit from diversification: when financial assets traded on more sectors, each household is willing to invest more in risky assets in total.
- Complementarity between set of traded assets and investments: key in dynamics of economic development.

Equilibrium VIII

- Find the set of sectors that are active: finding a threshold sector $n^*(t)$ such that all $j \leq n^*(t)$ can meet their minimum size requirements while no additional sector can enter.
- Figure above: plot level of investment for each sector in a balanced portfolio, $I^*(n^*(t))$ given by (49), together with the minimum size requirement, $M(j)$ given by (36).
- First curve is "demand for assets" and curve for (36) is "supply of assets".
- Figure shows a unique intersection between the two curves.
- Both curves are upward-sloping: more than one intersection is possible in general.
- Condition $Q \geq (2-\gamma)\, q$ is sufficient to ensure a unique intersection
- If violated, multiple equilibria, equilibrium with more active sectors gives higher ex ante utility to all households.
- Focus on $Q \geq (2-\gamma)\, q$: static equilibrium uniquely defined.

Equilibrium IX

Proposition Suppose that $Q \geq (2-\gamma)\, q$ and that $K(t)$ is given. Then there exists a unique time t equilibrium in which all sectors $j \leq n^*(t) = n^*[K(t)]$ are open and those $j > n^*[K(t)]$ are shut, where

$$\begin{aligned} n^*[K(t)] = {} & (Q+q\gamma)/2q \\ & - \{(Q+q\gamma)^2 \\ & -4q\left[D^{-1}(Q-q)(1-\gamma)\Gamma K(t)^{\alpha} + \gamma Q\right]\}^{1/2}/2q \end{aligned} \tag{50}$$

if $K(t) \leq D^{1/\alpha}\Gamma^{-1/\alpha}$, and $n^*[K(t)] = 1$ if $K(t) > D^{1/\alpha}\Gamma^{-1/\alpha}$ with Γ defined as $\Gamma \equiv (1-\alpha)\beta(1+\beta)^{-1}$. In this equilibrium,

$$s^*(t) = \frac{\beta}{1+\beta}(1-\alpha)K(t),$$

and $X^*(t)$ and $I^*(j,t)$ are given by (48) and (49) with $n^*(t) = n^*[K(t)]$.

Equilibrium X

- Note equilibrium threshold sector $n^* [K]$ is increasing in K.
  - More capital: economy is able to open more intermediate sectors.
  - Contributes to the complementarity in the behavior since (49), in turn, implies that when more open sectors, investment in each sector will increase.

Equilibrium Dynamics I

- Law of motion for the capital stock, $K(t)$, will be given by a simple Markov process.
- Recall risky sectors will be successful with probability $n^*[K(t)]$ when capital stock is $K(t)$
- Thus law of motion for the capital stock:

$$K(t+1) = \begin{cases} \frac{q(1-n^*[K(t)])}{Q-qn^*[K(t)]} Q\Gamma K(t)^{\alpha} & \text{with probability } 1-n^*[K(t)] \\ Q\Gamma K(t)^{\alpha} & \text{with probability } n^*[K(t)] \end{cases} \tag{51}$$

where $n^*[K(t)]$ is given by equation (50) and recall that $\Gamma \equiv (1-\alpha)\beta(1+\beta)^{-1}$.

- First line of (51) less than the second, which refers to case in which investments have been successful.

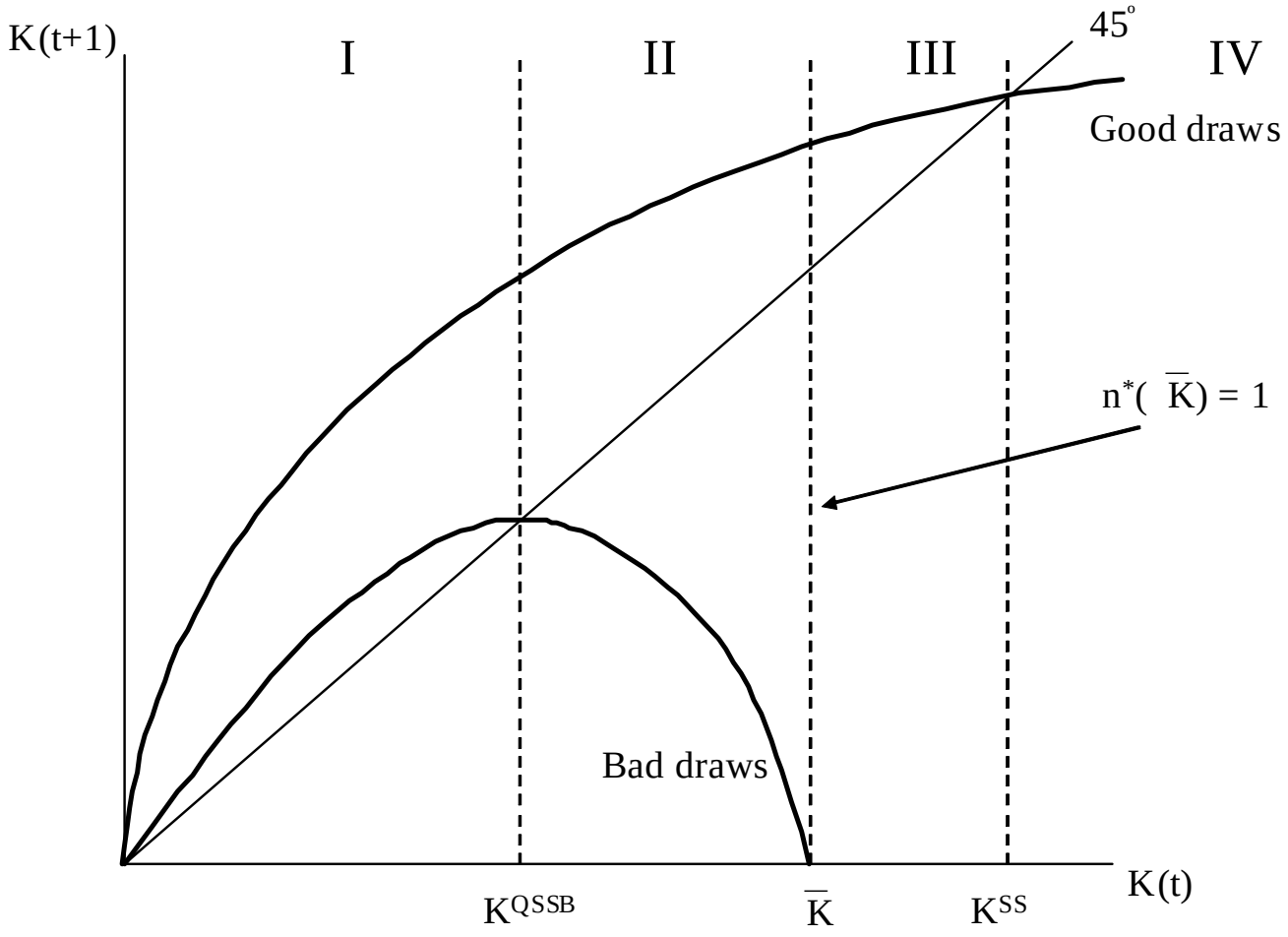

Figure: The stochastic correspondence of the capital stock.

Equilibrium Dynamics II

- Equation (51) is simple Markov process: given $K(t)$, $K(t+1)$ can only take two values.
- But a Markov process not a Markov chain: for different $K(t)$, values of $K(t+1)$ belong to the entire $\mathbb{R}_+$.
- Diagram for (51):
  - Here, only values exactly on the two curves plotted in the figure are possible.
  - First curve is $Q\Gamma K(t)^{\alpha}$: result if households followed equilibrium investment strategies (48) and (49), and at each date, economy turned out to be lucky
  - Second, inverse U-shaped curve is $q(1-n^*[K(t)])Q\Gamma K(t)^{\alpha} / (Q - qn^*[K(t)])$: applies if the economy is unlucky at each date.
  - Both curves start above the 45^0 line because (34) satisfies the Inada conditions.
  - Economy will be on the upper curve with probability $n^*[K(t)]$ and on the lower curve with probability $1-n^*[K(t)]$.

Equilibrium Dynamics III

- Not only probabilities of success change with $K(t)$ but so does average productivity.
- Define *expected total factor productivity* (TFP) conditional on the proportion of intermediate sectors that are open:

$$\sigma^e(n^*[K(t)]) = (1 - n^*[K(t)])\frac{q(1 - n^*[K(t)])}{Q - qn^*[K(t)]}Q + n^*[K(t)]Q. \tag{52}$$

- Differentiation establishes that $\sigma^e(n^*[K(t)])$ is strictly increasing in $n^*[K(t)]$.
- As the economy develops opens more intermediate sectors, productivity will endogenously increase.
- $n^*[K]$ is increasing in K, so average productivity is increasing $K(t)$.

Proposition The expected total factor productivity of the economy $\sigma^e(n^*[K]))$ is increasing in n^* and thus increasing in K.

Equilibrium Dynamics IV

- Figure also suggests that two levels of capital stock are special and useful:

1. K^{QSSB}: "quasi steady state" of economy which *always has unlucky draws*.
2. K^{QSSG} : "quasi steady state" of an economy which *always receives good news*.

- These levels are plotted in the figure and are also easy to compute::

$$K^{QSSB} = \left[\frac{q\left(1 - n^{*}\left[K^{QSSB}\right]\right)}{Q - qn^{*}\left[K^{QSSB}\right]} Q\Gamma \right]^{\frac{1}{1-\alpha}} \quad \text{and} \quad K^{QSSG} = (Q\Gamma)^{\frac{1}{1-\alpha}}. \tag{53}$$

Equilibrium Dynamics V

- K^{QSSG} : economy never faces any risk and thus similar to standard neoclassical growth model.
- If, in equilibrium, $n^* \left[K^{QSSG}\right] = 1$, K^{QSSG} becomes a proper steady state: economy accumulates sufficient capital to open all intermediate sectors, eliminates all risk and would always be on the upper curve.
- Equations (50) and (53): condition for $n^* \left[K^{QSSG}\right] = 1$: saving level corresponding to K^{QSSG} be sufficient to ensure a balanced portfolio of investments, of at least D:

$$D < \Gamma^{\frac{1}{1-\alpha}} Q^{\frac{\alpha}{1-\alpha}}. \tag{54}$$

- When (54) is satisfied, denote K^{QSSG} by K^{SS}.
- Under the assumption that (54) is satisfied, Figure draws $n^* \left[K^{SS}\right]$.

Equilibrium Dynamics VI

- Divides the range of capital stocks into four regions
  - Region I: $K(t)$ low, both curves above 45° line, economy grows regardless of luck.
  - Region II: economy grows if positive shocks but crisis if not.
    - ★ Between these two regions lies the bad quasi steady state K^{QSSB}:
    - ★ when $K < K^{QSSB}$, economy will definitely grow towards K^{QSSB}.
    - ★ when $K > K^{QSSB}$, the economy may grow or contract.
    - ★ $n^*[K]$ is increasing in K: right-hand side neighborhood of K^{QSSB}, highest probability of contracting.
    - ★ Economy tends to spend a long time in region II.
  - Region III: economy travels towards steady state K^{SS}.
    - ★ If a sequence of good news in region II: ultimately exit from to region III.
    - ★ $\bar{K}$ such that $n^*[\bar{K}] = 1$: once reach $\bar{K}$, enough capital to open all the sectors.
    - ★ All risk is diversified: dynamics as canonical model without uncertainty.
  - Region IV: so much $K(t)$ that even with positive shocks economy will contract.

Equilibrium Dynamics VII

- Economy as achieving *takeoff* as in Rostow's account, in two senses:
  1. After takeoff successfully diversifies all risk: growth progresses steadily rather than with fluctuations
  2. Aggregate (labor and total factor) productivity will increase after this level of capital.
- Also, manages risks better: more sectors being open, equivalently more financial intermediaries
  - financial and economic development go hand-in-hand, not one causing the other.

Proposition Suppose that condition (54) holds, then the stochastic process $\{K(t)\}_{t=1}^{\infty}$ converges to the point K^{SS} with probability 1.

Equilibrium Dynamics VIII

- Thus variability will *eventually* decline and disappear.
- Amplitudes of economic fluctuations and level of capital stock or output in the economy: cross-sectional and time-series comparisons suggest that poorer nations suffer from greater economic variability.
- Conditional variance of TFP (expected value was defined in (52) above).
- Define $\sigma(n^* [K(t)])$ as a random variable that takes the values $q(1 - n^* [K(t)])Q / (Q - qn^* [K(t)])$ and Q with respective probabilities $(1 - n^* [K(t)])$ and $n^* [K(t)]$.
- Expectation of this random variable is $\sigma^e(n^* [K(t)]))$ as defined in (52).
- Taking logs, we can rewrite (51) as

$$\triangle \log(K(t+1)) = \log \Gamma - (1-\alpha) \log(K(t)) + \log(\sigma(n^* [K(t)])). \tag{55}$$

- Capital (and output) growth volatility, after removing deterministic "convergence effects", determined by stochastic component σ.

Equilibrium Dynamics IX

- Denote the (conditional) variance of $\sigma(n^*[K(t)]$ given $K(t)$ by V_n.

Proposition Let $\mathcal{V}_n \equiv \text{Var}(\sigma(n^*) \mid n^*) = n^*(1-n^*)\left[Q(Q-q)/(Q-qn^*)\right]^2$. Then we have that

- If $\gamma \geq Q/(2Q-q)$, then $\partial\mathcal{V}_n/\partial K \leq 0$ for all $K \geq 0$.
- If $\gamma < Q/(2Q-q)$, then there exists $\tilde{K}$ defined such that $n^*(\tilde{K}) = Q/(2Q-q) < 1$ and.

$$\begin{aligned} \frac{\partial \mathcal{V}_n}{\partial K} &\leq 0 \text{ for all } K \geq \tilde{K} \\ \frac{\partial \mathcal{V}_n}{\partial K_t} &> 0 \text{ for all } K < \tilde{K}. \end{aligned}$$

- Counteracting effects of two forces:
  1. As economy develops, more savings are invested in risky assets; and
  2. As more sectors are opened, idiosyncratic risks are better diversified.
- If $\gamma \geq Q/(2Q-q)$: second effect always dominates.
- If $\gamma < Q/(2Q-q)$: first dominates for sufficiently low $K(t)$; after $\tilde{K}$, second dominates.

Efficiency I

- Though all agents are price takers, equilibrium is not Pareto Optimal.
- Results from an economically meaningful *pecuniary externality*.
- Though all agents are price takers, not an Arrow-Debreu economy: set of traded commodities is determined endogenously by a zero profit condition.
- Ignore any potential source of intertemporal inefficiency (which may arise in overlapping generations economies).
- Thus take a particular level of savings $s(t)$, or equivalently $K(t)$, as given.
- Look whether way savings are allocated across different sectors of the economy is (constrained) efficient.

Efficiency II

- Social planner's problem:

$$\max_{n(t),X(t),[I(j,t)]_{0\leq j\leq n(t)}} \left\{ \begin{array}{c} \int_0^{n(t)} \log(qX(t) + QI(j,t))dj \\ +(1-n(t))\log(qX(t)) \end{array} \right\} \tag{56}$$

subject to

$$X(t) + \int_0^{n(t)} I(j,t)\,dj \leq s(t).$$

- Chooses: set of sectors that are active, $[0, n(t)]$, amount invested in the safe sector $X(t)$ and allocation among other sectors $[I(j,t)]_{0\leq j\leq n(t)}$.
- Could have chosen active sectors not to be an interval of the form $[0, n(t)]$, but no loss of generality in imposing this form.
- Main difference with problem (40): social planner also chooses $n(t)$, representative household took available assets as given.

Efficiency III

Proposition Let $n^*[K(t)]$ be given by (50), and $s(t)$ and $K(t)$ denote current level of savings and capital stock available to the social planner. Then, the unique solution to the maximization problem in (56) is:

- For all $s(t) < D$, the set of active sectors is given by $\left[0, n^S[K(t)]\right]$, where $n^S[K(t)] > n^*[K(t)]$. The amount of investment in the safe sector is given by $X^S(t)$, where $X^S(t) < X^*(K(t))$. Finally, there exists a sector $j^*(t) \in \left(0, n^S[K(t)]\right)$ such that the portfolio of risky sectors for each household takes the form

$$\begin{array}{ll} I^S(j,t) = M(j^*) > M(j) & \text{for} \quad j < j^*(t) \\ I^S(j,t) = M(j) & \text{for } j \in \left[j^*(t), n^S[K(t)]\right] \\ I^S(j,t) = 0 & \text{for} \quad j > n^S(K(t)) \end{array} . \tag{57}$$

- For all $s(t) \geq D$, $n^S[K(t)] = n^*[K(t)] = 1$ and $I^S(j,t) = s(t)$ for all $j \in [0,1]$.

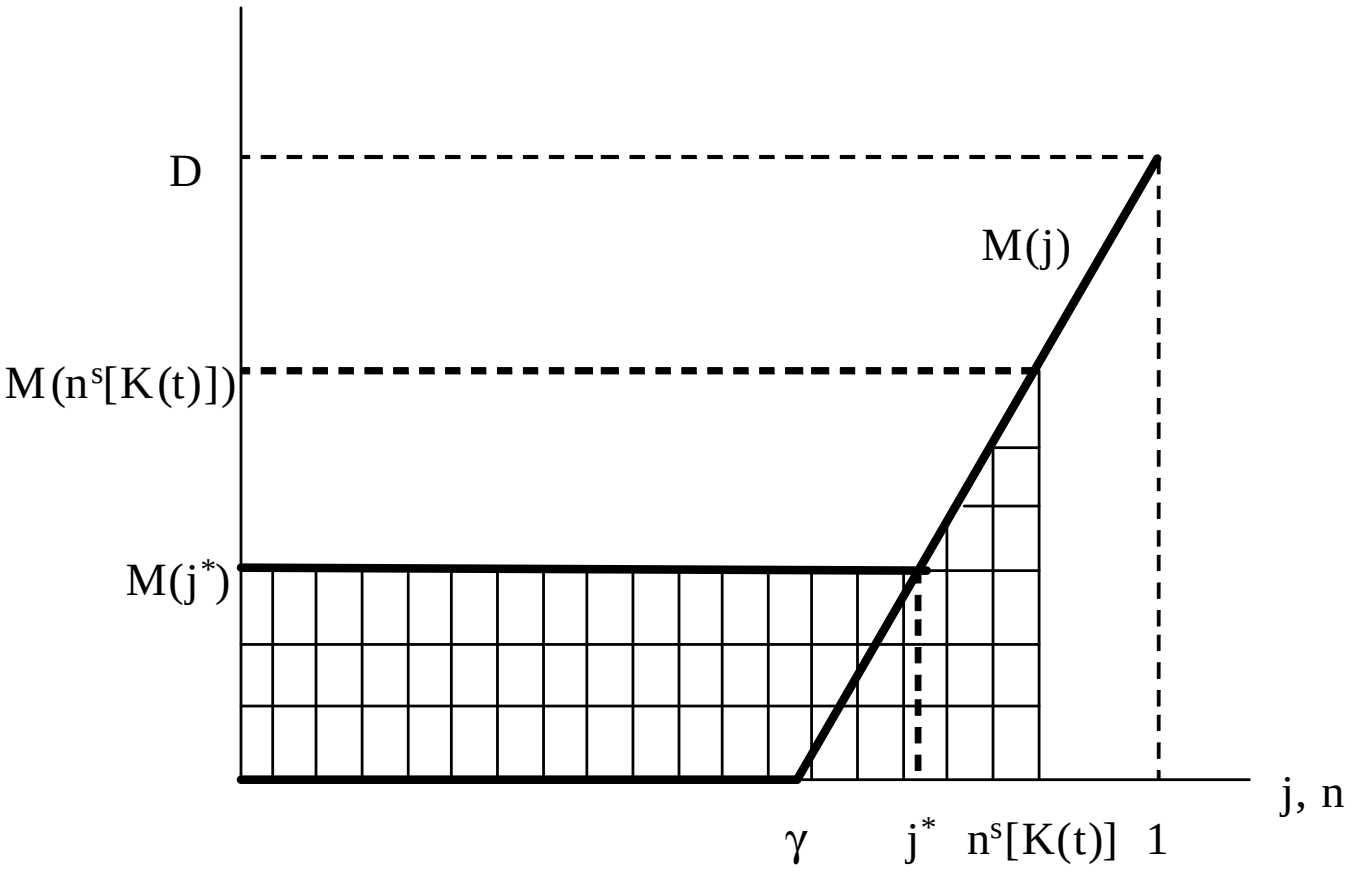

Figure: The efficient portfolio allocation.

Efficiency IV

- Before full diversification, social planner opens more sectors than decentralized equilibrium.
- Deviating from *balanced portfolio*:
  - invest less in sectors without the minimum size requirement.
  - *cross-subsidizing* sectors with high a minimum size requirements
- Starting with balanced portfolio, opening a few more sectors benefits consumers, to achieve better risk diversification.
- Intuition for inefficiency in descentralized solution:
  1. *Pecuniary externality* of marginal dollar of investment in high minimum size requirements sector
     - ★ Makes it possible for sector to be active, and provide better risk diversification to all other agents.
     - ★ Each household taking prices as given ignore this pecuniary externality
  2. Because households take prices as given and in equilibrium $P(j,t) = 1$ for all active sectors, always hold a balanced portfolio.
     - ★ But Pareto optimal involves cross-subsidization.

Efficiency V

- First Welfare Theorem fails because of endogenously incomplete markets: no "competitive pricing" for commodities that are not traded.
- In Arrow-Debreu equilibrium, even commodities not traded in equilibrium are priced.
- Not an Arrow-Debreu equilibrium, which does not exist here because of nonconvexity in the production set.
- Equilibrium concept: all commodities that are traded in equilibrium are priced competitively and set of traded commodities determined by a free entry condition.

Inefficiency with Alternative Market Structures I

- Can market failure in portfolio choices be overcome with some financial institution?
- Imagine funds are intermediated through a financial coalition-intermediary.
- Intermediary can collect all the savings and offer to each saver a *complex* security that pays $QI^S(j,t) + qX^S(t)$ in each state j, where $I^S(j,t)$ and $X^s(t)$ are as in the optimal portfolio.
- Holding this security would make each consumer better off compared to the equilibrium.
- But unless strong assumptions are made about the set of contracts that financial intermediary can offer, equilibrium from competition among intermediaries will be identical to equilibrium allocation
- Model more complex financial intermediaries as "intermediary-coalitions:" set of households who join their savings together and invest in a particular portfolio intermediate sectors.

Inefficiency with Alternative Market Structures II

- Coalitions may be organized by a specific household, and if it is profitable for other households to join the coalition, the organizer of the coalition can charge a premium.
- Free entry into financial intermediation or coalition-building,
- Adopt the following assumptions.
  1. Coalitions maximize a weighted utility of their members at all points in time. In particular, a coalition cannot commit to a path of action that will be against the interests of its members in the continuation game.
  2. Coalitions cannot exclude other agents (or coalitions) from investing in a particular project.

Proposition The set of equilibria of the financial intermediation game described above is always non-empty and all equilibria have exactly the same structure as the competitive equilibrium.

Inefficiency with Alternative Market Structures III

- Intuition:
  - Pareto optimal allocation involves a non-balanced portfolio and cross-subsidization.
  - Thus shadow price of investing in some sectors should be higher than in others, even though the cost is equal to 1.
  - Differences in shadow prices will then support a non-balanced portfolio.
  - Sectors with no or low minimum size requirements are being implicitly taxed in this allocation.
  - Cross-subsidization difficult to sustain in equilibrium:
    - ★ deviate towards slightly reducing investments in coalitions/intermediaries that engage in cross-subsidization and move portfolio towards a balanced one.
  - At the end, only allocations without cross-subsidization can survive as equilibria.
- Even with unrestricted financial intermediaries inefficiency from endogenously incomplete markets cannot be prevented.

Conclusions

- Introduced a number of workhorse models of macroeconomics, such as the neoclassical growth model under uncertainty and the basic Bewley model.
- Also, stochastic models can significantly enrich the analysis of economic growth and economic development.
  - model of take off into sustained and steady growth provides a good approximation to the economic development process that much of Western Europe underwent over the past 700 years or so.
  - random elements and luck can matter for the timing of takeoff among countries that satisfies some prerequisites for takeoff.
- Introduces important ideas related to incomplete markets.