Advanced Economic Growth: Lecture 21: Stochastic Dynamic Programming and Applications

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Stochastic Growth

- Stochastic growth models: useful for two related reasons:
  1. Range of problems involve either aggregate uncertainty or individual level uncertainty interacting with investment and growth process.
  2. Wide range of applications in macroeconomics and in other areas of dynamic economic analysis.
- Dynamic optimization under uncertainty is considerably harder.
- Continuous-time stochastic optimization methods are very powerful, but not used widely in macroeconomics
- Focus on discrete-time stochastic models.

Stochastic Dynamic Programming I

- Introduction to basic stochastic dynamic programming.
- To avoid measure theory: focus on economies in which stochastic variables take finitely many values.
- Enables to use Markov chains, instead of general Markov processes, to represent uncertainty.
- Then indicate how the results can be generalized to stochastic variables represented by continuous, or mixture of continuous and discrete, random variables.

Dynamic Programming with Expectations I I

- Introduce *stochastic* (random) variable $z(t) \in \mathcal{Z} \equiv \{z_1, ..., z_N\}$.
- Note $\mathcal{Z}$ is finite and thus compact.
- Let instantaneous payoff at time t be $U(x(t), x(t+1), z(t))$, where $x(t) \in X \subset \mathbb{R}^K$ for some $K \geq 1$ and $U : X \times X \times \mathcal{Z} \to \mathbb{R}$.
- Returns discounted by discount factor $\beta \in (0, 1)$.
- Initial value $x(0)$ is given.
- Think of $x(t)$ as the *state variable* (state vector) and of $x(t+1)$ as the *control variable* (control vector) at time t.
- Constraint on $x(t+1)$ incorporates the stochastic variable $z(t)$:

$$x(t+1) \in G(x(t), z(t)),$$

Dynamic Programming with Expectations II

- $G(x, z)$ is a set-valued mapping or a correspondence:

$$G : X \times \mathcal{Z} \rightrightarrows X.$$

- $z(t)$ follows a (first-order) *Markov chain*: current value of $z(t)$ only depends on its last period value, $z(t-1)$:

$$\Pr[z(t) = z_j \mid z(0), ..., z(t-1)] \equiv \Pr[z(t) = z_j \mid z(t-1)].$$

- Simplest example: finitely many values and is independently distributed over time:

$$\Pr[z(t) = z_j \mid z(0), ..., z(t-1)] = \Pr[z(t) = z_j].$$

- But Markov chains enable modelling stochastic shocks correlated over time.

Dynamic Programming with Expectations III

- Markov property allows simple notation for the probability distribution of $z(t)$.
- Can also represent a Markov chain as:

$$\Pr\left[z(t) = z_j \mid z(t-1) = z_{j'}\right] \equiv q_{jj'},$$

for any any $j, j' = 1, ..., N$, where $q_{jj'} \geq 0$ for all j, j' and

$$\sum_{j=1}^{N} q_{jj'} = 1 \text{ for each } j' = 1, ..., N.$$

- $q_{jj'}$ is also referred to as a *transition probability*.

Example: Optimal Growth Problem I

- Objective is to maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c\left(t\right)\right).$$

- Take expectations: future values of consumption per capita is stochastic (depend on future z's).
- Production function (per capita):

$$y\left(t\right) = f\left(k\left(t\right), z\left(t\right)\right),$$

- $z\left(t\right) \in \mathcal{Z} \equiv \left\{z_1, ..., z_N\right\}$, follows a Markov chain.
- Most natural interpretation of $z\left(t\right)$: TFP term, so one might write $y\left(t\right) = z\left(t\right) f\left(k\left(t\right)\right)$.

Example: Optimal Growth Problem II

- Constraint facing problem at time t:

$$k(t+1) = f(k(t), z(t)) + (1-\delta) k(t) - c(t), \qquad (1)$$

  $k(t) \geq 0$ and given $k(0)$
- Formulation implies at time $c(t)$ is chosen, $z(t)$ has been realized.
- Thus $c(t)$ is a random variable depending on the realization of $z(t)$.
- More generally, $c(t)$ may depend on the entire history of the random variables.
- Define

$$z^t \equiv (z(0), z(1), ...z(t))$$

  as the *history* of variable $z(t)$ up to date t.
- Let $\mathcal{Z}^t \equiv \mathcal{Z} \times ... \times \mathcal{Z}$ (the t-times product), so that $z^t \in \mathcal{Z}^t$.

Example: Optimal Growth Problem III

- For given $k(0)$, level of consumption at time t can be most generally written as

$$c(t) = \tilde{c}\left[z^t\right],$$

- Clearly, $c(t)$ cannot depend on future realizations of z—values have not been realized, not be feasible.
- But also not all functions $\tilde{c}\left[z^t\right]$ could be admissible as feasible plans.
- No point in making $c(t)$ function of the history of $k(t)$, since those are endogenously determined by the choice of past consumption levels and by the realization of past stochastic variables.
- In recursive formulation will write $c(t)$ as function of current capital stock and current value of the stochastic variable.

Example: Optimal Growth Problem IV

- Let $x(t) = k(t)$, so that

$$\begin{aligned} x(t+1) &= k(t+1) \\ &= f(k(t), z(t)) + (1-\delta) k(t) - \tilde{c}\left[z^t\right] \\ &\equiv \tilde{k}\left[z^t\right], \end{aligned}$$

- Feasibility: note

$$k(t+1) \equiv \tilde{k}\left[z^t\right]$$

depends only on history of stochastic shocks up to time t and not on $z(t+1)$.

- In addition, feasibility requires that $\tilde{k}[\cdot]$ satisfies

$$\begin{aligned} \tilde{k}\left[z^t\right] &\leq f(\tilde{k}\left[z^{t-1}\right], z(t)) + (1-\delta)\tilde{k}\left[z^{t-1}\right] \\ \text{for all } z^{t-1} &\in \mathcal{Z}^{t-1} \text{ and } z(t) \in \mathcal{Z}. \end{aligned}$$

Example: Optimal Growth Problem V

- Maximization problem:

$$\max_{\left\{\tilde{c}[z^t],\tilde{k}[z^t]\right\}_{t=0}^{\infty}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u\left(\tilde{c}\left[z^t\right]\right) \mid z\left(0\right)\right]$$

subject to

$$\begin{aligned} \tilde{k}\left[z^t\right] &\leq f(\tilde{k}\left[z^{t-1}\right], z\left(t\right)) + \left(1-\delta\right)\tilde{k}\left[z^{t-1}\right] - \tilde{c}\left[z^t\right] \\ \text{for all } z^{t-1} &\in \mathcal{Z}^{t-1} \text{ and } z\left(t\right) \in \mathcal{Z}, \end{aligned}$$

and starting with the initial conditions $\tilde{k}\left[z^{-1}\right] = k\left(0\right)$ and $z\left(0\right)$.

- Or, using function $U\left(x\left(t\right), x\left(t+1\right), z\left(t\right)\right)$ above:

$$\max_{\left\{\tilde{k}[z^t]\right\}_{t=0}^{\infty}} \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t U\left(\tilde{k}\left[z^{t-1}\right], \tilde{k}\left[z^t\right], z\left(t\right)\right),$$

where:

$U\left(x\left(t\right), x\left(t+1\right), z\left(t\right)\right) =$
$u\left(f(k\left(t\right), z\left(t\right)) - k\left(t+1\right) + \left(1-\delta\right)k\left(t\right)\right).$

Example: Optimal Growth Problem VI

- Timing convention:
  - $\tilde{k}\left[z^{t-1}\right]$=value of capital stock at time t, inherited from the investments at $t-1$, thus depends on z^{t-1},
  - $\tilde{k}\left[z^{t}\right]$=choice of capital stock for next period made at time t given z^{t}.
- Recursive formulation: Since $z(t)$ follows Markov chain: $z(t)$ contains information about available resources and about stochastic distribution of $z(t+1)$.
- Thus might expect policy function of the form:

$$k(t+1) = \pi(k(t), z(t)). \tag{2}$$

- And recursive characterization of the form:

$$V(k,z) = \sup_{y \in [0, f(k,z)+(1-\delta)k]} \left\{ \begin{array}{c} u(f(k,z)+(1-\delta)k-y) \\ +\beta \mathbb{E}\left[V(y,z') \mid z\right] \end{array} \right\}, \tag{3}$$

Example: Optimal Growth Problem VII

- $\mathbb{E}[\cdot \mid z]$ denotes the expectation conditional on current value of z and incorporates the fact that z is a Markov chain.
- Suppose this program has a solution, i.e. exists a feasible plan that achieves the value $V(k, z)$ starting with k and z.
- Then: set of next date's capital stock that achieve this maximum can be represented by a correspondence $\Pi(k, z) \subset X$ for each $k \in \mathbb{R}_+$ and $z \in \mathcal{Z}$.
- For any $\pi(k, z) \in \Pi(k, z)$,

$$\begin{aligned} V(k, z) &= u(f(k, z) + (1-\delta)k - \pi(k, z)) \\ &\quad + \beta \mathbb{E}\left[V\left(\pi(k, z), z'\right) \mid z\right]. \end{aligned}$$

- When $\Pi(k, z)$ is single valued, $\pi(k, z)$ would be uniquely defined and optimal choice capital stock can be represented as in (2).

Dynamic Programming with Expectations I

- Let a *plan* be denoted by $\tilde{x}\left[z^t\right]$.
- Plan specifies the value of the vector $x \in \mathbb{R}^K$ for time $t+1$, i.e., $x(t+1) = \tilde{x}\left[z^t\right]$, for any $z^t \in \mathcal{Z}^t$.
- Sequence problem takes the form:

$$\begin{aligned}
&\textbf{Problem B1} \quad : \\
V^*(x(0), z(0)) &= \sup_{\{\tilde{x}[z^t]\}_{t=-1}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(\tilde{x}\left[z^{t-1}\right], \tilde{x}\left[z^t\right], z(t)\right) \\
&\quad \text{subject to} \\
\tilde{x}\left[z^t\right] &\in G(\tilde{x}\left[z^{t-1}\right], z(t)), \qquad \text{for all } t \geq 0 \\
&\quad \tilde{x}\left[z^{-1}\right] = x(0) \text{ given,}
\end{aligned}$$

- Expectations at time $t = 0$, $\mathbb{E}_0$, are taken over the possible infinite sequences of $(z(0), z(1), z(2), z(3), \ldots)$.

Dynamic Programming with Expectations II

- Adopt convention that $\tilde{x}\left[z^{-1}\right] = x(0)$ and write maximization problem with respect to $\{\tilde{x}[z^t]\}_{t=-1}^{\infty}$ (starts at $t = -1$ and $\tilde{x}\left[z^{-1}\right] = x(0)$ is introduced as an additional constraint).
- V^* is conditioned on $x(0) \in \mathbb{R}^K$, taken as given, and on $z(0)$, since choice of $x(1)$ is made after $z(0)$ is observed.
- First constraint in Problem B1 ensures that the sequence $\{\tilde{x}[z^t]\}_{t=-1}^{\infty}$ is feasible.
- Functional equation corresponding to the recursive formulation:

$$\begin{aligned} \textbf{Problem B2} &: \\ V(x,z) &= \sup_{y \in G(x,z)} \left\{ U(x,y,z) + \beta \mathbb{E}\left[V(y,z') \mid z \right] \right\}, \quad (4) \\ \text{for all } x &\in X \text{ and } z \in \mathcal{Z} \end{aligned}$$

- $V : X \times \mathcal{Z} \rightarrow \mathbb{R}$ is a real-valued function.

Dynamic Programming with Expectations III

- $y \in G(x, z)$: constraint on next period's state vector as a function of realization of z.
- Can also write Problem B2 as

$$\begin{aligned} V(x,z) &= \sup_{y \in G(x,z)} \left\{ U(x,y,z) + \beta \int V(y,z') Q\left(z, dz'\right) \right\}, \\ \text{for all } x &\in X \text{ and } z \in \mathcal{Z}, \end{aligned}$$

- $\int f\left(z'\right) Q\left(z_0, dz'\right)$=Lebesgue integral of f with respect to Markov process for z given last period's value z_0.
- Want to establish conditions under which the solutions to Problems B1 and B2 coincide.
- Set of feasible *plans* starting with $x\left(t\right)$ and $z\left(t\right)$:

$$\begin{aligned} \Phi(x\left(t\right), z\left(t\right)) &= \{\{\tilde{x}\left[z^s\right]\}_{s=t-1}^{\infty} : \tilde{x}\left[z^s\right] \in G(\tilde{x}\left(z^{s-1}\right), z\left(s\right)), \\ \text{for } s &= t-1, t, t+1, ...\}. \end{aligned}$$

Dynamic Programming with Expectations IV

- Denote a generic element of $\Phi(x(0), z(0))$ by $\mathbf{x} \equiv \{\tilde{x}[z^t]\}_{t=-1}^{\infty}$.
- Elements of $\Phi(x(0), z(0))$: not infinite sequences of vectors in $\mathbb{R}^K$, but infinite sequences of feasible plans $\tilde{x}[z^t]$ that assign a value $x \in \mathbb{R}^K$ for any history $z^t \in \mathcal{Z}^t$ for any $t = 0, 1, \ldots$.
- We are interested in when the:
  1. solution $V(x, z)$ to the Problem B2 coincides with the solution $V^*(x, z)$; and
  2. set of maximizing plans $\Pi(x, z) \subset \Phi(x, z)$ also generates an optimal feasible plan for Problem B1 (presuming both have feasible plans attaining supremums).
- Set of maximizing plans $\Pi(x, z)$: for any $\pi(x, z) \in \Pi(x, z)$,

$$V(x, z) = U(x, \pi(x, z), z) + \beta \mathbb{E}\left[V(\pi(x, z), z') \mid z\right]. \quad (5)$$

Dynamic Programming with Expectations V

ASSUMPTION 16.1. $G(x, z)$ is nonempty for all $x \in X$ and $z \in \mathcal{Z}$. Moreover, for all $x(0) \in X$, $z(0) \in \mathcal{Z}$, and $\mathbf{x} \in \Phi(x(0), z(0))$, $\lim_{n\to\infty} \mathbb{E}\left[\sum_{t=0}^{n} \beta^t U(\tilde{x}\left[z^{t-1}\right], \tilde{x}\left[z^t\right], z(t)) \mid z(0)\right]$ exists and is finite.

ASSUMPTION 16.2. X is a compact subset of $\mathbb{R}^K$, G is nonempty, compact-valued and continuous. Moreover, let
$\mathbf{X}_G = \{(x, y, z) \in X \times X \times \mathcal{Z} : y \in G(x, z)\}$ and suppose that
$U : \mathbf{X}_G \to \mathbb{R}$ is continuous.

- 16.1 only imposes compactness of X; $\mathcal{Z}$ is already compact.
- Continuity of U in (x, y, z) is equivalent to continuity in (x, y); $\mathcal{Z}$ is a finite set, can endow it with discrete topology so continuity is automatic.

Dynamic Programming with Expectations VI

Theorem **(Equivalence of Values)** Suppose Assumptions 16.1 and 16.2 hold. Then for any $x \in X$ and any $z \in \mathcal{Z}$, any $V^*(x, z)$ defined in Problem B1 is a solution to Problem B2. Moreover, any solution $V(x, z)$ to Problem B2 that satisfies $\lim_{t\to\infty} \beta^t \mathbb{E}\left[V\left(\tilde{x}\left[z^{t-1}\right], z(t)\right)\right] = 0$ for any $\{\tilde{x}[z^t]\}_{t=-1}^{\infty} \in \Phi(x(0), z(0))$, and any $\tilde{x}\left[z^{-1}\right] = x(0) \in X$ and $z \in \mathcal{Z}$ is a solution to Problem B1, so that $V^*(x, z) = V(x, z)$ for any $x \in X$ and any $z \in \mathcal{Z}$.

Dynamic Programming with Expectations VII

Theorem **(Principle of Optimality)** Suppose Assumptions 16.1 and 16.2 hold. For $x(0) \in X$ and $z(0) \in \mathcal{Z}$, let $\mathbf{x}^* \equiv \{\tilde{x}^*[z^t]\}_{t=-1}^{\infty} \in \Phi(x(0), z(0))$ be a feasible plan that attains $V^*(x(0), z(0))$ in Problem B1. Then we have

$$V^*(\tilde{x}^*\left[z^{t-1}\right], z(t)) = U(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)) + \beta\mathbb{E}\left[V^*(\tilde{x}^*\left(z^t\right), z(t+1)) \mid z(t)\right] \quad (6)$$

for $t = 0, 1, ...$

Moreover, if any $\mathbf{x}^* \in \Phi(x(0), z(0))$ satisfies (6), then it attains the optimal value in Problem B1.

Theorem **(Existence of Solutions)** Suppose that Assumptions 16.1 and 16.2 hold. Then the unique function $V : X \times \mathcal{Z} \to \mathbb{R}$ that satisfies (4) is continuous and bounded in x for each $z \in \mathcal{Z}$. Moreover, an optimal plan $\mathbf{x}^* \in \Phi(x(0), z(0))$ exists for any $x(0) \in X$ and any $z(0) \in \mathcal{Z}$.

Dynamic Programming with Expectations VIII

ASSUMPTION 16.3. U is strictly concave: for any $\alpha \in (0,1)$ and any (x,y,z), $(x',y',z) \in \mathbf{X}_G$:

$$U\left(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)\,y', z\right) \geq \alpha U(x,y,z) + (1-\alpha)U(x',y',z),$$

and if $x \neq x'$,

$$U\left(\alpha x + (1-\alpha)x', \alpha y + (1-\alpha)\,y', z\right) > \alpha U(x,y,z) + (1-\alpha)U(x',y',z).$$

Moreover, $G(x,z)$ is convex in x: for any $z \in \mathcal{Z}$, any $\alpha \in [0,1]$, and any $x, x' \in X$, whenever $y \in G(x,z)$ and $y' \in G(x',z)$, then

$$\alpha y + (1-\alpha)y' \in G\left(\alpha x + (1-\alpha)x', z\right).$$

ASSUMPTION 16.4. For each $y \in X$ and $z \in \mathcal{Z}$, $U(\cdot, y, z)$ is strictly increasing in its first K arguments, and G is monotone, i.e. $x \leq x'$ implies $G(x,z) \subset G(x',z)$ for each $z \in \mathcal{Z}$.

ASSUMPTION 16.5. $U(x,y,z)$ is continuously differentiable in x in the interior of its domain $\mathbf{X}_G$.

Dynamic Programming with Expectations IX

Theorem **(Concavity of the Value Function)** Suppose that Assumptions 16.1, 16.2 and 16.3 hold. Then the unique function V that satisfies (4) is strictly concave in x for each $z \in \mathcal{Z}$. Moreover, the optimal plan can be expressed as $\tilde{x}^* [z^t] = \pi (x^* (t), z (t))$, where the policy function $\pi : X \times \mathcal{Z} \to X$ is continuous in x for each $z \in \mathcal{Z}$.

Theorem **(Monotonicity of the Value Function I)** Suppose that Assumptions 16.1, 16.2 and 16.4 hold and let $V : X \times \mathcal{Z} \to \mathbb{R}$ be the unique solution to (4). Then for each $z \in \mathcal{Z}$, V is strictly increasing in x.

Dynamic Programming with Expectations X

Theorem **(Differentiability of the Value Function)** Suppose that Assumptions 16.1, 16.2, 16.3 and 16.5 hold. Let π be the policy function defined above and assume that $x' \in \text{Int}X$ and $\pi(x', z) \in \text{Int}G(x', z)$ at $z \in \mathcal{Z}$, then $V(x, z)$ is continuously differentiable at (x', z), with derivative given by

$$D_x V(x', z) = D_x U(x', \pi(x', z), z). \tag{7}$$

- Since the value function now also depends on z, an additional monotonicity result can also be obtained.

Dynamic Programming with Expectations XI

Assumption 16.6. (i) G is monotone in z in the sense that $z \leq z'$ implies $G(x, z) \subset G(x, z')$ for each any $x \in X$ and $z, z' \in \mathcal{Z}$ such that $z \leq z'$.
(ii) For each $(x, y, z) \in \mathbf{X}_G$, $U(x, y, z)$ is strictly increasing in z.
(iii) The Markov chain for z is monotone in the sense that for any nondecreasing function $f : \mathcal{Z} \to \mathbb{R}$, $\mathbb{E}[f(z') \mid z]$ is also nondecreasing in z.

- To interpret the last part suppose that $z_j \leq z_{j'}$ whenever $j < j'$.
- Then this condition will be satisfied if and only if we have that for any $\bar{j} = 1, ..., N$ and any $j'' > j'$, $\sum_{j=\bar{j}}^{N} q_{jj''} \geq \sum_{j=\bar{j}}^{N} q_{jj'}$.

Dynamic Programming with Expectations XII

Theorem **(Monotonicity of the Value Function II)** Suppose that Assumptions 16.1, 16.2 and 16.6 hold and let $V : X \times \mathcal{Z} \to \mathbb{R}$ be the unique solution to (4). Then for each $x \in X$, V is strictly increasing in z.

Proofs of the Stochastic Dynamic Programming Theorems I

- For any feasible $\mathbf{x} \equiv \{\tilde{x}[z^t]\}_{t=-1}^{\infty}$, and any initial $x(0) \in X$ and $z(0) \in \mathcal{Z}$, define

$$\mathbf{U}(\mathbf{x}, z(0)) \equiv \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t U\left(\tilde{x}\left[z^{t-1}\right], \tilde{x}\left[z^t\right], z(t)\right) \mid z(0)\right]$$

- Note that for any $x(0) \in X$ and $z(0) \in \mathcal{Z}$,

$$V^*(x(0), z(0)) = \sup_{\mathbf{x} \in \Phi(x(0), z(0))} \mathbf{U}(\mathbf{x}, z(0)).$$

Proofs of the Stochastic Dynamic Programming Theorems II

- Assumption 16.1 ensures all values are bounded; it follows by definition that

$$V^*(x(0), z(0)) \geq \mathbf{U}(\mathbf{x}, z(0)) \text{ for all } \mathbf{x} \in \Phi(x(0), z(0)) \quad (8)$$

and

$$\begin{aligned} \text{for any } \varepsilon &> 0, \text{ there exists } \mathbf{x}' \in \Phi(x(0), z(0)) \quad (9) \\ \text{s.t. } V^*(x(0), z(0)) &\leq \mathbf{U}(\mathbf{x}', z(0)) + \varepsilon \end{aligned}$$

Proofs of the Stochastic Dynamic Programming Theorems III

- Conditions for $V(\cdot,\cdot)$ to be a solution to Problem B2 are similar.
- For any $x(0) \in X$ and $z(0) \in \mathcal{Z}$,

$$\begin{aligned} V(x(0), z(0)) &\geq U(x(0), y, z) + \beta \mathbb{E}\left[V(y, z(1)) \mid z(0)\right], \quad (10) \\ \text{all } y &\in G(x(0), z(0)), \end{aligned}$$

- Also

$$\begin{aligned} \text{for any } \varepsilon &> 0, \text{ there exists } y' \in G(x(0), z(0)) \quad (11) \\ \text{s.t.} V(x(0), z(0)) &\leq U(x(0), y', z(0)) \\ &\quad + \beta \mathbb{E}\left[V(y, z(1)) \mid z(0)\right] + \varepsilon. \end{aligned}$$

Proofs of the Stochastic Dynamic Programming Theorems IV

Lemma Suppose that Assumption 16.1 holds. Then for any $x(0) \in X$, any $z(0) \in \mathcal{Z}$, any $\mathbf{x} \equiv \{\tilde{x}[z^t]\}_{t=-1}^{\infty} \in \Phi(x(0), z(0))$, we have that

$$\begin{aligned} \mathbf{U}(\mathbf{x}, z(0)) &= U(x(0), \tilde{x}[z^0], z(0)) \\ &\quad + \beta \mathbb{E}\left[\mathbf{U}(\{\tilde{x}[z^t]\}_{t=0}^{\infty}, z(1)) \mid z(0)\right]. \end{aligned}$$

Proof of Equivalence of Values Theorem I

- If $\beta = 0$, Problems B1 and B2 are identical, thus the result follows immediately.
- Suppose $\beta > 0$ and take an arbitrary $x(0) \in X$ and an arbitrary $z(0) \in \mathcal{Z}$.
- First, note that U continuous over $X \times X \times \mathcal{Z}$ (with$\mathcal{Z}$ endowed with the natural discrete topology).
- Assumptions 16.1 and 16.2 imply that the objective function in Problem B1 is continuous in the product topology and the constraint set is compact.
- By Weierstrass's Theorem, a solution to this maximization problem exists and thus $V^*(x(0), z(0))$ is well defined.

Proof of Equivalence of Values Theorem II

- Berge's Maximum Theorem implies that $V^*(x(0), z(0))$ is continuous and thus bounded over the compact set $X \times \mathcal{Z}$.
- Now consider some $x(1) \in G(x(0), z(0))$. Another application of Weierstrass's Theorem implies that there exists $\mathbf{x}' \equiv \{\tilde{x}'[z^t]\}_{t=0}^{\infty} \in \Phi(x(1), z(1))$ attaining $V^*(x(1), z(1))$ for any $z(1) \in \mathcal{Z}$ (and with $\tilde{x}'[z^0] = x(1)$).
- This implies:

$$\mathbb{E}[V^*(x(1), z(1)) \mid z(0)] = \sum_{j=1}^{N} q_{jj'} V^*(x(1), z_j)$$

for j' defined by $z(0) = z_{j'}$.

Proof of Equivalence of Values Theorem III

- Next, since $(x(0), \mathbf{x}') \in \Phi(x(0), z(0))$ and $V^*(x(0), z(0))$ is the supremum in Problem B1 starting with $x(0)$ and $z(0) \in \mathcal{Z}$, the Lemma above implies:

$$\begin{aligned} V^*(x(0), z(0)) &\geq U\left(x(0), \tilde{x}'\left[z^0\right], z(0)\right) \\ &\quad +\beta\mathbb{E}\left[\mathbf{U}\left(\left\{\tilde{x}'\left[z^t\right]\right\}_{t=0}^{\infty}, z(1)\right) \mid z(0)\right], \\ &= U\left(x(0), \tilde{x}'\left[z^0\right], z(0)\right) \\ &\quad +\beta\mathbb{E}\left[V^*(x(1), z(1)) \mid z(0)\right], \end{aligned}$$

  and establishes (10).

- Next, take an arbitrary $\varepsilon > 0$. By (9), there exists $\mathbf{x}'_\varepsilon = \left(x(0), \tilde{x}'_\varepsilon\left[z^0\right], \tilde{x}'_\varepsilon\left[z^1\right] \ldots\right) \in \Phi(x(0), z(0))$ such that

$$\mathbf{U}\left(\mathbf{x}'_\varepsilon, z(0)\right) \geq V^*(x(0), z(0)) - \varepsilon.$$

Proof of Equivalence of Values Theorem IV

- By the feasibility of $\mathbf{x}'_{\varepsilon}$, we have $\mathbf{x}''_{\varepsilon} = \left(\tilde{x}'_{\varepsilon}\left[z^0\right], \tilde{x}'_{\varepsilon}\left[z^1\right], ...\right) \in \Phi\left(\tilde{x}'_{\varepsilon}\left[z^0\right], z(1)\right)$ for any $z(1) \in \mathcal{Z}$.
- Moreover, also by definition $V^*\left(\tilde{x}'_{\varepsilon}\left[z^0\right], z(1)\right)$ is the supremum in Problem B1 starting with the initial conditions $\tilde{x}'_{\varepsilon}\left[z^0\right]$ and $z(1)$.
- Then the Lemma above implies that for any $\varepsilon > 0$,

$$\begin{aligned} V^*(x(0), z(0)) - \varepsilon &\leq U\left(x(0), \tilde{x}'_{\varepsilon}\left[z^0\right], z(0)\right) \\ &\quad + \beta \mathbb{E}\left[\mathbf{U}\left(\left\{\tilde{x}\left[z^t\right]\right\}_{t=0}^{\infty}, z(1)\right) \mid z(0)\right] \\ &= U\left(x(0), \tilde{x}'_{\varepsilon}\left[z^0\right], z(0)\right) \\ &\quad + \beta \mathbb{E}\left[V^*\left(\tilde{x}'_{\varepsilon}\left[z^0\right], z(1)\right) \mid z(0)\right], \end{aligned}$$

  so that (11) is satisfied.
- This establishes that any solution to Problem B1 satisfies (10) and (11), and is thus a solution to Problem B2.

Proof of Equivalence of Values Theorem V

- To establish the converse, (10) implies for any $\tilde{x}\left[z^0\right] \in G\left(x(0), z(0)\right)$,

$$\begin{aligned} V\left(x(0), z(0)\right) \geq\ & U\left(x(0), \tilde{x}\left[z^0\right], z(0)\right) \\ & +\beta\mathbb{E}\left[V\left(\tilde{x}\left[z^0\right], z(1)\right) \mid z(0)\right]. \end{aligned}$$

- Substituting recursively for $V\left(\tilde{x}\left[z^0\right], z(1)\right)$, $V\left(\tilde{x}\left[z^1\right], z(2)\right)$, etc., and taking $\mathbb{E}$

$$\begin{aligned} V\left(x(0), z(0)\right) \geq\ & \mathbb{E}\left[\sum_{t=0}^{n} U\left(\tilde{x}\left[z^{t-1}\right], \tilde{x}\left[z^t\right], z(t)\right) \mid z(0)\right] \\ & +\beta^{n+1}\mathbb{E}\left[V\left(\tilde{x}\left[z^n\right], z(n+1)\right) \mid z(0)\right]. \end{aligned}$$

- By definition: $\lim_{n\to\infty} \mathbb{E}\left[\sum_{t=0}^{n} U\left(\tilde{x}\left[z^{t-1}\right], \tilde{x}\left[z^t\right], z(t)\right) \mid z(0)\right] = \mathbf{U}\left(\mathbf{x}, z(0)\right)$ B
- By the hypothesis of the theorem $\lim_{n\to\infty} \beta^{n+1}\mathbb{E}\left[V\left(\tilde{x}\left[z^n\right], z(n+1)\right) \mid z(0)\right] = 0$,
- So (8) is verified.

Proof of Equivalence of Values Theorem VI

- Let $\varepsilon > 0$ be a positive scalar. From (11), for any $\varepsilon' = \varepsilon(1-\beta) > 0$, exists $\tilde{x}_\varepsilon[z^0] \in G(x(0), z(0))$:

$$\begin{aligned} V(x(0), z(0)) &\leq U\left(x(0), \tilde{x}_\varepsilon\left[z^0\right]\right) \\ &\quad + \beta \mathbb{E} V\left(\tilde{x}_\varepsilon\left[z^0\right], z(1) \mid z(0)\right) + \varepsilon'. \end{aligned}$$

- Let $\tilde{x}_\varepsilon[z^t] \in G\left(\tilde{x}_\varepsilon\left[z^{t-1}\right], z(t)\right)$, with $\tilde{x}_\varepsilon\left[z^{-1}\right] = x(0)$, and define $\mathbf{x}_\varepsilon \equiv \left(x(0), \tilde{x}_\varepsilon\left[z^0\right], \tilde{x}_\varepsilon\left[z^1\right], \tilde{x}_\varepsilon\left[z^2\right] \ldots\right)$.

Proof of Equivalence of Values Theorem VII

- Substituting recursively $V\left(\tilde{x}_\varepsilon\left[z^1\right]\right)$, $V\left(\tilde{x}_\varepsilon\left[z^t\right]\right)$, etc. and taking expectations.

$$
\begin{aligned}
V\left(x\left(0\right), z\left(0\right)\right) \quad \leq & \quad \mathbb{E}\left[\sum_{t=0}^{n} U\left(\tilde{x}_\varepsilon\left[z^{t-1}\right], \tilde{x}_\varepsilon\left[z^t\right], z\left(t\right)\right) \mid z\left(0\right)\right] \\
& + \beta^{n+1}\mathbb{E}\left[V\left(\tilde{x}_\varepsilon\left[z^n\right], z\left(n+1\right)\right) \mid z\left(0\right)\right] \\
& + \varepsilon' + \varepsilon'\beta + \ldots + \varepsilon'\beta^n \\
\leq & \quad \mathbf{U}\left(\mathbf{x}_\varepsilon, z\left(0\right)\right) + \varepsilon,
\end{aligned}
$$

- Last step follows using $\varepsilon = \varepsilon' \sum_{t=0}^{\infty} \beta^t$ and that as $\lim_{n\to\infty} \mathbb{E}\left[\sum_{t=0}^{n} U\left(\tilde{x}_\varepsilon\left[z^{t-1}\right], \tilde{x}_\varepsilon\left[z^t\right], z\left(t\right)\right) \mid z\left(0\right)\right] = \mathbf{U}\left(\mathbf{x}_\varepsilon, z\left(0\right)\right)$.
- Thus V satisfies (9) and completes the proof.

Proof of Principle of Optimality Theorem I

- Suppose $\mathbf{x}^* \equiv \left(x(0), \tilde{x}^*\left[z^0\right], \tilde{x}^*\left[z^1\right], \tilde{x}^*\left[z^2\right], ...\right) \in \Phi(x(0), z(0))$ is a feasible plan attaining solution to Problem B1.
- Let $\mathbf{x}_t^* \equiv \left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], \tilde{x}^*\left[z^{t+1}\right], ...\right)$ be the continuation of this plan from time t.
- First show that for any $t \geq 0$, $\mathbf{x}_t^*$ attains the supremum starting from $\tilde{x}^*\left[z^{t-1}\right]$ and any $z(t) \in \mathcal{Z}$, that is,

$$\mathbf{U}(\mathbf{x}_t^*, z(t)) = V^*\left(\tilde{x}^*\left[z^{t-1}\right], z(t)\right). \quad (12)$$

- Proof is by induction: hypothesis is trivially satisfied for $t = 0$ since, by definition, $\mathbf{x}_0^* = \mathbf{x}^*$ attains $V^*(x(0), z(0))$.

Proof of Principle of Optimality Theorem II

- Next suppose that the statement is true for t, so that $\mathbf{x}_t^*$ attains the supremum starting from $\tilde{x}^*\left[z^{t-1}\right]$ and any $z(t) \in \mathcal{Z}$, or equivalently (12) holds for t and for $z(t) \in \mathcal{Z}$.
- Now using this relationship we will establish that (12) holds and $\mathbf{x}_{t+1}^*$ attains the supremum starting from $\tilde{x}^*\left[z^t\right]$ and any $z(t+1) \in \mathcal{Z}$.
- Equation (12) implies that

$$
\begin{aligned}
V^*(\tilde{x}^*\left[z^{t-1}\right], z(t)) &= \mathbf{U}(\mathbf{x}_t^*, z(t)) \qquad (13)\\
&= U\left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)\right) \\
&\quad + \beta \mathbb{E}\left[\mathbf{U}(\mathbf{x}_{t+1}^*, z(t+1)) \mid z(t)\right].
\end{aligned}
$$

- Let $\mathbf{x}_{t+1} = \left(\tilde{x}^*\left[z^t\right], \tilde{x}\left[z^{t+1}\right], ...\right) \in \Phi(\tilde{x}^*\left[z^t\right], z(t+1))$ be any feasible plan starting with state vector $\tilde{x}^*\left[z^t\right]$ and stochastic variable $z(t+1)$.

Proof of Principle of Optimality Theorem III

- By definition,
  $\mathbf{x}_t = \left(\tilde{x}^* \left[z^{t-1}\right], \mathbf{x}_{t+1}\right) \in \Phi\left(\tilde{x}^* \left[z^{t-1}\right], z\left(t\right)\right)$.
- By the induction hypothesis, $V^* \left(\tilde{x}^* \left[z^{t-1}\right], z\left(t\right)\right)$ is the supremum starting with $\tilde{x}^* \left[z^{t-1}\right]$ and $z\left(t\right)$:

$$\begin{aligned} V^* \left(\tilde{x}^* \left[z^{t-1}\right], z\left(t\right)\right) & \geq \mathbf{U}(\mathbf{x}_t, z\left(t\right)) \\ & = U\left(\tilde{x}^* \left[z^{t-1}\right], \tilde{x}^* \left[z^{t}\right], z\left(t\right)\right) \\ & \quad + \beta \mathbb{E}\left[\mathbf{U}(\mathbf{x}_{t+1}, z\left(t+1\right)) \mid z\left(t\right)\right] \end{aligned}$$

  for any $\mathbf{x}_{t+1}$.
- Combining this inequality with (13):

$$\begin{aligned} \mathbb{E}\left[V^*(\tilde{x}^* \left[z^{t}\right], z\left(t+1\right)) \mid z\left(t\right)\right] & = \mathbb{E}\left[\mathbf{U}(\mathbf{x}^*_{t+1}, z\left(t+1\right)) \mid z\left(t\right)\right] \\ & \geq \mathbb{E}\left[\mathbf{U}(\mathbf{x}_{t+1}, z\left(t+1\right)) \mid z\left(t\right)\right] \end{aligned}$$

  for all $\mathbf{x}_{t+1} \in \Phi(\tilde{x}^* \left[z^{t}\right], z\left(t+1\right))$.

Proof of Principle of Optimality Theorem IV

- Next, complete the proof that $\mathbf{x}_{t+1}^*$ attains supremum starting from $\tilde{x}^*\left[z^t\right]$ and any $z\left(t\right) \in \mathcal{Z}$ and equation (12) holds starting from $\tilde{x}^*\left[z^t\right]$ and any $z\left(t\right) \in \mathcal{Z}$.
- Suppose, to a obtain contradiction, that this is not the case.
- Then there exists $\hat{\mathbf{x}}_{t+1} \in \Phi(\tilde{x}^*\left[z^t\right], z\left(t+1\right))$ for some $z\left(t+1\right) = \hat{z}$ such that
$$\mathbf{U}(\mathbf{x}_{t+1}^*, \hat{z}) < \mathbf{U}(\hat{\mathbf{x}}_{t+1}, \hat{z}).$$
- Then construct the sequence $\hat{\mathbf{x}}_{t+1}^* = \mathbf{x}_{t+1}^*$ if $z\left(t\right) \neq \hat{z}$ and $\hat{\mathbf{x}}_{t+1}^* = \hat{\mathbf{x}}_{t+1}$ if $z\left(t\right) = \hat{z}$.
- Since $\mathbf{x}_{t+1}^* \in \Phi(\tilde{x}^*\left[z^t\right], \hat{z})$ and $\hat{\mathbf{x}}_{t+1} \in \Phi(\tilde{x}^*\left[z^t\right], \hat{z})$, we also have $\hat{\mathbf{x}}_{t+1}^* \in \Phi(\tilde{x}^*\left[z^t\right], \hat{z})$.

Proof of Principle of Optimality Theorem V

- Then without loss of generality taking $\hat{z} = z_1$,

$$\begin{aligned}
\mathbb{E}\left[\mathbf{U}(\hat{\mathbf{x}}^*_{t+1}, z(t+1)) \mid z(t)\right] &= \sum_{j=1}^{N} q_{jj'} \mathbf{U}(\hat{\mathbf{x}}^*_{t+1}, z_j) \\
&= q_{1j'} \mathbf{U}(\hat{\mathbf{x}}_{t+1}, z_j) + \sum_{j=2}^{N} q_{jj'} \mathbf{U}(\mathbf{x}^*_{t+1}, z_j) \\
&> q_{1j'} \mathbf{U}(\mathbf{x}^*_{t+1}, z_j) + \sum_{j=2}^{N} q_{jj'} \mathbf{U}(\mathbf{x}^*_{t+1}, z_j) \\
&= \mathbb{E}\left[\mathbf{U}(\mathbf{x}^*_{t+1}, z(t+1)) \mid z(t)\right],
\end{aligned}$$

contradicting (**??**) and completing the induction step, which establishes that $\mathbf{x}^*_{t+1}$ attains the supremum starting from $\tilde{x}^*[z^t]$ and any $z(t+1) \in \mathcal{Z}$.

Proof of Principle of Optimality Theorem VI

- Equation (12) then implies that

$$\begin{aligned} V^*\left(\tilde{x}^*\left[z^{t-1}\right], z(t)\right) &= \mathbf{U}(\mathbf{x}_t^*, z(t)) \\ &= U\left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)\right) \\ &\quad + \beta \mathbb{E}\left[\mathbf{U}(\mathbf{x}_{t+1}^*, z(t+1)) \mid z(t)\right] \\ &= U\left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)\right) \\ &\quad + \beta \mathbb{E}\left[V^*(\tilde{x}^*\left(z^t\right), z(t+1)) \mid z(t)\right], \end{aligned}$$

  establishing (6) and thus completing the proof of the first part.
- Now suppose that (6) holds for $\mathbf{x}^* \in \Phi(x(0), z(0))$.

Proof of Principle of Optimality Theorem VII

- Then substituting repeatedly for $\mathbf{x}^*$:

$$\begin{aligned} V^*(x(0), z(0)) &= \sum_{t=0}^{n} \beta^t U\left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)\right) \\ &\quad + \beta^{n+1}\mathbb{E}\left[V^*(\tilde{x}^*(z^n), z(n+1)) \mid z(0)\right]. \end{aligned}$$

- Since V^* is bounded, $\lim_{n\to\infty} \beta^{n+1}\mathbb{E}\left[V^*(\tilde{x}^*(z^n), z(n+1)) \mid z(0)\right] = 0$ and thus

$$\begin{aligned} \mathbf{U}(\mathbf{x}^*, z(0)) &= \lim_{n\to\infty} \sum_{t=0}^{n} \beta^t U\left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)\right) \\ &= V^*(x(0), z(0)), \end{aligned}$$

- Thus $\mathbf{x}^*$ attains the optimal value in Problem B1.
- This completes the proof of the second part of the theorem.

Proof of Existence Theorem I

- Consider Problem B2. In view of Assumptions 16.1 and 16.2, there exists some $M < \infty$, such that $|U(x, y, z)| < M$ for all $(x, y, z) \in \mathbf{X}_G$.
- This $|V^*(x, z)| \leq M/(1-\beta)$, all $x \in X$ and all $z \in \mathcal{Z}$.
- Consequently, consider the function $V^*(\cdot, \cdot) \in \mathbf{C}(X \times \mathcal{Z})$.
- $\mathbf{C}(X \times \mathcal{Z})$: set of continuous functions defined on $X \times \mathcal{Z}$, where X is endowed with the sup norm, $\|f\| = \sup_{x \in X} |f(x)|$ and $\mathcal{Z}$ is endowed with the discrete topology.
- Moreover, all functions in $\mathbf{C}(X \times \mathcal{Z})$ are bounded because they are continuous and both X and $\mathcal{Z}$ are compact.

Proof of Existence Theorem II

- Now define the operator T

$$TV(x,z) = \max_{y \in G(x,z)} \left\{ U(x,y,z) + \beta \mathbb{E}\left[V\left(y,z'\right) \mid z \right] \right\}. \qquad (14)$$

- Suppose that $V(x,z)$ is continuous and bounded.
- Then $\mathbb{E}\left[V\left(y,z'\right) \mid z\right]$ is also continuous and bounded, since it is simply given by

$$\mathbb{E}\left[V\left(y,z'\right) \mid z \right] \equiv \sum_{j=1}^{N} q_{jj'} V\left(y,z_j\right),$$

  with j' defined such that $z = z_{j'}$.
- Moreover, $U(x,y,z)$ is also continuous and bounded over $\mathbf{X}_G$.
- A fixed point of the operator T, $V(x,z) = TV(x,z)$, will then be a solution to Problem B2 for given $z \in \mathcal{Z}$.

Proof of Existence Theorem III

- T is well defined: Maximization problem (14): max. continuous function over compact set, by Weierstrass's Theorem it has a solution.
- Also satisfies Blackwell's sufficient conditions for a contraction.
- Contraction Mapping Theorem: unique fixed point $V \in \mathbf{C}(X \times \mathcal{Z})$ to (14) exists and this is also the unique solution to Problem B2.
- Now consider maximization in Problem B2.
- Since U and V are continuous and $G(x, z)$ is compact-valued, Weierstrass's Theorem implies that $y \in G(x, z)$ achieving the maximum exists.
- This defines the set of maximizers $\Pi(x, z) \subset \Phi(x, z)$ for Problem B2.
- Let $\mathbf{x}^* \equiv \left(x(0), \tilde{x}^*\left[z^0\right], \tilde{x}^*\left[z^1\right], \tilde{x}^*\left[z^2\right], \ldots\right) \in \Phi(x(0), z(0))$ with $\tilde{x}^*\left[z^t\right] \in \Pi\left(\tilde{x}^*\left[z^{t-1}\right], z(t)\right)$ for all $t \geq 0$ and each $z(t) \in \mathcal{Z}$. Then from the previous two Theorems , $\mathbf{x}^*$ is also an optimal plan for Problem B1. □

Stochastic Euler Equations I

- Use $*$'s to denote optimal values and D for gradients.
- Using Assumption 16.5 and differentiability of Value function Theorem, necessary conditions for an interior optimal plan:

$$D_y U(x, y^*, z) + \beta \mathbb{E}\left[D_x V\left(y^*, z'\right) \mid z\right] = 0, \tag{15}$$

  - $x \in \mathbb{R}^K$=current value of the state vector,
  - $z \in \mathcal{Z}$=current value of the stochastic variable, and
  - $D_x V\left(y^*, z'\right)$= gradient of the value function evaluated at next period's state vector y^*.

- Using the stochastic equivalent of the Envelope Theorem for dynamic programming and differentiating (5) with respect to the state vector, x::

$$D_x V\left(x, z\right) = D_x U(x, y^*, z). \tag{16}$$

Stochastic Euler Equations II

- No expectations, since equation is conditioned on the realization of $z \in \mathcal{Z}$.
- Note y^* here is a shorthand for $\pi(x, z)$.
- Combining these two equations, stochastic Euler equation:

$$D_y U(x, \pi(x, z), z) + \beta \mathbb{E}\left[D_x U\left(\pi(x, z), \pi\left(\pi(x, z), z'\right), z'\right) \mid z\right] = 0,$$

  - $D_x U$: gradient vector of U with respect to its first K arguments, and
  - $D_y U$: with respect to the second set of K arguments.

- In notation more congruent with the sequence version:

$$\begin{aligned} & D_y U(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)) \\ & +\beta \mathbb{E}\left[D_x U\left(\tilde{x}^*\left[z^t\right], \tilde{x}^*\left[z^{t+1}\right], z(t+1)\right) \mid z(t)\right] \\ = & \; 0, \end{aligned} \tag{17}$$

for $z^{t-1} \in \mathcal{Z}^{t-1}$.

Stochastic Euler Equations III

- Transversality condition? Discounted marginal return from state variable to tend to zero as planning horizon goes to infinity.
- Stochastic environment: look at expected returns, but what information to condition upon? In general,

$$\lim_{t\to\infty} \beta^t \mathbb{E}\left[\begin{array}{c} D_x U(\tilde{x}^*\left[z^{s+t-1}\right], \tilde{x}^*\left[z^{s+t}\right], z(s+t)) \\ \cdot \tilde{x}^*\left[z^{s+t-1}\right] \mid z(s) \end{array}\right] = 0 \qquad (18)$$

for all $z(s) \in \mathcal{Z}$ and $z^{s-1} \in \mathcal{Z}^{s-1}$.

Theorem (Euler Equations and the Transversality Condition) Let $X \subset \mathbb{R}^K_+$ and suppose that Assumptions 16.1-16.5 hold. Then the sequence of feasible plans $\{\tilde{x}^*[z^t]\}_{t=-1}^{\infty}$, with $\tilde{x}^*[z^t] \in \text{Int}\, G(\tilde{x}^*[z^{t-1}], z(t))$ for each $z(t) \in \mathcal{Z}$ and each $t = 0, 1, \ldots,$ is optimal for Problem B1 given $x(0)$ and $z(0) \in \mathcal{Z}$ if it satisfies (17) and (18).

Proof of Theorem: Sufficiency of Euler Equations and Transversality Conditions I

- Consider an arbitrary $x(0) \in X$ and $z(0) \in \mathcal{Z}$, and let $\mathbf{x}^* \equiv \{\tilde{x}^*[z^t]\}_{t=-1}^{\infty} \in \Phi(x(0), z(0))$ be a feasible plan satisfying (17) and (18).
- We first show that $\mathbf{x}^*$ yields a higher value than any other $\mathbf{x} \equiv \{\tilde{x}[z^t]\}_{t=-1}^{\infty} \in \Phi(x(0), z(0))$.
- For any $\mathbf{x} \in \Phi(x(0), z(0))$ and any $z^\infty \in \mathcal{Z}^\infty$ define

$$\Delta_{\mathbf{x}}(z^\infty) \equiv \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t \Big[U\left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)\right) - U\left(\tilde{x}\left[z^{t-1}\right], \tilde{x}\left[z^t\right], z(t)\right)\Big]$$

- i.e., the difference of the *realized* objective function between the feasible sequences $\mathbf{x}^*$ and $\mathbf{x}$.

Proof of Theorem: Sufficiency of Euler Equations and Transversality Conditions II

- From Assumptions 16.2 and 16.5, U is continuous, concave, and differentiable, so that for any $z^\infty \in \mathcal{Z}^\infty$ and any $\mathbf{x} \in \Phi(x(0), z(0))$

$$\begin{aligned}\Delta_{\mathbf{x}}(z^\infty) \geq & \lim_{T\to\infty} \sum_{t=0}^{T} \beta^t [D_x U\left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)\right) \\ & \cdot \left(\tilde{x}^*\left[z^{t-1}\right] - \tilde{x}\left[z^{t-1}\right]\right) \\ & + D_y U\left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)\right) \cdot \left(\tilde{x}^*\left[z^t\right] - \tilde{x}\left[z^t\right]\right)].\end{aligned}$$

- Since this is true for any $z^\infty \in \mathcal{Z}^\infty$, we can take expectations on both sides to obtain

$$\begin{aligned}& \mathbb{E}\left[\Delta_{\mathbf{x}}(z^\infty) \mid z(s)\right] \\ \geq & \lim_{T\to\infty} \mathbb{E}\left[\begin{array}{c} \sum_{t=0}^{T} \beta^t [D_x U\left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)\right) \\ \cdot \left(\tilde{x}^*\left[z^{t-1}\right] - \tilde{x}\left[z^{t-1}\right]\right) \mid z(s) \end{array}\right] \\ & + \lim_{T\to\infty} \mathbb{E}\left[\begin{array}{c} \sum_{t=0}^{T} \beta^t D_y U\left(\tilde{x}^*\left[z^{t-1}\right], \tilde{x}^*\left[z^t\right], z(t)\right) \\ \cdot \left(\tilde{x}^*\left[z^t\right] - \tilde{x}\left[z^t\right]\right) \mid z(s) \end{array}\right]\end{aligned}$$

Proof of Theorem: Sufficiency of Euler Equations and Transversality Conditions III

- Rearranging the previous expression, we obtain

$$
\begin{aligned}
&\mathbb{E}\left[\Delta_{\mathbf{x}}\left(z^{\infty}\right) \mid z\left(s\right)\right] \geq \\
&\lim_{T\to\infty} \mathbb{E}\left[\begin{array}{c} \sum_{t=0}^{T} \beta^{t} D_{y} U\left(\tilde{x}^{*}\left[z^{t-1}\right], \tilde{x}^{*}\left[z^{t}\right], z\left(t\right)\right) \\ \cdot\left(\tilde{x}^{*}\left[z^{t}\right] - \tilde{x}\left[z^{t}\right]\right) \mid z\left(s\right) \end{array}\right] \\
&\lim_{T\to\infty} \mathbb{E}\left[\begin{array}{c} \sum_{t=0}^{T} \beta^{t+1} D_{x} U\left(\tilde{x}^{*}\left[z^{t}\right], \tilde{x}^{*}\left[z^{t+1}\right], z\left(t+1\right)\right) \\ \cdot\left(\tilde{x}^{*}\left[z^{t}\right] - \tilde{x}\left[z^{t}\right]\right) \mid z\left(s\right) \end{array}\right] \\
&-\lim_{T\to\infty} \mathbb{E}\left[\begin{array}{c} \beta^{T+1} D_{x} U\left(\tilde{x}^{*}\left[z^{T}\right], \tilde{x}^{*}\left[z^{T+1}\right], z\left(T+1\right)\right) \\ \cdot\tilde{x}^{*}\left[z^{T}\right] \mid z\left(s\right) \end{array}\right] \\
&+\lim_{T\to\infty} \mathbb{E}\left[\begin{array}{c} \beta^{T+1} D_{x} U\left(\tilde{x}\left[z^{T}\right], \tilde{x}\left[z^{T+1}\right], z\left(T+1\right)\right) \\ \cdot\tilde{x}\left[z^{T}\right] \mid z\left(s\right) \end{array}\right].
\end{aligned}
$$

Proof of Theorem: Sufficiency of Euler Equations and Transversality Conditions IV

- Since $\mathbf{x}^* \equiv \{\tilde{x}^*[z^t]\}_{t=-1}^{\infty}$ satisfies (17), the terms in first and second lines are all equal to zero.
- Moreover, since $\mathbf{x}^* \equiv \{\tilde{x}^*[z^t]\}_{t=-1}^{\infty}$ satisfies (18), the third line is also equal to zero.
- Finally, since U is increasing in x, $D_x U \geq 0$, and $x \geq 0$, the fourth line is nonnegative, establishing that $\mathbb{E}[\Delta_{\mathbf{x}}(z^\infty) \mid z(s)] \geq 0$ for any $\mathbf{x} \in \Phi(x(0), z(0))$ and any $z(s) \in \mathcal{Z}$.
- Consequently, $\mathbf{x}^*$ yields higher value than any feasible $\mathbf{x} \in \Phi(x(0), z(0))$, and is therefore optimal.

Generalization to Markov Processes I

- What if z does not take on finitely many values?
- Simplest example: one-dimensional stochastic variable $z(t)$ given by the process $z(t) = \rho z(t-1) + \sigma \varepsilon(t)$, where $\varepsilon(t)$ has a standard normal distribution.
- Most of the results we care about generalize to such cases.
- But greater care in formulating in the sequence form of Problem B1 and in the recursive form of Problem B2.
- Need to ensure existence of feasible plans, which now need to be "measurable" with respect to the information set available at the time.
- To avoid long detour, assume both $\mathcal{Z}$ and X are compact and that the function $\tilde{x}[z^t]$ is "well-defined"—in particular, finite-valued and measurable.

Generalization to Markov Processes II

- Again representing all integrals with the expectations, we can state the main theorems for stochastic dynamic programming with general Markov processes.
- Define $\mathcal{Z}$ as a compact subset of $\mathbb{R}$ ($\mathcal{Z}$ as finite number of elements and $\mathcal{Z}$ as an interval are special cases).
- Let $z(t) \in \mathcal{Z}$ represent the uncertainty, and suppose its probability distribution can be represented as a Markov process,

$$\Pr[z(t) \mid z(0), ..., z(t-1)] \equiv \Pr[z(t) \mid z(t-1)].$$

- Again use the notation $z^t \equiv (z(0), z(1), ..., z(t))$ to represent the history of the realizations of the stochastic variable.
- Objective function and the constraint sets are represented as before: $\tilde{x}[z^t]$ again denotes a *feasible plan*.
- Set of feasible plans after history z^t denoted by $\Phi(\tilde{x}[z^{t-1}], z(t))$.
- Set of feasible plans starting with $z(0) \equiv z^0$ is then $\Phi(x(0), z^0)$.

Generalization to Markov Processes III

- Whenever there exists a function V that is a solution to Problem B2, define $\Pi(x,z) \subset \Phi(x,z)$ such that any $\pi(x,z) \in \Pi(x,z)$ satisfies
$$V(x,z) = U(x, \pi(x,z), z) + \beta \mathbb{E}\left[V(\pi(x,y), z') \mid z\right].$$
- Same assumptions as before but now require relevant functions to be *measurable* and correspondence $\Phi(x(t), z^t)$ to always admit *a measurable selection* for all $x(t) \in X$ and $z^t \in \mathcal{Z}^t$ (refer to these assumptions with a *).

Theorem **(Existence of Solutions)** Suppose that $\Phi\left(x(0), z^0\right)$ is nonempty for all $z^0 \in \mathcal{Z}$ and all $x(0) \in X$. Suppose also that for any $\mathbf{x} \in \Phi\left(x(0), z^0\right)$, $\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t U\left(\tilde{x}\left[z^{t-1}\right], \tilde{x}\left[z^t\right], z(t)\right) \mid z(0)\right]$ is well-defined and finite-valued. Then any solution $V(x,z)$ to Problem B2 coincides with the solution $V^*(x,z)$ to Problem B1. Moreover, if $\Pi(x,z)$ is non-empty for all $(x,z) \in X \times \mathcal{Z}$, then any $\pi(x,z) \in \Pi(x,z)$ achieves $V^*(x,z)$.

- Note imposes stronger requirements than Assumption 16.1.

Generalization to Markov Processes IV

Theorem **(Continuity of Value Functions)** Suppose the hypotheses in the Existence of Solutions Theorem are satisfied and Assumption 16.2* holds. Then there exists a unique function $V : X \times \mathcal{Z} \rightarrow \mathbb{R}$ that satisfies (4). Moreover, V is continuous and bounded. Finally, an optimal plan $\mathbf{x}^* \in \Phi(x(0), z(0))$ exists for any $x(0) \in X$ and any $z(0) \in \mathcal{Z}$.

Theorem **(Concavity of Value Functions)** Suppose the hypotheses in the Existence of Solutions Theorem are satisfied and Assumptions 16.2* and 16.3* hold. Then the unique function V that satisfies (4) is strictly concave in x for each $z \in \mathcal{Z}$. Moreover, the optimal plan can be expressed as $\tilde{x}^*[z^t] = \pi(x(t), z(t))$, where the policy function $\pi : X \times \mathcal{Z} \rightarrow X$ is continuous in x for each $z \in \mathcal{Z}$.

Generalization to Markov Processes* V

Theorem **(Monotonicity of Value Functions)** Suppose the hypotheses in the Existence of Solutions Theorem are satisfied and Assumptions 16.2* and 16.4* hold. Then the unique value function $V : X \times \mathcal{Z} \to \mathbb{R}$ that satisfies (4) is strictly increasing in x for each $z \in \mathcal{Z}$.

Theorem **(Differentiability of Value Functions)** Suppose the hypotheses in the Existence of Solutions Theorem are satisfied and Assumptions 16.2*, 16.3* and 16.5* hold. Let π be the policy function defined above and assume that $x' \in \text{Int} X$ and $\pi(x', z) \in \text{Int} G(x', z)$ for each $z \in \mathcal{Z}$, then $V(x, z)$ is continuously differentiable at x', with derivative given by

$$D_x V(x', z) = D_x U(x', \pi(x', z), z). \qquad (19)$$

Applications: The Permanent Income Hypothesis I

- Consider a consumer maximizing discounted lifetime utility

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c(t)),$$

- To start with assume that $u(\cdot)$ is strictly increasing, continuously differentiable and concave and denote its derivative by $u'(\cdot)$.
- Will shortly look at the case in which $u(\cdot)$ is given by a quadratic.
- Consumer can borrow and lend freely at a constant interest rate $r > 0$, lifetime budget constraint:

$$\sum_{t=0}^{\infty} \frac{1}{(1+r)^t} c(t) \leq \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} w(t) + a(0), \qquad (20)$$

- $a(0)$ denotes his initial assets and $w(t)$ is his labor income.

Applications: The Permanent Income Hypothesis II

- Assume $w(t)$ is random and takes values from the set $\mathcal{W} \equiv \{w_1, ..., w_N\}$.
- Suppose that $w(t)$ is distributed independently over time and the probability that $w(t) = w_j$ is q_j (naturally with $\sum_{j=1}^{N} q_j = 1$).
- Lifetime budget constraint (20) is a stochastic constraint: require it to hold *almost surely*, i.e. with probability 1.
- That lifetime budget constraint must hold with probability 1 imposes *endogenous borrowing constraints*.
- For example, suppose $w_1 = 0$ and $q_1 > 0$: then there is a positive probability that the individual will receive zero income for any sequence of periods of length $T < \infty$.
- Hence if he ever chooses $a(t) < 0$ there will be a positive probability of violating lifetime budget constraint, even with zero consumption in all future periods.

Applications: The Permanent Income Hypothesis III

- Thus, endogenous borrowing constraint:

$$a(t) \geq -\sum_{s=0}^{\infty} \frac{1}{(1+r)^s} w_1 \equiv -b_1,$$

  with w_1 denoting the minimum value of w within the set $\mathcal{W}$ and the last relationship defining b_1.
- First solve as a sequence problem: choosing sequence of feasible plans $\{\tilde{c}[w^t]\}_{t=0}^{\infty}$.
- Lagrangian: even though a single lifetime budget constraint (20), not a unique Lagrange multiplier λ.
- Consumption plans are made conditional on the realizations of events up to a certain date.
- In particular, consumption at time t will be conditioned on the history of shocks up to that date, $w^t \equiv (w(0), w(1), ..., w(t))$.

Applications: The Permanent Income Hypothesis IV

- Notation $\tilde{c}\left[w^t\right]$ emphasizes consumption at t is a mapping from the history of income realizations, w^t.
- Lagrange multiplier, representing marginal utility of money, is also a random variable and can depend only on w^t.
- Therefore write multiplier as $\tilde{\lambda}\left[w^t\right]$.
- The first-order conditions for this problem:

$$\beta^t u'\left(\tilde{c}\left[w^t\right]\right) = \frac{1}{(1+r)^t}\tilde{\lambda}\left[w^t\right], \tag{21}$$

- (Discounted) marginal utility of consumption after history w^t equated to the (discounted) marginal utility of income after history w^t, $\tilde{\lambda}\left[w^t\right]$.
- Economically interpretable, but not particularly useful unless we know law of motion of $\tilde{\lambda}\left[w^t\right]$.

Applications: The Permanent Income Hypothesis V

- Not straightforward to derive: formulation where prices for all possible claims to consumption contingent on any realization of history are introduced is more convenient for this.
- For now, formulate the same problem recursively.
- Flow budget constraint of the individual:

$$a' = (1+r)(a+w-c),$$

- Conversely, this implies $c = a + w - (1+r)^{-1} a'$.
- Value function conditioned on current asset holding a and current realization of the income shock w:

$$V(a,w) = \max_{a' \in [-b_1,(1+r)(a+w)]} \left\{ \begin{array}{c} u\left(a+w-(1+r)^{-1}a'\right) \\ +\beta \mathbb{E} V(a',w') \end{array} \right\},$$

Applications: The Permanent Income Hypothesis VI

- Used that w is distributed independently across periods: expectation of the continuation value not conditioned on current w.
- Need to restrict the set of feasible asset levels to be able to apply Theorems.
- Take $\bar{a} \equiv a(0) + w_N / r$, where w_N is the highest level of labor income.
- Impose that $a(t) \in [0, \bar{a}]$ and verify the conditions under which this has no effect on the solution.
- First-order condition for the maximization problem:

$$\frac{1}{1+r} u'(c(t)) = \beta \mathbb{E}_t \frac{\partial V(a(t+1), w(t+1))}{\partial a}. \tag{22}$$

- Noting that $\partial V(a', w') / \partial a$ is also the marginal utility of income, this equation is very similar to (21).

Applications: The Permanent Income Hypothesis VII

- But additional mileage now comes from the envelope condition from the diferentiability Theorem:

$$\frac{\partial V\left(a\left(t\right),w\left(t\right)\right)}{\partial a}=u'\left(c\left(t\right)\right).$$

- Combining this equation with (22), obtain the famous stochastic Euler equation of stochastic permanent income hypothesis:

$$u'\left(c\left(t\right)\right)=\beta\left(1+r\right)\mathbb{E}_{t}u'\left(c\left(t+1\right)\right). \tag{23}$$

- Equation becomes even simpler and perhaps more insightful when utility function is quadratic:

$$u\left(c\right)=\phi c-\frac{1}{2}c^{2},$$

with ϕ sufficiently large that in the relevant range $u\left(\cdot\right)$ is increasing in c.

Applications: The Permanent Income Hypothesis VIII

- Using this quadratic form with (23), Hall's famous stochastic equation:

$$c(t) = (1-\kappa)\phi + \kappa \mathbb{E}_t c(t+1), \tag{24}$$

  where $\kappa \equiv \beta(1+r)$.

- Striking prediction: variables such as current or past income should not predict future consumption growth.
  - Large empirical literature tests this focusing on *excess sensitivity*: if future consumption growth depends on current income, this is interpreted as evidence for excess sensitivity, rejecting (24).
  - Rejection often considered as evidence in favor of credit constraints
  - But excess sensitivity can also emerge when the utility function is not quadratic (see, for example, Zeldes, 1989, Caballero, 1990).

- Equation (24) takes an even simpler form when $\beta = (1+r)^{-1}$, i.e., when the discount factor is the inverse of the gross interest rate.

Applications: The Permanent Income Hypothesis IX

- In this case, $\kappa = 1$ and $c(t) = \mathbb{E}_t c(t+1)$ or $\mathbb{E}_t \Delta c(t+1) = 0$, so that the expected value of future consumption should be the same as today's consumption.
- Referred to as "martingale" property: random variable $z(t)$ is a martingale with respect to some information set Ω_t if $\mathbb{E}[z(t+1) \mid \Omega_t] = z(t)$.
- It is a submartingale, if $\mathbb{E}[z(t+1) \mid \Omega_t] \geq z(t)$ and supermartingale if $\mathbb{E}[z(t+1) \mid \Omega_t] \leq z(t)$.
- Thus whether consumption is a martingale, submartingales or supermartingale depends on the interest rate relative to the discount factor.

Applications: Search for Ideas I

- Problem of a single entrepreneur, with risk-neutral objective function

$$\sum_{t=0}^{\infty} \beta^t c(t).$$

- Entrepreneur's consumption given by the income he generates in that period (there is no saving or borrowing):

$$y(t) = a'(t)$$

- $a'(t)$ is the quality of the technique he has available for production.
- At $t = 0$, entrepreneur starts with $a(0) = 0$.
- At each date, can either engage in production using one of the techniques already or spend searching for a new technique.

Applications: Search for Ideas II

- Each period in search, he gets an independent draw from a time-invariant distribution function $H(a)$ defined over a bounded interval $[0, \bar{a}]$.
- Consumption decision is trivial: no saving or borrowing, has to consume current income, $c(t) = y(t)$.
- Write the maximization problem facing the entrepreneur as a sequence problem.
- Let $\mathbf{a}^t \in \mathbf{A}^t \equiv [0, \bar{a}]^t$=sequence of techniques observed by the entrepreneur over past t periods, with $a(s) = 0$, if at s engaged in production.
- Write $\mathbf{a}^t = (a(0), ..., a(t))$.
- Then a decision rule for this individual would be

$$q(t) : \mathbf{A}^t \to \{a(t)\} \cup \{\text{search}\},$$

Applications: Search for Ideas III

- $\mathcal{P}_t$: set of functions from $\mathbf{A}^t$ into $a(t) \cup \{\text{search}\}$, and $\mathcal{P}^\infty$ the set of infinite sequences of such functions.
- Individual's problem:

$$\max_{\{q(t)\}_{t=0}^{\infty} \in \mathcal{P}^\infty} \mathbb{E} \sum_{t=0}^{\infty} \beta^t c(t)$$

subject to $c(t) = 0$ if $q(t) =$"search" and $c(t) = a'$ if $q(t) = a'$ for $a(s) = a'$ for some $s \leq t$.

- Problem looks complicated but dynamic programming formulation quite tractable.
- Two observations from fact problem is stationary:
  1. Can denote value of an agent who has just sampled a technique $a \in [0, \bar{a}]$ by $V(a)$: can discard all techniques sampled except last one.
  2. Once start producing at technique a', continue forever: if willing produce at a' would also do so at time $t + 1$.

Applications: Search for Ideas IV

- Thus if production at some technique a' at date t, $c(s) = a'$ for all $s \geq t$.
- Thus value on accepting technique a':

$$V^{accept}(a') = \frac{a'}{1-\beta}.$$

- Therefore:

$$\begin{aligned} V(a') &= \max_{q \in \{0,1\}} qV^{accept}(a') + (1-q)\beta \mathbb{E}V \\ &= \max\left\{V^{accept}(a'), \beta \mathbb{E}V\right\} \\ &= \max\left\{\frac{a'}{1-\beta}, \beta \mathbb{E}V\right\}, \end{aligned} \tag{25}$$

  - q is acceptance decision ($q = 1$ is acceptance) and expected continuation value of not producing at available techniques is:

$$\mathbb{E}V = \int_0^{\bar{a}} V(a)\, dH(a) \tag{26}$$

A slight digression I

- Special structure of search problem enables a direct solution, but optimal policies can be derived with Contraction Mpping Techniques.
- For this, combine the two previous equations and write

$$
\begin{aligned}
V(a') &= \max\left\{\frac{a'}{1-\beta}, \beta\int_0^{\bar{a}} V(a)\,dH(a)\right\}, \qquad (27)\\
&= TV(a'),
\end{aligned}
$$

  where the second line defines the mapping T.
- Now (27) is in a form to which we can apply the above theorems.
- Blackwell's sufficiency theorem applies: T is a contraction since it is monotonic and satisfies discounting.
- Next, let $V \in \mathbf{C}([0, \bar{a}])$, i.e., the set of real-valued continuous (hence bounded) functions defined over the set $[0, \bar{a}]$, which is a complete metric space with the sup norm.
- Contraction Mapping Theorem implies unique value function $V(a)$ exists in this space.

A slight digression II

- Thus dynamic programming formulation immediately leads to existence of an optimal solution (and thus optimal strategies).
- Moreover, can apply Theorems on properties of contraction mappings, taking S' to be the space of nondecreasing continuous functions over $[0, \bar{a}]$, which is a closed subspace of $\mathbf{C}([0, \bar{a}])$.
- Therefore, $V(a)$ is nondecreasing.
- Could also prove that $V(a)$ is piecewise linear with first a flat portion and then an increasing portion.
- Let the space of such functions be S'', which is another subspace of $\mathbf{C}([0, \bar{a}])$, but is not closed.
- Starting with any nondecreasing function $V(a)$, $TV(a)$ will be a piecewise linear function starting with a flat portion.
- Theorems on properties of contraction mappings imply that the unique fixed point, $V(a)$, must have this property too.

Applications: Search for Ideas V

- The digression used Theorems on properties of contraction mappings to argue that $V(a)$ would take a piecewise linear form.
- Can also be deduced directly from (27): $V(a)$ is a maximum of two functions, one of them flat and the other one linear.
- Therefore $V(a)$ must be piecewise linear, with first a flat portion.
- Now determine the optimal policy using the recursive formulation of Problem B2.
- The fact that $V(a)$ is linear (and strictly increasing) after a flat portion immediately tells us that the optimal policy will take a *cutoff rule.*
- I.e., there will exist a cutoff technology level R such that all techniques above R are accepted and production starts.

Applications: Search for Ideas VI

- $V(a)$ is strictly increasing after some level: if some a' is accepted, all technologies with $a > a'$ will also be accepted.
- Moreover, this cutoff rule must satisfy:

$$\frac{R}{1-\beta} = \int_0^{\bar{a}} \beta V(a)\, dH(a), \tag{28}$$

- Also since $a < R$ are turned down, for all $a < R$

$$\begin{aligned} V(a) &= \beta \int_0^{\bar{a}} V(a)\, dH(a) \\ &= \frac{R}{1-\beta}, \end{aligned}$$

- And for all $a \geq R$, we have

$$V(a) = \frac{a}{1-\beta}.$$

Applications: Search for Ideas V

- Using these observations:

$$\int_0^{\bar{a}} V(a)\, dH(a) = \frac{RH(R)}{1-\beta} + \int_{a \geq R} \frac{a}{1-\beta} dH(a).$$

- Combining this equation with (28), we have

$$\frac{R}{1-\beta} = \beta \left[\frac{RH(R)}{1-\beta} + \int_{a \geq R} \frac{a}{1-\beta} dH(a) \right]. \tag{29}$$

- Manipulating this equation, we obtain

$$R = \frac{\beta}{1-\beta H(R)} \int_R^{\bar{a}} a dH(a),$$

- Equation (29) can be rewritten in a more useful way as follows:

$$\frac{R}{1-\beta} = \beta \left[\int_{a<R} \frac{R}{1-\beta} dH(a) + \int_{a \geq R} \frac{a}{1-\beta} dH(a) \right].$$

Applications: Search for Ideas VI

- Now subtracting $\beta R / (1-\beta) = \beta R \int_{a<R} dH(a) / (1-\beta) + \beta R \int_{a \geq R} dH(a) / (1-\beta)$ from both sides, we obtain

$$R = \frac{\beta}{1-\beta} \left[\int_{R}^{\bar{a}} (a-R) \, dH(a) \right], \tag{30}$$

- Left-hand side=cost of foregoing production with a technology R.
- Right-hand side=expected benefit of one more round of search.
- At the cutoff, have to be equal.
- Define the right-hand side of (30):

$$\gamma(R) \equiv \frac{\beta}{1-\beta} \left[\int_{R}^{\bar{a}} (a-R) \, dH(a) \right].$$

Applications: Search for Ideas VII

- Suppose also that H has a continuous density, denoted by h.
- Then we have

$$\begin{aligned}\gamma'(R) &= -\frac{\beta}{1-\beta}(R-R)\,h(R) - \frac{\beta}{1-\beta}\left[\int_R^{\bar{a}} dH(a)\right] \\ &= -\frac{\beta}{1-\beta}\left[1-H(R)\right] < 0\end{aligned}$$

- This implies that equation (30) has a unique solution.
- Higher β, by making the entrepreneur more patient, increases the cutoff threshold R.

Other Applications

1. *Asset Pricing*:
   - Lucas (1978): economy in which a set of identical agents trade claims on stochastic returns of a set of given assets ("trees").
   - Each agent solves a consumption smoothing problem similar but has to save in assets with stochastic returns rather than at a constant interest rate.
   - Market clearing will be achieved when the total supply of assets is equal to total demand: each agent is happy to hold the appropriate amount of claims on the returns from these assets.
2. *Investment under Uncertainty*.
3. *Optimal Stopping Problems*: search model discussed is an example.