

The Wald Problem and the Relation of Sequential Sampling and Ex-Ante Information Costs

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Abstract

We consider the two state sequential sampling problem of [Wald \(1945\)](#). We show that any distribution of posteriors can be attained and that the ex-ante cost of attaining any probability distribution over posteriors equals the expected change of the log likelihood ratio. With many states, not all distributions can be attained but we provide a characterization of those that can be attained and show that the characterization of ex-ante cost generalizes naturally to those that can be attained.

The sequential sampling model of [Wald \(1945\)](#) is arguably one of the oldest and most widely used models of information acquisition. We provide a novel reduced form characterization of Wald's model that relates it to flexible information acquisition model a la [Sims \(2003\)](#). In the sequential sampling problem of [Wald](#), a decision maker sequentially observes signals at a cost and dynamically decides when to stop acquiring information. Any stopping rule gives rise to a probability distribution over posteriors and we show that in the two state case any distribution over posteriors can be induced by some stopping rule. The ex-ante cost of acquiring information can then be identified with the expected stopping time. We show that the expected stopping time, and thus the ex-ante cost of information, corresponding to the Wald model, is proportional to the expected change between the prior and posterior log likelihood ratio

$$q \log \left(\frac{q}{1-q} \right) + (1-q) \log \left(\frac{1-q}{q} \right).$$

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This expression generalizes naturally to the many state case, where the decision maker observes a diffusion whose drift depends on the state. In this case, the cost of information is given by the normalized expected change between the prior and the posterior weighted log likelihood ratio.

$$\sum_{i,j} \frac{q_i \log\left(\frac{q_i}{q_j}\right)}{(n-1)(m_i - m_j)^2},$$

where the summations are over states and m_i is drift in state i . Thus, the cost of distinguishing nearby states i and j , where m_i is close to m_j , is higher than the cost of distinguishing far away states. This expression gives the cost of attaining any distribution over posteriors, if that distribution is feasible, independent of the stopping rule that gives rise to it. However, in the many state case, it is no longer possible to attain any distribution over posteriors. We provide a characterization of which distributions over posteriors may arise which follows from a result of [Rost \(1971\)](#) on Skorokhod embeddings for Markov processes.

We thus identify an ex-ante cost of acquiring information that arises from the classical sequential sampling approach proposed in [Wald \(1945\)](#). Ex-ante, or static, cost functions of information are widely used in economics. [Sims \(2003\)](#) proposed the relative entropy cost function to model attention costs, reflecting the capacity necessary to transmit the information; but the same functional form has also been widely used to represent the cost of information acquisition. More generally, a static cost function is said to be posterior-separable ([Caplin and Dean, 2013](#)) if it is the expectation of a strictly convex function of posteriors $\phi : [0, 1] \rightarrow \mathbb{R}_+$. The cost function that we identify for the Wald problem is an example of a posterior-separable cost function.

We show that all posterior-separable cost functions also have a sequential sampling foundation. A generalization of Wald's sampling problem (for the two state case) allows for a posterior dependent sequential sampling cost $c : [0, 1] \rightarrow \mathbb{R}_+$, where $c(p)$ is the flow cost of continuing to sample when the current posterior equals p . We characterize a one-to-one functional which maps sequential sampling flow costs to posterior-separable ex-ante cost functions. This functional maps the sequential sampling cost $p(1-p)$ to the two state entropy cost ([Sims, 2010](#)), which corresponds to the expected change in entropy

$$q \log q + (1-q) \log(1-q).$$

We also provide a many state analogue of this result.

We motivate our results by spelling out how existing economic results taking a sequential sampling approach to information acquisition are equivalent to taking a ex-ante approach to the cost of information. We describe the equivalence between Bayesian per-

suasion with sequential sampling (as in [Brocas and Carrillo \(2007\)](#)) or with ex-ante information acquisition (as in [Kamenica and Gentzkow \(2011\)](#)). The Wald cost function is linear-in-precision, and we contrast the economic implications of the Wald cost function with the more concave entropy cost function in investment problems.¹

Our work relates to [Hébert and Woodford \(2017\)](#) and [Zhong \(2017a\)](#) in connecting dynamic information acquisition models to ex-ante cost of information. While [Hébert and Woodford \(2017\)](#) analyze a model where the signals in each period are endogenously chosen by the agent, we analyze the classical model introduced in [Wald \(1945\)](#) where signals are exogenously fixed. [Hébert and Woodford \(2017\)](#) rule out the Wald model through continuity and finiteness assumptions on the cost of sampling.² Closely related [Zhong \(2017a\)](#) provides conditions on the cost function in a problem with endogenous information choice, two states, and discounting, under which the posterior belief in the continuous time limit becomes a pure jump process. Interestingly, he shows that the belief under the optimal information acquisition policy evolves deterministically before an action is taken. [Zhong \(2017b\)](#) argues that with endogenous information choice and linear waiting cost the dynamic information acquisition problem can be reduced into a static one. [Caplin and Dean \(2015\)](#); [Chambers, Liu, and Rehbeck \(2017\)](#); [Andrew Caplin and Leahy \(2017\)](#) provide revealed preference characterizations of the static information acquisition problem, which we show to be equivalent to a generalization of the Wald problem if there are only two states.

Finally, [Pomatto, Strack, and Tamuz \(2018\)](#) provide an axiomatic characterization of all ex-ante cost functions with constant marginal cost of information. The ex-ante cost corresponding to the Wald problem turns out to be a special case. Our result complements their work as it suggests a specific well founded function in the larger space of constant marginal cost functions they identify.

1 The Sequential Learning Problem with Two States

We begin by considering the case of only two states which much of the literature focused on and will extend our result to the case of many states in Section 2. There are two states $\theta \in \{-1, +1\}$ and a single agent who wants to learn about the state. Time is continuous and at every instant in time the agent observes a process $(Z_t)_{t \geq 0}$ to learn about the state θ . We denote by p_0 the prior probability the agent assigns to the state $\theta = +1$. The change

¹[van Nieuwerburgh and Veldcamp \(2010\)](#) studied this comparison restricting attention to normal signals; we study a two state analogue where the linear-in-precision Wald cost function is micro-founded and arbitrary distributions over posteriors are possible.

²While ruling out the Wald case, [Hébert and Woodford \(2017\)](#) also describe a general equivalence between posterior-separable cost functions and some non-constant flow costs. The representation in the current version of their paper was obtained simultaneously with our characterization.

of the process Z is the sum of the state plus a noise term which is the increment of a Brownian motion $(W_t)_{t \geq 0}$

$$dZ_t = \theta dt + \sigma dW_t. \quad (1)$$

Denote by p_t the posterior probability the agent assigns to the high state at time t

$$p_t \triangleq \mathbb{P}[\theta = +1 \mid (Z_s)_{s \leq t}].$$

At every point in time the agent can decide to either continue or stop observing the signal Z , depending on the information she has already received. For every instant in time where the agent observes the signal she pays a bounded and positive flow cost $c(\cdot)$ which depends on her posterior belief, i.e. if the agent observes the signals until time t the total cost she pays for observing the signal is given by

$$\int_0^t c(p_s) ds.$$

In the classical formulation of the problem proposed in Wald (1945) this cost is constant. A strategy of the agent is a stopping time τ adapted to the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ generated by the process Z . Every such stopping time τ generates a distribution over final values of the belief process p , which we denote by F_τ

$$F_\tau(\cdot) \triangleq \mathbb{P}[p_\tau \leq \cdot].$$

We define the ex ante cost of distribution $G \in \Delta([0, 1])$ of posterior beliefs³ to be the minimal cost at which the agent can generate that distribution:

$$C(G) \triangleq \inf_{\tau: F_\tau = G} \mathbb{E} \left[\int_0^\tau c(p_s) ds \right],$$

where we set C to infinity whenever the set $\{\tau: F_\tau = G\}$ is empty and there is no sampling strategy that achieves the distribution over posteriors G .

Our goal is to characterize the ex-ante cost $C(G)$ as a function of the distribution over posterior beliefs G . We start by stating the dynamics of the belief process:

Lemma 1. *The belief process $(p_t)_{t \geq 0}$ solves the stochastic differential equation (SDE)*

$$dp_t = p_t(1 - p_t) \frac{2}{\sigma} dB_t,$$

³Throughout we denote by $\Delta([0, 1])$ the space of cumulative distribution functions on $[0, 1]$, i.e. $G \in \Delta([0, 1])$ is equivalent to $G: [0, 1] \rightarrow [0, 1]$ and G non-decreasing.

where $B_t \triangleq \frac{1}{\sigma} \{X_t - \int_0^t (2p_s - 1) ds\}$ is a Brownian motion with respect to the agent's information.

Let us denote the mean of a distribution G by $\mu_G \triangleq \int_0^1 q dG(q)$. Next, we observe that for a given distribution G with support on $[0, 1]$ there exists a strategy τ in the sequential sampling problem, which leads to that distribution of posterior beliefs if and only if G is consistent with the prior belief p_0 .

Proposition 1. *There exists τ with $F_\tau = G$ if and only if $p_0 = \mu_G$.*

While Proposition 1 shows that any distribution of posterior beliefs can be achieved by some sampling strategy not every distribution can be achieved by learning only for a finite time. For example, to learn the state perfectly, i.e. achieve the posterior distribution that assigns mass only to 0 and 1 the agent needs to never stop learning. More generally, a distribution of posterior beliefs can be achieved using a *finite* stopping strategy if and only if the distribution assigns no mass to 0 and 1.

Lemma 2. *F_τ assigns strictly positive probability to either 0 or 1 if and only if $\tau = \infty$ with strictly positive probability.*

It is worth noting that there will be in general multiple learning strategies which will lead to a given distribution of posteriors. One case where the strategy is unique is if the distribution G consists only of two mass points. In this case the strategy constructing this distribution is the first leaving time of the interval with end-points equal to the mass points.

One force that keeps the agent from learning forever are the sampling costs she pays. In the following section we will characterize the sampling costs only as a function of the posterior distribution.

Define the function $\phi_c : [0, 1] \rightarrow \mathbb{R}$ by

$$\phi_c(q) \triangleq \int_{p_0}^q \int_{p_0}^x \frac{c(y)\sigma^2}{2[y(1-y)]^2} dy dx. \quad (2)$$

Our first result shows that the sampling cost is given by the expected value of ϕ_c under F_τ .

Proposition 2 (Characterization of Sampling Cost). *For every a.s. finite stopping strategy τ the sampling cost satisfies*

$$\int_0^1 \phi_c(q) dF_\tau(q) = \mathbb{E} \left[\int_0^\tau c(p_s) ds \right].$$

Proposition 2 characterizes the cost associated with any strategy in the sequential sampling problem and shows that it only depends on the induced posterior belief distribution F_τ . Any sampling strategy that leads to a given posterior distribution will have the same cost. It thus does not matter which sampling strategy the agent uses to attain a given distribution over posteriors and for most posterior distributions the optimal sampling strategy will not be unique. In combination with Proposition 1 this leads to a characterization of the ex ante cost $C(G)$ associated with a given distribution of posterior beliefs G :

Theorem 1 (Equivalence of Static Cost and Sequential Sampling).

1. The ex ante cost $C : \Delta([0, 1]) \rightarrow \mathbb{R}_+$ of attaining a posterior belief distribution G is given

$$C(G) = \begin{cases} \int_0^1 \psi(q) dG(q) & \text{if } \mu_G = p_0 \\ \infty & \text{else} \end{cases}, \quad (3)$$

with $\psi = \phi_c$ defined in (2).

2. Conversely, any posterior-separable ex-ante cost C that solves (3) with ψ twice differentiable is induced by a Wald problem with prior p_0 and flow cost $c(\cdot)$ of

$$c(q) = \psi''(q) \frac{2[q(1-q)]^2}{\sigma^2}. \quad (4)$$

Theorem 1 establishes a one-to-one relation between ex-ante cost functions $C(\cdot)$ and dynamic flow cost $c(\cdot)$ in the sequential sampling problem. For every flow cost in the sequential sampling problem there exists a unique function $\phi_c(\cdot)$ defined in (2) such that $C(G) = \int_0^1 \phi_c(q) dG(q)$ is an equivalent static cost function. Conversely, every affine cost function which is sufficiently smooth can be represented by some flow cost in the dynamic sequential sampling problem.

1.1 Equivalence of Wald's Sequential Sampling Model to Log-Likelihood Cost

In this section we show that Theorem 1 implies that the classical sequential sampling problem introduced in the seminal paper by Wald (1945) where the flow cost is constant is equivalent to a problem where the agent can choose any distribution $G \in \Delta([0, 1])$ of posterior beliefs and pays ex-ante cost $C_{Wald}(G)$ equal to the expected change in log-likelihood-ratio. We define the expected log-likelihood-ratio (LLR) $L : [0, 1] \rightarrow \mathbb{R}$ by

$$L(q) \triangleq q \log \left(\frac{q}{1-q} \right) + (1-q) \log \left(\frac{1-q}{q} \right). \quad (5)$$

The cost which correspond to the cost of information acquisition in the sequential sampling problem turns out the proportional to the average change in LLR from the prior p_0 to the posterior, which distributed according to G .

Proposition 3 (Reformulation of Wald’s Sequential Sampling). *If c is constant, C is given by*

$$C_{Wald}(G) = \begin{cases} \frac{c\sigma^2}{2} \left\{ \int_0^1 L(q) dG(q) - L(p_0) \right\} & \text{if } \mu_G = p_0 \\ \infty & \text{else} \end{cases}. \quad (6)$$

It is worth noting that (6) assigns infinite cost to any distribution G that assigns positive mass to the posterior beliefs zero or one. This reflects the fact that to learn the state with certainty (with some non-zero probability), the agent has to collect a sample of infinite expected size, which, as the cost per unit of time is constant in the Wald model, leads to infinite costs. An immediate consequence of Proposition 3 is that the Wald model is in terms of behavior equivalent to a model with posterior-separable ex-ante cost (as introduced in [Caplin and Dean, 2013](#)).

1.2 Equivalence of Entropy Cost to Posterior Variance Flow Cost

We next engage in the opposite exercise and determine the dynamic flow cost corresponding to the commonly used ex-ante entropy cost proposed by [Sims \(2003\)](#). Sims proposed to model the cost of information as the change in the negative of the entropy $H : [0, 1] \rightarrow \mathbb{R}$ given by

$$H(q) = q \log(q) + (1 - q) \log(1 - q). \quad (7)$$

We can ask the inverse question, which (belief dependent) cost function $c : [0, 1] \rightarrow \mathbb{R}_+$ in the sequential sampling problem is equivalent to entropy cost. Plugging in (4) yields that the answer to this turns out to be

$$c_{Entropy}(q) = \frac{2}{\sigma^2} q(1 - q). \quad (8)$$

Proposition 4 (Reformulation of Entropy Cost as Sequential Sampling). *When the cost function in the sequential sampling problem is given by $c_{Entropy}$ then*

$$C(G) \triangleq \begin{cases} \int_0^1 H(q) dG(q) - H(p_0) & \text{if } \mu_G = p_0 \\ \infty & \text{else} \end{cases}.$$

Note, that the posterior variance when the agent posterior equals q is given by

$$\text{var}(q) = \mathbb{E}_q \left[(\theta - \mathbb{E}_q[\theta])^2 \right] = 4q(1 - q)$$

and the flow cost corresponding to the entropy model

$$c_{Entropy}(q) = \frac{var(q)}{2\sigma^2}$$

is thus proportional to the variance of the posterior. This means that whenever the agent is relatively sure about the state (and the posterior variance is low) the cost of sampling another signal is low as well. The marginal cost of learning is thus decreasing: the more an agent already knows the cheaper it becomes to acquire further information. While this concavity might describe some situations well where the cost represent the burden of mental processing it seems less natural in situations of physical information acquisition.

1.3 Illustration: Comparing the Ex-Ante and Sequential-Sampling Approaches in Economic Applications

Consider an agent who is deciding what information, i.e. distribution over posterior beliefs G , to generate. Suppose that the decision maker's indirect utility of posteriors is given by the function $v : [0, 1] \rightarrow \mathbb{R}_+$. We will first be agnostic on the origin of the indirect utility function and discuss how solve the decision maker's problem with the ex-ante information cost function and then discuss the sequential sampling of the same problem.

Given a posterior-separable ex-ante cost function of information $C(G) = \alpha \int_0^1 \psi(q) dG(q)$ the decision problem is to solve

$$\max_G \int_0^1 (v(q) - \alpha \psi(q)) dG(q) \tag{9}$$

subject to the belief consistency constraint that $\mu_G = p_0$. This problem can be solved graphically by considering the concavification of the function $(q \mapsto v(q) - C(q))$. [Kamenica and Gentzkow \(2011\)](#) pointed this out in the case where there was no cost of information and [Gentzkow and Kamenica \(2014\)](#) extended this observation to the case of posterior separable cost. A first contribution of our result is that we complement [Gentzkow and Kamenica \(2014\)](#) by providing the cost function that is micro-founded in the Wald problem and can be used under this solution method.

Our equivalence result means that the solution to the problem with ex-ante cost is the same as the solution to the sequential sampling problem with the equivalent flow cost where the agent commits to a stopping rule. [Brocas and Carrillo \(2007\)](#) have examined a discrete time version of this problem, both with no cost of sampling and - in an extension in their Proposition 2 - with a constant flow cost. Our model is the continuous time limit. In this sense, a second contribution of our result is to shows an equivalence - for the two state case - between the analysis of [Brocas and Carrillo \(2007\)](#) with costless and

constant flow cost and the analysis of [Kamenica and Gentzkow \(2011\)](#) and [Gentzkow and Kamenica \(2014\)](#), respectively, when we use our derived expected log likelihood cost function for the posterior-separable cost function.

This analysis of optimal information acquisition is valid whatever the source of the indirect utility function $v(\cdot)$. The agent making the information acquisition decision might be using the information to choose an action of his own (the single agent case). But the agent might be a “persuader” who can commit to make the outcome of the information acquisition public and seeks to influence a different decision maker (the conflict of interest case). This latter interpretation is the focus of the work of [Brocas and Carrillo \(2007\)](#) and [Kamenica and Gentzkow \(2011\)](#). We emphasize that the analysis is exactly the same in either case.

We have discussed an equivalence between the static information acquisition problem and the sequential sampling problem when the agent commits to a stopping rule ex-ante. A natural question is: what if the agent cannot commit to a stopping rule and makes an interim optimal choice? In the single decision maker case, this clearly does not make a difference. But what about the conflict of interest case? A third contribution of our result is that it highlights why it cannot make a difference. To see why, consider the stopping rule that the decision maker would want to commit to. This rule will be of the form that either the decision maker stops immediately, or keeps sampling until the posterior hits a lower or upper bound. But the decision maker will have no incentive to deviate from that strategy, as continuing to sample within the interval is optimal and stopping is optimal once either the lower or upper bounds are reached. [Henry and Ottaviani \(2018\)](#) have already highlighted this equivalence.

Given the equivalence between the ex ante information acquisition and the interim sampling analysis, we see that the methods to solve optimal sampling and stopping problems must tie in to the concavification analysis described above. In fact, [Dayanik and Karatzas \(2003\)](#) have established this equivalence and propose to use concavification methods to solve optimal stopping problems. A final contribution of our result is that it highlights this connection.⁴ While we have not yet done so, it seems likely that working directly with our closed form expression for the Wald cost function can simplify the analysis and improve the interpretation of results in this literature.

⁴[Henry and Ottaviani \(2018\)](#) note that the solution to their “Wald persuasion” optimal stopping problem (Proposition 1) must reduce to the two state special case of [Kamenica and Gentzkow \(2011\)](#) when there is no discounting and no cost of information (in part c). We can add the observation that the solution must reduce to the two state special case of [Gentzkow and Kamenica \(2014\)](#) if there is no discounting but a positive cost of information.

1.4 Illustration: The Wald cost function versus Entropy cost function

We now discuss the comparison between the Wald cost function and the entropy cost function. An immediate observation is that if the value function v is bounded, because the Wald cost function is unbounded, it will never be optimal to acquire perfect information. But because the entropy cost function is bounded, it is possible that it will sometimes be optimal to acquire perfect information. This alludes to the broader point that cost of information acquisition vanishes around certainty for Sims model while it is non vanishing in the Wald model.

We can illustrate this difference in a classical economic insurance and consumption problem. Suppose that an agent must decide how to allocation consumption between state -1 or +1; if she consumes x in state 1, she can consume $1 - x$ in state -1 . Suppose that she is an expected utility maximizer with constant relative risk aversion $\rho > 0$ and thus von Neumann Morgenstern utility function

$$u_{\rho}(x) = \begin{cases} \frac{x^{1-\rho}-1}{1-\rho} & \text{if } \rho \neq 1 \\ \log(x) & \text{if } \rho = 1 \end{cases}.$$

Now the value function is defined to be

$$v_{\rho}(q) \triangleq \max_x (q u_{\rho}(x) + (1 - q) u_{\rho}(1 - x))$$

and calculation shows that

$$v_{\rho}(q) = \begin{cases} \frac{\left(q^{\frac{1}{\rho}} + (1-q)^{\frac{1}{\rho}}\right)^{\rho} - 1}{1-\rho} & \text{if } \rho \neq 1 \\ q \log(q) + (1-q) \log(1-q) & \text{if } \rho = 1 \end{cases}. \quad (10)$$

Thus the value function of an agent with log utility has the the same functional form as entropy cost function.⁵ Unsurprisingly, we will see that therefore it will be important for a decision maker with entropy cost of information whether his relative risk aversion is greater than or less than 1. For a general convex symmetric value function $v : [0, 1] \rightarrow \mathbb{R}$ and cost function $\psi : [0, 1] \rightarrow \mathbb{R}$ we can define the ratio between the marginal benefit of

⁵Entropy measures the value of information that arises in a number of other settings: for an agent seeking to avoid ruin in an investment problem (Cabrales et al., 2013) and for an investors choosing a multidimensional portfolio in a CARA-normal setting (van Nieuwerburgh and Veldcamp, 2010).

information and the marginal cost of information at a posterior belief q by

$$M(q) \triangleq \frac{v''(q)}{\psi''(q)}.$$

M measures the benefit cost ratio in the sense that by Ito's lemma the expected gain from sampling a marginal unit of time equals $q(1-q)\frac{2}{\sigma}v''(q)$ while the cost equals $\alpha q(1-q)\frac{2}{\sigma}\psi''(q)$.

We say that the *benefit cost ratio is decreasing in certainty* if $M(q)$ is strictly increasing for $q \in (0, 1/2)$ and by symmetry strictly decreasing for $q \in (1/2, 1)$. We say that the benefit cost ratio is increasing in certainty if the opposite conclusion holds.

Proposition 5. *Consider the information acquisition problem given in (9). Suppose that the benefit cost ratio is decreasing in certainty then:*

1. *If the cost of information is small $\alpha < M(1)$ the agent learns the state perfectly.*
2. *If the cost of information is intermediate $\alpha \in (M(1), M(1/2))$ there exists an interval $(q^*, 1 - q^*)$ such that the agent acquires information if and only if $q \in (q^*, 1 - q^*)$. In this case the signal he acquires will lead to a posterior q^* or $1 - q^*$.*
3. *If the cost of information is large $\alpha > M(1/2)$ the agent will not acquire information.*

Suppose that the benefit cost ratio is increasing in certainty then:

1. *If the cost of information is small $\alpha < M(1/2)$ the agent learns the state perfectly.*
2. *If the cost of information is intermediate $\alpha \in (M(1/2), M(1))$ there exists an interval $(q^*, 1 - q^*)$ such that the agent acquires information if and only if $q \notin [q^*, 1 - q^*]$. In this case the signal he acquires will lead to a posterior of either 0 or q^* if $q < q^*$ and either $1 - q^*$ or 1 if $q > q^*$.*
3. *If the cost of information is large $\alpha > M(1)$ the agent will not acquire information.*

Proposition 5 shows that the optimal information acquisition strategy looks qualitatively different depending on whether the cost benefit ratio M is increasing or decreasing. If it is increasing towards certainty then the agent optimally acquires information if and only if her prior is close to certainty if it is decreasing the agent acquires information if she is relatively uncertain about the state of the world, ie. her prior is close to $1/2$.

Whether the benefit cost ratio is increasing or decreasing depends on the shape of the cost function. The following lemma shows that the cost function corresponding to the Wald model leads to qualitatively different predictions from entropy cost.

Lemma 3. *The benefit cost ratio M*

1. *is increasing (decreasing) towards certainty under entropy cost and relative risk aversion ρ less than 1 (greater than 1),*
2. *and is decreasing towards certainty under the Wald cost given in (5).*

Lemma 3 establishes that the learning strategies under Wald and entropy cost are very different. Under entropy the decision maker will acquire more information when she is already relatively *certain* about the state of the world (q close to 0 or 1), while under Wald cost the decision maker will acquire information if she is relative uncertain about the state of the world (q close to $1/2$). Intuitively, as entropy cost has decreasing marginal cost of information it is more attractive to acquire information if one is already well informed.

The Wald model delivers a cost function that is linear in the precision of posteriors. There is a long tradition of considering a linear cost in the precision of information when information comes in the form of normal signals : see [Wilson \(1975\)](#) for an early reference, [van Nieuwerburgh and Veldcamp \(2010\)](#) for a recent application discussed further below and [Veldcamp \(2011\)](#) for a review of this literature. The assumption that the cost of normal signals is linear in the precision can be naturally given a Wald-like motivation: suppose that a decision maker can choose ex ante how many signals to acquire and there is a constant cost of signals. Our cost function is more general in that it allows the decision maker to condition his decision to acquire more information on what he has learnt so far. [van Nieuwerburgh and Veldcamp \(2010\)](#) compare linear cost and entropy cost in an investment problem with normal signal and (while they focus on a different question)⁶ our analysis for the two state example parallels theirs in showing how agents' investment problems will have interior optima with the linear-in-precision cost function but have corner solutions with the more concave entropy cost function.

2 The Sequential Learning Problem with Many States

We now generalize our results beyond the two state case. Consider the generalization of the Wald model to the case with $n \geq 2$ states denoted by $m_1, \dots, m_n \in \mathbb{R}$. As in the two state case we assume that the signal $(Z_t)_{t \geq 0}$ equals the state $\theta \in \{m_1, \dots, m_n\}$ plus a noise term which is the increment of a Brownian motion $(W_t)_{t \geq 0}$

$$dZ_t = \theta dt + \sigma dW_t. \tag{11}$$

We denote by $p_t^i = \mathbb{P}[\theta = m_i \mid (Z_s)_{s \leq t}]$ the posterior probability assigned to state i conditional on the signal realization up to time t . Optimal stopping in this generalization of the

⁶They consider a multi-asset portfolio choice problem and consider how an investor allocates a fixed amount of attention across assets.

Wald model has been considered in the statistical literature (e.g. Chernoff 1961; Bather and Walker 1962) under the assumption that the prior p_0 is Normally distributed. Our next lemma characterizes the evolution of the posterior belief:

Lemma 4. *The belief process $(p_t^i)_{t \geq 0}$ solves the SDE*

$$dp_t^i = p_t^i \left(m_i - \sum_{j=1}^n p_t^j m_j \right) \frac{1}{\sigma} dB_t,$$

where $B_t \triangleq \frac{1}{\sigma} \{X_t - \int_0^t (2p_s - 1) ds\}$ is a Brownian motion with respect to the agent's information.

Feasible Posterior Distributions and Learning Strategies We start out by describing the set of feasible posterior distributions, i.e. the set of distributions G such that there exists a (potentially randomized) stopping time τ with $G \sim p_\tau$. We first state a tractable sufficient condition. To derive this condition we first characterize the posterior log-likelihood ratio between any pair of states:

Lemma 5. *The posterior log-likelihood between state i and j equals*

$$\log \frac{p_\tau^i}{p_\tau^j} = \log \frac{p_0^i}{p_0^j} + (m_i - m_j) \left(Z_t - t \frac{m_i + m_j}{2} \right). \quad (12)$$

An immediate consequence of (12) is the following necessary condition for a distribution to be implementable:

Proposition 6. *If there exists a stopping time τ such that $p_\tau \sim G$ then for every belief q in the support of G we have that there exists constants $z \in \mathbb{R}$ and $t \geq 0$ such that for all states i, j*

$$\log \frac{q^i}{p_0^i} - m_i \left(z - t \frac{m_i}{2} \right) = \log \frac{q^j}{p_0^j} - m_j \left(z - t \frac{m_j}{2} \right). \quad (13)$$

Proposition (6) implies that the support of G is contained in a two-dimensional manifold. Note, that as the belief space is of dimension $n - 1$ this immediately implies that not all distributions of posterior beliefs can be the result of some learning strategy τ when there are at least 4 states. Notably, (13) of Proposition (6) implies a uniform continuity restriction on the posterior distribution if the prior probabilities of two states are close and their associated mean signals m_i and m_j are close then their posterior log-likelihood must be close.

While Proposition (6) states a necessary condition it is not sufficient. The following proposition provides a sharp characterization of all distributions which can be the result of some learning strategy.

Proposition 7 (Rost, 1971). *There exists a (potentially randomized) strategy τ such that $G = F_\tau$ if and only if for every non-negative function $h : [0, 1]^n \rightarrow \mathbb{R}_+$*

$$\mathbb{E} \left[\int_0^\infty h(p_t) \mid p_0 \sim G \right] \leq \mathbb{E} \left[\int_0^\infty h(p_t) \mid p_0 \right]. \quad (14)$$

That the condition of Proposition 7 is necessary is an immediate consequence of the strong Markov property of the belief process as the distribution F_τ of the posterior belief at time τ equals G and h is non-negative

$$\begin{aligned} \mathbb{E} \left[\int_0^\infty h(p_t) \mid p_0 \right] &= \mathbb{E} \left[\int_0^\tau h(p_t) \mid p_0 \right] + \mathbb{E} \left[\int_\tau^\infty h(p_t) \mid p_0 \right] \\ &= \mathbb{E} \left[\int_0^\tau h(p_t) \mid p_0 \right] + \mathbb{E} \left[\int_0^\infty h(p_t) \mid p_0 \sim F_\tau \right] \\ &\geq \mathbb{E} \left[\int_0^\infty h(p_t) \mid p_0 \sim G \right]. \end{aligned}$$

That it is also sufficient is not immediate and was shown for general Markov processes in Rost (1971). From now on we denote by

\mathcal{G}

the set of all distributions which satisfy (14).

Sampling Cost As in the two state case we assume that for every instant in time where the agent observes the signal she pays a bounded and positive flow cost $c(\cdot)$ which depends on her posterior belief, i.e. if the agent observes the signals until time t the total cost she pays for observing the signal is given by

$$\int_0^t c(p_s) ds.$$

While there are now many distributions of the posterior belief for which no sampling strategy exists the cost associated with a given posterior distribution F_τ can be characterized using the same argument we used in the case of two states. To this end assume that there exists a function $\phi_c : \mathbb{R}^n \rightarrow \mathbb{R}$, with $\phi_c(p_0) = 0$ such for all p^7

$$c(p) = \frac{1}{2\sigma^2} \sum_{i,j=1}^n \phi_{ci,j}(p) p^i p^j \left(m_i - \sum_{j=1}^n p^j m_j \right) \left(m_j - \sum_{j=1}^n p^j m_j \right). \quad (15)$$

⁷We denote by $\phi_{ci,j}(p) = \frac{\partial^2}{\partial p^i \partial p^j} \phi_c(p)$ the cross derivative with respect to p^i and p^j .

Note, that in the case of two states $n = 2$ and $m_1 = +1, m_2 = -1$ one can express ϕ_c as a function of the first variable only $\phi_c(p^1, p^2) = \phi_c(p^1)$. Then, condition (15) simplifies to

$$c(p^1) = \frac{1}{2\sigma^2} \phi_c''(p^1) (p^1)^2 (1 - \{p^1(+1) - (1-p^1)(-1)\})^2 = \frac{2}{\sigma^2} \phi_c''(p^1) (p^1(1-p^1))^2$$

$$\Rightarrow \phi_c''(p^1) = \frac{\sigma^2 c(p)}{2(p^1(1-p^1))^2},$$

which is equivalent to (2). Thus, (15) generalizes the definition of ϕ_c given in the two state model.

Our next result shows that the cost of acquiring a posterior distribution F_τ is characterized by a result analogous to the two state case:

Proposition 8 (Characterization of Sampling Cost). *For every a.s. finite stopping strategy τ the sampling cost satisfies*

$$\int_0^1 \phi_c(q) dF_\tau(q) = \mathbb{E} \left[\int_0^\tau c(p_s) ds \right].$$

Proposition 2 characterizes the cost associated with any sampling strategy in the Wald problem and shows that it only depends on the induced posterior belief distribution F_τ . As in the two state case, any sampling strategy that leads to a given posterior distribution will have the same cost. It thus does not matter which sampling strategy the agent uses to attain a given distribution over posteriors and for many posterior distributions the optimal sampling strategy will not be unique.

As in the two state case Proposition 2 immediately leads to a static cost function representing the cost dynamic cost in Wald's sequential sampling problem: We define the distance weighted expected log-likelihood-ratio (LLR) $L : [0, 1] \rightarrow \mathbb{R}$ by

$$L(q) \triangleq \sum_{i,j} \frac{q_i \log\left(\frac{q_i}{q_j}\right)}{(n-1)(m_i - m_j)^2}. \quad (16)$$

Proposition 9 (Reformulation of Wald's Sequential Sampling). *If c is constant, the equivalent static cost function C is given by*

$$C_{Wald}(G) = \begin{cases} 2c\sigma^2 \left\{ \int_0^1 L(q) dG(q) - L(p_0) \right\} & \text{if } G \in \mathcal{G} \\ \infty & \text{else} \end{cases}. \quad (17)$$

The multi-state version of Shannon entropy $H : \mathbb{R}^n \rightarrow \mathbb{R}$ is given by

$$H(q) = \sum_{i=1}^n q_i \log(q_i). \quad (18)$$

We can again ask the inverse question, which (belief dependent) cost function $c : \mathbb{R}^n \rightarrow \mathbb{R}_+$ in the sequential sampling problem is equivalent to entropy. Plugging in (15) yields that the answer to this turns out to be exactly the posterior variance

$$c_{Entropy}(q) = \frac{1}{2\sigma^2} \sum_{i=1}^n q_i \left(m_i - \sum_{j=1}^n q^j m_j \right)^2.$$

Proposition 10 (Reformulation of Entropy Cost as Sequential Sampling). *When the cost function in the sequential sampling problem is given by $c_{Entropy}$ then the equivalent static cost function equals entropy*

$$C(G) \triangleq \begin{cases} \int_0^1 H(q) dG(q) - H(p_0) & \text{if } G \in \mathcal{G} \\ \infty & \text{else} \end{cases}.$$

This result may seem paradoxical, as the entropy cost function does not depend on the difference between states, yet the corresponding flow cost $c_{Entropy}$ depends on the drift m_i corresponding to the different states. The resolution is that the dependence of the flow cost on drift exactly counteracts the role of the difference between states, so this drops out of the final expression for cost.

Applications for the many state case are more complicated because not all distributions of posteriors can arise. Our characterization of the Wald cost function could be used to put bounds on solutions to optimization problems if we ignore feasibility constraints.

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Proofs

Proof of Lemma 1. Follows from [Liptser and Shiryaev \(2013\)](#) Theorem 9.1. \square

Proof of Proposition 1. Theorem 2 in [Ankirchner, Hobson, and Strack \(2015\)](#) implies that there exists a stopping time τ such that $F_\tau = G$ if G has mean p_0 . The first part of Proposition 2 in [Ankirchner, Hobson, and Strack \(2015\)](#) shows that F_τ must have mean p_0 . \square

Proof of Lemma 2. By Feller's test for explosions (c.f. Theorem 5.29 in [Karatzas and Shreve, 2012](#)) the belief process $(p_t)_t$ hits neither 0 nor 1 in finite time. Hence, it follows that $\tau = \infty$ with strictly positive probability whenever G assigns positive probability to either 0 or 1.

To see that the converse is also true observe that $(p_t)_t$ converges almost surely to 0 or 1 and thus whenever $\tau = \infty$, the final distribution over posterior must assign strictly positive probability to 0 or 1. \square

Proof of Proposition 2. Let τ be an arbitrary a.s. finite stopping time. To simplify notation we define $\eta(q) \triangleq \frac{2}{\sigma}q(1-q)$. By Ito's Lemma, we have that

$$\phi_c(p_\tau) = \phi_c(p_0) + \int_0^\tau \phi_c'(p_s)\eta(p_s)dB_s + \int_0^\tau \phi_c''(p_s)\frac{\eta(p_s)^2}{2}ds \quad (19)$$

By definition $\phi_c(p_0) = 0$. Define,⁸

$$r_n \triangleq n \wedge \inf \left\{ t : \int_0^t |\phi_c'(p_s)|^2 \eta(p_s)^2 ds \geq n \right\}. \quad (20)$$

Consider the sequence of random variables $(\phi_c(p_{\tau \wedge r_n}))_n$. As, due to the definition of r_n (given in (20)), the integral over $|\phi_c'|^2 \eta^2$ is bounded, the second summand in (19) is a Martingale and thus vanishes when taking expectations

$$\mathbb{E}[\phi_c(p_{\tau \wedge r_n})] = \mathbb{E} \left[\int_0^{\tau \wedge r_n} \phi_c''(p_s)\frac{\eta(p_s)^2}{2} ds \right] = \mathbb{E} \left[\int_0^{\tau \wedge r_n} c(p_s) ds \right]. \quad (21)$$

Here, the last equality follows, by the definition of ϕ_c

$$\phi_c''(p_s) = \frac{2c(p_s)}{\eta(p_s)^2}. \quad (22)$$

Note, that the sequence r_n satisfies $r_n \rightarrow \infty$ for $n \rightarrow \infty$. As p is a bounded Martingale it

⁸We set the infimum over the empty set to be infinity.

a.s. converges. Consequently, $p_{\tau \wedge r_n}$ converges to p_τ for $n \rightarrow \infty$. By Fatou's Lemma

$$\begin{aligned} \mathbb{E}[\phi_c(p_\tau)] &= \mathbb{E}\left[\liminf_{n \rightarrow \infty} \phi_c(p_{\tau \wedge r_n})\right] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[\phi_c(p_{\tau \wedge r_n})] = \liminf_{n \rightarrow \infty} \mathbb{E}\left[\int_0^{\tau \wedge r_n} c(p_s) ds\right] \\ &\leq \mathbb{E}\left[\int_0^\tau c(p_s) ds\right]. \end{aligned}$$

As $p_\tau \sim F_\tau$ it follows, that $\int_0^1 \phi_c(q) dF_\tau(q) \leq \mathbb{E}\left[\int_0^\tau c(p_s) ds\right]$.

To complete the proof and it remains to be shown that equality holds. Assume that

$$\int_0^1 \phi_c(q) dF_\tau(q) < \infty$$

as otherwise the equality is an immediate conclusion of the above inequality. As ϕ_c is convex and $(p_t)_t$ is a Martingale it follows from Jensen's inequality that

$$\phi_c(p_{\tau \wedge r_n}) \leq \mathbb{E}[\phi_c(p_\tau) \mid \mathcal{F}_{\tau \wedge r_n}]. \quad (23)$$

As the right-hand side of (23) is a conditional expectation of $\phi_c(p_\tau)$ and $\mathbb{E}[\phi_c(p_\tau)] < \infty$ it follows from the law of iterated expectations that

$$\mathbb{E}[\mathbb{E}[\phi_c(p_\tau) \mid \mathcal{F}_{\tau \wedge r_n}]] = \mathbb{E}[\phi_c(p_\tau)] < \infty$$

the right-hand side of (23) is uniformly integrable. As the right-hand side is uniformly integrable the left-hand side of (23) is uniformly integrable as well. Consequently,

$$\lim_{n \rightarrow \infty} \mathbb{E}[\phi_c(p_{\tau \wedge r_n})] = \mathbb{E}\left[\lim_{n \rightarrow \infty} \phi_c(p_{\tau \wedge r_n})\right] = \mathbb{E}[\phi_c(p_\tau)] = \int_0^1 \phi_c(q) dF_\tau(q).$$

We thus have that,

$$\int_0^1 \phi_c(q) dF_\tau(q) = \lim_{n \rightarrow \infty} \mathbb{E}\left[\int_0^{\tau \wedge r_n} c(p_s) ds\right].$$

As $c \geq 0$ and r_n is non-decreasing the sequence on the right-hand side is increasing and we can apply the monotone convergence theorem to get that

$$\lim_{n \rightarrow \infty} \mathbb{E}\left[\int_0^{\tau \wedge r_n} c(p_s) ds\right] = \mathbb{E}\left[\lim_{n \rightarrow \infty} \int_0^{\tau \wedge r_n} c(p_s) ds\right] = \mathbb{E}\left[\int_0^\tau c(p_s) ds\right]. \quad \square$$

Proof of Theorem 1. We begin by characterizing the static cost function $C(\cdot)$ which is equivalent to the dynamic cost in the Wald problem. First, note that by Proposition 1 the sampling cost are infinite if $\mu_G \neq p_0$ and there exists a stopping time τ such that $F_\tau = G$

whenever $\mu_G = p_0$. By Proposition 2 $C(G) = \int_0^1 \phi_c(q) dG(q)$. Thus, for every dynamic cost in the Wald problem there exist an equivalent static cost function.

To show the converse we note, plugging in the flow cost c given in (6) yields that

$$\phi_c(q) = \psi(q) - \psi(p_0) - (q - p_0)\psi'(p_0).$$

Taking the expectation over the posterior $q \sim G$, with mean $p_0 = \mu_G$ yields that

$$\int_0^1 \phi_c(q) dG(q) = \int_0^1 \psi(q) dG(q) - \psi(p_0),$$

which completes the proof. \square

Proof of Proposition 3. The result follows from Theorem 1 by observing that the function ϕ_c defined in (2) for constant cost c is given by

$$\phi_c(q) = \frac{c \sigma^2}{2} \{L(q) - L(p_0) - (q - p_0)L'(p_0)\}.$$

As the mean of G is given by p_0 the second term vanishes when taking the expectation with respect to G . \square

Proof of Proposition 4. The result follows from the second part of Theorem 1 with $\psi = H$. \square

Proof of Proposition 5. The agent choose an information structure that solves

$$\int_0^1 (v(q) - \alpha \psi(q)) dG(q).$$

If $M(q) = \frac{v''(q)}{\psi''(q)}$ is decreasing towards certainty then $h(q) \triangleq v(q) - \alpha \psi(q)$ changes it's second derivative at most once from positive to negative for $q \in (0, 1/2)$. This implies that there exists $q^* \in [0, 1/2]$ such that h is strictly concave in $(0, q^*)$ and convex in $(q^*, 1/2)$. Thus, no information acquisition is optimal in $(0, q^*)$ and information acquisition is optimal in $(q^*, 1/2)$. If $\alpha < M(1)$ then $h''(0) = h''(1) > 0$ and thus $q^* = 0$ and the agent acquires information for every prior. If $\alpha > M(1/2)$ then $h''(1/2) < 0$ and thus $q^* = 1/2$ and the agent never acquires information. The proof if M increases towards certainty is completely analogous. \square

Proof of Lemma 3. We begin by considering entropy cost $\psi(q) = q \log(q) + (1 - q) \log(1 -$

q). In this case we have that

$$M(q) = \frac{v''(q)}{\psi''(q)} = \frac{2(\rho - 1)((1 - q)q)^{\frac{1}{\rho} - 1} (q^{1/\rho} + (1 - q)^{1/\rho})^{\rho - 2}}{(2^\rho - 2)\rho}.$$

We thus have that

$$M'(q) = \frac{(\rho - 1)}{(2^{\rho - 1} - 1)\rho^2} ((1 - q)q)^{\frac{1}{\rho} - 2} (q^{1/\rho} + (1 - q)^{1/\rho})^{\rho - 3} \\ \times \left(q^{1/\rho}(\rho q - 1) - (\rho(1 - q) - 1)(1 - q)^{1/\rho} \right).$$

It follows that the sign of $M'(q)$ equals the sign of

$$q^{1/\rho}(\rho q - 1) - (\rho(1 - q) - 1)(1 - q)^{1/\rho} = h(q) - h(1 - q),$$

where $h(q) = q^{1/\rho}(\rho q - 1)$. Note that the derivative of h is given by

$$h'(q) = \frac{q^{\frac{1}{\rho} - 1}(\rho q(\rho + 1) - 1)}{\rho}.$$

Hence, for $\rho \leq 1$ the derivative is negative for $q \in (0, 1/2)$ and thus $h(q) - h(1 - q) = M'(q) \geq 0$ for $q \in (0, 1/2)$. We next consider the case of $\rho > 1$. In this case we note that

$$q^{1/\rho}(\rho q - 1) - (\rho(1 - q) - 1)(1 - q)^{1/\rho} = \left[(1 - q)^{1/\rho} - q^{1/\rho} \right] - \rho \left[(1 - q)^{1/\rho + 1} - q^{1/\rho + 1} \right] \\ \leq \left[(1 - q)^{1/\rho} - q^{1/\rho} \right] - \left[(1 - q)^{1/\rho + 1} - q^{1/\rho + 1} \right] \\ = (1 - q)^{1/\rho} [1 - (1 - q)] - q^{1/\rho} [1 - q] \\ = q(1 - q) \left[(1 - q)^{1/\rho - 1} - q^{1/\rho - 1} \right] \leq 0.$$

It thus follows that $M'(q) \leq 0$ for $q \in (0, 1/2)$ when $\rho > 1$. □

We next consider Wald cost $\psi(q) = q \log\left(\frac{q}{1 - q}\right) + (1 - q) \log\left(\frac{1 - q}{q}\right)$. In this case we have that

$$M(q) = \frac{v''(q)}{\psi''(q)} = \frac{2(\rho - 1)((1 - q)q)^{1/\rho} (q^{1/\rho} + (1 - q)^{1/\rho})^{\rho - 2}}{(2^\rho - 2)\rho}.$$

We thus have that

$$M'(q) = -\frac{2(\rho-1)}{(2\rho-2)\rho^2}((1-q)q)^{\frac{1}{\rho}-1} \left(q^{1/\rho} + (1-q)^{1/\rho} \right)^{\rho-3} \\ \times \left((\rho(q-1)+1)q^{1/\rho} + (\rho q-1)(1-q)^{1/\rho} \right).$$

It follows that the sign of $M'(q)$ equals the sign of

$$- \left[(1-\rho(1-q))q^{1/\rho} - (1-\rho q)(1-q)^{1/\rho} \right] = h(q) - h(1-q), \quad (24)$$

where $h(q) = -(1-\rho(1-q))q^{1/\rho}$. As h is decreasing in q it follows that $M'(q) = h(q) - h(1-q) \geq 0$ for all $q \in (0, 1/2)$.

Proof of Lemma 4. Follows from [Liptser and Shiryaev \(2013\)](#) Theorem 9.1. □

The proof of Proposition 8 is mostly parallel to the proof of Proposition 2 with the main difference that instead of Ito's Lemma for single dimensional Markov processes we make use of it's multi-dimensional generalization as the belief process lives in \mathbb{R}^n :

Proof of Proposition 8. Let τ be an arbitrary a.s. finite stopping time. To simplify notation we define

$$\eta_i(q) \triangleq 1/\sigma p_i^j \left(m_i - \sum_{j=1}^n p_i^j m_j \right).$$

Recall that we denote by ϕ_{ci} the first partial derivative of ϕ_c with respect to p^i and by $\phi_{ci,j}$ the cross-derivative with respect to p^i and p^j . By Ito's Lemma, we have that

$$\phi_c(p_\tau) = \phi_c(p_0) + \int_0^\tau \sum_{i=1}^n \phi_{ci}(p_s) \eta_i(p_s) dB_s + \int_0^\tau \sum_{i,j=1}^n \phi_{ci,j}(p_s) \frac{\eta_i(p_s)\eta_j(p_s)}{2} ds \quad (25)$$

By definition $\phi_c(p_0) = 0$. Define,⁹

$$r_n \triangleq n \wedge \inf \left\{ t : \int_0^t \sum_{i=1}^n |\phi_{ci}(p_s)|^2 \eta_i(p_s)^2 ds \geq n \right\}. \quad (26)$$

Consider the sequence of random variables $(\phi_c(p_{\tau \wedge r_n}))_n$. As, due to the definition of r_n (given in (26)), the integral over $\sum_{i=1}^n |\phi_{ci}'|^2 \eta^2$ is bounded, the second summand in (25) is

⁹We set the infimum over the empty set to be infinity.

a Martingale and thus vanishes when taking expectations

$$\begin{aligned}
\mathbb{E}[\phi_c(p_{\tau \wedge r_n})] &= \mathbb{E} \left[\int_0^{\tau \wedge r_n} \sum_{i,j=1}^n \phi_{ci,j}(p_s) \frac{\eta_i(p_s) \eta_j(p_s)}{2} ds \right] \\
&= \frac{1}{2\sigma^2} \sum_{i,j=1}^n \phi_{ci,j}(p) p^i p^j \left(m_i - \sum_{j=1}^n p^j m_j \right) \left(m_j - \sum_{j=1}^n p^j m_j \right) \\
&= \mathbb{E} \left[\int_0^{\tau \wedge r_n} c(p_s) ds \right]. \tag{27}
\end{aligned}$$

Here, the last equality follows, by the definition of ϕ_c , i.e. (15). Note, that the sequence r_n satisfies $r_n \rightarrow \infty$ for $n \rightarrow \infty$. As p is a bounded Martingale it a.s. converges. Consequently, $p_{\tau \wedge r_n}$ converges to p_τ for $n \rightarrow \infty$. By Fatou's Lemma

$$\begin{aligned}
\mathbb{E}[\phi_c(p_\tau)] &= \mathbb{E} \left[\liminf_{n \rightarrow \infty} \phi_c(p_{\tau \wedge r_n}) \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[\phi_c(p_{\tau \wedge r_n})] = \liminf_{n \rightarrow \infty} \mathbb{E} \left[\int_0^{\tau \wedge r_n} c(p_s) ds \right] \\
&\leq \mathbb{E} \left[\int_0^\tau c(p_s) ds \right].
\end{aligned}$$

As $p_\tau \sim F_\tau$ it follows, that $\int_0^1 \phi_c(q) dF_\tau(q) \leq \mathbb{E} \left[\int_0^\tau c(p_s) ds \right]$.

To complete the proof and it remains to be shown that equality holds. Assume that

$$\int_0^1 \phi_c(q) dF_\tau(q) < \infty$$

as otherwise the equality is an immediate conclusion of the above inequality. As ϕ_c is convex and $(p_t)_t$ is a Martingale it follows from Jensen's inequality that

$$\phi_c(p_{\tau \wedge r_n}) \leq \mathbb{E}[\phi_c(p_\tau) \mid \mathcal{F}_{\tau \wedge r_n}]. \tag{28}$$

As the right-hand side of (28) is a conditional expectation of $\phi_c(p_\tau)$ and $\mathbb{E}[\phi_c(p_\tau)] < \infty$ it follows from the law of iterated expectations that

$$\mathbb{E}[\mathbb{E}[\phi_c(p_\tau) \mid \mathcal{F}_{\tau \wedge r_n}]] = \mathbb{E}[\phi_c(p_\tau)] < \infty$$

the right-hand side of (28) is uniformly integrable. As the right-hand side is uniformly integrable the left-hand side of (28) is uniformly integrable as well. Consequently,

$$\lim_{n \rightarrow \infty} \mathbb{E}[\phi_c(p_{\tau \wedge r_n})] = \mathbb{E} \left[\lim_{n \rightarrow \infty} \phi_c(p_{\tau \wedge r_n}) \right] = \mathbb{E}[\phi_c(p_\tau)] = \int_0^1 \phi_c(q) dF_\tau(q).$$

We thus have that,

$$\int_0^1 \phi_c(q) dF_\tau(q) = \lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^{\tau \wedge r_n} c(p_s) ds \right].$$

As $c \geq 0$ and r_n is non-decreasing the sequence on the right-hand side is increasing and we can apply the monotone convergence theorem to get that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_0^{\tau \wedge r_n} c(p_s) ds \right] = \mathbb{E} \left[\lim_{n \rightarrow \infty} \int_0^{\tau \wedge r_n} c(p_s) ds \right] = \mathbb{E} \left[\int_0^\tau c(p_s) ds \right]. \quad \square$$

Proof of Proposition 6. As a consequence of (12) we have that

$$\begin{aligned} \log \frac{p_\tau^i}{p_\tau^j} &= \log \frac{p_0^i}{p_0^j} + (m_i - m_j) \left(Z_t - t \frac{m_i + m_j}{2} \right) \\ \Rightarrow \log \frac{q^i}{p_0^i} - m_i \left(z - t \frac{m_i}{2} \right) &= \log \frac{q^j}{p_0^j} - m_j \left(z - t \frac{m_j}{2} \right) - m_i t \frac{m_j}{2} + m_j t \frac{m_i}{2} \\ \Rightarrow \log \frac{q^i}{p_0^i} - m_i \left(z - t \frac{m_i}{2} \right) &= \log \frac{q^j}{p_0^j} - m_j \left(z - t \frac{m_j}{2} \right). \end{aligned} \quad \square$$

Proof. Proof of Proposition 9: We have to show that $2\sigma^2 cL(p)$ (where L is defined in (16)) satisfies (15) with constant c . We have that the second partial derivatives of L are given by

$$\begin{aligned} L_{i,i}(p) &= \sum_{j \neq i} \frac{p_i + p_j}{p_i^2} \beta_{ij} \\ L_{i,j}(p) &= -\beta_{i,j} \left[\frac{1}{p_i} + \frac{1}{p_j} \right] \end{aligned}$$

where $\beta_{ij} \triangleq \frac{1}{(n-1)(m_i-m_j)^2}$. Thus, the right-hand-side of (15) simplifies to

$$\begin{aligned}
& \frac{1}{2\sigma^2} \sum_{i,j=1}^n 2\sigma^2 c L_{i,j}(p) p^i p^j \left(m_i - \sum_{j=1}^n p^j m_j \right) \left(m_j - \sum_{j=1}^n p^j m_j \right) \\
&= c \sum_i \sum_{j \neq i} -\beta_{i,j} \left[\frac{1}{p_i} + \frac{1}{p_j} \right] p^i p^j \left(m_i - \sum_{k=1}^n p^k m_k \right) \left(m_j - \sum_{k=1}^n p^k m_k \right) \\
&\quad + c \sum_i \sum_{j \neq i} \frac{p_i + p_j}{p_i^2} \beta_{ij} (p^i)^2 \left(m_i - \sum_{k=1}^n p^k m_k \right)^2 \\
&= c \sum_i \sum_{j \neq i} -\beta_{i,j} [p^i + p^j] \left(m_i - \sum_{k=1}^n p^k m_k \right) \left(m_j - \sum_{k=1}^n p^k m_k \right) \\
&\quad + c \sum_i \sum_{j \neq i} [p_i + p_j] \beta_{ij} \left(m_i - \sum_{k=1}^n p^k m_k \right)^2 \\
&= c \sum_i \sum_{j \neq i} \beta_{i,j} [p^i + p^j] \left(m_i - \sum_{k=1}^n p^k m_k \right) \left[\left(m_i - \sum_{k=1}^n p^k m_k \right) - \left(m_j - \sum_{k=1}^n p^k m_k \right) \right].
\end{aligned}$$

Plugging in the definition of β_{ij} yields that that

$$\begin{aligned}
& c \sum_{i,j=1}^n L_{i,j}(p) p^i p^j \left(m_i - \sum_{j=1}^n p^j m_j \right) \left(m_j - \sum_{j=1}^n p^j m_j \right) \\
&= c \sum_i \sum_{j \neq i} \frac{1}{(n-1)(m_i-m_j)^2} [p^i + p^j] \left(m_i - \sum_{k=1}^n p^k m_k \right) [m_i - m_j] \\
&= c \sum_i \sum_{j \neq i} \frac{[p^i + p^j]}{(n-1)(m_i-m_j)} \left(m_i - \sum_{k=1}^n p^k m_k \right) \\
&= c \sum_i \sum_{j \neq i} \frac{p^i (m_i - \sum_{k=1}^n p^k m_k)}{(n-1)(m_i-m_j)} - \frac{p^j (m_j - \sum_{k=1}^n p^k m_k)}{(n-1)(m_j-m_i)} + \frac{p^j (m_i - m_j)}{(n-1)(m_i-m_j)} \\
&= c \sum_i \sum_{j \neq i} \frac{p^j (m_i - m_j)}{(n-1)(m_i-m_j)} = c \sum_i \frac{1-p^i}{(n-1)} = c \sum_i \frac{1-p^i}{(n-1)} = c.
\end{aligned}$$

Thus, $\phi(p) = 2\sigma^2 c L(p)$ solves satisfies (15) with constant c and the result follows from Proposition 8. \square

Proof of Proposition 10. As the cross derivatives of H defined in (18) equal zero we have

that

$$\begin{aligned}
 & \frac{1}{2\sigma^2} \sum_{i,j=1}^n H_{i,j}(q) q^i q^j \left(m_i - \sum_{k=1}^n q^k m_k \right) \left(m_j - \sum_{k=1}^n q^k m_k \right) \\
 &= \frac{1}{2\sigma^2} \sum_{i=1}^n H_{i,i}(q) (q^i)^2 \left(m_i - \sum_{k=1}^n q^k m_k \right)^2 \\
 &= \frac{1}{2\sigma^2} \sum_{i=1}^n \left[\frac{1}{q^i} \right] (q^i)^2 \left(m_i - \sum_{k=1}^n q^k m_j \right)^2 \\
 &= \frac{1}{2\sigma^2} \sum_{i=1}^n q^i \left(m_i - \sum_{k=1}^n q^k m_j \right)^2 .
 \end{aligned}$$

Thus, $c_{Shannon}$ in combination with H solves satisfies (15) and the result follows from Proposition 8. □