Separable Preferences

- Let *I* be a finite (for now) set of indices (e.g. time periods, fruits, states). (*We will see a representation theorem for countably many time periods, it needs more assumptions. And the expected utility representations extend to uncountable state spaces, this also needs more structure.*)

- For each $i \in I$ there is a set X_i, let $X := \times_{i \in I} X_i$.

- Analyst observes complete transitive preference $\succsim$ on *X.*

- **Definition:** $\succsim$ has an *additively separable representation* if there are $u_i : X_i \to \mathbb{R}$ s.t. $U(x_1,...,x_n) = u_1(x_1) + ... + u_n(x_n)$ represents $\succsim$.

- In an additively separable representation, the tradeoff between any x_i and x_j is independent of the other components, i.e. of $X_{-i,j}$.

- For any $E \subseteq I$ and any $x, y \in X$ define $x_E y \in X := \begin{cases} x_i & i \in E \\ y_i & i \notin E \end{cases}$.

- **Definition:** $\succsim$ is *singleton separable* if for all $i \in I$ and all $x, y, z, z' \in X$, $x_i z \succsim y_i z \leftrightarrow x_i z' \succsim y_i z'$. (*this should remind you of the independence axiom of expected utility*!)

- Singleton separability implies that for each index *i* we have a complete transitive preference $\succsim_i$ on X_i that is independent of the other components: $x_i \succsim_i y_i$ if $x_i z \succsim y_i z$ for some *z.*

- But it doesn't yet imply that the tradeoff between any x_i and x_j is independent of the other components. (E.g. in the discounting application, we need the tradeoff between consumption in periods *t* and *s* to be independent of consumption in other periods.)

- For example if $X_1 = \{1,2,3\}, X_2 = \{1,3,5\}$, the preference induced by $u(x_1.x_2) = x_1 x_2 + x_1^{x_2}$ is strictly increasing in x_1 for each x_2 and vice versa. But it does not have an additive representation (HW).

- So we use a stronger condition:

  $\succsim$ has *jointly separable indices* if for any $E \subseteq I$ and all $x, y, z, z' \in X$, $x_E z \succsim y_E z \leftrightarrow x_E z' \succsim y_E z'$. (*Strzalecki calls this "separable."*).

- With 3 or more indices this say the tradeoffs between x_i and x_j don't depend on the level of some 3rd index *k.*

- For this to have any bite we need 3 indices that "matter."

- *Reason:* with only 2 indices, jointly separable indices reduces to singleton separability and as we saw that doesn't imply an additive representation. And having 3 with one that doesn't matter is like having 2.

- **Definition:** An index *i* is *null* if for all $x, y, z \in X$, $x_i z \sim y_i z$.

- The next theorem will ask that every index is non-null, and also that each X_i is connected; together these two conditions mean that each coordinate has a continuum of elements.

- Since additively separable representations have a utility function, we expect to need a continuity condition when *X* isn't finite.

- "Technical condition": Assume each X_i is a connected subset of $\mathbb{R}^k$ (or more generally a connected topological space) and that $\succsim$ on $X := \times_{i \in I} X_i$ is continuous w.r.t. the product topology.

Theorem (Debreu [1960], Wakker *J Math Pyschology* [1988]): Suppose complete transitive $\succsim$ satisfies the technical condition and has at least three non-null indices. Then it has jointly separable indices iff it has an additively separable representation by continuous utility functions $u_i : X_i \to \mathbb{R}$ s.t. u_i is constant whenever *i* is null. Moreover, if $v_1,...,v_n$ also represent $\succsim$ then there are $\alpha > 0$ and β_i s.t. $v_i = \alpha u_i + \beta_i$.

Proof: omitted.

Finite Horizon Consumption Streams

- Start with the finite-horizon case, indices $t \in \mathcal{T} := \{0,1,..,T\}$ for some $T > 1$.

- Assume that $X = Z^{T+1}$ where Z is a connected subset of $\mathbb{R}^n$, and X has the product topology.

- Debreu's theorem will let us get an additively separable representation, but the representation $U(z_0, z_1,..) = \sum_t \delta^t u(z_t)$ needs more assumptions to get the same function u in every period and a constant discount factor.

- **Definition:** $\succsim$ on $X = Z^{T+1}$ is *stationary* if for all $c \in Z, x, y \in Z^T$ $(c, x_0,..., x_{T-1}) \succsim (c, y_0,..., y_{T-1})$ iff $(x_0,..., x_{T-1,} c) \succsim (y_0,..., y_{T-1}, c)$.

- **Definition:** $\succsim$ on $X = Z^{T+1}$ is *sensitive* if all of the indices are non-null.

Theorem (Fishburn [1970], *Utility for Decision Making*): Complete transitive preference $\succsim$ on $X := \times_{t\in\mathcal{T}} Z$ is continuous, stationary, sensitive, and has jointly separable indices, iff there is a number $\delta > 0$ and a continuous non-constant function $u : Z \to \mathbb{R}$ s.t. $\succsim$ is represented by $U(z_0, z_1, ..) = \sum_t \delta^t u(z_t)$. Moreover δ is unique and u is unique up to affine transformations.

Proof sketch for sufficiency:

- Debreu [1960] implies there are non-constant and continuous functions $v_0, ..., v_1 : Z \to \mathbb{R}$ s.t. $U = \sum_{t=0}^{T} v_t(z_t)$ represents $\succsim$.
- Fix an arbitrary $e \in Z$ and rescale all the v_t so that $v_t(e) = 0$ for all *t.*
- Now define preference relation $\succsim^*$ on the "one period shorter" space Z^T by $(x_0, ..., x_{T-1}) \succsim^* (y_0, ..., y_{T-1})$ iff $(x_0, ..., x_{T-1}, e) \succsim (y_0, ..., y_{T-1}, e)$; this preference can be represented by $\sum_{t=0}^{T-1} v_t(z_t)$ from the last step.

- From stationarity

$$(x_0,\ldots,x_{T-1},e) \succsim (y_0,\ldots,y_{T-1},e)$$

iff

$$(e,x_0,\ldots,x_{T-1}) \succsim (e,y_0,\ldots,y_{T-1}).$$

- So $\succsim^*$ can also be represented by $\sum_{t=0}^{T-1} v_{t+1}(z_t)$.
- Affine uniqueness of the representation and sensitivity implies there exist $\delta_t > 0$ and b_t s.t. $v_t = \delta_t v_{t+1} + b_t$.

- And since $v_t(e) = 0$ for all *t*, every b_t is 0 as well, and from stationarity $\delta_t = \delta$.

- Define $u = v_0$, then $v_t = \delta^t u$. ■

Interpreting δ

- δ can be less, equal to, or even more than 1: the agent could care more about periods further in the future.

- Can interpret $\delta < 1$ either as pure time preference- prefer utility now to later- or as uncertainty that the consumption stream will continue.

- $\delta = 1$ gives the time average criterion $U(z_0, z_1, .., z_T) = \sum_{t=0}^{T} u(z_t)/(T+1)$.

- In some settings we expect δ to depend on the period length Δ according to $\delta = \exp(-\rho\Delta)$, though that's not part of this representation theorem.

- Conditions of the representation theorem are consistent with $\delta \approx 0$, but parents try to convince their kids that this is a bad idea.

Infinite Horizon Consumption Streams

- Now let $\mathcal{T} = \{0,1,...\}$, $X = Z^{\infty}$.
- The representation $U(z_0, z_1,..) = \sum_{t=0}^{\infty} \delta^t u(z_t)$ corresponds to similar conditions: again it needs stationarity, joint separability, and some forms of non-null indices and continuity.

- But things are more complicated here due to the possibility that the discounted sum diverges, and apparently technical conditions like continuity can have more substantive impact than you might have expected.

- It is also harder to say what we mean by "complete patience" or "time neutrality", as $\lim_{T\to\infty} \sum_{t=0}^{T} u(z_t)/(T+1)$ need not exist. (*this has been of special interest in modelling society's preferences as opposed to those of individuals, maybe talk about social time preference last lecture).*

- Two alternate “completely patient” criteria:

  - $\liminf_{T\to\infty}\sum_{t=0}^{T}u(z_t)/(T+1)$

  - the *overtaking criterion*: $z \succsim z'$ if there is a τ s.t. for all $T \geq \tau$ $\sum_{t=0}^{T}u(z_t) \geq \sum_{t=0}^{T}u(z_t^{'})$.

- **Definition:** Preference $\succsim$ is *stationary* if for all $c \in Z, x, y \in Z^{\infty}$, $(x_0, x_1, \ldots) \succsim (y_0, y_1, \ldots)$ iff $(c, x_0, x_1, \ldots) \succsim (c, y_0, y_1, \ldots)$.

- **Definition:** Preference $\succsim$ is *sensitive to the initial period* if index 0 is non-null.

- If we assume $Z = \mathbb{R}$ and set $u(z) = z$, time averaging is not sensitive to the initial period, the overtaking criterion is.

- **Definition:** $\succsim$ on $X = Z^{\infty}$ is *initially separable* if for all $a,b,c,d \in Z$ and $x,y \in Z^{\infty}$, $abx \succsim cdx$ iff $aby \succsim cdy$.

  This says that the tradeoff between first two periods doesn't depend on what comes afterwards.

- *Claim:* If $\succsim$ on $X = Z^{\infty}$ is stationary, then it is sensitive and jointly separable iff it is sensitive to the initial period and initially separable. *Proof :* HW

- **Definition:** $\succsim$ satisfies *constant equivalence* if for any $x \in Z^{\infty}$ there is $c \in Z$ s.t. $x \sim (c,c,\ldots) := \vec{c}$.

  Note: this jointly constrains the set Z *and how agent feels about "tail consumption"*, as $x = (1,2,4,\ldots)$ *might be better than any constant path.*

- **Definition:** $\succsim$ is *tail continuous* if for any $c \in Z$ and $x \in Z^{\infty}$ s.t.
  - (a) If $x \succ \vec{c}$, there exists τ s.t. for all $t \geq \tau$ $(x_0, \ldots x_t, c, c, \ldots) \succ \vec{c}$ and
  - (b) If $x \prec \vec{c}$, there exists τ s.t. for all $t \geq \tau$ $(x_0, \ldots x_t, c, c, \ldots) \prec \vec{c}$.

- Given the separability assumption, this is a 1-player version of the "continuity at infinity" condition that many of you saw in 14.122.

- Neither time average utility nor the overtaking criterion is tail continuous.

- This is related to the fact that the solution and maximized payoff to a finite-time optimization problem with time average payoff can be very different to the solution to the problem with an infinite horizon, while with tail continuity truncation doesn't matter much:

When to Harvest a Tree:

Suppose the agent owns a tree that grows at rate 1/year; if she chops it down at time T gets flow utility of 1 from period T to 2T-1.

So her menu of choices corresponds to this menu of utility flows:

$$\{(0,1,0,0....),(0,0,1,1,0,0,...),(0,0,0,1,1,1,0,0,...),(\underbrace{0,}_{\text{k times}}\underbrace{1,}_{\text{k times}}0,...),...\}.$$

With discount factor δ the payoff to chopping at T is $\dfrac{\delta^T(1-\delta^T)}{1-\delta}$; the best time is finite and increases in δ.

With the overtaking criterion the best time is "as late as possible," so no maximum if the time is unbounded.

And with time averaging no choice yields an improvement on chop at once.

Lemma: If $\succsim$ is continuous in the product topology and Z is compact, $\succsim$ is tail continuous.

Theorem If $\succsim$ on $X = Z^{\infty}$ is stationary, sensitive, and jointly separable, continuous in the product topology, and Z is connected and compact, then $\succsim$ can be represented by $U(z_0, z_1, ..) = \sum_{t=0}^{\infty} \delta^t u(z_t)$ with $\delta \in (0,1)$. Moreover δ is identified uniquely and u is cardinally unique.

Proof: omitted

Remarks:

- Tail continuity forces $\delta < 1$.

- Theorem doesn't need the same Z each period, but does need the max possible u to not grow "too quickly" compared to δ.
- One proof technique is to prove that the representation works on "eventually constant" paths and then extend it to all paths using tail continuity as in Strzalecki's notes and Bleichrodt, Rohde, and Wakker (*J. Math. Psych.* [2008]).

Let $\vec{c}$ denote a constant path $(c, c, \ldots)$.

Definition: $\succsim$ *likes consumption smoothing more than* $\succsim'$ if for all $c \in Z$ and $x \in X$, $\vec{c} \succsim' x \rightarrow \vec{c} \succsim x$, and also $\vec{c} \succ' x \rightarrow \vec{c} \succ x$:

If $\succsim'$ likes (or strictly prefers) a constant consumption plan more than a given alternative, so does $\succsim$.

Proposition (Strzalecki) If $\succsim$ likes consumption smoothing more than $\succsim'$ and they both have discounting representations (u, δ) and (u', δ') with *u, u'* continuous and strictly increasing, then $\delta = \delta'$ and $u' = \phi \circ u$ for some ϕ that is strictly increasing and concave.

- This is a partial order on preferences.

- A strict order on $u(z) = -\exp(-Az)$ or $u(z) = z^{\alpha}$: higher *A* or lower α likes smoothing more.

- Suppose the agent is maximizing $\sum_{t=0}^{\infty} \delta^t u(z_t)$ given a sequence of prices and $u(z) = z^\alpha$. Then the agent has a *constant elasticity of intertemporal substitution,* i.e. the response of z_{t+1} / z_t to changes in prices is constant.

- If we extend the additively separable discounting representation to expected utility, it links the agent's risk aversion and preference for smoothing over time.

- This has led to interest in more flexible specifications such as Epstein-Zinn: $V_t(z) = \left((1-\delta)z_t^\rho + \delta V_{t+1}(z)^{\rho/\alpha}\right)^{\alpha/\rho}$; will discuss this next class.

Dynamic Choice

- So far this lecture has only considered period-0 preference.

- Additive discounting is recursively consistent if time-t preference maximizes $\sum_{\tau=t}^{\infty} \delta^{\tau-t} u(x_\tau) := V_t(x)$.

- Here we have the familiar dynamic programming equation

$$V_t(x) = u(x_t) + \delta V_{t+1}(x).$$

- *Note: we can represent the same choices with the "normalized" value function* $U_t(x) =: (1-\delta)\sum_{\tau=t}^{\infty} \delta^{\tau-t} u(x_\tau)$ so that $U_t = (1-\delta)u(z_t) + \delta U_{t+1}$ and *value is measured in per-period utility.*

- Koopmans *Ema* [1960] defined more general *recursive preferences*:

$$V_t(x) = W(x_t, V_{t+1}(x)) \text{ ,}$$

with W strictly increasing in its 2nd argument, and each V_t only depending on $(x_t, x_{t+1}, ...)$ (and not on x_τ for $\tau < t$, as it might with e.g. habit formation.)

This says that the preferred choice today depends on current consumption (which I set to 0 in the special 2-period model of last lecture) and the value of the continuation problem.

- Period-0 preferences $\succsim_0$ satisfy *tail separability* if for any $t > 0, c, d \in Z^{t+1}$, and $x, y \in Z^\infty$, $(c_0,...,c_t,x) \succsim_0 (c_0,...,c_t,y)$ iff $(d_0,...,d_t,x) \succsim_0 (d_0,...,d_t,y)$: preferences from period *t+1* don't depend on consumption through period t.

- All Koopmans preferences are tail separable.

- For tail-separable preferences we can define time-t preferences by $x \succsim_{t+1} y$ if $(c_0,...,c_t,x) \succsim_0 (c_0,...,c_t,y)$ for some $(c_0,...c_t)$.

- And with tail separability the family of preferences $\{\succsim_t\}_t$ is recursively consistent- this makes them intuitive/easy to work with.

- Koopmans preferences are used in macro and growth (e.g. Lucas and Stokey *JET* [1984], Straub and Werning [2015]) both to test the generality of conclusions and to better fit some data.

- Straub and Werning identify $\partial W / \partial V$ with the "local discount factor"- this is the sensitivity of the aggregator to tomorrow's value.

- Some of their results assume this is non-constant, which rules out the usual constant-discount-factor representation.

- One special case of Koopmans preferences is Uzawa [1968] preferences, where V is additively separable between today's utility and the continuation value

  $V_t(x) = u(x_t) + \delta(u(x_t))V_{t+1}(x)$, so $\partial W / \partial V$ is independent of V but depends on *x.*

  Then $V_0(x) = u(x_0) + \delta(u(x_0))u(x_1) + \delta(u(x_0))\delta(u(x_1))u(x_2) + ...$

- Uzawa preferences let impatience depend on current consumption, but only through its impact on current flow utility.

- Literature assumes δ monotone and compares "increasing marginal impatience" and "decreasing marginal impatience."

- Used in growth and macro, e.g. Obstfeld *J Mon E* [1990], which studied the implied optimal consumption and savings paths, and how they respond to taxes and to transitory shocks.

Reading for next time: Strzalecki 6.1-6.5; Tversky and Kahneman *Science* [1974]. *Optional:* read Strzalecki 8.1-8.2