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Bidder-Optimal Information Structures in Auctions*

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Abstract

We characterize the bidders' surplus maximizing information structure in an optimal auction for a single unit good and related extensions to multi-unit and multi-good problems. The bidders seek to find a balance between participation (and the avoidance of exclusion) and efficiency. The information structure that maximizes the bidders' surplus is given by a generalized Pareto distribution at the center of demand distribution, and displays complete information disclosure at either end of the Pareto distribution.

JEL CLASSIFICATION: D44, D47, D83, D84.

KEYWORDS: Optimal Auctions, Welfare Bounds, Information Structure

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1 Introduction

1.1 Motivation

In many significant allocation problems, the choice of information provision and information disclosure interact strongly with the mechanism that guides the allocation. A leading example is the world of digital advertising. The central problem here is that the advertisers are bidding for display or sponsored product advertisements. The match between advertisers and viewers on the websites of the publishers is often made through intermediaries. A given demand side platform seeks to make bids across advertisements so as to maximize the surplus of the advertisers to be as attractive as possible for the advertisers. A given supply side platform seeks to design the auction so as to be attractive to the publishers and hence to maximize the profit from the auction. (Both demand and supply platforms typically receive a commission of the surplus and thus their objective function is roughly aligned with this description.)

A prominent tool of the demand platform is to manage the match information between advertisers and viewers through the design of bidding categories and characteristics. Thus, the demand side platform seeks to influence the information regarding values and bids whereas the supply side platform chooses the optimal auction format.

With this perspective, we ask what is the optimal information structure that the bidders should have to maximize the equilibrium surplus of the bidders anticipating that the supply side platform will choose the optimal auction format to extract as much surplus as possible.

Our analysis is most immediately concerned with the information design in auctions. But the method allows us to consider related problems in nonlinear pricing. Suppose we have a given supply of quantities as in Loertscher and Muir (2022) then this imposes restrictions on the allocation function similar to the ones imposed by competition in an auction. We can then ask what is the bidder or in this instance representative consumer optimal value representation. We address this with current results by allowing for a general allocation probability or allocation quality distribution. This would correspond to a recommender or search algorithm on a digital marketplace that makes recommendations while independent sellers choose prices optimally given the nature of the recommendation.

1.2 Results

The first central result that we obtain is that the optimal information structure will always generate a regular distribution (Theorem 1). Moreover, the regular distribution will have the feature that if it suggests exclusion and a reserve price above the lowest value in the support, then the bidders will receive complete information about their values below the reserve price. An implication of the regularity is that the optimal auction can always be implemented by a second price auction with a reserve price (Proposition 4).

With this restriction on the set of candidate solutions, we then proceed to the determination of the optimal solution. The second central result offers a complete characterization of the optimal information structure (Theorem 2). The information structure generates a distribution of posterior expectations of the bidders. This distribution will have at most three segments: (i) a lower segment may be excluded and the bidders receive complete information in this segment that agrees with their prior distribution, (ii) the intermediate segment has a shifted Pareto distribution with constant profit elasticity, and (iii) an upper segment where each bidder has again complete information. Thus, compared to the single buyer version of third-degree price discrimination as in Bergemann, Brooks, and Morris (2015) or Roesler and Szentes (2017), the constant profit elasticity is only maintained in the intermediate segment. Moreover, the profit elasticity can be below one rather than exactly one as in Roesler and Szentes (2017). The competition generated by the auction is sufficiently strong that the bidders cannot secure sales with probability one as in the single bidder version of the problem. On the other hand, the competition is sufficiently strong that the bidders will receive complete information in the upper tail of the distribution. Here, the auctioneer will not want to distort the auction for profit considerations and will be guided by social efficiency alone.

The appearance of the Pareto distribution together with a smaller support below the Pareto distribution increases the bidders' surplus while avoiding inefficient exclusion as much as possible. In any case, in the lower and upper segments of the demand distribution there is no distortion of information and all the bidders learn their value completely.

Theorem 1 and 2 are the main results of the paper. In the remainder, we use the sharp characterization of Theorem 2 to show how the nature of the allocation problem influences the information design. We first consider two special allocation problems that relate to earlier works, namely the single bidder environment (Proposition 6 and 7) of Roesler and Szentes (2017) and the many bidder case when the seller has to sell, thus an absolute auction (Proposition 8). These special cases lead us to Theorem 3 which provides a characterization of when the seller excludes bidders in the optimal auction.

Finally, we discuss how our analysis extends to other allocation problems. First, we discuss settings where the objective function gives some weight to the profit of the seller. This may range from the welfare maximization solution where profit receives equal weight as the bidders' surplus does to any, possibly negative profit weight below 1 (Proposition 15). Then we extend the analysis to a model with many goods of different qualities and many bidders. This model encompasses the model of nonlinear pricing with exogenous supply as in Loertscher and Muir (2022) or the multi-unit auction in Kleiner, Moldovanu, and Strack (2021). What is the relation to the nonlinear pricing problem? Suppose we have a given supply of quantities or qualities, then this imposes restrictions on the allocation function very similar to those in the symmetric auction setting, and we have corresponding characterization of the optimal information structure (Proposition 14 and 15).

In our analysis, we restrict attention to symmetric and independent information structures across bidders. This seems appropriate in many applications and guarantees a uniform and fair solution across all participating bidders who are ex-ante symmetric. This is with some loss of generality. For example, we will find that with many bidders, competition will eventually depress the bidder surplus to near zero. If we were to choose asymmetric information structures, then a solution that would increase the profit is an asymmetric information structure that leaves all but one bidder with their prior information and chooses the remaining information structure in such a way as to maximize the surplus of this singular bidder. With a large number of bidders, this asymmetric solution would improve the aggregate surplus of the bidders relative to the symmetric solution (Proposition 12). We could offer a symmetric version of this information structure in which we randomize across the identity of the informed bidder. The resulting information structure would be symmetric, but would by necessity display correlation across the bidders. Here we focus on independent and symmetric information structures.

1.3 Related Literature

The single bidder version of the unit demand problem was analyzed in Roesler and Szentes (2017). They showed that if there is common knowledge of the gains of trade then one optimal solution of the information design is a Pareto distribution with shape parameter 1 that always guarantees efficient trade. When we consider many bidders we find that: (a) it is optimal to inefficiently exclude values, (b) it is optimal to conflate values and generate inelastic demands (the distribution stays within the class of generalized Pareto distributions), and (c) it is always optimal to perfectly inform high values (which in Roesler and Szentes (2017) was undetermined). Condorelli and Szentes (2020) consider the consumer maximizing demand in the same single-unit demand problem in the presence of lower and upper bounds on values only, but absent any majorization constraints. We offer a similar generalization here to nonlinear allocation problems.

In a recent paper, Bergemann, Heumann, Morris, Sorokin, and Winter (2022) we considered the information design problem in a fixed auction format, namely the second price auction without reserve prices. We separately analyzed the seller profit maximizing information structure as well as the bidder surplus maximizing structure. These two posterior demand distributions were mirror images of each other. Either the upper or the lower part of the distribution is compressed in a single mass point, and the remaining part of the value distribution leads to complete revelation of the values. By allowing the seller to adapt to the choice of the posterior distribution with an optimal mechanism, we find it beneficial to: (a) limit the amount of conflating to prevent the seller from conflating allocation, (b) give additional information to lower reservation price, and (c) provide complete information to the excluded values.

The nature of our problem is related to but different from zero-sum games that have been used to formulate robust auction problems. Bergemann, Brooks, and Morris (2017), Brooks and Du (2021), and Brooks and Du (2023) consider robust auction design problems where the seller is choosing a mechanism to maximize the profit whereas nature is choosing an information structure (and equilibrium) to minimize the profit. The problem that we are analyzing is not a zero-sum game as the objective functions of profit maximization and bidders' surplus maximization are not merely the negative of each other. We consider a sequential version of the game where the bidders choose the information structure (first) in anticipation of an optimal auction being adopted by the seller. It is an open question whether our problem allows for a saddle point and how it would differ from the zero-sum game formulation of the above cited papers.

Bergemann and Pesendorfer (2007) offered a solution when the seller can both choose the profit maximizing auction and information structure. The analysis there maintained the independence assumption across bidders but allowed for asymmetric information structures across bidders. Similarly, in Bergemann, Heumann, and Morris (2023c), we considered the canonical nonlinear pricing problem where the seller could jointly optimize the information structure and the menu pricing.

2 Model

2.1 Payoffs

There is a seller of a single indivisible good to N bidders. Each bidder $i = 1, 2, ..., N$ has a net utility given by:

$$u(v_i, q_i, p_i) \triangleq v_i q_i - p_i,$$

where $v_i \in \mathbb{R}_+$ is the value, $q_i \in [0, 1]$ is the probability and $p_i \in \mathbb{R}_+$ is the payment of bidder i . Bidders i's value v_i is distributed according to an absolutely continuous distribution F on a compact support $[\underline{v}, \bar{v}] \subset \mathbb{R}_+$, identically and independently distributed across bidders.

Bidders do not observe their own value, rather they observe a signal $s(v)$ about their own value. The bidders' information structure is symmetric and is summarized by the distribution of expected values of an individual bidder, which we denote by G. By Blackwell (1951), Theorem 5, there exists an information signal that induces a distribution G of expected values if and only if G is a mean-preserving contraction of F, i.e.,

$$\int_v^{\overline{v}} F(x)dx \leq \int_v^{\overline{v}} G(x)dx, \forall v \in [\underline{v}, \bar{v}],$$

with equality for $v = \underline{v}$. If G is a mean-preserving contraction of F (or G majorizes F), we write $G \succ F$. It follows from Shaked and Shanthikumar (2007), Theorem 3.A.1 that the majorization relationship is equivalent to the convex stochastic order, thus $G \succ F \Leftrightarrow F \succ_{cx} G$.

We present much of the analysis in the quantile space $t \in [0, 1]$ rather than the value space $v \in \mathbb{R}_+$, with $v = F^{-1}(t)$. We denote the respective inverses as follows:

$$V(t) \triangleq F^{-1}(t) \quad \text{and} \quad W(t) \triangleq G^{-1}(t). \tag{1}$$

Following Shaked and Shanthikumar (2007), Theorem 3.A.5, we have that:

$$G \succ F \Leftrightarrow F^{-1} \succ G^{-1} \Leftrightarrow V \succ W.$$

2.2 Mechanism

The seller chooses a *symmetric* mechanism $M = (Q, P)$ to maximize profit. A direct and symmetric (interim) mechanism is denoted by:

$$Q, P : [\underline{v}, \bar{v}] \to [0, 1] \times \mathbb{R}_+,$$

where $Q(v)$ denotes the probability of winning and $P(v)$ denotes the payment. It is convenient to write the allocation rule in terms of quantiles:

$$r(t) \triangleq Q(W(t)). \tag{2}$$

Of course, not every function $Q(v)$ is feasible in a symmetric environment (for example, $Q(v) = 1$ cannot be implemented if there is more than one bidder). To write the feasibility condition, we define the efficient allocation rule $q(t)$ that assigns the object with probability one to the bidder in the highest quantile (or value):

$$q(t) \triangleq t^{N-1}. \tag{3}$$

A quantile allocation rule $r(t)$ is feasible if and only if:

$$\int_t^1 r(s)ds \leq \int_t^1 q(s)ds, \text{ for all } t \in [0,1], \tag{4}$$

in which case we write $r \prec_w q$. We add a subscript "w" which stands for "weak" since the precedence in (4) does not need to be satisfied with equality at $t = 0$.

As it is standard in the literature, the allocation rule pins down the payments via the Envelope condition. For any information structure and allocation rule the seller's profit is given by:

$$\Pi \triangleq N\left(\int_0^1 (1-t)W(t)dr(t) + r(0)W(0)\right).$$

Thus, the surplus of an individual bidder is:

$$U \triangleq \int_0^1 r(t)W(t)dt - \frac{\Pi}{N}.$$

We wish to find the information structure that maximizes the bidders' surplus:

$$U^* \triangleq \max_{\substack{\{W:W\prec V\} \\ \{r:r\prec_w q\}}} \int_0^1 r(t)W(t)dt - \left(\int_0^1 W(t)(1-t)dr(t) + r(0)W(0)\right) \tag{5}$$

$$\text{subject to: } r \in \arg\max_{\{\hat{r}:\hat{r}\prec_w q\}} \int_0^1 (1-t)W(t)d\hat{r}(t) + \hat{r}(0)W(0). \tag{6}$$

The maximization problem is decomposed into an inner problem (6) and an outer problem (5). The inner problem is an optimal auction design problem with one majorization constraint, namely $\{\hat{r} : \hat{r} \prec_w q\}$, which requires that allocation is feasible. The outer problem maximizes the difference between the social surplus and the profit of the seller. It is subject to two majorization constraints, the feasibility constraints $\{r : r \prec_w q\}$ and the mean-preserving contraction constraint, $\{W : W \prec V\}$.

Note that we have introduced the model as if the seller is restricted to a symmetric mechanism. This is inconsequential as the optimal mechanism is symmetric when the distribution of expected values is symmetric (see Myerson (1981)). Hence, the non-trivial assumption is that the information structure is symmetric.

3 Optimality Without Majorization Constraint

We begin the analysis with a relaxed version of bidders' surplus problem (5). In this section, we omit the majorization constraint on the distribution of values[1]:

$$\{W : W \prec V\},$$

and replace with the weaker constraint that the distribution W of values has compact support:

$$w \in [m, 1],$$

for some $m \in [0, 1)$. We refer to this as a majorization-free problem or a problem without majorization. Thus we fix any m from now on and consider the following problem:

$$U^* \triangleq \max_{\substack{\{W \in \Delta[m,1]\} \\ \{r : r \prec_w q\}}} \int_0^1 r(t)W(t)dt - \left(\int_0^1 W(t)(1-t)dr(t) + r(0)W(0)\right) \tag{7}$$

$$\text{subject to: } r \in \arg\max_{\{\hat{r} : \hat{r} \prec_w q\}} \int_0^1 (1-t)W(t)d\hat{r}(t) + \hat{r}(0)W(0).$$

If the lower bound m is close to 1, then each bidder has only very limited private information. After all, the seller already knows that the value is between m and 1. If one were to consider more general support restrictions of the form $[m, M]$ the analysis would go through unchanged by simply rescaling.

3.1 Profit-Optimal Mechanism in Quantile Space

It is useful to characterize the profit that the seller can obtain for any fixed information structure. We do this directly in the quantile rather than the value space. Towards this end, we define a profit

[1] Notice that here the distribution of values essentially refers to the distribution of expected values and if without confusion, we henceforth use these two terms interchangeably.

function from an individual bidder as follows:

$$\pi(t) \triangleq F^{-1}(t)(1-t). \tag{8}$$

Thus $\pi(t)$ is the (expected) profit that the seller would obtain if she was selling to a single bidder and the price $F^{-1}(t)$ excluded the lowest fraction t of values. Bulow and Roberts (1989) used the quantile space, or in their word, the quantity space (as the sales quantity is $q = 1-t$) to state the "simple economics of auctions", see also Dhangwatnotai, Roughgarden, and Yan (2015) and Hartline (2017). To make the notation more compact we define the profit function under W by:

$$\pi_w(t) \triangleq W(t)(1-t). \tag{9}$$

Hence, π_w is the counterpart of π when evaluated at information structure W. The marginal revenue in the quantile space identifies the virtual utility:

$$\frac{\pi_w(t)}{dt} = (1-t)W'(t) - W(t).$$

Since the quantile

$$t = 1 - G(W(t)) \Leftrightarrow W'(t) = \frac{1}{g(W(t))},$$

we have that

$$(1-t)W'(t) - W(t) = -\left(W(t) - \frac{1-G(W(t))}{g(W(t))}\right).$$

To describe the seller's profit maximization problem, we denote by $\text{cav}[\pi_w]$ the concavification of π_w. We denote by t_x a critical quantile below which the seller excludes bidders, and thus assigns zero probability to the bidder t receiving the object:

$$t_x \triangleq \max\{t \,|r(t) = 0\}. \tag{10}$$

Proposition 1 (Seller's Profit)
For any given information structure W, the seller's profit is given by:

$$\max_{\{r:r\prec_w q\}} \int_0^1 \pi_w(t)dr(t) = q(t_x)\pi_w(t_x) + \int_{t_x}^1 \text{cav}[\pi_w](t)dq(t), \tag{11}$$

where $\text{cav}[\cdot]$ *is the concavification and:*

$$t_x \in \arg\max_t \pi_w(t). \tag{12}$$

Proof of Proposition 1. The proof corresponds to concavifying the profit function (as originally analyzed by Myerson (1981) and more recently by Kleiner, Moldovanu, and Strack (2021)). We give a short outline of the proof for completeness and to provide the main elements of the analysis that we will use later.

We denote by $\phi(t)$ the derivative of $-\text{cav}[\pi_w](t)$:

$$\phi(t) \triangleq -\frac{d \ \text{cav}[\pi_w](t)}{dt}, \tag{13}$$

which is the virtual value of quantile t. By construction $\text{cav}[\pi_w](t)$ is concave and so $\phi(t)$ is non-decreasing. We denote by $\{[\underline{t}_i, \bar{t}_i]\}_{i=1}^{I}$ a collection of intervals such that:

$$\text{cav}[\pi_w](t) = \pi_w(t) \iff t \in [\underline{t}_i, \bar{t}_i].$$

We then have that an optimal allocation rule $r\,(t)$ given by:

$$r(t) = \begin{cases} 0, & \text{if } \phi(t) < 0; \\ \frac{\int_{\bar{t}_i}^{\underline{t}_{i+1}} q(x)dx}{\underline{t}_{i+1} - \bar{t}_i}, & \text{if } \phi(t) \geq 0 \text{ and } t \notin [\bar{t}_i, \underline{t}_{i+1}]; \\ q(t), & \text{if } \phi(t) \geq 0 \text{ and } t \in [\underline{t}_i, \bar{t}_i]. \end{cases}$$

That is, the optimal allocation rule consists of excluding types with negative virtual values, and then conflating types whenever the profit function is smaller than its concavification. This allocation rule generates profit (11). ■

The optimal mechanism is such that the seller sets a reserve price that excludes bidders whose expected value corresponds to a quantile below t_x . Furthermore, the good is not necessarily allocated to the bidder with the highest expected value above the reserve price. Whenever $\pi_w(t) < \text{cav}[\pi_w](t)$, the seller can increase its profit by conflating the allocations of different values. This conflating increases profit but also generates inefficiencies, and so it decreases total surplus. This way, for any given distribution of values that generates profit function π_w , the seller can obtain a profit corresponding to the concavification of π_w.

3.2 Positive Regular Information Structures

We now consider the following set of regular distributions (in the terminology of Myerson (1981)):

$$\mathcal{W}_+ \triangleq \{W \in \Delta[m, 1] : W(t)(1-t) \text{ is decreasing and concave}\}.$$

Note that the set of distributions in $\mathcal{W}_+$ does not need to be majorized by V. The distributions in $\mathcal{W}_+$ are those that generate positive and increasing virtual values. Hence, for any distribution $W \in \mathcal{W}_+$, the profit-maximizing mechanism is a second-price auction without a reserve price.

Proposition 2 (Regular Information Structures are Optimal)
An information structure W^* solves (7) only if $W^* \in \mathcal{W}_+$.

Proof. We prove this by contradiction. Suppose it is not regular so there is some t such that $\pi_w^*(t) < \mathrm{cav}[\pi_w^*](t)$ or $t_x > 0$ (with t_x as defined in Proposition 1), namely there is some t such that $\pi_w^*(0) < \mathrm{cav}[\pi_w^*](t)$. We would then be able to generate more bidder surplus by considering the distribution of values:

$$\widehat{W}(t) = \begin{cases} \frac{\mathrm{cav}[\pi_w^*](t)}{1-t}, & \text{if } t \geq t_x; \\ \frac{\mathrm{cav}[\pi_w^*](t_x)}{1-t}, & \text{if } t < t_x. \end{cases}$$

We now verify that $\widehat{W}$ generates higher bidder surplus than W.

By construction, the seller's profit is the same when the distribution of values is W^* or $\widehat{W}$. To check this, let the profit function associated with $\widehat{W}$ be $\widehat{\pi}_w$, and note that for all $t \geq t_x$, $\widehat{\pi}_w(t) = \mathrm{cav}[\pi_w^*](t)$; and for all $t < t_x$, $\widehat{\pi}_w(t) = \mathrm{cav}[\pi_w^*](t_x)$. We also have that $\widehat{W}$ first-order stochastically dominates W. Furthermore, an efficient mechanism is a profit-maximizing mechanism when the distribution of values is $\widehat{W}$ (since $\widehat{W}$ generates monotone virtual values at non-excluded quantiles). Hence, the total surplus generated when the distribution of values is $\widehat{W}$ is larger than when the distribution of values is W^*. Hence, we find that $\widehat{W}$ generates higher bidder surplus than W^*, thus reaching a contradiction. This proves that π_w^* must be (weakly) concave and (weakly) decreasing. ■

We can now characterize the optimal distribution of values by simply maximizing over $\mathcal{W}_+$. As the seller employs an efficient mechanism $q\left(t\right)$ and collecting terms from (7) we obtain the following problem:

$$\widehat{W} \in \underset{W \in \mathcal{W}_+}{\arg\max} \left\{ \left(\int_0^1 W(t) \frac{ds(t)}{dt} dt \right) \right\}, \tag{14}$$

where

$$s(t) \triangleq -q(t)(1-t). \tag{15}$$

Expression (14) corresponds to (7) but replacing the allocation $r(t)$ in the objective function with the efficient allocation $q(t)$. The new term $s\left(t\right)$ represents the difference between the first and second order statistics of the type draws (that is, the highest and the second highest realizations of the N independent type draws). The integral (14) tells us that the bidders' surplus is the integral of the change in the probability differential between the first and second order statistics at quantile t weighted by the value $W(t)$ at quantile t.

In an *absolute second-price auction*, the social surplus is determined by the first-order statistic of the values of the N bidders and the seller's profit by the corresponding second-order statistic. Therefore, the bidders' surplus is determined by the difference in the distribution of the first-order and second-order statistics. For any information structure W we denote by $w_{(1)}$ and $w_{(2)}$ the first and second-order statistics. For any $t \in [0,1]$:

$$\mathbb{P}\{w_{(1)} \leq W(t)\} = t^N \text{ and } \mathbb{P}\{w_{(2)} \leq W(t)\} = Nt^{N-1} - (N-1)t^N. \tag{16}$$

The difference between these order statistics at quantile t is $s(t)$:

$$s(t) = \left(\mathbb{P}\{w_{(1)} \leq W(t)\} - \mathbb{P}\{w_{(2)} \leq W(t)\}\right)/N = -q(t)(1-t). \tag{17}$$

Hence $s(t)$ is the probability that the highest bidder is below $W(t)$ minus the probability that the second-highest bidder is below $W(t)$.

We can gain intuition for the bidder-optimal information structure by analyzing the probability difference $s(t)$. This function describes how the competition between bidders affects bidder surplus (abstracting away from the effects that the information structure has on the seller-optimal reserve price). It is easy to verify that s is quasiconvex with a unique minimum and a unique inflection point denoted as follows:

$$t_s \triangleq \operatorname*{arg\,min}_{t \in [0,1]} s(t); \quad t_I \triangleq \operatorname*{arg\,min}_{t \in [0,1]} \frac{ds(t)}{dt}. \tag{18}$$

The minimum and the inflection point only relate to the number of bidders:

$$t_I = \frac{N-2}{N} < \frac{N-1}{N} = t_s. \tag{19}$$

The probability difference is therefore concave at low quantiles and convex at high quantiles. (If $N = 2$ it is completely convex). The shape of the function $s(t)$ is displayed in Figure 1 below. We can now explain how the slope and curvature of the probability differential affect the incentives to conflate values.

The slope of $s(t)$ gives the incentives to reduce or increase the values of bidders. The bidders benefit from decreasing the values of quantiles where the slope of $s(t)$ is negative (low quantiles); the bidders benefit from increasing the values of quantiles where the slope of $s(t)$ is positive (high quantiles). Typically, the slope of s would not play a relevant role in an information-design problem,

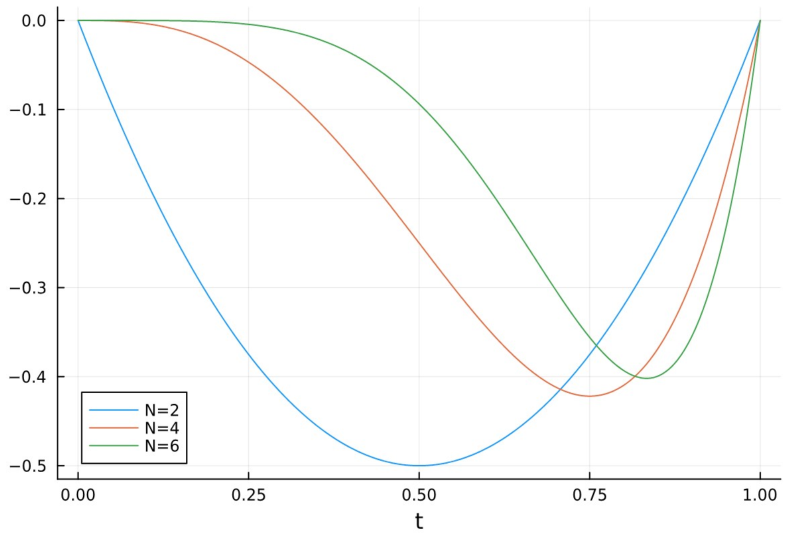

Figure 1: Probability Differential of First and Second Order Statistics

as bidders cannot change the ex-ante distribution of values: as governed by the Bayes plausibility condition, it is possible to conflate or separate values but not increase or decrease them. However, when the optimal information structure induces the seller to exclude some quantiles $t \in [0, t_x]$, the value of these quantiles is smaller than the value of the quantiles that are not excluded. Hence, the information structure can conflate excluded values with non-excluded values to effectively reduce the values of the bidders that are not excluded.

We state this formally.

Corollary 1 (Regular Information Structures are Optimal)
An information structure W^* solves (7) only if it solves (14)*.*

Proof. Following Proposition 2, we have that $W^* \in \mathcal{W}_+$. Thus, the profit-maximizing allocation is $r(t) = q(t)$. Replacing into the objective function of (7) we obtain the result. ■

3.3 Generalized Pareto Distribution

We can now find the solution to (14). For this we define a class of truncated generalized Pareto distributions on the interval $[m, 1]$ which are defined by a threshold quantile t_z:

$$G(w\,|t_z) = \begin{cases} 1 - \frac{(1-t_z)(1-\alpha)}{w-\alpha}, & \text{if } m \leq w < 1; \\ 1, & \text{if } w = 1. \end{cases} \tag{20}$$

The parameter α is determined as a function of the quantile t_z and the given support restriction m as follows:

$$\alpha = \begin{cases} 0, & \text{if } 1 - t_z \geq m; \\ \frac{m-(1-t_z)}{t_z}, & \text{if } 1 - t_z < m. \end{cases} \tag{21}$$

The quantile t_z represents the probability that the distribution $G(w\,|t_z)$ attains before the truncation at 1, thus

$$\lim_{w \uparrow 1} G(w\,|t_z) = t_z.$$

With the generalized Pareto function $G(w\,|t_z)$ being the value distribution, we can write the value $W(\cdot)$ as a function of the quantile as follows:

$$W(t\,|t_z) = \alpha + (1-\alpha)\frac{1-t_z}{1-t}, \tag{22}$$

and the associated profit function is

$$\pi(t\,|t_z) = \begin{cases} (1-t_z) + \alpha(t_z - t), & \text{if } t \leq t_z; \\ (1-t), & \text{if } t \geq t_z. \end{cases} \tag{23}$$

Thus we find that under the generalized Pareto function the profit function is piecewise linear in the quantile t, and the slope of the linear function depends on t_z, in other words the size of the mass point $1 - t_z$.

We thus have that $\pi(t\,|t_z)$ is continuous at t_z, it is equal to $(1-t)$ at quantiles larger than t_z and it generates a linear profit function at quantiles $t \leq t_z$. To understand how the parameter α is determined recall that the optimal profit function is non-increasing, so $\alpha \geq 0$ is a necessary constraint for optimality. If the lower bound m does not bind (when $(1-t_z) > m$), then the non-negativity constraint on α binds and so $\alpha = 0$. If $(1-t_z) < m$, then the lower bound m binds, and so α is the slope of the linear segment that connects the profit $\pi(0\,|t_z) = m$ and $\pi(t_z\,|t_z) = (1-t_z)$.

The generalized Pareto distribution (see Johnson, Kotz, and Balakrishnan (1994)) is defined by three parameters: location μ, scale σ and shape ξ:

$$F_\xi(z) = \begin{cases} 1 - (1+\xi z)^{-\frac{1}{\xi}} & \text{if } \xi \neq 0; \\ 1 - e^{-z} & \text{if } \xi = 0. \end{cases} \tag{24}$$

The related location-scale family is obtained by replacing z with

$$z = \frac{w-\mu}{\sigma}. \tag{25}$$

In our analysis, the shape parameter will always satisfy $\xi = 1$ and the solution will determine the location parameter μ and the scale parameter σ. The location parameter μ shifts the distribution on $x-$axis as the value w has to satisfy $w \geq \mu$. The scale parameter σ then controls the scale (or compression) of the distribution. A feature of the generalized Pareto distribution with shape parameter $\xi = 1$ is that the value w as a function of the quantile t is a reciprocal function:

$$t = 1 - \frac{1}{1 + \frac{w-\mu}{\sigma}} \Leftrightarrow w = \mu - \sigma\left(\frac{1}{t-1} + 1\right).$$

Moreover, the profit function is expressed in the quantile space:

$$\pi_w(t) = (1-t)\,w = (1-t)\left(\mu - \sigma\left(\frac{1}{t-1}+1\right)\right) = \mu - t(\mu - \sigma),$$

is linear in t and constant if location and scale agree, or $\mu = \sigma$. Moreover, if location is larger than scale, or $\mu - \sigma > 0$, then the profit function is decreasing in the quantile t.

The class of generalized Pareto function expressed in (20) have location μ and scale σ, respectively:

$$\mu = 1 - t_z(1-\alpha), \quad \sigma = (1-t_z)(1-\alpha). \tag{26}$$

3.4 Bidder Surplus Maximizing Distribution

We can now give a complete description of the bidder surplus maximizing information structure.

Proposition 3 (Optimal Information Structure)
An information structure W^* solves the bidders' surplus maximizing problem (7) only if $W^*(t) = W(t\,|t_z)$ for some $t_z \in [0,1]$.

Proof. We recall that $ds(t)/dt < 0$ for all $t < t_s$ and $ds(t)/dt > 0$ for all $t > t_s$. We fix some $W \in \mathcal{W}_+$, and consider two cases.

(Case 1) We first analyze the case $W(t_s)(1-t_s) \geq m$. Since $\pi_w(t)$ is non-increasing, we have that for all $t \leq t_s$:

$$W(t)(1-t) \geq W(t_s)(1-t_s). \tag{27}$$

Since we are considering the case $W(t_s)(1-t_s) \geq m$, we have that this condition also implies that $W(0) \geq m$. The fact that $\pi_w(t)$ is non-increasing also implies that for all $t \geq t_s$:

$$W(t)(1-t) \leq W(t_s)(1-t_s). \tag{28}$$

Using the definition of t_s (see (18)) and the fact that $s(t)$ is quasi-convex we have that, for all $W \in \mathcal{W}_+$:

$$\int_0^1 W(t)\frac{ds(t)}{dt}dt \leq \int_0^{t_s} \left(\frac{W(t_s)(1-t_s)}{1-t}\right)\frac{ds(t)}{dt}dt + \int_{t_s}^1 \min\{\frac{W(t_s)(1-t_s)}{1-t},1\}\frac{ds(t)}{dt}dt.$$

The bound is obtained because $ds(t)/dt < 0$ for all $t < t_s$, so we get an upper bound by replacing $W(t)$ with the bound in (27) and $ds(t)/dt > 0$ for all $t > t_s$, so we get an upper bound by replacing $W(t)$ with the bound in (28). The minimum in the second integral appears because (28) provides one upper bound on $W(t)$ but the support of values is bounded by 1. We can see that the inequality is tight if and only if $W(t) = W(t\,|t_z\,)$ for some parameter t_z with $\alpha = 0$ since

$$W(t)(1-t) = W(t_s)(1-t_s),$$

using (22).

(Case 2) We now analyze the case $W(t_s)(1-t_s) < m$. Since $\pi_w(t)$ is weakly concave and it must satisfy that the support of values is bounded by m (that is, $W(0) \geq m$), we have that for all $t \leq t_s$:

$$W(t)(1-t) \geq W(t_s)(1-t_s) + \alpha(t_s - t),$$

where

$$\alpha = -\frac{W(t_s)(1-t_s)-m}{t_s}.$$

By construction, we have that the right-hand-side of the inequality is equal to m when evaluated at $t = m$. The fact that $\pi_w(t)$ is weakly concave, also implies that for all $t \geq t_z$

$$W(t)(1-t) \leq W(t_s)(1-t_s) + \alpha(t_s - t).\}.$$

As in the previous case, using the definition of t_s and the fact that s is quasi-concave we have that, for all $W \in \mathcal{W}_+$.

$$\int_0^1 W(t)\frac{ds(t)}{dt}dt \leq \int_0^{t_s} \frac{W(t_s)(1-t_s)+\alpha(t_s-t)}{1-t}\frac{ds(t)}{dt}dt + \int_{t_s}^1 \min\{\frac{W(t_s)(1-t_s)+\alpha(t_s-t)}{1-t},1\}\frac{ds(t)}{dt}dt.$$

We can see that the inequality is tight if and only if $W(t) = W(t\,|t_z\,)$ (for some parameter t_z with $\alpha > 0$). ■

We have shown that the optimal information structure is one where all values have a positive probability of winning, and thus there is no exclusion. As the number of bidders increases, the bidders can secure themselves information rents only by spreading the support of the distribution downwards to create more dispersion in the values.

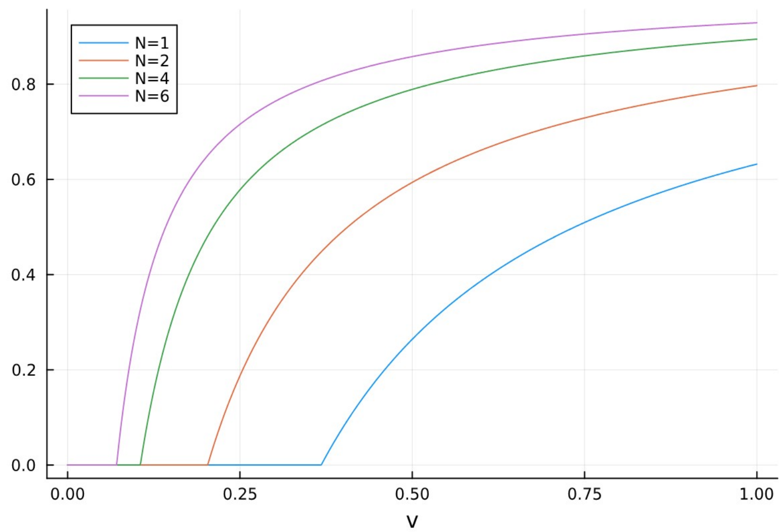

Figure 2: Optimal Distribution Function $G(t\,|t_z)$ with $m = 0$.

3.5 Impact of Support Restriction on Optimal Solution

We illustrate the above result by displaying the optimal distribution and associated profit function. We start with the case when the support restriction is given by the unit interval $[0, 1]$, thus $m = 0$. In Figure 2 we display how the optimal solution changes as we increase the number of bidders N.We observe that as the number N of bidders grows, the support of the optimal distribution $G(t\,|t_z)$ expands downwards, and the size $1 - t_z$ of the atom at 1 shrinks, and conversely the probability t_z of a value below 1 increases.

Associated with the distribution $G(t\,|t_z)$ is the profit function $\pi(t\,|t_z)$ with $N = 2, 4, 6$ which we display in Figure 3.

In consequence, the profit function from a single bidder falls as the number of bidders increases but the aggregate revenue increases. Now as we increase the lower bound m of the support restriction, the ability of the bidders to retain information rent decreases as the scope for private information decreases. Below we plot the resulting optimal distribution and profit function for $m = 0.15$ and $m = 0.4$.

With a lower bound of $m = 0.15$, the optimal distribution $G(t\,|t_z)$ remains unconstrained for a small number of bidders, or $N = 2$. But as the number of bidders increases, the optimal solution would like to create a larger support of the values to create more dispersions but are constrained by the lower bound m (as displayed in Figure 4). In consequence the associated profit function $\pi(t\,|t_z)$ changes its shape as the lower bound becomes a constraint. As displayed in Figure 5, the

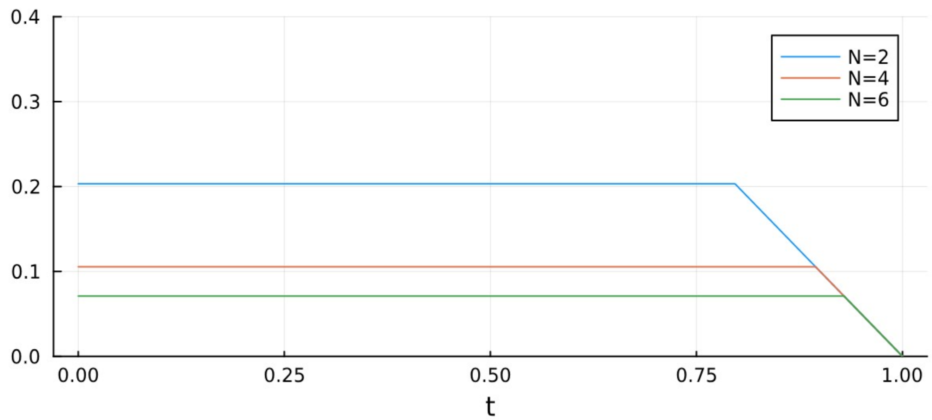

Figure 3: Profit Function $\pi\left(t\,|t_z\right)$ with $m = 0$.

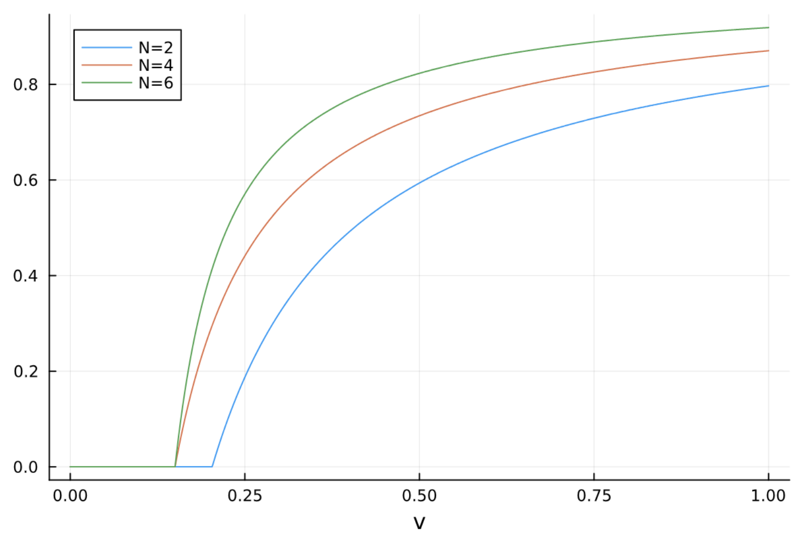

Figure 4: Distribution Function $G\left(t\,|t_z\right)$ with $m = 0.15$.

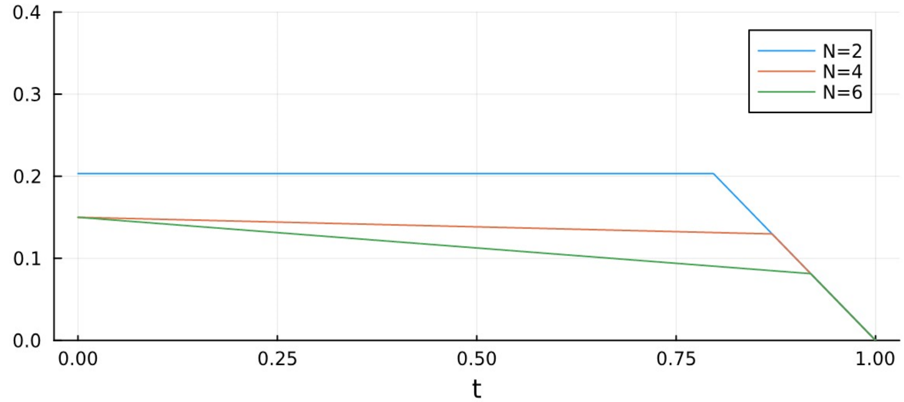

Figure 5: Profit Function $\pi(t\,|t_z)$ with $m = 0.15$.

profit function remains flat for $N = 2$, but then displays a decline even at the very beginning for $N = 4$ and $N = 6$.

As the lower bound increases to $m = 0.4$, it becomes a constraint for the bidders in their attempt to generate information rents. Now, the optimal distribution always begins at the lower bound (see Figure 6).

The impact of the constraint then appears in the profit function which is not flat anymore, that is it does not display the profit indifference anymore, but rather it leads to linearly decreasing profit functions (see **??**). Moreover, the slope of the profit function is decreasing in the number of bidders N. We assumed the support of the values to be $[m, 1]$ for some $m \geq 0$. For $m = 0$, the optimal solution is always a Pareto distribution whose scale and location agree and there is a constant revenue. As m (and N) increases the optimal information structure cannot maintain a constant profit anymore. The optimal solution is now a linearly decreasing profit function rather than a constant profit function in the quantile t. This description of the optimal information structure and associated linear profit function will also be the key in the environment with the majorization constraint as we see next.

The additional complexity brought about by the majorization constraint will arise from low and high values in the distribution F. For low values, the bidders will have to concede the possibility of exclusion, and for high values, efficiency considerations will outweigh revenue considerations. Thus, we will have complete disclosure for low and high values, and in between there will be the generalized Pareto distribution that emerges as the solution to the majorization-free problem in Proposition 3.

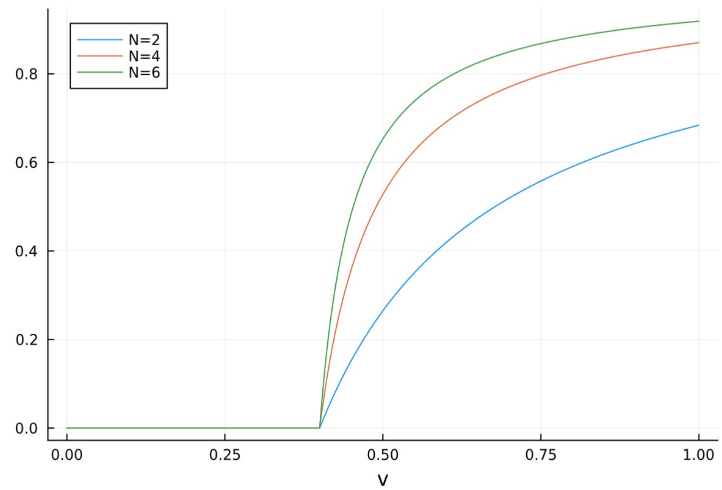

Figure 6: Distribution Function $G(t\,|t_z)$ with $m = 0.4$.

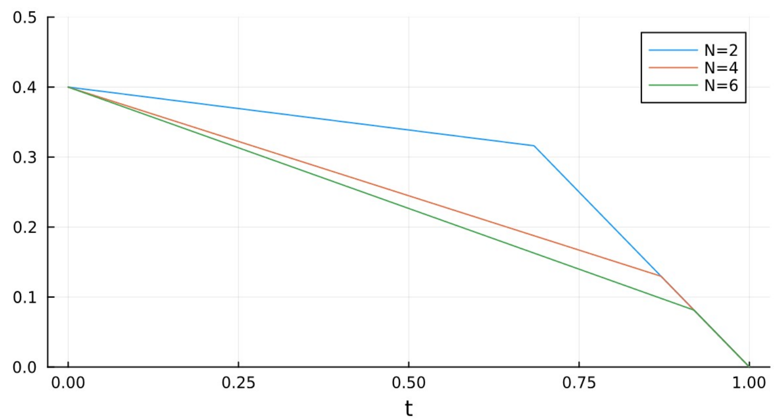

Figure 7: Profit Function $\pi(t\,|t_z)$ with $m = 0.4$.

4 Optimality With Majorization Constraint

We now return to the original problem of maximizing the bidders' surplus in the presence of the majorization constraint given by $W \prec V$. In the absence of any constraint V, we showed in Proposition 3 that the optimal information structure W is a positive regular distribution. For the remainder of the analysis, we assume that the profit function $\pi(t)$ is quasi-concave and concave wherever it is decreasing. This assumption is a minor generalization of the classic assumption that the virtual values are non-decreasing (that is, requiring that $\pi(t)$ is concave), that is V is regular. As we eventually transform the given distribution F of values and hence the profit function π through the choice of mean-preserving contraction G, we may ask why we assume the quasiconcavity of the original problem. In short, this guarantees that any conflating that arises in the information design can be attributed to a positive impact on the bidders' surplus rather than an attempt by the seller to conflate due to the need of ironing. In the presence of an irregular distribution F, we expect that a more complicated characterization of the optimal information structure would prevail, but then the need for conflating intervals would arise from two very different and conflicting sources.

The construction that guides us to the bidders' optimal bidding structure in the presence of the majorization constraint is now similar to the one without the constraint, but some additional steps are needed to account for the possibility of exclusion and the prevalence of efficiency consideration for high values.

4.1 Regular Information Structure

We now show that the bidder-optimal information structure dissuades the seller from introducing inefficiencies to the mechanism that arise from conflating allocation. For this, we introduce a class of information structures that are parametrized by a critical quantile $t_x \in [0,1]$. The threshold t_x will be the critical quantile that determines the exclusion level of the profit-maximizing mechanism (consistent with the notation used in Proposition 1). We formally define:

$$\mathcal{W}(t_x) \triangleq \left\{ W \prec V : \begin{array}{c} \forall t \leq t_x : W(t) = V(t) \text{ and } V(t)(1-t) \leq W(t_x)(1-t_x); \\ \forall t > t_x : W(t)(1-t) \text{ is non-increasing and weakly concave.} \end{array} \right\}. \tag{29}$$

In other words, any distribution in $\mathcal{W}(t_x)$ has the following properties: (i) the profit-maximizing quantile is t_x; as we vary t_x we can span all possible maximizing quantiles; (ii) any information structure in this set is decreasing and concave for quantiles larger than the profit-maximizing quantile t_x and (iii) the information structure is complete information for quantiles below t_x. The relevance of this set is immediate from the following proposition.

Any element $W \in \mathcal{W}(t_x)$ thus generates a regular problem or regular distribution in the sense of Section 5 of Myerson (1981). The additional property is that the quantile values $W(t)$ coincide with the value $V(t)$ at the lower end of the quantiles, that is for $t \in [0, t_x]$. We could therefore refer to the information structure as *composite regular*, but for simplicity refer to it as regular information structure.

Proposition 4 (Implication of Regular Information Structures)
If $W \in \mathcal{W}(t_x)$ then, (i) a second-price auction with reserve price $W(t_x)$ maximizes profit and (ii) bidders whose expected values are below the reserve price know their ex post values. Furthermore, any W satisfying these two properties also satisfies $W \in \mathcal{W}(t_x)$, for some $t_x \in [0,1]$.

Proof. Following the proof of Proposition 1, when $W(t)(1-t)$ is locally concave, it corresponds to the case when virtual values are locally increasing. Hence, if $W^* \in \mathcal{W}(t_x)$ whenever the virtual value is positive it is increasing. Hence, the optimal mechanism is a second-price auction with a reserve price (see Myerson (1981)). The converse follows in an analogous way. ■

Hence, when the information structure is in $\mathcal{W}(t_x)$ the seller can use a mechanism that allocates the good interim efficiently to bidders that have expected values above the reserve price. It is interim efficient because the bidder with the highest *interim* value wins the object (conditional on being above the reserve price), but the winner might differ from the bidder with the highest *ex post* value.

We can then rewrite the problem of maximizing the bidders' surplus as stated earlier in (5) for regular information structures as follows:

Corollary 2 (Computing the Bidder-Optimal Regular Information Structure)
A regular information structure W solves (5) if and only if it solves:

$$W^* \in \underset{t_x \in [0,1], W \in \mathcal{W}(t_x)}{\arg\max} \left(\int_{t_x}^{1} W(t) \frac{ds(t)}{dt} dt - (1-t_x) q(t_x) W(t_x) \right). \tag{30}$$

Expression (30) corresponds to (5) but replacing the allocation $r(t)$ in the objective function with the following allocation:

$$r(t) = \begin{cases} 0 & \text{if } t < t_x; \\ q(t) & \text{if } t \geq t_x. \end{cases}$$

The integral (30) tells us that the bidders' surplus is the integral of the change in the probability differential between the first and second order statistics at quantile t weighted by the value $W(t)$ at quantile t.

While conflating is never optimal in the optimum, this type of variation is useful to show that, when there is exclusion $t_x > 0$, then the seller must be indifferent between multiple reserve prices. It is this indifference that precludes the possibility of further conflating that reduces the expected value of bidders that are not excluded. This is the seller's optimal allocation because, following Theorem 1, the optimal information structure satisfies $W^* \in \mathcal{W}(t_x)$. Hence, we are left with an optimization over a single distribution $W \in \mathcal{W}(t_x)$ and the cutoff t_x.

4.2 Optimality of Regular Information Structures

We can now formally state that a bidder-optimal information structure is indeed a regular distribution.

Theorem 1 (Regular Information Structures are Optimal)
An information structure W^* solves (5) *only if $W^* \in \mathcal{W}(t_x)$ for some $t_x \in [0,1]$.*

The theorem shows that in any bidder-optimal information structure, the distribution of values will be regular. The intuition is that whenever the seller conflates the allocation this decreases the total surplus and increases profit. Hence, it is detrimental to bidder surplus. Hence, the bidder-optimal information structure induces the seller to allocate the good interim efficiently among those bidders that have expected values above the reserve price.

The proof of Theorem 1 proceeds in several steps which are laid out in detail in the Appendix. First, we characterize the most-efficient profit-maximizing mechanism. This will be the mechanism used by the seller (which obviously depends on the information structure). We then characterize the distribution of expected values of bidders who do not buy the good. We then characterize the distribution of values in any non-regular interval. Finally, we show that bidder surplus is maximized by a regular distribution.

4.3 Characterization of Optimal Information Structure

We now provide the bidder-optimal information structure. For this, we first introduce the family of distributions of expected values that will turn out to be optimal. The family of distributions will be parametrized by three parameters (t_x, t_z, α). The first parameter is the exclusion level t_x. The second parameter is a threshold t_z and an associated value v_z:

$$v_z \triangleq V(t_z). \tag{31}$$

If a bidder's value is below $V(t_x)$ or above $V(t_z)$, then the bidder learns his value; bidders whose values are in between have expected values that are drawn according to a different distribution that is determined by α. When $\alpha = 0$, then this distribution is a Pareto distribution (with shape parameter 1); when $\alpha > 0$, then the distribution of expected values below the cutoff follows a generalized Pareto distribution as defined earlier in (20). We define:

$$G(w\,|t_x, t_z) \triangleq \begin{cases} 1-(1-F(v_z))\frac{v_z-\alpha}{w-\alpha}, & \text{if } w \in [F^{-1}(t_x), F^{-1}(t_z)]; \\ F(w), & \text{if } w \notin [F^{-1}(t_x), F^{-1}(t_z)]. \end{cases} \tag{32}$$

We observe that the above distribution $G(w\,|t_x, t_z)$ is closely related to the distribution $G(w\,|t_z)$ that we introduced in the analysis of the majorization-free environment, see (20). The additional elements in the definition of $G(w\,|t_x, t_z)$ relative to the distribution $G(w\,|t_z)$ reflect the presence of the majorization constraints: (i) without the majorization constraint, the solution did not display any exclusion and thus resulted in $t_x = 0$; (ii) with majorization constraint, there is complete disclosure above the quantile threshold t_z. Thus the distribution $G(w\,|t_x, t_z)$ has to match the distribution $F(v)$ at the critical quantile $t_z = F(v_z)$ and the threshold t_z is attained at v_z rather than at $v = 1$ as in the majorization-free problem.

We illustrate some examples of distributions in Figure 8 and their respective densities in Figure 9, which are optimal for a given N (as stated in the respective figures). The distribution is determined by three parameters (t_x, t_z, α). However, it must also satisfy $F(\cdot) \prec G(\cdot\,|t_x, t_z)$, so there are only two "free parameters", namely (t_x, t_z), while α is endogenously determined to satisfy the majorization constraint. This is the reason we do not add α as an argument of $G(w\,|t_x, t_z)$. We shall later see that α can be interpreted as the shadow cost of allocating the good in the optimal information structure. Note also that not all pairs $(t_x, t_z) \in [0,1] \times [0,1]$ are feasible.

Lemma 1 (Free Parameters)
For every (t_x, t_z), there exists at most one α such that $F(\cdot) \prec G(\cdot\,|t_x, t_z)$.

Proof. We define:

$$H(v\,|t_x, t_z, \alpha) \triangleq \begin{cases} 1-(1-F(v_z))\frac{v_z-\alpha}{v-\alpha}, & \text{if } v \in [F^{t-1}(t_x), F^{-1}(t_z)]; \\ F(v), & \text{if } v \notin [F^{-1}(t_x), F^{-1}(t_z)]. \end{cases}$$

This is the same as the definition of $G(v\,|t_x, t_z)$ but we allow α to vary independently of (t_x, t_z). The distribution $H(v\,|t_x, t_z, \alpha)$ is well defined only if $\alpha < F^{-1}(t_x)$. We can then see that for any $\alpha' < \alpha$:

$$H(v\,|t_x, t_z, \alpha) < H(v\,|t_x, t_z, \alpha').$$

Hence, $H(v\,|t_x, t_z, \alpha)$ first-order stochastically dominates $H(v\,|t_x, t_z, \alpha')$. Hence, there can be only one α that satisfies:

$$\int_0^1 H^{-1}(t\,|t_x, t_z, \alpha)dt = \int_0^1 F^{-1}(t)dt.$$

Thus, for any given (t_x, t_z), there exists at most one α such that the majorization constraint is satisfied. ■

We now characterize the bidder-optimal information structures.

Theorem 2 (Bidder Optimal Information Structure)
Every bidder-optimal information structure satisfies

$$G^*(w) = G(w\,|t_x, t_z),$$

for some parameters (t_x, t_z). Furthermore, $\alpha \geq 0$ and, if $t_x > 0$, then $\alpha = 0$.

We can also present the optimal information structure in the quantile space and write it directly in terms of the values $W(t)$ or the profit function $\pi_w(t)$. The value $W(t)$ in the quantile space is given by

$$W(t\,|t_x, t_z) \triangleq \alpha + (v_z - \alpha)\frac{1 - t_z}{1 - t}, \tag{33}$$

which generalizes the expression (22) to the majorization constraint problem.

Proposition 5 (Bidder Optimal Information Structure: Profit Function)
Any bidder-optimal profit function π_w^* is given by:

$$\pi_w^*(t) \triangleq \begin{cases} \pi(t_z) + \alpha(t_z - t), & \text{if } t \in [t_x, t_z]; \\ \pi(t), & \text{if } t \notin [t_x, t_z]; \end{cases} \tag{34}$$

for some parameters (t_x, t_z). Furthermore, $\alpha \geq 0$ and, if $t_x > 0$, then $\alpha = 0$.

We illustrate some examples of profit functions in Figure 10, which correspond to the profit functions of the distributions in Figure 8. Hence, the distribution below the cutoff t_z is designed

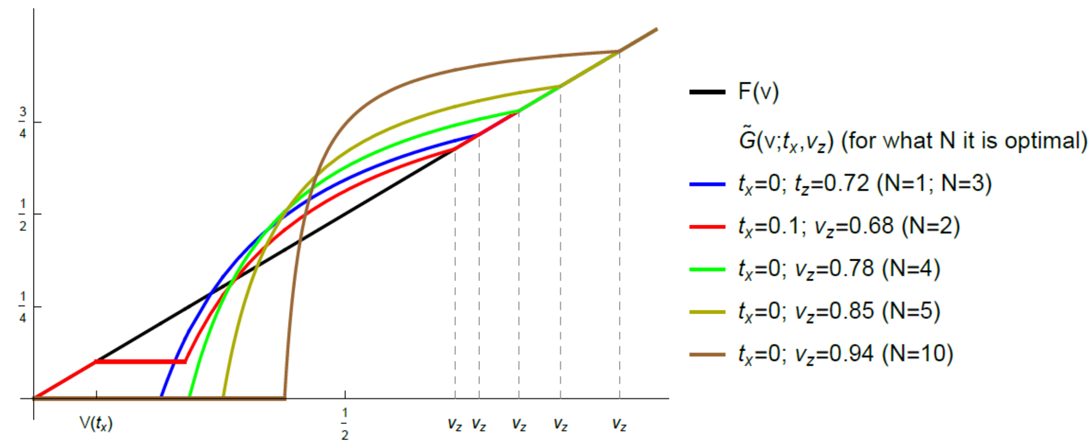

Figure 8: Prior and Posterior Distributions in Value Space v, w

to generate a linear profit function with slope $-\alpha$. When $\alpha = 0$, the seller is indifferent between allocating or discarding the good and is indifferent on how to allocate the good among bidders whose expected values are below v_z. When $\alpha > 0$, the seller finds it strictly optimal to allocate the good (so it is never discarded), but the seller is still indifferent on how to allocate the good among bidders whose values are below v_z.

When $\alpha > 0$ the distribution of expected values below v_z follows a generalized Pareto distribution. The generalized Pareto distribution is a Pareto distribution whose values are shifted upwards. Hence, discarding the good introduces more inefficiencies than with the classic Pareto distribution, so it is never optimal to discard the good. However, the cost-benefit analysis of any re-allocation among bidders of different values remains the same as with the classic Pareto distribution, so the seller is indifferent on how to allocate the good to bidders whose values are below v_z.

We can interpret the distribution of expected values as a demand function, where the mass of bidders willing to pay w per unit is:

$$D^*(w) \triangleq 1 - G^*(w). \tag{35}$$

The generalized Pareto distribution generates demand functions with elasticity:

$$\frac{w}{D^*(W)} \frac{dD^*(W)}{dw} = -\frac{w}{w-\alpha}. \tag{36}$$

Hence, when $\alpha = 0$ we get the classic unit elasticity. In contrast, when $\alpha > 0$, the demand is elastic (less than -1) and it is increasing in w (the distribution will always satisfy $w \geq \alpha$).

If the prior distribution $F(v)$ is the uniform distribution on the unit interval, critical values v_z and critical quantiles t_z happen to coincide.

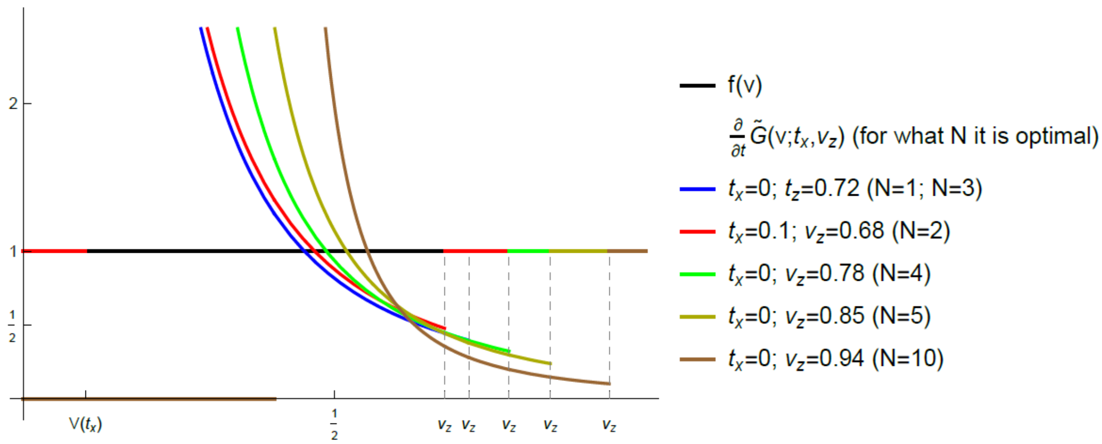

Figure 9: Prior and Posterior Densities in Value Space v, w

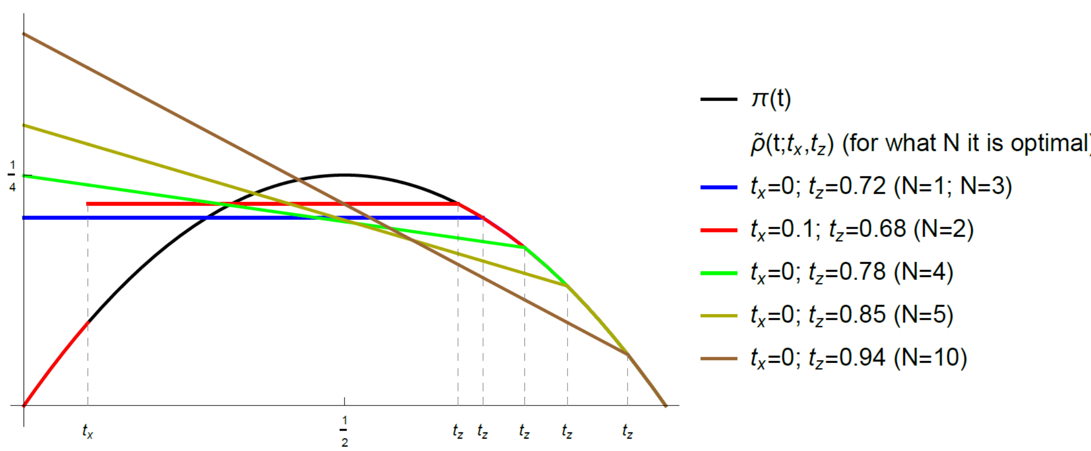

Figure 10: Profit Functions in Quantile Space for Uniform Distribution $F(v) = v$.

5 Two Reasons to Conflate Values

Theorem 2 characterizes the bidder-optimal information structure. The characterization is complete up to two parameters (t_x, t_z) that are endogenously determined by the distribution of values F and the number of bidders N. We now explain how these two endogenous parameters are determined by the parameters of the model.

5.1 Exclusion and Competition

Conflating values can increase bidder surplus for two reasons. First, conflating values allows to reduce the reserve price in the seller-optimal mechanism (that is, reducing t_x). Second, conflating values can decrease competition at the bottom of the distribution which increases bidder surplus (captured by the fact that $s(t)$ is concave at low quantiles). We begin by studying two auxiliary problems that shut down one of the two reasons to conflate values.

We first analyze the bidder-optimal information structure when there is a single bidder. In this case, the seller's problem is completely determined by the reserve price as established in Roesler and Szentes (2017). They showed that the bidder-optimal information structure will minimize this reserve price and no value will be excluded. Naturally, the solution to this problem depends only on the distribution of values (as we are fixing the number of bidders). The solution will be determined by a quantile t_1 that determines when the generated profit function is non-increasing. Roesler and Szentes (2017) first provided an analysis of this problem. We offer a different approach and solution here for two reasons. First, for completeness, it is useful to explain the nature of the solution in the quantile space. Second, the original problem has multiple solutions and the solution we provide here will not coincide with the one given by Roesler and Szentes (2017). In particular, the distribution of values G^* that we derive will replace the mass point in the truncated Pareto distribution with a segment of complete information disclosure, and thus G^* will coincide with the original distribution F on the upper end of the support.

We then analyze the problem where the seller is constrained to choose a mechanism in which the good is assigned with probability one, that is, constrained to use a "must-sell" mechanism. Of course, even if the seller is constrained to sell the good with probability one, he can use inefficient mechanisms. We show the solution to this alternative problem is also determined by a single quantile t_m. The most remarkable property is that the optimal cutoff t_m depends only on the number of bidders but not on the distribution of values F. Furthermore, it is increasing in N.

Finally, we connect the solution to both problems with the bidder-optimal information structure characterized in Theorem 2. We show that, if the bidder-optimal information structure when the seller is constrained to sell the good is indeed feasible in the general problem (that is, the solution would not induce the seller to exclude bidders even if he were allowed to do so), then this is the solution to the unconstrained problem. Otherwise, the solution will be designed to minimize the reserve price, possibly with some exclusion.

We illustrate the nature of the solution in Figures 11-12. For this, it is visually simpler to see the solution by allowing N to take continuous values. While there is no economic interpretation for non-integer values of N in the auction setting we have studied so far, optimization problem (5) remains well defined. Furthermore, we will see later that this problem can be interpreted naturally in a different economic environment. In Figure 11 we illustrate t_m (dependent only on N) that is increasing in N, and the cutoff t_1 (dependent only on F). We denote the intersection by $\widehat{N}$, and illustrate some parameters of the solution in Figure 12. If $N \geq \widehat{N}$, then the solution is the same as when the seller is constrained to sell the good: there is no exclusion ($t_x = 0$) and typically the distribution of expected values does not generate a demand with unit elasticity ($\alpha > 0$). If $N \leq \widehat{N}$, then the solution will consist of an exclusion level t_x and a cutoff t_z such that the distribution of expected values generates a demand with unit elasticity in quantiles $[t_x, t_z]$. In the former case ($N \geq \widehat{N}$) conflating arises to reduce competition at the bottom of the distribution of expected values; in the latter case ($N \leq \widehat{N}$) conflating arises to reduce the reserve price (possibly generating some exclusion). Note that when $N \leq \widehat{N}$ the solution does not necessarily coincide with the single-bidder solution because it may be optimal to allow for some exclusion. The reason exclusion may be optimal is that conflating introduces inefficiencies even if the seller allocates the good interim efficiently (in general, interim efficiency will differ from ex-post efficiency).

Throughout this section, we assume that $\pi(t)$ is concave at every $t \in [0, 1]$. We recall our results so far relied on a mildly weaker assumption (that is, π being quasi-concave and concave on its decreasing part). The former assumption is equivalent to requiring that the virtual values are non-decreasing, while the latter assumption is equivalent to requiring that the virtual values are non-decreasing whenever they are positive.

The bidder-optimal information structure is determined by the parameters (t_x, t_z), which are endogenously determined by the distribution of values and the number of bidders. A substantive part of the analysis in this section consists of understanding how the parameters of the model change t_z for a fixed level of exclusion t_x. To provide an alternative interpretation of this exercise,

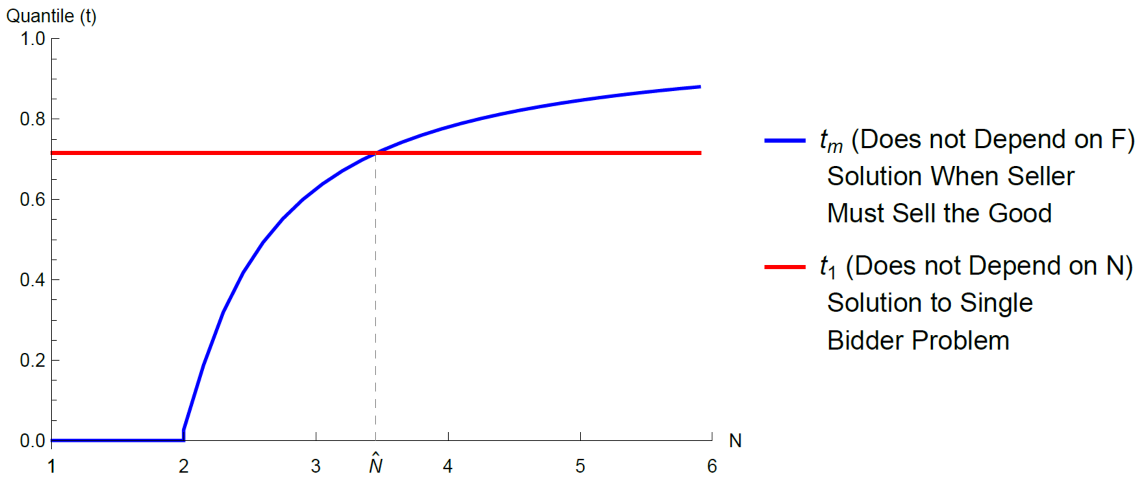

Figure 11: Threshold Values of the Solution to the Restricted Problems: (*i*) N=1 and (*ii*) Must Sell Mechanism.

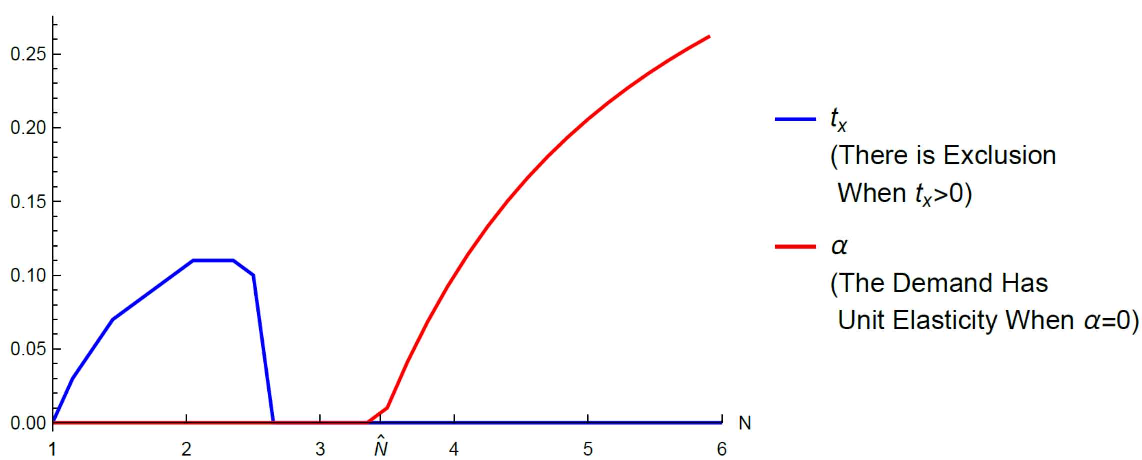

Figure 12: Solution to General Problem with $N > 1$

we characterize how t_z changes the informativeness of the information structure.

Lemma 2 (Effect of Threshold t_z)
For all $t_x, t_z, t_z' \in [0,1]$ with $t_z < t_z'$, $W(\cdot\,|t_x, t_z) \prec W(\cdot\,|t_x, t_z')$.

Proof. There is a simple sufficient condition for $W \prec V$ that we repeatedly use throughout the paper. We write

$$\text{sign}(W - V) \triangleq (-,+), \tag{37}$$

if there exists a unique $\hat{t} \in [0,1]$ such that $W(t) - V(t)$ is non-positive for quantiles $t \leq \hat{t}$ and it is non-negative for quantiles $t \geq \hat{t}$. A sufficient condition for $V \succ W$ is that V and W have the same mean, that is,

$$\int_0^1 W(t)dt = \int_0^1 V(t)dt,$$

and $\text{sign}(W - V) = (+,-)$ (see Theorem 3.A.44, Shaked and Shanthikumar (2007)).

We observe that for all t_z, t_z' we have that:

$$\int_0^1 W(t\,|t_x, t_z)dt = \int_0^1 W(t\,|t_x, t_z')dt.$$

This implies that

$$\frac{\partial \alpha}{\partial t_z} > 0,$$

where α is the parameter that determines $\pi_w(t\,|t_x, t_z)$ in (34). We thus have that $\text{sign}(W(t\,|t_x, t_z) - W(t\,|t_x, t_z')) = (+,-)$, hence $W(t\,|t_x, t_z) \prec W(t\,|t_x, t_z')$, which proves the result. ■

As we provide a comparative statics analysis of how t_z changes with the parameters of the model, we can thus compare how much information is conveyed in information structure $W(\cdot\,|t_x, t_z)$ versus $W(\cdot\,|t_x, t_z')$.

5.2 Single Bidder Problem

We now analyze the problem of finding the bidder optimal information structure when there is a single bidder. Since there is a single bidder, the only inefficiency that can arise is that the object remains unsold. Formally, we solve the following problem:

$$W^* \in \underset{t_x \in [0,1], W \in \mathcal{W}(t_x)}{\arg\max} \frac{1}{N}\left(\int_{t_x}^1 W(t)dt - (1 - t_x)W(t_x)\right). \tag{38}$$

This corresponds to (30) but replacing $q(t) \in [0,1]$ with $q(t) = 1$ for all $t \in [0,1]$.

We now define t_1 implicitly as follows. If $\pi(t)$ is decreasing at every $t \in [0,1]$, then we set $t_1 = 0$. Otherwise, we define $t_1 > 0$ as the unique solution to the following equation:

$$\int_0^{t_1} \frac{V(t_1)(1-t_1)}{1-t} dt = \int_0^{t_1} V(t) dt. \tag{39}$$

The threshold t_1 identifies the critical upper threshold t_z in the special case of a single bidder. By construction, we have that, if $t_1 > 0$, then $t_x = 0$, and thus $W(t\,|0,t_1)(1-t)$ is constant in $t \in [0,1]$, where $W(t\,|0,t_1)$ is the information structure expressed in values given earlier in (33).

Proposition 6 (Optimal Information Structure for a Single Bidder)
A solution to the single bidder problem (38) *is* $W(t\,|0,t_1)$. *Furthermore, if* $\pi(t)$ *is increasing at* $t = 0$, *then* $t_1 > 0$.

We can provide comparative statics to describe how t_1 changes with F. In general, if F is "greater and less variable" than $\widehat{F}$, then $t_1 \leq \widehat{t}_1$. We now formalize this intuition. For this we define:

$$\Phi(x\,|V) \triangleq \int_0^x \frac{V(x)(1-x)}{1-t} dt - \int_0^x V(t) dt.$$

We note that $\Phi(x\,|V)$ is quasi-concave in x and $\Phi(x\,|V)$ is increasing at $x = 0$ if and only if $\pi(x)$ is increasing at $x = 0$. Hence, at $t = t_1$ we have that $\Phi(t\,|V)$ is decreasing.

We then have that, for any $k \in \mathbb{R}_+$ and distribution of values $\widehat{V}(t) = V(t) + k$, we have that:

$$\Phi\left(\hat{t}\,\middle|\widehat{V}\right) \leq \Phi(\hat{t}\,|V),$$

and hence $\widehat{t}_1 \leq t_1$. Hence, translating a distribution to higher values leads to a lower cutoff t_1.

However, the cutoff t_1 is not necessarily decreasing in the first-order stochastic dominance. Consider for example the distribution F being uniform in $[1/4, 1/2]$ and the distribution $\widehat{F}$ being uniform in $[1/2, 3/2]$. We have that $\widehat{F}$ first-order stochastically dominates F, however, $t_1 < \widehat{t}_1$. Since first-order stochastic dominance is a sufficient condition for second-order stochastic dominance, the latter order is also insufficient to guarantee the monotonicity of the cutoff t_1.

Definition 1 (Dispersive Order)
$\widehat{F}$ *is smaller than* F *in the dispersive order if*

$$\widehat{V}(\beta) - \widehat{V}(\alpha) \leq V(\beta) - V(\alpha),$$

for all $0 \leq \alpha < \beta \leq 1$ *(see Definition 3.B.1., Shaked and Shanthikumar (2007)).*

The dispersive order requires the difference between any two quantiles of $\widehat{F}$ to be smaller than the difference between the corresponding quantiles of F. If the quantile functions are differentiable, then a necessary and sufficient condition for a dispersive order is that:

$$\frac{d\widehat{V}(t)}{dt} \leq \frac{dV(t)}{dt}.$$

This is a notion of variability. For example, if two random variables have the same mean and $\widehat{F}$ is smaller than F in the dispersive order then $\widehat{F}$ is also smaller than F in the convex order. It is also translation invariant, that is, if $\widehat{F}$ is smaller than F in the dispersive order, then the order will be preserved if one of the random variables is translated by a constant. In the example above, the random variable that is uniform in $[1/2, 3/2]$ is greater in the dispersive order than the random variable that is uniform in $[1/4, 1/2]$. Hence, in this case, the higher variability leads to a higher cutoff $\widehat{t}_1 \leq t_1$.

Proposition 7 (Comparative Statics t_1)
If $\widehat{F}$ is smaller than F in the dispersive order and $\widehat{F}$ first-order stochastically dominates F, then $\widehat{t}_1 \leq t_1$.

The proposition shows that first-order stochastic dominance jointly with the dispersive order can provide sufficient conditions for obtaining a monotone comparative static on t_1. To prove the result, we note that we can write Φ as follows:

$$\Phi(x\,|V) \triangleq \int_0^x \frac{1}{1-t} \int_t^x -\phi(z)dzdt,$$

where ϕ is defined as in (13) and can be explicitly written as follows:

$$\phi(z) = V(z) - \frac{dV(z)}{dz}(1-z).$$

If $\widehat{F}$ is smaller than F in the dispersive order, then $d\widehat{V}(z)/dz \leq dV(t)/dz$. If $\widehat{F}$ first-order stochastically dominates F, then $V(t) \leq \widehat{V}(z)$. This proves the result.

We can alternatively interpret the proof as stating if $\widehat{F}$ is smaller than F in the dispersive order and $\widehat{F}$ first-order stochastically dominates F, then the virtual values in quantile space generated by $\widehat{F}$ are larger than the virtual values in quantile space generated by F. Hence, if $\widehat{F}$ is smaller than F in the dispersive order and $\widehat{F}$ first-order stochastically dominates F, then the solution to the single-bidder optimal-pricing problem exhibits less exclusion under $\widehat{F}$ than under F.

5.3 Must-Sell Mechanisms

We now consider the following problem:

$$U_m \triangleq \max_{t_z \in [0,1]} \frac{1}{N} \int_0^1 W(t\,|0, t_z) \frac{ds(t)}{dt} dt; \tag{40}$$

the maximizing cutoff that solves this maximization problem is denoted by t_m. The subscript *"m"* is an acronym for *must sell*. We consider an auxiliary problem in which the seller is constrained to transfer the good but can use any mechanism. Formally, the problem is written as in (5)-(6), except the maximization is over all allocations subject to $r \prec q$ instead of $r \prec_w q$. If the majorization constraint is not weak, then the mechanism will be non-wasteful. We then have that, if $t_m \geq t_1$, then $W(t\,|0, t_m)$ solves the problem where the seller is constrained to sell the good. We begin by providing properties of the optimization problem.

Lemma 3 (Characterization of Threshold t_m)
The objective function of the must-sell mechanism (40) *is quasi-concave in t_z with a unique interior optimum (denoted t_m). Furthermore, t_m is determined independent of the distribution of values (that is, it depends on N but not on F).*

The most remarkable aspect of this lemma is that the distribution of values does not play any role in determining the cutoff.

Proposition 8 (Comparative Statics with Respect to N and Limit)
The cutoff t_m is increasing in N, $t_m = 0$ when $N = 2$, and in the limit $N \to \infty$, $t_m \to 1$.

As N increases and the seller must sell the good, the optimal information structure becomes less informative. In the limit $N \to \infty$ the information structure converges to an information structure that gives no information (generates a distribution of expected values that is an atom), except at the very top of the distribution.

5.4 The Determinants of the Optimal Information Structure

We use the above two benchmarks and in particular the thresholds t_1 and t_m to characterize the bidder-optimal information structure with N bidders, while allowing the seller to exclude bidders.

Theorem 3 (Bidder-Optimal Information Structure)
If $t_1 < t_m$, then the optimal information structure is given by $W(t\,|0, t_m)$. Otherwise, the optimal mechanism displays $\alpha = 0$ (possibly with exclusion level $t_x > 0$).

The above theorem gives a sharp characterization of the nature of the solution. If the solution to the must-sell mechanism (40) is feasible, then this is the solution. These are the situations in which avoiding exclusion is not a concern when designing the bidder-optimal information structure. In this case, low values are conflated to decrease competition at the bottom of the distribution, which increases consumer surplus.

If the solution to (40) is not feasible, then the solution is shaped by the information structure that can best avoid exclusion. Hence, lower values are conflated following a Pareto distribution, which allows to lower the reserve price. In this case, the planner has two instruments to prevent exclusion. First, the seller can increase the amount of conflating (increase t_z), but this leads to too much conflating relative to t_m. The other instrument is to accept some exclusion ($t_x > 0$), which allows to reduce the amount of conflating.

We now immediately obtain the nature of the solution when $t_m = 0$ or $t_1 = 0$. First, we can find situations in which the solution to the bidder-optimal information structure when the seller must sell the good is feasible.

Proposition 9 (When the Must-Sell Mechanism is Optimal)
If π is non-increasing or if there is a sufficiently large number of bidders N, then the optimal information structure is given by $W(t\,|0, t_m)$. Furthermore, the resulting optimal mechanism exhibits no exclusion.

This proposition characterizes when the solution corresponds to (40). In these cases, the cutoff t_z does not depend on the distribution of values. In these cases, there is also no exclusion. Note that when $N = 1$ the optimal mechanism also does not exhibit exclusion. Hence, exclusion only arises for intermediate values of N. These are the situations in which the seller wants only a moderate level of conflating to dampen competition of low values. In these cases, exclusion arises because avoiding exclusion completely introduces too many inefficiencies. Note that even if π is non-increasing, the profit function generated by a given information structure might be non-monotonic.

Proposition 10 (Two Bidders)
If $N = 2$, then in the optimal mechanism $\alpha = 0$.

Proof. If $N = 2$, we have that $s(t)$ is convex and thus $t_m = 0$. If π is non-increasing, then $t_1 = 0$. The result then follows from applying Theorem 3. ■

6 Related Allocation Problems

Throughout the proofs in the paper we have used the following facts: (i) the distribution of values satisfies that $\pi(t)$ is quasi-concave and concave wherever it is decreasing, (ii) any variation of the information structure that increases total surplus and reduces profit will lead to a higher bidder surplus, (iii) $s(t)$ has a unique inflection point, being concave for low quantiles, (iv) $s(t)$ is quasiconvex, and (v) $q(0) = 0$. We can now apply the results to alternative allocation problems and alternative objective functions.

6.1 Other Objective Functions

We now assume that the information structure is chosen to maximize a linear combination of total surplus and a negative weight on profit $\lambda \in (0, \infty)$:

$$W_\lambda \triangleq TS - \lambda \Pi,$$

where TS is the total surplus. We formally write the problem as follows, where the only variation is that we have added a weight λ to profit.

$$W_\lambda^* \triangleq \max_{\substack{\{W:W\prec V\}\\ \{r:r\prec_w q\}}} \int_0^1 r(t)W(t)dt - \lambda \left(\int_0^1 W(t)(1-t)dr(t) + r(0)W(0) \right)$$
$$\text{subject to (6)}$$

The case we have studied so far is the case $\lambda = 1$; the case $\lambda \to \infty$ corresponds to the case where the information structure minimizes total profit; the case $\lambda = 0$ is when the objective is to maximize total surplus. Note that the analysis requires that $\lambda \geq 0$ as the arguments rely on variations of the information structure that increase the total surplus and reduce the profit will lead to a higher bidder surplus. When $\lambda < 0$, this is no longer valid.

To explain how results are extended, we define:

$$s_\lambda(t) \triangleq \int_t^1 \left[q(x)dx - \lambda \int_t^1 q(x) - q(t) \right] dx.$$

The analysis goes through unchanged as long as s_λ is concave for low quantiles and convex for high quantiles.

To verify this condition we explicitly calculate the second derivative:

$$\frac{d^2 s_\lambda}{dt^2} = -(1+\lambda)q'(t) + \lambda(1-t)q''(t).$$

If $q'' \leq 0$, then s_λ is convex and the condition is immediately satisfied ($q''(t) \leq 0$ corresponds to assuming the distribution of qualities has increasing density). We also have that s_λ is convex if $\lambda \geq 0$ and the distribution of qualities has increasing hazard rate (that is, if $(1-t)q'(t)$ is decreasing). If $q(t) = t^{N-1}$, then there is a unique inflection point at:

$$t_I = \frac{\lambda(N-2)}{\lambda(N-1)+1}.$$

So the condition is also satisfied. We can then conclude the following theorem.

Theorem 4 (Welfare-Maximizing Information Structure)
For every $\lambda \geq 0$, any welfare-optimal information structure W^*_λ satisfies

$$G^*_\lambda(t) = G(t\,|t_x, t_z\,),$$

for some parameters (t_x, t_z). Furthermore, $\alpha \geq 0$ and, if $t_x > 0$, then $\alpha = 0$.

A particular instance of the above problem is when $\lambda = 0$. In this case, the information is chosen to maximize the total surplus. Note that the mechanism is endogenously chosen, so complete information might still not be the surplus-maximizing information structure. However, it is easy to see s_0 is always convex, so the problem is completely analogous to that studied in Section 9.2.1.

Corollary 3 (Surplus-Maximizing Information Structure)
If the objective is to maximize total surplus, then for any optimal information structure W^*_0, there exists (t_x, t_z) such that:

$$G^*_0(t) = G(t\,|t_x, t_z\,).$$

Furthermore, $G^*_0(t)$ generates a unit-elasticity demand in $[t_x, t_z]$ (and possibly $t_x > 0$).

6.2 Multi-Unit Auction

Suppose instead of a single good for sale, the seller has N goods of qualities $\theta_1 \leq \ldots \leq \theta_N$. If a bidder with value v_i buys a good of quality θ_j, he gets a utility $v_i\theta_j$. The model studied so far corresponds to the case $\theta_1 = \ldots = \theta_{N-1} = 0$ and $\theta_N = 1$.

Following Kleiner, Moldovanu, and Strack (2021), for any information structure, the seller's profit is given by:

$$r \in \underset{\{\hat{r}:\hat{r}\prec_w q\}}{\arg\max} \int_0^1 (1-t)W(t)d\hat{r}(t) + \hat{r}(0)W(0), \tag{41}$$

where now $q(t)$ is given by:

$$q(t) = \sum_{j=1}^{N} \theta_j \left((1-t)^{N-j} t^{j-1} \frac{(N-1)!}{(j-1)!(N-j)!} \right).$$

Thus, the set of feasible allocations are those that are weakly majorized by the efficient allocation, written in terms of quantiles.

We solve:

$$U^* \triangleq \max_{\substack{\{W: W \prec V\} \\ \{r: r \prec_w q\}}} \int_0^1 r(t) W(t) dt - \left(\int_0^1 W(t)(1-t) dr(t) + r(0)W(0) \right), \tag{42}$$

subject to (41).

The results of Theorem 1 and 2 go through as long as (i) $s(t)$ has a unique inflection point, being concave for low quantiles and convex for high quantiles, (ii) $s(t)$ is quasiconvex, and (iii) $q(0) = 0$. If the first condition is satisfied but either of the last two conditions is not satisfied, then a weaker characterization can be found (see Proposition 15 in the Appendix).

6.3 Large Market

Consider now the situation in which there is a seller that has a unit mass of goods for sale. Goods have various qualities with distribution F_q. There is also a mass 1 of bidders with a value distribution F_v. We now denote the inverse of F_q by:

$$q(t) = F_q^{-1}(t).$$

As before, we consider the situation in which first the bidders choose their information structure and then the seller chooses the optimal mechanism. The payoff environment studied here is a particular instance of the payoff environment studied by Bergemann, Heumann, and Morris (2023c). The main difference is that here we study the consumer-optimal information structure while there we considered the seller-optimal information structure.

In the large market setting, we recover the same problem as in (5) except that now $q(t)$ is determined by the exogenous distribution of qualities (the derivation can be found in Bergemann, Heumann, and Morris (2023c)). Note that we can now interpret the distribution of qualities of the form $q(t) = t^{N-1}$ even for non-integer values of N. In the limit $N \to 1$, we have that $q(t)$ converges to 1 pointwise.

As before, the results of Theorem 1 and 2 go through as long as (i) $s(t)$ has a unique inflection point, being concave for low quantiles and convex for high quantiles, (ii) $s(t)$ is quasiconvex, and (*iii*) $q(0) = 0$. If $s(t)$ is convex, then we do not need to impose $s(0) = 0$.[2] If the first condition is satisfied but either of the last two conditions is not satisfied, then a weaker characterization can be found (see Proposition 15 in the Appendix).

We can, for example, easily characterize the optimal mechanism when F_q has increasing hazard rate.

Proposition 11 (Increasing Hazard Rate)
If F_q has increasing hazard rate, then for any bidder-optimal information structure W^*, there exists (t_x, t_z) such that:

$$G^*(t) = G(t|t_x, t_z).$$

Furthermore, $G^*(t)$ generates a unit-elasticity demand in $[t_x, t_z]$ (and possibly $t_x > 0$).

Proof. By definition, F_q has increasing hazard rate if and only if:

$$\frac{F_q'(x)}{(1 - F_q(x))},$$

is increasing. Taking the derivative, we obtain:

$$\frac{(F_q'(x))^2}{(1 - F_q(x))^2} + \frac{F_q''(x)}{(1 - F_q(x))} \geq 0.$$

Writing this condition in quantile space, we obtain:

$$\frac{1}{(q'(t))^2(1-t)^2} - \frac{q''(t)}{(q'(t))^3(1-t)} \geq 0.$$

On the other hand,

$$s''(t) = 2q'(t) - (1-t)q''(t) \geq q'(t) \geq 0.$$

We thus obtain that $s(t)$ is convex. Thus, we obtain the same solution as with $N = 2$ (we also study this case in Section 9.2.1 in the Appendix). ■

We can provide a corollary showing when full disclosure is optimal.

Corollary 4 (When Full Disclosure is Optimal)
If $F_q(v)$ has increasing hazard rate and $\pi(t)$ is non-increasing, then full disclosure is the optimal information structure $G^* = F$..

[2]If $s(t)$ is convex and $s(0) \neq 0$, we can approximate $s(t)$ by a sequence of convex functions $s_k(t)$ satisfying that $s_k(0) = 0$ and they converge to $s(t)$ pointwise for every $t \neq 0$.

7 Asymmetric and Correlated Information Structures

We now analyze asymmetric and correlated information structures. We first show that under some conditions an asymmetric information structure may generate higher bidder surplus than the optimal symmetric information. Under the asymmetric information structure there will be only 1 bidder who gets positive bidder surplus. By randomizing ex ante over all agents (to identify which agent might get positive rents), we find that a correlated information structure can replicate the asymmetric information structure, and thus yield higher bidder surplus than the optimal symmetric information structure.

We make the following assumptions. We assume that π is weakly concave and non-increasing and there is an upper bound $\bar{v}$ on the support of values. To simplify the algebra, we also assume there is an atom (arbitrarily small) at $\bar{v}$ (the analysis goes through unchanged if we relax this last assumption). We compare the information structure in two situations.

We first obtain an upper bound on the bidder surplus generated by the optimal symmetric information structure by considering a situation in which the seller is constrained to use an efficient mechanism. That is, we consider first a situation in which the seller must sell the good via the second-price auction. Following Bergemann, Heumann, Morris, Sorokin, and Winter (2022), the optimal information structure consists of conflating all values below quantile $t_q = (N-2)/(N-1)$. It is simple to check that in the limit $N \to \infty$ there is an infinite number of bidders with expected value μ_v and the number of bidders with value $\bar{v}$ is distributed according to Poisson distribution with parameter $\lambda = 1$. The bidders get positive rents only if there is 1 and only 1 bidder with value $\bar{v}$, which occurs with probability $1/e$. Hence, the bidder surplus is bounded as follows:

$$U^* \leq \frac{\bar{v} - \mu_v}{e}.$$

We remark that this bound holds only for a large enough N.

We now consider the following asymmetric information structure. There are $N-1$ bidders that obtain no information and so they have an expected value μ_v and there is one bidder that has complete information. Since π is weakly concave and non-increasing the seller will find it optimal to use a second-price auction. In this situation, the bidder's surplus will be:

$$U^A \triangleq \mathbb{E}[\max\{v - \mu_v, 0\}].$$

That is, when the bidder who has complete information has a value above μ_v he gets positive rents equal to the difference between his value and the second-highest bid that is always equal to μ_v.

We compare the profit generated by the asymmetric information structure with the upper bound for symmetric information structures, and if:

$$\frac{\bar{v} - \mu_v}{e} < \mathbb{E}[\max\{v - \mu_v, 0\}], \tag{43}$$

then the asymmetric information structure generates strictly higher rents than the optimal symmetric information structure does.

Proposition 12

As $N \to \infty$, if (43) *is satisfied, then there exists an asymmetric information that generates higher bidder surplus than the optimal symmetric information structure.*

8 Conclusion

In this paper, we studied the bidder-optimal information structure when the seller responds with the seller-optimal allocation. In Theorem 3 we explored when the nature of solution remains qualitatively the same under varying objective functions. In concurrent work, Bergemann, Heumann, and Morris (2023b), we study the profit-minimizing information structure for a large number of bidders and given supply of quantities. This corresponds to a nonlinear pricing problem and we show that a saddle point exists. We use the saddle point to construct the corresponding robust mechanism that guarantees the highest profit across all information structures in a large market with a continuum of bidders. The main connection we find is that the profit-minimizing information structure is always a regular distribution (in fact, this holds without having to make any assumptions on the distribution of values).

The interpretation that we offered here is that the bidders can influence the surplus property of the bids by suppressing some information about their values for the object. A different, and perhaps more elementary exercise is how to combine bundles of different objects with different information structures so as to raise the profit and or the surplus. In Bergemann, Heumann, and Morris (2023a) we pursue this line of reasoning in the presence of assets with common and idiosyncratic payoff shocks. The analysis still relies on majorization arguments, but now only along a single dimension of common vs. idiosyncratic shock rather than the two-dimensional analysis in terms of values and quantities that are at the core of this paper and Bergemann, Heumann, and Morris (2023b). Loertscher and Muir (2023) and Sadzik and Woolnough (2023) are recent contributions that also investigate how the composition of the object influences the profit. Loertscher and Muir (2023) consider a model where the private information is the location on the Hotelling line and the seller chooses the composition of the goods offered in terms of a weighted sum of their locations at the extremes of the Hotelling line. Sadzik and Woolnough (2023) consider a strategic trading model where there are informed and uninformed traders. A central issue is to determine conditions under which strategic traders do not wish to issue assets in complete markets, and thus support the bundling of goods in equilibrium.

9 Appendix

The Appendix collects the proofs and auxiliary results that were omitted in the main body of the paper.

9.1 Proof of Theorem 1

We now proceed to prove this theorem in several steps. First, we characterize the most-efficient revenue-maximizing mechanism. This will be the mechanism used by the seller (which obviously depends on the information structure). We then characterize the distribution of expected values of bidders who do not buy the good. We then characterize the distribution of values in any non-regular interval. Finally, we show that bidder surplus is maximized by a regular distribution.

9.1.1 The Most-Efficient Revenue-Maximizing Allocation

While the characterization of the seller's revenue is standard in the literature, to characterize the bidder surplus, we need to describe explicitly the allocation function that maximizes revenue for a fixed information structure, i.e., we describe the quantile allocation r that solves the revenue maximization problem:

$$r \in \underset{\{\tilde{r}:\tilde{r} \prec_w q\}}{\arg\max} \int_0^1 W(t)(1-t)d\tilde{r}(t). \tag{44}$$

There may be multiple revenue-maximizing allocations only if there exists some interval $[t_1, t_2]$ such that for every quantile $t \in [t_1, t_2]$: (a) $\text{cav}[\pi_w](t) = \pi_w(t)$, and (b) $\pi_w(t)$ is linear. For a "generic" information structure $W(t)$, maximization problem (44) has a unique solution, which gives the revenue described in Proposition 1. However, since the information structure is endogenous, we might have that there are multiple solutions to (44). If multiple allocation functions maximize revenue, the most efficient one will be the one that generates the highest bidder surplus. Hence, we now characterize the most-efficient revenue-maximizing allocation function.

For any fixed profit function π_w we define:

$$t_m(\pi_w) \triangleq \min\{t \in [0,1] : t \in \arg\max \text{cav}[\pi_w](t)\}. \tag{45}$$

That is, $t_m(\pi_w)$ is the smallest quantile that maximizes revenue. Consider $\{[\underline{t}_i, \bar{t}_i]\}_{i\in I}$ being a collection of monotonic disjoint intervals in $[t_m(\pi_w), 1]$ such that $\text{cav}[\pi_w](t) = \pi_w(t)$ for every $t \in [\underline{t}_i, \bar{t}_i]$ and $\text{cav}[\pi_w](t) > \pi_w(t)$ for every $t \in (\bar{t}_i, \underline{t}_{i+1})$.

Proposition 13 (Most Efficient Optimal Allocation Rule)
The following allocation rule r solves the revenue maximization problem (44)*:*

1. *For all $t < t_m(\pi_w)$, $r(t) = 0$;*

2. *For all $t \in [\underline{t}_i, \bar{t}_i]$, $r(t) = q(t)$;*

3. *For all $t \in ([)\bar{t}_i, \underline{t}_{i+1})$, $r(t) = \frac{\int_{\bar{t}_i}^{t_{i+1}} q(s)ds}{\underline{t}_{i+1} - \bar{t}_i}$.*

Furthermore, this allocation rule generates higher total surplus than any other allocation rule $\hat{r}$ that solves (44)*.*

Proof of Proposition 13. The first part of the statement corresponds to showing that the allocation rule generates the same revenue as the ones that appear in Proposition 1. But this corresponds to the proof of Proposition 1. The second part of the proof corresponds to showing that this is the most efficient optimal allocation rule. We consider a different allocation rule $\hat{r}(t)$ that solves (44) and show it generates less total surplus than r.

We first show that for every $t \in [0,1]$:

$$\int_t^1 r(s)ds \geq \int_t^1 \hat{r}(s)ds. \tag{46}$$

This corresponds to showing that r weakly majorizes $\hat{r}$ (i.e., $\hat{r} \prec_w r$). For every $t \in [\underline{t}_i, \bar{t}_i]$ with $t \geq t_m(\pi_w)$ we have that:

$$\int_t^1 r(s)ds = \int_t^1 q(s)ds \geq \int_t^1 \hat{r}(s)ds.$$

If there exists $t \in [\bar{t}_i, \underline{t}_{i+1}]$ such that

$$\int_t^1 r(s)ds < \int_t^1 \hat{r}(s)ds,$$

we necessarily have that:

$$\int_{\bar{t}_i}^1 r(t)dt < \int_{\bar{t}_i}^1 \hat{r}(t)dt \quad \text{or} \quad \int_{\underline{t}_{i+1}}^1 r(t)dt < \int_{\underline{t}_{i+1}}^1 \hat{r}(t)dt. \tag{47}$$

To check this, note for any t such that $\pi_w(t) < \text{cav}[\pi_w](t)$ we must have that

$$\frac{dr}{dt} = \frac{d\hat{r}}{dt} = 0.$$

We thus have that the majorization constraint would be violated. Finally, for every $t \leq t_m(\pi_w)$ we must have that $\hat{r}(t) = 0$. Hence, we also get that (46) is satisfied for $t \leq t_m(\pi_w)$. Hence, r weakly majorizes $\hat{r}$.

The total surplus generated by r is given by:

$$\int_0^1 r(t)W(t)dt = \int_0^1 \int_t^1 r(s)dsdW(t) + W(0)\int_0^1 r(s)ds, \tag{48}$$

where the equality is obtained by integrating by parts. The analogous expression holds for $\hat{r}$. We have that (46) implies that total surplus generated by r will be larger than that generated by $\hat{r}$. Hence, r generates a higher total surplus and the same profit as $\hat{r}$. ■

Henceforth, we assume that the seller always implements the allocation described in Proposition 13. Hence, the quantiles $\{[\underline{t}_i, \bar{t}_i]\}_{i\in I}$ are defined as in Proposition 13.

9.1.2 Excluded Types

We now characterize the distribution on the domain of values that are excluded by the seller's mechanism.

Lemma 4 (Excluded Types)
If W^* is a bidder-optimal information structure, then the majorization constraint binds at $t_m(\pi_w^*)$:

$$\int_{t_m(\pi_w^*)}^1 V(s)ds = \int_{t_m(\pi_w^*)}^1 W^*(s)ds.$$

Furthermore, without loss of generality, there is an optimal information structure in which $W^*(t) = V(t)$ for all $t \in [0, t_m(\pi_w^*)]$.

Proof of Lemma 4. Suppose π_w^* is an optimal information structure and consider $t_x = t_m(\pi_w^*)$ (defined as in (45)). Suppose that the majorization constraint does not bind at t_x, i.e.,

$$\int_{t_x}^1 W^*(t)dt < \int_{t_x}^1 V(t)dt.$$

We then consider the following information structure:

$$\widehat{W}(t) = \begin{cases} W^*(t), & \text{for all } t \in [t_x, 1]; \\ \frac{W^*(t_x)}{1-t}, & \text{for all } t \in [t_x - \eta, t_x]; \\ V(t), & \text{for all } t \in [0, t_x - \eta]; \end{cases}$$

where η is such that:

$$\int_{t_x-\eta}^{1} \widehat{W}(t)dt = \int_{t_x-\eta}^{1} V(t)dt.$$

We then have that $\widehat{W}$ is monotonic and satisfies the majorization constraint. We have that the total surplus generated is larger and the revenue is the same, so bidder surplus is larger. ∎

There will be a natural indeterminacy in the bidder-optimal information structure because the bidder surplus (and total surplus) does not depend on the information that is provided to quantiles that are excluded by the revenue-maximizing mechanism. Hence, the bidder-optimal information structure may prescribe any information for quantiles that are excluded. Hence, in what follows, we say $W(t) = \widehat{W}(t)$ if both distributions are the same for quantiles that are not excluded, that is,

$$W(t) = \widehat{W}(t) \iff t_m(W) = t_m(\widehat{W}) \text{ and } W(t) = \widehat{W}(t), \text{ for all } t \geq t_m(W).$$

Of course, this only disciplines the information of quantiles that do not buy the good.

9.1.3 Structure of Irregular Intervals

We now show that, if the bidder-optimal information structure is not regular (i.e., $\pi_w^*(t) < \text{cav}[\pi_w^*](t)$ for some t), then the quantiles that are being conflated have a constant expected value. That is, the information structure generates an atom of expected values.

Lemma 5 (Atoms in Pooling Intervals)
In every conflating interval $(\bar{t}_i, \underline{t}_{i+1})$:

$$W^*(t) = W^*(\bar{t}_i), \textit{ for all } t \in (\bar{t}_i, \underline{t}_{i+1}).$$

Proof. Suppose there exists an interval $(\bar{t}_i, \underline{t}_{i+1})$ in which:

$$W^*(\bar{t}_i) < \lim_{t \uparrow \underline{t}_{i+1}} W^*(t),$$

where the limit is taken from below. We show that W^* does not maximize bidder surplus.

We denote by m the slope of $\text{cav}[\pi_w^*]$ in this interval:

$$m = \frac{\pi_w^*(\underline{t}_{i+1}) - \pi_w^*(\bar{t}_i)}{\underline{t}_{i+1} - \bar{t}_i}.$$

Then, consider the following information structure:

$$\widehat{W}(t) = \begin{cases} W^*(t), & \text{if } t \notin (\bar{t}_i, \underline{t}_{i+1}); \\ \frac{W^*(\bar{t}_i)(1-\bar{t}_i)+m(t-\bar{t}_i)}{1-t}, & \text{if } t \in (\bar{t}_i, \bar{t}_i+\eta]; \\ \frac{W^*(\bar{t}_i)(1-\bar{t}_i)+m\eta}{1-\bar{t}_i-\eta}, & \text{if } t \in [\bar{t}_i+\eta, \underline{t}_{i+1}); \end{cases}$$

where η is implicitly defined to satisfy:

$$\int_{\bar{t}_i}^{\underline{t}_{i+1}} W^*(t)dt = \int_{\bar{t}_i}^{\underline{t}_{i+1}} \widehat{W}(t)dt. \tag{49}$$

We first show such η exists. When $\eta = 0$,

$$\int_{\bar{t}_i}^{\underline{t}_{i+1}} W^*(t)dt > \int_{\bar{t}_i}^{\underline{t}_{i+1}} W^*(\bar{t}_i)dt = \int_{\bar{t}_i}^{\underline{t}_{i+1}} \widehat{W}(t)dt;$$

when $\eta = \underline{t}_{i+1} - \bar{t}_i$,

$$\int_{\bar{t}_i}^{\underline{t}_{i+1}} W^*(t)dt = \int_{\bar{t}_i}^{\underline{t}_{i+1}} \frac{\pi_w^*(t)}{1-t}dt < \int_{\bar{t}_i}^{\underline{t}_{i+1}} \frac{\text{cav}[\pi_w^*](t)}{1-t}dt = \int_{\bar{t}_i}^{\underline{t}_{i+1}} \widehat{W}(t)dt.$$

Since the integrals are continuous in η, there exists η such that (49) is satisfied.

We now prove that $\widehat{W}$ is feasible, and it generates higher total surplus and the same revenue as W. Hence, $\widehat{W}$ generates higher bidder surplus.

(Feasibility) First, note that for all $t \in (\bar{t}_i, \bar{t}_i + \eta)$,

$$\widehat{W}(t)(1-t) = \text{cav}[\pi_w^*](t) > W^*(t)(1-t).$$

We also have that $\widehat{W}$ and W^* cross only once in $[\bar{t}_i + \eta, \underline{t}_{i+1})$ (since $\widehat{W}$ is constant in this range). We thus have that $\text{sign}(W^* - \widehat{W}) = (-, +)$. Second, note that $\widehat{W}(t)$ is clearly non-decreasing in each of the segments of the definition. Also, it is continuous at $\bar{t}_i + \eta$, and we have that:

$$\lim_{t \uparrow \underline{t}_{i+1}} \widehat{W}(t) < \lim_{t \uparrow \underline{t}_{i+1}} W^*(t) \leq W^*(\underline{t}_{i+1}).$$

Hence it is non-decreasing. Hence W^* is a mean-preserving spread of $\widehat{W}$.

(Equal Revenue) Since $\pi_w^*(t) = \hat{\pi}_w(t)$ for all $t \notin (\bar{t}_i, \underline{t}_{i+1})$, we have that $\text{cav}[\pi_w^*](t) \leq \text{cav}[\hat{\pi}_w](t)$. Nevertheless, by construction, we have that:

$$\hat{\pi}_w(t) \begin{cases} = \text{cav}[\pi_w^*](t), & \text{for all } t \in (\bar{t}_i, \bar{t}_i + \eta]; \\ < \text{cav}[\pi_w^*](t), & \text{for all } t \in (\bar{t}_i + \eta, \underline{t}_{i+1}). \end{cases}$$

Thus, $\text{cav}[\pi_w^*] = \text{cav}[\widehat{\pi}_w]$.

(Higher Total Surplus) Let r^* and $\hat{r}$ be the profit-maximizing allocation rules when the information structure is W^* and $\widehat{W}$, respectively. We have that $r^*(t)$ is constant in the domain $(\bar{t}_i, \underline{t}_{i+1})$, while $\hat{r}$ is constant in $[\bar{t}_i + \eta, \underline{t}_{i+1})$ and screens quantiles in $(\bar{t}_i, \bar{t}_i + \eta]$. So $\hat{r}$ is a mean-preserving spread of r^*. We thus have that:

$$\int_{\bar{t}_i}^{\underline{t}_{i+1}} r^*(t) W^*(t) dt = \int_{\bar{t}_i}^{\underline{t}_{i+1}} r^*(t) \widehat{W}(t) dt < \int_{\bar{t}_i}^{\underline{t}_{i+1}} \hat{r}(t) \widehat{W}(t) dt.$$

The first equality follows from the fact that r^* is constant in the interval $(\bar{t}_i, \underline{t}_{i+1})$ and (49) is satisfied; the inequality follows from the fact that $\hat{r}$ is a mean-preserving spread of r^* (and $\widehat{W}$ is not constant in this interval so the inequality is strict). Thus, $\widehat{W}$ generates higher total surplus. ■

9.1.4 Regular Information Structures are Optimal

We now conclude the proof by showing that the bidder-optimal information structure is regular, that is, $W^* \in \mathcal{W}(t_x)$ for some $t_x \geq 0$. We assume that the bidder-optimal information structure W^* contains a conflating interval and we reach a contradiction.

We begin by fixing some interval conflating interval $(\bar{t}_i, \underline{t}_{i+1})$ and denote by $[t_1, t_2]$ an interval such that: (a) $(\bar{t}_i, \underline{t}_{i+1}) \subset [t_1, t_2]$, (b) $\text{cav}[\pi_w^*](t)$ is affine in $[t_1, t_2]$, and (c) $\text{cav}[\pi_w^*](t)$ is strictly concave at the limits $t \in \{t_1, t_2\}$. That is, $[t_1, t_2]$ is an interval in which $\text{cav}[\pi_w^*](t)$ is affine, it contains a conflating interval and in any strictly larger interval $\text{cav}[\pi_w^*](t)$ is not affine. Furthermore, without loss of generality, we assume that $\pi_w^*(t) = \text{cav}[\pi_w^*](t)$ for all $[t_1, \bar{t}_i]$ (this amounts to considering the "left-most" non-regular interval in $[t_1, t_2]$). As before, the slope of $\text{cav}[\pi_w^*](t)$ in the interval $[\bar{t}_i, \underline{t}_{i+1}]$ is denoted by:

$$m = \frac{\pi_w^*(\underline{t}_{i+1}) - \pi_w^*(\bar{t}_i)}{\underline{t}_{i+1} - \bar{t}_i}. \tag{50}$$

We show that there exists another information structure $\widehat{W}$ that generates higher bidder surplus, thus reaching a contradiction.

We first provide properties as to where the majorization constraint binds.

Lemma 6 (Binding Constraints at the Lower Bound of Affine Interval)
If a non-regular interval exists, then there exists $t_b \in [t_1, \bar{t}_i]$ such that the majorization constraint binds at t_b and $W^*(t_b) = V(t_b)$.

Proof. Recall that in the interval $[t_1, \bar{t}_i]$, $\text{cav}[\pi_w^*](t) = \pi_w^*(t)$. We assume that the majorization constraint does not bind in $[t_1, \bar{t}_i]$ or it binds at t_b and $W^*(t_b) \neq V(t_b)$. We reach a contradiction by proving that there exists $\widehat{W}$ that generates higher bidder surplus.

We consider information structure $\widehat{W}$ parameterized by $\varepsilon > 0$ as follows:

$$\widehat{W}(t) = \begin{cases} W^*(t), & \text{for all } t \in [0, \hat{t}] \cup [\underline{t}_{i+1}, 1]; \\ \frac{\pi_w^*(\underline{t}_{i+1}) + (m+\varepsilon)(t - \underline{t}_{i+1})}{1-t}, & \text{for all } t \in (\hat{t}, \bar{t}_i']; \\ \frac{\pi_w^*(\underline{t}_{i+1}) + (m+\varepsilon)(\bar{t}_i - \underline{t}_{i+1})}{1-\bar{t}_i}, & \text{for all } t \in [\bar{t}_i', \underline{t}_{i+1}); \end{cases}$$

where $\bar{t}_i'$ and $\hat{t}$ are defined as follows. First, we have that:

$$\hat{t} \triangleq \min\{t \in [0, \bar{t}_i] : \pi_w^*(t) \geq \pi_w^*(\underline{t}_{i+1}) + (m+\varepsilon)(t - \underline{t}_{i+1})\}.$$

Note that

$$\pi_w^*(\bar{t}_i) = \pi_w^*(\underline{t}_{i+1}) + m(\bar{t}_i - \underline{t}_{i+1}) > \pi_w^*(\underline{t}_{i+1}) + (m+\varepsilon)(\bar{t}_i - \underline{t}_{i+1}),$$

and $\pi_w^*(t)$ is upper-hemicontinuous. Hence $\hat{t}$ exists and $\hat{t} \leq \bar{t}_i$. Second, $\bar{t}_i'$ is implicitly defined so that:

$$\int_0^1 W^*(t)dt = \int_0^1 \widehat{W}(t)dt. \tag{51}$$

We prove such $\bar{t}_i'$ exists (when ε is small enough) when we prove $\widehat{W}$ is feasible.

We now prove that $\widehat{W}$: (i) is feasible (when ε is small enough), (ii) generates higher total surplus than W^*, and (iii) generates lower revenue than W^*.

(Feasibility) We first show that there exists $\bar{t}_i'$ such that (51) is satisfied appealing to the Intermediate Value Theorem. We first note that for all $t \notin (\hat{t}, \underline{t}_{i+1})$, $\widehat{W}(t) = W(t)$ so proving that the integrals are the same with limits $\{\hat{t}, \underline{t}_{i+1}\}$ (instead of limits $\{0, 1\}$) suffices to prove the result. If $\bar{t}_i' = \underline{t}_{i+1}$, then we have that

$$\int_{\hat{t}}^{\underline{t}_{i+1}} W^*(t)dt < \int_{\hat{t}}^{\underline{t}_{i+1}} \widehat{W}(t)dt. \tag{52}$$

To prove this, we note that:

$$\lim_{\varepsilon \to 0} \int_{\hat{t}}^{\underline{t}_i} \widehat{W}(t)dt = \int_{\hat{t}}^{\underline{t}_i} \text{cav}[\pi_w^*](t)(1-t)dt > \int_{\hat{t}}^{\underline{t}_i} \pi_w^*(t)\ (1-t)dt = \int_{\hat{t}}^{\underline{t}_i} W^*(t)dt.$$

Hence, we can find a ε small enough such that (52) is satisfied. We now define $\tilde{t}$ as follows:

$$\tilde{t} \triangleq \max\{t \in [0, \underline{t}_{i+1}] : \pi_w^*(t) \geq \pi_w^*(\underline{t}_{i+1}) + (m+\varepsilon)(t - \underline{t}_{i+1})\}.$$

If $\bar{t}'_i = \tilde{t}$, then we have that:

$$\int_{\hat{t}}^{\underline{t}_i} W^*(t)dt > \int_{\hat{t}}^{\underline{t}_i} \widehat{W}(t)dt, \tag{53}$$

which follows from the fact that in this case $\widehat{W}(t) \leq W^*(t)$ for every t (and strict for some t). Since the integrals are continuous in $\bar{t}'_i$, we have that there exists $\bar{t}'_i$ such that (51) is satisfied. We also note that $\widehat{W}$ is increasing.

Finally, we prove the majorization constraint is satisfied, that is,

$$\int_t^1 W^*(s)ds \leq \int_t^1 V(s)ds.$$

We have that, for all $t \notin (\hat{t}, \underline{t}_{i+1})$,

$$\int_t^1 \widehat{W}(s)ds = \int_t^1 W^*(s)ds \leq \int_t^1 V(s)ds.$$

Hence, we now check the majorization constraint is also satisfied by quantiles $t \in (\hat{t}, \underline{t}_{i+1})$. For this, note that we must have that $\hat{t} \to t_1$ as $\varepsilon \to 0$ and possibly $\hat{t} = t_1$ for a small enough ε if W^* is discontinuous at t_1.

Consider first the case in which the majorization constraint does not bind in $[t_1, \bar{t}_i]$. In the limit $\varepsilon \to 0$ we have that $\widehat{W}(t) \to W^*(t)$ for every t. If the majorization constraint is not binding at any $t \in [t_1, \bar{t}_i]$, then the majorization constraint will not be violated by $\widehat{W}$ when ε is small enough.

On the other hand, if the majorization constraint binds at some $t_b \in [t_1, \bar{t}_i]$ and $W^*(t_b) \neq V(t_b)$ then $W^*(t) \geq V(t)$ for all t in some neighborhood $[t_b, t_b + \delta]$ (otherwise, the majorization constraint would be violated in some neighborhood $[t_b, t_b + \delta]$). Furthermore, if $W^*(t_b) > V(t_b)$, then $W^*(t_b)$ must be discontinuous at t_b (otherwise, the majorization constraint would be violated in some neighborhood $[t_b - \delta, t_b]$). Hence, we consider the case that $W^*(t) > V(t)$ and use the fact that in this case $W^*(t)$ is discontinuous at t_b to prove $\widehat{W}$ is feasible. Since $W^*(t)$ is discontinuous at t_b we have that $t_b = t_1$ and the majorization constraint does not bind in $(t_1, \bar{t}_i]$. We then define:

$$\Psi^*(t) \triangleq \int_0^t \left[W^*(s) - V(s)\right] ds \quad \text{and} \quad \widehat{\Psi}(t) \triangleq \int_0^t \left[\widehat{W}(s) - V(s)\right] ds.$$

We have that $\Psi^*(t) > 0$ for all $t \in (t_1, \bar{t}_i]$, $\widehat{\Psi}(t) \to \Psi^*(t)$ for every t, and $\hat{\Psi}'(t_1) > 0$. Hence, for a ε small enough $\widehat{\Psi}(t) > 0$ for all $t \in (t_1, \bar{t}_i]$. Hence $\widehat{W}$ satisfies the majorization constraint.

(Lower Revenue) By construction, we have that $\text{cav}[\pi_w^*](t)$ can be written as follows:

$$\text{cav}[\pi_w^*](t) = \begin{cases} \pi_w^*(t), & \text{if } t \in (\hat{t}, \bar{t}_i]; \\ \pi_w^*(\bar{t}_i) + m(\bar{t}_i - t), & \text{if } t \in [\bar{t}_i, \underline{t}_{i+1}); \\ \text{cav}[\pi_w^*](t), & \text{if } t \notin (\hat{t}, \underline{t}_{i+1}). \end{cases}$$

An analogous expression holds for $\text{cav}[\pi_w]$ (but using $\underline{t}_i',\ \bar{t}_i',\ \underline{t}_{i+1}$). We now note that,

$$\text{cav}[\widehat{\pi}_w](t) = \begin{cases} \pi_w^*(\underline{t}_{i+1}) + (m+\varepsilon)(t - \underline{t}_{i+1}), & \text{if } t \in (\hat{t}, \underline{t}_{i+1}); \\ \text{cav}[\pi_w^*](t), & \text{if } t \in [0, \hat{t}] \cup [\underline{t}_{i+1}, 1]. \end{cases}$$

As explained above, for all $t \in (\bar{t}_i, \underline{t}_{i+1})$,

$$\pi_w^*(\underline{t}_{i+1}) + (m+\varepsilon)(\underline{t}_{i+1} - t) < \pi_w^*(\underline{t}_{i+1}) + m(\underline{t}_{i+1} - t),$$

and for all $t \in (\hat{t}, \bar{t}_i]$,

$$\pi_w^*(\underline{t}_{i+1}) + (m+\varepsilon)(\underline{t}_{i+1} - t) < \pi_w^*(t).$$

We thus have that $\text{cav}[\widehat{\pi}_w](t) \leq \text{cav}[\pi_w^*](t)$. Finally, for all $t \notin (\hat{t}, \underline{t}_{i+1})$, we have that $\pi_w^*(t) = \widehat{\pi}_w(t)$. Hence, for all $t \notin (\hat{t}, \underline{t}_{i+1})$ $\text{cav}[\pi_w^*](t) \geq \text{cav}[\widehat{\pi}_w](t)$. Hence, $\widehat{\pi}_w$ generates lower total revenue.

(Higher Total Surplus) We note that $\text{sign}(\hat{W}(t) - W^*(t)) = (-,+)$ so $W^* \prec \widehat{W}$ (see Shaked and Shanthikumar (2007)). We also have that $\bar{t}_i < \bar{t}_i'$. Hence, $\hat{r}$ is a mean-preserving spread of r. Hence, $\widehat{W}$ generates a higher total surplus. ■

Lemma 7 (Quantiles in the Convex Zone ($t \geq t_I$))
If there exists $t_b \geq t_I$ such that (a) the majorization constraint binds at t_b, (b) $W^*(t_b) = V(t_b)$, and (c) $\text{cav}[\pi_w^*](t_b) =_w^* (t_b)$, then $V(t) = W^*(t)$ for all $t \in [t_b, 1]$.

Proof. Suppose otherwise that there exists t_b satisfying the above conditions and $W^*(t) \neq V(t)$ for some $t \geq t_b$. Then the following information structure is feasible and generates higher bidder surplus than W:

$$\widehat{W}(t) = \begin{cases} W^*(t), & \text{if } t_b \leq t; \\ V(t), & \text{if } t_b \geq t. \end{cases}$$

The information structure $\widehat{W}$ is clearly feasible: the majorization constraint is not binding at quantiles $t \geq t_b$ and for every $t \leq t_b$:

$$\int_t^1 W^*(s)ds = \int_t^1 \widehat{W}(s)ds.$$

Hence, if W^* satisfies the majorization constraint so does $\widehat{W}$.

We also note that, for every $t \geq t_b$, $\text{cav}[\pi^*_w](t) = \pi^*_w(t)$. We can see this by noting that $\pi(t)$ is concave for every $t > t_b$. At $t = t_b$, $\pi^*_w(t)$ must also be concave. We can see this by noting that, if the derivative of $\pi^*_w(t)$ were smaller than the derivative of $\pi(t)$ at t_b, then the majorization constraint would be violated in some neighborhood $[t_b, t_b + \delta]$. That is, we must have that:

$$\frac{d\pi(t_b)}{dt} < \frac{d\pi^*_w(t_b)}{dt}.$$

Hence, $\pi^*_w(t)$ is concave at $t = t_b$.

Finally, we check that $\widehat{W}$ generates higher bidder surplus. We note that the revenue-maximizing allocation at quantiles $t \geq t_b$ when the information structure is $\widehat{W}$ is equal to the efficient allocation. Hence, the difference in bidder surplus can be written as follows:

$$U^* - \hat{U} = \int_{t_b}^{1} W^*(t) d(r^*(t)(t-1)) - \left(\int_{t_b}^{1} V(t) d(q(t)(t-1)) \right).$$

Here we used that $\widehat{W}(t) = V(t)$ for all $t \geq t_b$. For any given information structure, the bidder surplus is weakly larger when the seller uses the efficient allocation as opposed to the revenue-maximizing allocation (the efficient allocation generates by construction higher total surplus and lower revenue). We thus have that:

$$U^* - \hat{U} < \int_{t_b}^{1} (W^*(t) - V(t)) \frac{s(t)}{dt} dt = \int_{t_b}^{1} \frac{d^2 s(t)}{dt^2} \int_{t}^{1} (W^*(\ell) - V(\ell)) d\ell dt.$$

Note that $s(t)$ is strictly convex for all $t > t_I$ and the majorization constraint implies that

$$\int_{t}^{1} (W^*(\ell) - V(\ell)) d\ell \leq 0.$$

If $W^*(t) \neq V(t)$ for some $t \geq t_b$, then the inequality will be strict in some open interval of types. Hence, if $W^*(t) \neq V(t)$ for some $t \geq t_b$, then $\hat{U}$ generates more bidder surplus. We thus reach a contradiction. ■

Lemma 8 (Quantiles in the Concave Zone ($t \leq t_I$))
Suppose that there exists a non-regular interval $(\bar{t}_i, \underline{t}_{i+1})$ such that $\underline{t}_{i+1} < t_I$, then $\text{cav}[\pi^*_w]$ must be linear in $[\underline{t}_{i+1}, t_I]$.

Proof. Suppose there exists $\hat{t} \in [\underline{t}_{i+1}, t_I]$ such that $\text{cav}[\pi^*_w]$ is strictly concave at $\hat{t}$. To make the notation more compact, we will write the proof assuming that $\underline{t}_{i+1} < \hat{t}$, but the proof goes through essentially the same way when $\underline{t}_{i+1} = \hat{t}$ (we explain the adjustment required at the end of the proof).

Let $\hat{m}$ be the derivative of $\text{cav}[\pi^*_w]$ at $\hat{t}$. If it is non-differentiable at $\hat{t}$ we take the average of the right and left limits of the derivatives:

$$\hat{m} = \lim_{\Delta \to 0} \frac{\text{cav}[\pi^*_w](\hat{t} + \Delta) - \text{cav}[\pi^*_w](\hat{t} - \Delta)}{2\Delta}.$$

We consider the following information structure:

$$\widehat{\pi}_w(t) = \begin{cases} \min\{\pi^*_w(t), \pi^*_w(\hat{t}) - \varepsilon + \hat{m}(t - \hat{t})\}, & \text{if } t \notin [\underline{t}_{i+1} - \eta, \underline{t}_{i+1}); \\ \pi^*_w(\underline{t}_{i+1}) + m(t - \underline{t}_{i+1}), & \text{if } t \in [\underline{t}_{i+1} - \eta, \underline{t}_{i+1}), \end{cases} \tag{54}$$

where m is defined in (50) and η is defined such that:

$$\int_0^1 W(t)dt = \int_0^1 \widehat{W}(t)dt. \tag{55}$$

Note that by construction we have that $\widehat{\pi}_w(\hat{t}) < \pi^*_w(\hat{t})$, and $\text{cav}[\pi^*_w]$ is strictly concave at $\hat{t}$, so $\widehat{\pi}_w(t) \leq \pi^*_w(t)$ only in some neighborhood around $\hat{t}$. More precisely, let $0 < \varepsilon_1 < \varepsilon_2$ be such that

$$\begin{aligned} \varepsilon_1 &= \inf\{x > 0 : \pi^*_w(\hat{t} - x) > \pi^*_w(\hat{t}) - \varepsilon - \hat{m} \cdot x\}, \\ \varepsilon_2 &= \sup\{x > 0 : \pi^*_w(\hat{t} + x) > \pi^*_w(\hat{t}) - \varepsilon + \hat{m} \cdot x\}. \end{aligned}$$

In other words, $\widehat{\pi}_w(t) \leq \pi^*_w(t)$ if and only if $t \in [\hat{t} - \varepsilon_1, \hat{t} + \varepsilon_2]$. In the limit $\varepsilon \to 0$, we have that $\varepsilon_1, \varepsilon_2 \to 0$.

We also have that, for every $t \in (\underline{t}_{i+1} - \eta, \underline{t}_{i+1})$:

$$\pi^*_w(t) < \text{cav}[\pi^*_w](t) = \widehat{\pi}_w(t).$$

Hence, for a ε small enough we can find η such that (55) is satisfied. We also have that $\text{sign}(W^* - \widehat{W}) = (-, +)$ so $\widehat{W}$ is a mean-preserving contraction of W.

We now verify that $\widehat{W}$ generates a higher bidder surplus. By definition, the revenue-maximizing allocation when the information structure is W^* conflates types in $(\bar{t}_i, \underline{t}_{i+1})$. A weakly higher bidder surplus is attained if types in $(\bar{t}_i, \underline{t}_{i+1} - \eta]$ are conflated and types in $[\underline{t}_{i+1} - \eta, \underline{t}_{i+1})$ are given the efficient allocation. More precisely, we consider the allocation:

$$\hat{r}(t) = \begin{cases} \frac{\int_{\bar{t}_i}^{\underline{t}_{i+1} - \eta} q(s)ds}{\underline{t}_{i+1} - \eta - \bar{t}_i}, & \text{if } t \in (\bar{t}_i, \underline{t}_{i+1} - \eta]; \\ q(t), & \text{if } t \in [\underline{t}_{i+1} - \eta, \underline{t}_{i+1}); \\ r^*(t), & \text{if } t \notin (\bar{t}_i, \underline{t}_{i+1}). \end{cases}$$

We note that $r^* \prec \hat{r}$, so:

$$U^* \leq \int_0^1 W^*(t) d((1-t)\hat{r}(t)).$$

We note that by construction $\text{cav}[\widehat{\pi}_w](t) = \widehat{\pi}_w(t)$ for all $t \in [\underline{t}_{i+1} - \eta, \underline{t}_{i+1})$. Hence, the revenue-maximizing allocation at quantiles $t \in [\underline{t}_{i+1} - \eta, \underline{t}_{i+1}) \cup [\hat{t} - \varepsilon_1, \hat{t} + \varepsilon_2]$ when the information structure is $\widehat{W}$ is equal to the efficient allocation. So $\hat{r}$ is also the revenue-maximizing allocation when the information structure is $\widehat{W}$.

We now define:

$$\widehat{\mathcal{W}} \triangleq [\underline{t}_{i+1} - \eta, \underline{t}_{i+1}] \cup [\hat{t} - \varepsilon_1, \hat{t} + \varepsilon_2].$$

We note that for every $t \notin \widehat{\mathcal{W}}$, we have that:

$$\widehat{\pi}_w(t) = \pi^*_w(t) \quad \text{and} \quad \text{cav}[\widehat{\pi}_w](t) = \text{cav}[\pi^*_w](t).$$

Hence, the information structures W^* and $\widehat{W}$ only differ in quantiles $t \in \widehat{\mathcal{W}}$. Hence, the difference in bidder surplus can be written as follows:

$$U^* - \hat{U} \leq \int_{t \in \widehat{\mathcal{W}}} W^*(t) d(\hat{r}(t)(t-1)) - \left(\int_{t \in \widehat{\mathcal{W}}} \widehat{W}(t) d(\hat{r}(t)(t-1)) \right).$$

Since for every $t \in \widehat{\mathcal{W}}$, $\hat{r}(t) = q(t)$, we can write the difference as follows:

$$U^* - \hat{U} \leq \int_{t \in \widehat{\mathcal{W}}} (W^*(t) - \widehat{W}(t)) \frac{s(t)}{dt} dt = \int_{t \in \widehat{\mathcal{W}}} \frac{d^2 s(t)}{dt^2} \int_t^1 (W^*(\ell) - \widehat{W}(\ell)) d\ell dt.$$

Note that $s(t)$ is strictly concave for all $t < t_I$ and $\widehat{W}$ is a mean-preserving contraction of W^* so we have that:

$$\int_t^1 (W^*(\ell) - \widehat{W}(\ell)) d\ell \geq 0,$$

and the inequality is strict at every $t \in \widehat{\mathcal{W}}$. Hence, $\hat{U}$ generates more bidder surplus. We thus reach a contradiction. This proves the result when $\hat{t} > \underline{t}_{i+1}$.

Finally, we explain the adjustment needed to make when $\underline{t}_{i+1} = \hat{t}$. In this case, we define $\widehat{\pi}_w$ as follows:

$$\widehat{\pi}_w(t) = \begin{cases} \min\{\pi^*_w(t), \pi^*_w(\hat{t}) - \varepsilon + \hat{m}(t - \hat{t})\}, & \text{if } t \notin [\underline{t}_{i+1} - \eta, \underline{t}_{i+1}); \\ \pi^*_w(\underline{t}_{i+1}) + \hat{m}(t - \underline{t}_{i+1}), & \text{if } t \in [\underline{t}_{i+1} - \eta, \underline{t}_{i+1}). \end{cases} \tag{56}$$

The only change is that we replaced m with $\hat{m}$ in the case $t \in [\underline{t}_{i+1} - \eta, \underline{t}_{i+1})$. The rest of the proof goes through without changes. ■

Lemma 9 (Properties of Non-Regular Information Structure)
Let $[t_1, t_2]$ be such that the concavification of the optimal information $\text{cav}[\pi_w^*]$ is affine in this interval and strictly concave at the limits $\{t_1, t_2\}$, and there is some non-regular interval $(\bar{t}_i, \underline{t}_{i+1}) \subset [t_1, t_2]$, then:

1. $t_1 < t_I$;
2. $t_I < t_2$;
3. $t_1 < \bar{t}_i$, *and*
4. *for every* $t \geq t_2$, $\text{cav}[\pi_w^*](t) = \pi_w^*(t)$.

Proof. We prove each of the items:

(1)If $t_I \leq t_1$, then following Lemma 6 and 7 this would not be an optimal information structure.

(2) If $t_I \geq t_2$, then following Lemma 8 this would not be an optimal information structure.

(3) Suppose we have that $t_1 = \bar{t}_i$ and following Proposition 6, we must have that $W^*(\bar{t}_i) = V(\bar{t}_i)$ and the majorization constraint binds at $\bar{t}_i$. Since there is an atom in quantiles $(\bar{t}_i, \underline{t}_{i+1})$, we must have that $W^*(t) < V(t)$ for all t in $(\bar{t}_i, \underline{t}_{i+1})$. We thus have that $W^*(t)$ will violate the majorization constraint at every quantile $t \in (\bar{t}_i, \underline{t}_{i+1})$.

(4) Following items (1) and (2), there can only be one interval $[t_1, t_2]$ in which $\pi_w^*(t) < \text{cav}[\pi_w^*](t)$.

■

Final Steps of the Proof of Theorem 1. We can now reach a contradiction. We fix an optimal information structure W^* and suppose a conflating interval $(\bar{t}_i, \underline{t}_{i+1})$ exists and let $\{t_1, t_2\}$ be as previously defined. Following Lemma 9, we assume that $t_I \leq t_2$. Without loss of generality we assume that for all $t \in [\underline{t}_{i+1}, t_2]$, $\pi_w^*(t) = \text{cav}[\pi_w^*](t)$ (this amounts to considering the "right-most" regular interval). We reach a contradiction by proving there is another information structure that generates higher bidder surplus. We first address the case $t_2 = \underline{t}_{i+1}$ and then address the case $t_2 > \underline{t}_{i+1}$.

We now consider the following information structure:

$$\widehat{\pi}_w(t) = \begin{cases} \pi_w^*(t), & \text{if } t \leq t_2; \\ \min\{\pi(t), \pi_w^*(\bar{t}_i) + \hat{m}(t - \bar{t}_i)\}, & \text{if } t \geq t_2; \end{cases}$$

where $\hat{m}$ is such that:

$$\int_0^1 W^*(t)dt = \int_0^1 \widehat{W}(t)dt. \tag{57}$$

We prove such $\hat{m}$ exists. If $\hat{m} = -W^*(\bar{t}_i)$, we would then have that $\widehat{\pi}(t) \leq \pi^*_w(t)$ for every $t \geq \bar{t}_i$. If $\hat{m} = m$, as defined in (50), we have that $\widehat{\pi}_w(t) \geq \pi^*_w(t)$ for every $t \geq \bar{t}_i$. Hence, there exists $\hat{m}$ such that (57) is satisfied.

We now recall that for all $t < t_2$, $W^*(t) = \widehat{W}(t)$ and for all $t \geq t_2$, $\pi^*_w(t) = \text{cav}[\pi^*_w](t)$. Hence, the difference in bidder surplus can be written as follows:

$$U^* - \hat{U} = \int_{t_2}^1 \int_t^1 (W^*(\ell) - \widehat{W}(\ell))d\ell \frac{d^2 s(t)}{dt^2} dt.$$

We also have that $t_I \leq t_2$, so in the interval $[t_2, 1]$:

$$\frac{d^2 s(t)}{dt^2} > 0.$$

Finally, we prove that for all $t \in [t_2, 1]$:

$$\int_t^1 \left[W^*(s) - \widehat{W}(s)\right] ds \leq 0. \tag{58}$$

To prove this we define $\hat{t}$ implicitly as follows:

$$V(\hat{t}) = W^*(\bar{t}_i) + \hat{m}(\hat{t} - \bar{t}_i).$$

We then have that for every $t \geq \hat{t}$

$$\int_t^1 \left[W^*(s) - \widehat{W}(s)\right] ds = \int_t^1 \left[W^*(s) - V(s)\right] ds \leq 0,$$

where the equality follows from the definition of $\widehat{W}$ and the inequality follows from the fact that W^* satisfies the majorization constraint.

We now prove that, for every $t \in [t_2, \hat{t}]$, (58) is satisfied. We proceed to prove this by contradiction. Suppose there exists $t_z \in [t_2, \hat{t}]$ such that (58) is not satisfied. Since (58) is satisfied at $t = \hat{t}$, we can consider t_z such that (58) is not satisfied and $\widehat{W}(t_z) < W^*(t_z)$. But (58) is satisfied at $t = t_2$, so there also exists $t'_z < t_z$ such that $\widehat{W}(t'_z) > \widehat{W}(t_z)$. We thus have that t_2, t'_z, and t_z satisfy:

$$\widehat{\pi}_w(t_2) \leq \pi^*_w(t_2); \quad \widehat{\pi}_w(t'_z) > \widehat{\pi}^*_w(t'_z); \quad \widehat{\pi}(t_z) < \pi^*_w(t_z).$$

Since $\widehat{\pi}$ is linear in this interval, this means that W^* is not concave. So we reach a contradiction, so (58) is satisfied. This implies that $\widehat{W}$ generates higher bidder surplus than W^* so we reach a contradiction.

The case $\underline{t}_{i+1} < t_2$ can be addressed in a completely analogous way. For this, we define

$$\tilde{t} = \max\{t_I, \underline{t}_{i+1}\},$$

and consider the following information structure:

$$\widehat{\pi}(t) = \begin{cases} \pi_w^*(t), & \text{if } t \leq \tilde{t}; \\ \min\{\pi(t), \pi_w(\tilde{t}) + \hat{m}(t - \tilde{t})\}, & \text{if } t \geq \tilde{t}. \end{cases}$$

The rest of the proof follows line-by-line the same way. ■

9.2 Proof of Theorem 2

We prove Theorem 2 by first proving Proposition 5 explicitly, and then Theorem 2 follows from the definition of the profit function. The proof proceeds in 3 steps. We first find the optimal information structure when s is convex everywhere (that is, when $N = 2$). We then find the optimal information structure using only that $s(t)$ has a unique inflection point. Since we are only going to use this property of s we will reach a weaker characterization of the bidder-optimal information function, but this is a natural stepping stone towards proving the main result. Finally, we use that s is quasiconvex and $q(0) = 0$. Throughout the proof, we fix the exclusion level t_x (and we can find the optimal exclusion level by maximizing over all t_x).

Before we begin the proof, we define:

$$\pi_w\left(t \,|t_x, t_z\right) = \begin{cases} \pi(t_z) - \alpha(t - t_z), & \text{if } t \in [t_x, t_z]; \\ \pi(t), & \text{if } t \notin [t_x, t_z]. \end{cases} \tag{59}$$

Thus the optimal information structure is $\pi_w^*(t) = \pi_w\left(t \,|t_x, t_z\right)$ for some optimally chosen parameters $(t_x, t_z) \in [0,1] \times [0,1]$.

9.2.1 Convex Objectives

We first consider the case when $s(t)$ is convex, which corresponds to the case $N = 2$. This is a natural benchmark to begin analyzing and it will allow us to introduce notation and concepts that are used when analyzing the general case.

We begin by considering the following class of information structures:

$$\pi_w(t\,|t_x) \triangleq \begin{cases} \pi(t), & \text{if } t \notin [t_x, t_v]; \\ \pi(t_v), & \text{if } t \in [t_x, t_v], \end{cases} \tag{60}$$

where t_v is defined implicitly as follows. If $\pi(t)$ is decreasing at every $t \in [t_x, 1]$, then we set $t_v = t_x$. Otherwise, we define $t_v > t_x$ as the unique solution to the following equation:

$$\int_{t_x}^{t_v} \frac{V(t_v)(1-t_v)}{1-t} dt = \int_{t_x}^{t_v} V(t)dt. \tag{61}$$

We note that t_v depends on t_x (we do not make this reference in the notation explicit because there will be no ambiguity). We prove the existence and uniqueness of such t_v as follows. We define:

$$\Psi(t_v) \triangleq \int_{t_x}^{t_v} \frac{V(t_v)(1-t_v)}{1-t} dt - \int_{t_x}^{t_v} V(t)dt,$$

and note that

$$\Psi'(t_v) = \int_{t_x}^{t_v} \frac{\pi'(t_v)}{1-t} dt.$$

Since π is quasi-concave and concave wherever it is decreasing, Ψ is quasi-concave (Ψ' is positive if and only if π' is positive), $\Psi(0) = 0$, and $\Psi(1) < 0$. Hence $\Psi(t_v)$ crosses 0 only once. Hence, $\pi_w(t\,|t_x) = \pi_w(t\,|t_x, t_z)$, where t_z is chosen such that $\alpha = 0$.

We next show that $\pi_w(t\,|t_x) \in \mathcal{W}(t_x)$. For this, note that by construction

$$\int_0^1 W(t|t_x)dt = \int_0^1 V(t)dt,$$

and $\text{sign}(\pi_w(t\,|t_x) - \pi(t)) = (+, -)$. Hence $W(t|t_x) \prec V$. We also have that $\pi'(t_v|t_x) \leq 0$. Hence, $\pi_w'(t|t_x) = 0$ for $t < t_v$ and $\pi_w'(t|t_x) \leq 0$ for $t > t_v$. Hence $\pi_w(t\,|t_x) \in \mathcal{W}(t_x)$. We now show this is the most informative information structure in $\mathcal{W}(t_x)$.

Lemma 10 (Most Informative Regular Information Structure)
Any profit function $\pi_w \in \mathcal{W}(t_x)$ satisfies $\pi_w \prec \pi_w(t\,|t_x)$.

Proof. For every $t \geq t_v$, we have that:

$$\int_0^{t_v} \pi_w(t\,|t_x)\,dt = \int_0^{t_v} \pi(t)dt \leq \int_0^{t_v} \pi_w(t)dt. \tag{62}$$

We now consider the quantiles $t < t_v$. Suppose there exists $\tilde{t} < t_v$ such that:

$$\int_0^{\tilde{t}} \pi_w\left(t\left|t_x\right.\right) dt > \int_0^{\tilde{t}} \pi_w(t) dt.$$

We then have that, for some $t \in [t_x, \tilde{t}]$

$$\pi_w\left(t\left|t_x\right.\right) > \pi_w(t).$$

But since (62) must be satisfied, there must also exist $t' \in [\tilde{t}, t_v]$ such that:

$$\pi_w(t'|t_x) < \pi_w(t').$$

Hence, π_w must be increasing in some part of the domain $[t_x, t_v]$. We thus have that $\pi_w(t_x) \notin \mathcal{W}(t_x)$. ■

We can thus conclude that this is the optimal information structure when $N = 2$.

Proposition 14 (Bidder-Optimal Information Structure)
If s is convex (that is, if $N = 2$), for every bidder-optimal information structure W^*, there exists t_x such that:

$$W^*(t) = \frac{\pi_w\left(t\left|t_x\right.\right)}{1-t}.$$

Proof. We write the difference between the revenues generated by W^* and by $W(t|t_x)$ as follows:

$$D \triangleq \left(\int_{t_x}^1 W^*(t)\frac{ds(t)}{dt}dt - (1-t_x)q(t_x)W^*(t_x)\right) - \left(\int_{t_x}^1 W(t|t_x)\frac{ds(t)}{dt}dt - (1-t_x)q(t_x)W(t_x|t_x)\right) \tag{63}$$

$$= \int_{t_x}^1 \int_t^1 (W^*(\ell) - W(\ell|t_x))d\ell \frac{d^2 s(t)}{dt^2} dt - (1-t_x)q(t_x)(W^*(t_x) - W(t_x|t_x)).$$

Since $W^* \prec W(t|t_x)$, we have that:

$$\int_t^1 (W^*(\ell) - W(\ell|t_x))d\ell \le 0,$$

and the inequality is strict at some t if $W^*(s) \neq W(s|t_x)$ at some s. We also have that

$$\frac{d^2 s(t)}{dt^2} > 0.$$

Hence, the first term is strictly negative if $W^*(s) \neq W(s|t_x)$ at some s. Since $W^*(t) \prec W(t|t_x)$, we also have that:

$$W^*(t_x) - W(t_x|t_x) \ge 0.$$

Thus, the second term is also weakly negative. We thus reach a contradiction, which implies

$$W^*(t) = W(t|t_x),$$

for some t_x. ■

9.2.2 Unique Inflection Point

We now provide a characterization of the optimal information structure using only that s has a unique inflection point. We introduce the following family of information structures:

$$\pi_w\left(t\left|t_x, \pi_0, t_z\right.\right) = \begin{cases} \min\{\pi(t_z) - \alpha(t - t_z), \pi_0\}, & \text{if } t \in [t_x, t_z]; \\ \pi(t), & \text{if } t \notin [t_x, t_z], \end{cases} \tag{64}$$

where (t_x, π_0, t_z) are parameters of the information structure that belong to a domain we specify next and α is implicitly defined to satisfy:

$$\int_0^1 \frac{\pi_w(t|t_x, \pi_0, t_z)}{1-t} dt = \int_0^1 \frac{\pi(t)}{1-t} dt.$$

As a notation, we later use W^{**} to denote the value associated with π_w. We prove such α exists in the proof of Proposition 15. In other words, the information structure is equal to the true distribution of values for quantiles larger than t_z and linear for quantiles below t_z but with a cap at π_0. When $\pi_0 \geq \pi(t_x) - \alpha(t_x - t_z)$ we have that $\pi_w\left(t\left|t_x, \pi_0, t_z\right.\right) = \pi_w\left(t\left|t_x, t_z\right.\right)$.

The parameters (t_x, π_0, t_z) are in the following domain: (i) $\pi_0 \in [\pi_w(t_v|t_x), \mu_v]$, (ii) $t_z \in [t_v, 1]$, and (iii) $t_x \in [0, 1]$, where the parameter μ_v is defined as follows:

$$\mu_v \triangleq \int_{t_x}^1 V(t) dt.$$

We can now show that every bidder-optimal information structure is of the form $W\left(t\left|t_x, \pi_0, t_z\right.\right)$, for some (t_x, π_0, t_z). This is a weaker result than Proposition 5 because we are allowing the profit function to be nonlinear in $[t_x, t_z]$.

Proposition 15 (Bidder-Optimal Information Structure)
For every bidder-optimal information structure W^*, there exists (t_x, π_0, t_z) such that:

$$W^*(t) = \frac{\pi_w\left(t\left|t_x, \pi_0, t_z\right.\right)}{1-t}.$$

Furthermore, $W\left(t|t_x, \pi_0, t_z\right)$ is linear in (t_I, t_z) (recall that t_I is defined in (18)).

We first prove that for every (t_x, π_0, t_z) in the specified domain, there exists α such that $\pi_w(t\,|t_x, \pi_0, t_z)$ is feasible. If $t_z = t_v$ then we obviously have that the information structure is feasible for $\alpha = 0$ (since in this case $\pi_w(t\,|t_x, \pi_0, t_z) = \pi_w(t \mid t_x)$ and the upper bound π_0 will not bind). Hence, we consider $t_z > t_v$.

If $\alpha = 0$, we have that:

$$\pi_w(t\,|t_x, \pi_0, t_z) \leq \pi_w(t\,|t_x), \text{ for every } t \in [0,1].$$

If $\alpha = \pi'(t_z)$ we have that

$$\pi_w(t\,|t_x, \pi_0, t_z) \geq \pi_w(t\,|t_x), \text{ for every } t \in [0,1].$$

Hence, there exists α such that:

$$\int_0^1 \pi_w(t\,|t_x, \pi_0, t_z)\,dt = \int_0^1 \pi(t)dt = \int_0^1 \pi_w(t\,|t_x)\,dt.$$

Furthermore, since $\alpha < -\pi'(t_z)$, we have that $\operatorname{sign}(\pi_w(t\,|t_x, \pi_0, t_z) - \pi(t)) = (+, -)$. Hence, $\pi_w(t\,|t_x, \pi_0, t_z)$ is a mean-preserving contraction of π.

We now fix some bidder-optimal information structure W^* such that $W^* \neq W^{**}(t_x, \pi_0, t_z)$ (for any (t_x, π_0, t_z)) and show it is suboptimal, thus reaching a contradiction. We denote by t_x the exclusion quantile of information structure W^* , that is, for all $t \in [0,1]$:

$$W^*(t)(1-t) \leq W^*(t_x)(1-t_x),$$

with strict inequality for all $t < t_x$. Throughout this section, we fix the parameter π_0 to be:

$$\pi_0 \triangleq W^*(t_x)(1-t_x),$$

and note that:

$$\pi_0 = \max_t \pi_w^*(t).$$

We begin by proving that there exists t_z such that $\pi_w(t\,|t_x, \pi_0, t_z)$ is more informative than any π_w for quantiles $t \geq t_I$ and less informative than any π_w for quantiles $t \leq t_I$.

Lemma 11 (Dominating Information Structure)
For any information structure $\widehat{W} \in \mathcal{W}(t_x)$, there exists t_z such that:

$$\int_0^t W\left(t\left|t_x, \pi_0, t_z\right.\right) dt \geq \int_0^t \widehat{W}(t)dt, \textit{ for all } t \leq t_I; \tag{65}$$

$$\int_0^t W\left(t\left|t_x, \pi_0, t_z\right.\right) dt \leq \int_0^t \widehat{W}(t)dt, \textit{ for all } t \geq t_I, \tag{66}$$

where $\pi_0 = \pi_w(t_x)$*. Furthermore, if* $\widehat{W}(t) \neq W\left(t\left|t_x, \pi_0, t_z\right.\right)$ *for some* t_z*, then some inequality is strict.*

Proof. We first prove that for any information structure $W \in \mathcal{W}(t_x)$, there exists t_z such that:

$$\int_0^{t_I} W\left(t\left|t_x, \pi_0, t_z\right.\right) dt = \int_0^{t_I} \widehat{W}(t)dt. \tag{67}$$

For this, we recall that π_w is the most informative information structures, which corresponds to $\pi_w\left(t\left|t_x, \pi_0, t_v\right.\right)$ (with t_v defined in (61)). We thus have that for all $t \in [0,1]$:

$$\int_0^t W\left(s\left|t_x, \pi_0, t_v\right.\right) ds \leq \int_0^t \widehat{W}(s)ds.$$

On the other hand, we must also have that for all $t \in [0,1]$:

$$\int_0^t W\left(s\left|t_x, \pi_0, 1\right.\right) ds \geq \int_0^t \widehat{W}(s)ds.$$

In particular, the two inequalities above also hold evaluated at $t = t_I$. Hence, by the Intermediate Value Theorem, there exists $\hat{t} \in [t_v, 1]$ such that (67) is satisfied.

We first prove that there exists at most two quantiles $\{t_1, t_2\}$ such that $\pi_w(t)$ and $\pi_w\left(t\left|t_x, \pi_0, t_z\right.\right)$ strictly cross. That is, there exists at most two quantiles $t_1, t_2 \in [t_x, t_z]$ such that the sign of

$$\pi_w(t) - \pi_w\left(t\left|t_x, \pi_0, t_z\right.\right)$$

changes. To prove this, note that π_w is concave and π_w^{**} is linear in the range $t \in [t_x, t_z]$ wherever $\pi_w\left(t\left|t_x, \pi_0, t_z\right.\right) < \pi_0$, so they cross at most twice in the range $[t_x, t_z]$ (that is, strict crossings).

Hence, there exist quantiles $\{t_1, t_2\}$ such that:

$$\int_0^t W\left(s\left|t_x, \pi_0, t_z\right.\right) ds - \int_0^t \widehat{W}(s)ds$$

is (weakly) increasing in $t \in [0, t_1] \cup [t_2, t_z]$ (wherever $W\left(t\left|t_x, \pi_0, t_z\right.\right) \geq \widehat{W}(t)$) and is (weakly) decreasing in $t \in [t_1, t_2]$ (wherever $W\left(t\left|t_x, \pi_0, t_z\right.\right) \leq \widehat{W}(t)$). By construction, we have that

$$\int_0^{t_I} W\left(t\left|t_x, \pi_0, t_z\right.\right) dt = \int_0^{t_I} \widehat{W}(t)dt.$$

We thus have that, for all $t \in [0, t_I]$:

$$\int_0^t W\left(s \left| t_x, \pi_0, t_z \right.\right) ds \geq \int_0^t \widehat{W}(s) ds, \tag{68}$$

and for all $t \in [t_I, t_z]$

$$\int_0^t W\left(s \left| t_x, \pi_0, t_z \right.\right) ds \leq \int_0^t \widehat{W}(s) ds. \tag{69}$$

This last inequality must also be satisfied in the range $[t_z, 1]$ since in this range $W\left(t \left| t_x, \pi_0, t_z \right.\right) = V(t)$. This proves the result. ■

We now write the difference between the revenues generated by W^* and by W^{**} as follows:

$$D \triangleq \left(-\int_{t_x}^1 W^*(t) \frac{ds(t)}{dt} dt - q(t_x) W^*(t_x) \right) - \left(-\int_{t_x}^1 W\left(t \left| t_x, \pi_0, t_z \right.\right) \frac{ds(t)}{dt} dt - q(t_x) W(t_x | t_x, \pi_0, t_z) \right) \tag{70}$$

$$= -\int_{t_x}^{t_I} (W^*(t) - W\left(t \left| t_x, \pi_0, t_z \right.\right)) \frac{ds(t)}{dt} dt - \int_{t_I}^1 (W^*(t) - W\left(t \left| t_x, \pi_0, t_z \right.\right)) \frac{ds(t)}{dt} dt \tag{71}$$

$$- q(t_x)(W^*(t_x) - W(t_x \left| t_x, \pi_0, t_z \right.)). \tag{72}$$

We now prove each of the terms is weakly negative with some being strictly negative. The last term is negative because:

$$W\left(t_x \left| t_x, \pi_0, t_z \right.\right) = \pi_w(t_x \left| t_x, \pi_0, t_z \right.) \leq \pi_0 = \pi_w^*(t_x) = W^*(t_x)(1 - t_x),$$

where the inequality follows from the construction of $\pi_w(t_x \left| t_x, \pi_0, t_z \right.)$. Integrating by parts the second term, we get:

$$-\int_{t_I}^1 (W^*(t) - W\left(t \left| t_x, \pi_0, t_z \right.\right)) \frac{ds(t)}{dt} dt = \int_{t_I}^1 \int_{t_I}^t (W^*(\ell) - W\left(\ell \left| t_x, \pi_0, t_z \right.\right)) d\ell \frac{d^2 s(t)}{dt^2} dt \leq 0,$$

where the inequality follows from (69) and the fact that s is convex in this domain. Finally, for the first term, we do the analogous calculation:

$$-\int_{t_x}^{t_I} (W^*(t) - W\left(t \left| t_x, \pi_0, t_z \right.\right)) \frac{ds(t)}{dt} dt = -\int_{t_x}^{t_I} \int_t^{t_I} (W^*(s) - W\left(s \left| t_x, \pi_0, t_z \right.\right)) ds \frac{d^2 s(t)}{dt^2} dt \leq 0.$$

Now the inequality follows from (68) and the fact that s is concave in this domain.

We thus conclude that $W^*(t) = W\left(t \left| t_x, \pi_0, t_z \right.\right)$ for some (t_x, π_0, t_z). Finally, we show that $W\left(t \left| t_x, \pi_0, t_z \right.\right)$ is linear in (t_I, t_z). If $W^*(t)$ is not linear in (t_I, t_z), we consider information structure

$W(t\,|t_x,\pi_0,t_z)$, where t_z is chosen such that $W(t\,|t_x,\pi_0,t_z)$ is linear in (t_I,t_z). We can find such t_z by noting that for all $t\in[0,1]$:

$$W\left(t\left|t_x,\pi_0,\pi^{-1}(\pi_0)\right.\right)\geq W^*(t)\text{ and }W(t\,|t_x,\pi_0,1)\leq W^*(t).$$

In other words, if we take $t_z=\pi_v^{-1}(\pi_0)$ we get a revenue function pointwise larger than W^* and if $t_z=1$ then we get a profit function pointwise smaller than W^*. Hence, such t_z exists.

The difference between the revenues generated can be written as follows:

$$D=-\int_{t_I}^{1}(W^*(t)-W(t\,|t_x,\pi_0,t_z))\frac{ds(t)}{dt}dt<0.$$

The equality follows the same way as in (63)-(72) but noting that:

$$W(t\,|t_x,\pi_0,t_z)=W^*(t)\text{ for all }t\in[0,t_I].$$

The inequality follows from the fact that sign$[W^*(t)-W(t\,|t_x,\pi_0,t_z)]=(-,+)$ (following the same steps as before) and by definition $ds(t)/dt$ is increasing for all $t\geq t_I$. We thus prove that $W(t\,|t_x,\pi_0,t_z)$ generates more consumer surplus than W^*, which is a contradiction. This proves that W^* must be linear in (t_I,t_z).

9.2.3 Quasi-concavity and $q(0)=0$

We now conclude the proof of Proposition 5. So far we have only used the curvature properties of $s(t)$. That is, it has a unique inflection point. We now use two additional properties of $s(t)$. Namely, that it is quasiconvex and that $q(0)=0$. We begin by defining:

$$t_s=\arg\max_{t\in[0,1]}s(t).$$

We show that π_w^* must be affine in $[t_x,t_s]$ whenever $t_x>0$.

Lemma 12 (Affine Section when there is Exclusion)
Suppose that $t_x>0$, then π_w^* is affine in $[t_x,t_s]$.

Proof. Suppose that there exists $t_1\in[t_x,t_s]$ such that π_w^* is strictly concave at t_1 and $t_x>0$. We reach a contradiction by proving that π_w^* is not a bidder-optimal information structure.

We consider the following information structure:

$$\widehat{\pi}_w(t)=\begin{cases}\min\{\pi_w^*(t),\pi_w^*(t_x)-\varepsilon_1\}, & \text{if } t\geq t_x;\\ \min\{\pi(t)+\varepsilon_2(1-t),\pi_w^*(t_x)-2\varepsilon_1\}, & \text{if } t\leq t_x,\end{cases}$$

where ε_1 and ε_2 are such that (i) the majorization constraint is satisfied, (ii) $\widehat{W}$ is increasing and (iii) $\widehat{\pi}_w(t) = \pi^*_w(t)$ for all $t \geq t_s$. We prove such $\varepsilon_1, \varepsilon_2$ can be found.

We begin by proving (iii), and for this we define t_2 implicitly as follows:

$$\pi^*_w(t_2) = \pi^*_w(t_x) - \varepsilon_1.$$

By construction, $\widehat{\pi}_w(t) = \pi^*_w(t)$ for all $t \geq t_2$. We now prove that $t_2 \leq t_s$. When ε_1 is small enough, t_2 varies continuously with ε_1 and $t_2 \to t_1$ as $\varepsilon_1 \to 0$. So we can obviously find ε_1 that is small enough such that the third condition is satisfied.

We now prove (ii). For this we write $\widehat{W}$ explicitly:

$$\widehat{W}(t) = \begin{cases} \min\{W^*(t), W^*(t_x) - \frac{\varepsilon_1(1-t_x)}{1-t}\}; & \text{if } t \geq t_x. \\ \min\{V(t) + \varepsilon_2, W^*(t_x) - \frac{2\epsilon_1(1-t_x)}{1-t}\}; & \text{if } t \leq t_x. \end{cases}$$

At t_x, $\widehat{W}$ is clearly increasing. Furthermore, at all $t \neq t_x$, $\widehat{W}$ is the minimum of increasing functions, so it is also increasing.

Finally, we prove (i). We denote by $t_3 \in [0, t_x]$ the solution to the following equation:

$$\pi(t_3) + \varepsilon_2(1 - t_3) = \pi^*_w(t_3) - 2\varepsilon_1.$$

We note that $\widehat{W}(t) \geq W^*(t)$ if and only if $t \leq t_3$. Hence, $\mathrm{sign}(\widehat{W} - W^*) = (+, -)$. Finally, if $\varepsilon_1 \to 0$ (keeping ε_2 fixed), we have that:

$$\int_0^1 \widehat{W}(t)dt > \int_0^1 W^*(t)dt.$$

On the other hand, if $\varepsilon_2 \to 0$ (keeping ε_1 fixed), we have that:

$$\int_0^1 \widehat{W}(t)dt < \int_0^1 W^*(t)dt.$$

Hence, we can find $\varepsilon_1, \varepsilon_2$ such that:

$$\int_0^1 \widehat{W}(t)dt = \int_0^1 W^*(t)dt.$$

Since $\mathrm{sign}(\widehat{W} - W^*) = (+, -)$ this means that $\widehat{W}$ satisfies the majorization constraint.

We now note that:

$$t_m(\pi^*_w) = t_m(\widehat{\pi}_w) = t_x.$$

So the quantile that maximizes revenue is not changed when $\varepsilon_1, \varepsilon_2$ are small enough. The difference in bidder surplus generated by W^* and $\widehat{W}$ is given by:

$$\begin{aligned} U^* - \hat{U} &= -\int_{t_x}^{1} (W^*(t) - \widehat{W}(t)) \frac{ds(t)}{dt} dt - q(t_x)(1-t_x)(W^*(t_x) - \widehat{W}(t_x)) \\ &= -\int_{t_x}^{t_2} (W^*(t) - \widehat{W}(t)) \frac{ds(t)}{dt} dt - q(t_x)(1-t_x)(W^*(t_x) - \widehat{W}(t_x)). \end{aligned}$$

We first note that $W^*(t_x) \geq \widehat{W}(t_x)$, so

$$-q(t_x)(1-t_x)(W^*(t_x) - \widehat{W}(t_x)) \leq 0.$$

We also note that for all $t \in [t_x, t_2]$, we have that $s'(t) > 0$ and $W^*(t) \geq \widehat{W}(t)$. We thus have that:

$$-\int_{t_x}^{t_2} (W^*(t) - \widehat{W}(t)) \frac{ds(t)}{dt} dt < 0.$$

We thus have that:

$$U^* - \hat{U} < 0.$$

We thus have that W^* is not a bidder-optimal information structure. We thus reach a contradiction, which proves the result. ■

Lemma 13 (Affine Section when there is No Exclusion)
Suppose that $t_x = 0$, then π_w^* is affine in $[0, t_z]$.

Proof. Suppose $t_x = 0$ and W^* is not affine in $[0, t_z]$. We reach a contradiction by proving that π_w^* is not a bidder-optimal information structure.

Consider information structure $\pi_w(t\,|0, t_z)$ where t_z is such that (65)-(66) are satisfied. The proof that such t_z exists follows the same way as in Lemma 11. We now write the difference between the revenues generated by W^* and by $W(t\,|0, t_z)$ as follows:

$$\begin{aligned} D &\triangleq \left(-\int_{t_x}^{1} W^*(t)\frac{ds(t)}{dt}dt - q(t_x)W^*(t_x)\right) - \left(-\int_{t_x}^{1} W(t\,|0,t_z)\frac{ds(t)}{dt}dt - q(t_x)W(t_x\,|0,t_z)\right) \quad (73) \\ &= -\int_{t_x}^{t_I} (W^*(t) - W(t\,|0,t_z))\frac{ds(t)}{dt}dt - \int_{t_I}^{1} (W^*(t) - W(t\,|0,t_z))\frac{ds(t)}{dt}dt \\ &\quad - q(t_x)(W^*(t_x) - W(t_x\,|0,t_z)). \end{aligned}$$

As in (71) we have that the first two terms after the second inequality are negative, while the third term is 0. Hence, we reach a contradiction. ■

9.3 Proofs of Results in Section 5

Proof of Theorem 3. As shorthand notation, we introduce:

$$U(W,r) \triangleq \int_0^1 r(t)W(t)dt - \left(\int_0^1 W(t)(1-t)dr(t) + r(0)W(0)\right);$$
$$m(W) \triangleq \underset{\{\hat{r}:\hat{r}\prec_w q\}}{\arg\max} \int_0^1 (1-t)W(t)d\hat{r}(t) + \hat{r}(0)W(0).$$

If multiple solutions exist so that $m(W)$ is not uniquely defined, we consider the solution characterized in Proposition 13. Using this notation, we have that:

$$U_m = \max_{t_z\in[0,1]} U(W\left(t\,|0,t_z\right), q).$$

We now prove that $U_m \geq U^*$.

For every $w_0 \in \mathbb{R}$, the function

$$(\pi(t) + w_0(1-t))$$

is concave, so Theorem 2 applies for every $w_0 \geq 0$. [3] Hence there exists $(t_x(w_0), t_z(w_0)) : \mathbb{R}_+ \to [0,1]\times[0,1]$ such that:

$$(W\left(t\,|t_x(w_0), t_z(w_0)\right) + w_0) \in \underset{W\prec(V+w_0)}{\arg\max}\ U(W, m(W)),$$

where we make explicit that the solution (in general) depends on w_0. To make the notation more compact, we use shorthand notation:

$$W\left(t_x\,|w_0\right) \triangleq W\left(t_x\,|t_x(w_0), t_z(w_0)\right).$$

Hence, we only make reference to the shift in values w_0.

We note that for all (W,r) and for all $w_0 \in \mathbb{R}_+$, we have that:

$$U(W,r) = U(W+w_0, r).$$

That is, if we keep the mechanism fixed and shift the distribution of values by a fixed amount, bidder surplus remains unchanged. We also have that for any $W \prec V$, $m(W) \prec_w m(W+w_0)$. That is, if the values are shifted upwards the optimal mechanism is less wasteful. Thus, for all W, we have that:

$$U(W, m(W+w_0)) \geq U(W, m(W)).$$

[3] If $\pi(t)$ is quasi-concave but not concave on its increasing segment, then $(\pi(t) + \bar{w}_0(1-t))$ might be decreasing and convex, which violates the assumption in Section 2.

And, in particular, we can conclude that:

$$U(W(t_x|w_0)+w_0, m(W(t_x|w_0)+w_0)) \geq U(W^*, m(W^*)) = U^*, \tag{74}$$

for all $w_0 \geq 0$.

We also have that in the limit $w_0 \to \infty$,

$$t_x(w_0) \to 0 \text{ and } m(W(t_x|w_0)+w_0) \to q.$$

To verify this, note that we must have that for any information structure $\widetilde{W}(t|t_x,t_z)$:

$$\widetilde{W}(0|t_x(w_0),t_z(w_0)) + w_0 \leq (\widetilde{W}(t_x|t_x(w_0),t_z(w_0)) + w_0)(1-t_x(w_0)).$$

But, in the limit, this is satisfied only when $t_x(w_0) \to 0$. We thus have that:

$$\lim_{w_0\to\infty} U(W(t_x(w_0)|w_0)+w_0, m(W(t_x(w_0)|w_0)+w_0)) = \max_{t_z\in[0,1]} U(W(t|0,t_z), q) = U_m. \tag{75}$$

Combining (75) and (74) we get $U_m \geq U^*$.

If $t_1 < t_m$, we have that $W(t|0,t_m)$ is non-increasing. We thus have that U_m is attainable and thus $U^* = U_m$. Since (40) is quasi-concave in t_z, we have that when $t_1 > t_m$, then

$$W(t|0,t_1) \geq W(t|0,t_z), \text{ for all } t_z \geq t_1.$$

Thus, the solution is $W(t|0,t_1)$ or $W(t|t_x,t_z)$ and in both cases $\alpha = 0$. ■

Proof of Lemma 3 and Proposition 8. If $N = 2$, we have that $s(t)$ is convex, so in this case we have that $t_m = 0$. We thus analyze $N > 2$. It is easy to see that $t_z = 0$ and $t_z = 1$ are not optimal, so we must have an interior optimum that satisfies the first-order condition. To check that $t_z = 1$ is not optimal it suffices to note that this information structure generates 0 bidder surplus. To check that $t_z = 0$ is not optimal it suffices to note that $s(t)$ is concave for low enough quantiles so any conflating of small quantiles increases bidder surplus.

We now verify that the first-order condition can be written as follows:

$$\frac{q(t_m)(t_m + (1-t_m)\log(1-t_m))}{t_m} - \int_0^{t_m} \frac{q(t)}{1-t}dt = 0. \tag{76}$$

Let α be the parameter that determines $\pi_w(t|t_x,t_z)$ in (59). For every t_z we have that:

$$\int_{t_x}^1 \frac{\pi_w(t|t_x,t_z)}{1-t}dt = \int_{t_x}^1 V(t)dt.$$

Taking the derivative with respect to t_z we get:

$$\int_{t_x}^{t_z} \frac{\pi'(t_z) + \alpha - \frac{\partial \alpha}{\partial t_z}(t - t_z)}{1-t} dt = 0.$$

We thus get:

$$\frac{\partial \alpha}{\partial t_z} = -\frac{\int_{t_x}^{t_z} \frac{\pi'(t_z)+\alpha}{1-t} dt}{\int_{t_x}^{t_z} \frac{-(t-t_z)}{1-t} dt}. \tag{77}$$

Taking the first derivative of (40), we get:

$$\frac{d}{dt_z} \int_0^1 W\left(t\,|0, t_z\right) ds(t) = -\int_0^{t_m} \frac{\pi'(t_z) + \alpha - \frac{\partial \alpha}{\partial t_z}(t - t_z)}{1-t} \frac{ds(t)}{dt} dt.$$

Replacing $\partial \alpha / \partial t_z$ with (77), we get:

$$\frac{d}{dt_z} \int_0^1 W\left(t\,|0, t_z\right) ds(t) = -(\pi'(t_z) + \alpha) \left(\int_0^{t_z} \frac{1}{1-t} \frac{ds(t)}{dt} dt + \frac{\int_0^{t_z} \frac{1}{1-t} dt}{\int_0^{t_z} \frac{-(t-t_z)}{1-t} dt} \int_0^{t_z} \frac{(t-t_z)}{1-t} \frac{ds(t)}{dt} dt \right).$$

We can write this more compactly as follows:

$$\frac{d}{dt_z} \int_0^1 W\left(t\,|0, t_z\right) ds(t) = -(\pi'(t_z)+\alpha) \left(\int_0^{t_z} \frac{1}{1-t} \left(1 - \frac{\log(1-t_z)}{t_z + (1-t_z)\log(1-t_z)} (t - t_z) \right) \frac{ds(t)}{dt} dt \right).$$

Integrating by parts, we get:

$$\frac{d}{dt_z} \int_0^1 W\left(t\,|0, t_z\right) ds(t) = -(\pi'(t_z) + \alpha) \left(q(t_z) - \frac{t_z}{t_z + (1-t_z)\log(1-t_z)} \int_0^{t_z} \frac{q(t)}{1-t} dt \right).$$

Rearranging terms, we get (76).

We thus define:

$$h(t_z) \triangleq \frac{q(t_z)(t_z + (1-t_z)\log(1-t_z)}{t_z} - \int_0^{t_z} \frac{q(t)}{1-t} dt.$$

Since the first-order condition must be satisfied by some t_z, we have that $h(t_z)$ has a root in $t_z \in (0, 1)$. We now prove it has a unique root. We note that:

$$\lim_{t_z \downarrow 0} h(t_z) = 0,$$

and

$$\frac{dh(t_z)}{dt_z} = -\frac{(N(t_z - 1) - t_z + 2)q(t_z)((t_z - 1)\log(1-t_z) - t_m)}{t_z(t_z - 1)}.$$

We thus have that:

$$\frac{dh(t_z)}{dt_z} = 0 \iff t_z = \frac{N-2}{N-1}.$$

Thus, $h(t_z)$ has a root and it has a unique root.

To verify the comparative statics, we note that:

$$\frac{d}{dN}\left(\frac{\int_0^{t_z} \frac{q(t)}{1-t} dt}{q(t_z)}\right) < 0.$$

Thus, the root is increasing in N. ■

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