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Location choice in two-sided markets with indivisible agents ☆

Robert M. Anderson [a], Glenn Ellison [b], Drew Fudenberg [c,*]

[a] *University of California, Berkeley, USA*
[b] *Massachusetts Institute of Technology, USA*
[c] *Harvard University, USA*

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Abstract

Consider a model of location choice by two sorts of agents, called "buyers" and "sellers": In the first period agents simultaneously choose between two identical possible locations; following this, the agents at each location play some sort of game with the other agents there. Buyers prefer locations with fewer other buyers and more sellers, and sellers have the reverse preferences. We study the set of possible equilibrium sizes for the two markets, and show that two markets of very different sizes can co-exist even if larger markets are more efficient. This extends the analysis of Ellison and Fudenberg [2003. Quart. J. Econ. 118, 1249–1278], who ignored the constraint that the number of agents of each type in each market should be an integer, and instead analyzed the "quasi-equilibria" where agents are treated as infinitely divisible.

JEL classification: C62; L14; O18; R11

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1. Introduction

In many economic models, agents benefit from interactions with agents whose characteristics are in some way different than their own. In exchange economies, agents gain from interactions with agents whose endowments or preferences are different; in economies with production, gains come about when producers interact with consumers, and in opposite-sex marriage "markets" gains arise from men meeting women. Many of these activities seem to be agglomerated: for example many industries are geographically concentrated, and trade in many goods takes place in a small number of marketplaces.

One explanation in the literature for the observed levels of agglomeration is that the agglomeration arises from "tipping" in the presence of increasing returns to scale. In the simplest such model, there is a continuum of identical

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* Corresponding author at: Harvard University, Department of Economics, Cambridge, MA, USA.
E-mail addresses: anderson@econ.berkeley.edu (R.M. Anderson), gellison@mit.edu (G. Ellison), dfudenberg@harvard.edu (D. Fudenberg).

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agents, who all prefer to be part of the largest market: Here the only equilibria are for all active markets to be the same size, and the only stable equilibria assign all of the agents to the same location. More strongly, these agglomerated or "completely tipped" outcomes are the only equilibria in models with a finite number of agents: The equal-market-shares outcomes are not equilibria when the agents have finite size, as a shift by a single agent makes the agent's new market larger and hence better than the one she left.

However, in many cases of interest, agents are not identical, and while all agents prefer larger markets, they also prefer markets with few agents of their own type, so that e.g. the buyer–seller ratio is more favorable. In this case, an agent who switches markets has an adverse "market impact" effect on the market she joins. When a seller contemplates a switch from market 1 to market 2, she should take into account that her joining market 2 will increase the seller–buyer ratio there, and thereby makes that market less attractive for all sellers, including herself. This raises the question of whether the market impact effect can generate equilibria with several active markets, or whether we should expect activity to concentrate in a single market absent any diseconomies (such as transportation costs) from doing so. This question cannot be resolved by working only with a continuum model, as often both the market impact effect and the benefits of agglomeration vanish as the market size becomes infinite. Instead, we study a model of location choice in a finite population composed of two sorts of agents, whom we will call "buyers" and "sellers."

The structure of the model is the same as in Ellison and Fudenberg (2003): In the first period of the model, agents simultaneously choose between two identical possible locations or markets. Following this, the agents at each location play some sort of game with the other agents there; the resulting payoffs are determined by the numbers of each type of agent who chose the same location.[1] We suppose that buyers prefer markets with fewer other buyers and more sellers, and that sellers have the reverse preferences. Thus, when the numbers of buyers and sellers are even, there is an equilibrium where the two markets are exactly the same size.[2]

The traditional view (see e.g. Krugman, 1991b) was that outcomes with two or more active markets are an unstable knife-edge whenever there are increasing returns, that is when larger markets are more efficient. Ellison and Fudenberg (2003) show that this view is incorrect, and that the "market impact effect" can allow several markets to remain open in equilibrium: instead of split market outcomes being an unstable knife-edge, there can be a "plateau" of such equilibria. As noted above, both the market impact effect and the benefit of increasing returns often vanish as the number of agents in each market grows to infinity. Ellison and Fudenberg (2003) argue that in many models of interest the effects vanish at the same rate. When this is true, Ellison and Fudenberg show that the incentive constraints for equilibrium are consistent with two large markets being active even when their sizes are unequal, and they give a lower bound on the width of this plateau. However, they ignore the constraint that the number of agents of each type in each market should be an integer, and instead analyze the "quasi-equilibria" in which agents are treated as infinitely divisible. For this reason, Ellison and Fudenberg's (2003) results say nothing about the existence of exact equilibria—outcomes that satisfy all of the incentive constraints and also assign an integer number of agents of each type to each market.

In the context of competing auction sites, Ellison et al. (2004) show that when the number of buyers is large and buyer values are uniformly distributed, for any quasi-equilibrium size ratio α of the game with B buyers and S sellers, there is a nearby S' for which the game with B buyers and S' sellers has an exact equilibrium with size ratio approximately α. This shows that exact equilibria exist in some large economies, but says nothing about the number of equilibria for a typical pair (S, B). Indeed, it leaves open the possibility that most pairs (S, B) have no exact equilibria other than complete tipping.

The point of this paper is to complete the analysis of tipping in large markets by providing a thorough analysis of equilibrium, paying attention to the constraint that each agent is indivisible. Put briefly, we have four main findings about economies (payoff functions and aggregate seller–buyer ratios) that satisfy the Ellison and Fudenberg (2003) assumptions.

[1] The assumption that buyers and sellers move simultaneously is most natural when the agents on the two sides of the market are roughly symmetric; some cases might be better described by asymmetric models where all the sellers move before any of the buyers. Note also that our model does not consider any actions by market makers such as eBay.

[2] More precisely, our results imply that when the numbers of buyers and sellers are even and sufficiently large, there is an equilibrium where the two markets are exactly the same size. We expect that typically this equilibrium will also be present even when the numbers of buyers and sellers is small, but have not tried to provide precise conditions under which this is true.

1. We sharpen the Ellison and Fudenberg (2003) result on quasi-equilibria by giving an upper bound on the size of the two-market plateau and a sharper lower bound. As S and B grow,[3] these bounds converge, so that in the limit we have a necessary and sufficient condition.
2. Under some natural specifications of payoffs there are sequences $\{S_n, B_n\}$ with $B_n \to \infty$ that completely tip for all n. That is, at each point in the sequence, the only pure strategy equilibria are the ones in which all agents choose the same location, so the plateau of two-market equilibria is empty.
   Despite the finding in point 2, it turns out that the sequences along which two-market equilibria fail to exist are quite special. Specifically,
3. For any $\varepsilon > 0$, there is a bound M such that if the aggregate buyer–seller and seller–buyer ratios are at least ε away from every integer, and the number of agents is at least M, then there is a range of exact equilibria. For fixed payoff functions the width of this range (expressed in terms of the proportion of agents in each market) is independent of the number of buyers and sellers when these are large.
4. Moreover, for generic sequences $\{S_n, B_n\}$ with $B_n \to \infty$ and S_n/B_n convergent, there is a plateau of exact equilibria for all large n. This plateau includes all of the size ratios predicted by the Ellison and Fudenberg (2003) result, and typically includes others as well. Moreover, the density of equilibria (i.e. the proportion of integers in a given range such that there is an equilibrium with that many buyers in one market, and the rest in the other) converges to a specific piecewise-linear function that we identify. When there are fewer buyers than sellers, for almost all integer numbers of buyers in the center of the plateau, there is an equilibrium with that many buyers in market 1; the symmetric statement is true when there are more buyers than sellers.

The reason that the result in point 1 delivers more size ratios than Ellison and Fudenberg (2003) is that their result provided a condition for the incentive constraints to be satisfied with exactly the same seller–buyer ratio in each market. The proof of the new result, like the Ellison et al. (2004) characterization of the quasi-equilibrium set in the case of uniformly distributed buyer values, allows the seller–buyer ratios in the two markets to be slightly unequal, which in some cases leads to a substantially larger range of size ratios.

The plateau of equilibria characterized in point 4 exists generically because it obtains whenever S_n/B_n has an irrational limit. Since the set of rational numbers has measure 0, our genericity result is analogous to the formulation of a generic sequence of purely competitive economies in, for example, Grodal (1975) and Dierker (1975). The reason that sequences with an irrational limit are better behaved is closely related to the fact that the map that sends x to $(x+y)_{\mathrm{mod}\ 1}$ is an ergodic transformation if and only if y is irrational. Fix a model with S sellers and B buyers overall, and let $\gamma = S/B$. Whether there is an equilibrium with an integer B_1 of buyers in market 1 depends on whether there is an integer S_1 satisfying certain inequalities. This in turn depends on whether $(\gamma B_1)_{\mathrm{mod}\ 1}$ lies in a certain interval in $[0, 1]$, where the length of the interval is the piecewise linear function of $\alpha = B_1/B$ referred to above. When B is large, the piecewise linear function changes slowly in response to changes in B_1, while $(\gamma B_1)_{\mathrm{mod}\ 1}$ has, by ergodicity, essentially the uniform distribution on $[0, 1]$. Consequently, the density of the set of integers B_1 such that $(\gamma B_1)_{\mathrm{mod}\ 1}$ lies in the interval converges to the length of the interval.

Another way to formulate the characterization in point 4 is as a statement about the probability that a randomly chosen large economy will have a broad plateau of equilibria with equilibria spread throughout it with the density H that we define. We show that if a sequence of economies is chosen at random by choosing some sequence $\{B_n\}$ with $B_n \to \infty$ and then choosing S_n to match as closely as possible a seller–buyer ratio drawn from some density function on a compact support, then the probability that the nth economy has a large plateau of equilibria with an equilibrium density that is within ε of H goes to one as $n \to \infty$.

The result in point 4 is proved using nonstandard analysis, which is a way of formalizing and manipulating infinitesimal, and infinitely large, quantities. The statement of the result, however, is entirely standard and can be understood without any knowledge of nonstandard analysis.[4] We believe that we could also demonstrate the generic existence of a plateau by extending the construction in point 3 to bound the buyer–seller ratios away from 1/2, 1/3, 2/3, and so

[3] The payoffs to buyers and sellers are defined as functions of $\gamma = S/B$ with domain $\gamma \in \Gamma$. The result requires that as S and B grow, γ stays away from the boundary of Γ.

[4] For more information about nonstandard analysis, see Hurd and Loeb (1985) and Anderson (1991). A methatheorem in nonstandard analysis guarantees the existence of a standard proof for the result; however, the standard proof is likely to be quite complex.

on through all of the rational numbers, but this seems too cumbersome to be worth pursuing given the availability of the result in point 4.

2. The geometry of quasi-equilibrium

Consider a simple two-stage model of location choice. In the first stage S sellers and B buyers simultaneously choose between market 1 or market 2. If S_i sellers and B_i buyers chose market i, then the market game gives the sellers in market i an expected payoff of $u_s(S_i, B_i)$ and the buyers an expected payoff of $u_b(S_i, B_i)$.[5] In a pure-strategy Nash equilibrium, there are four incentive constraints that must be satisfied: First, sellers in market 1 must be willing to stay in market 1, which is equivalent to the inequality

$$u_s(S_1, B_1) \geqslant u_s(S_2 + 1, B_2). \tag{S1}$$

We call this the $S1$ constraint. Then there are the analogous constraints that sellers in market 2 must be willing to stay in market 2, buyers in market 1 must be willing to stay in market 1, and buyers in market 2 must be willing to stay in market 2. We call these constraints $S2$, $B1$, and $B2$, respectively.

Given the total numbers S of sellers and B of buyers, an allocation (S_1, S_2, B_1, B_2) is a pure-strategy Nash equilibrium if and only if

(i) (*Incentive Compatibility*) *the four incentive constraints* $S1, S2, B1, B2$ *are satisfied.*
(ii) (*Feasibility*) $S_1 + S_2 = S$, $B_1 + B_2 = B$, *and* S_1, S_2, B_1, *and* B_2 *are all non-negative.*
(iii) (*Indivisibility*) S_1, S_2, B_1, *and* B_2 *are all integers.*

Ellison and Fudenberg (2003) call an allocation a *quasi-equilibrium* if it satisfies (i) and (ii).

Assumption 1. There is a nonempty interval $\Gamma = [\underline{\gamma}, \bar{\gamma}] \subset (0, \infty)$, continuously differentiable functions F_s and F_b, and continuous functions G_s and G_b on Γ with $F_s' < 0$ and $F_b' > 0$ such that the approximations

$$u_s(\gamma B, B) = F_s(\gamma) - \frac{G_s(\gamma)}{B} + o\left(\frac{1}{B}\right),$$
$$u_b(\gamma B, B) = F_b(\gamma) - \frac{G_b(\gamma)}{B} + o\left(\frac{1}{B}\right) \tag{1}$$

hold uniformly over $\gamma \in \Gamma$ in the limit as $B \to \infty$.

This assumption requires that the payoff functions converge to well-defined large-population limit at a rate of at least $1/B$, which will allow us to use calculus to approximate the incentive constraints when both markets are large. The assumption also implies that each agent's utility is strictly increasing in the proportion of agents of the other type when B is large.

We will maintain Assumption 1 throughout the paper. Ellison and Fudenberg (2003) argue that it is satisfied in many models of economic interest, including the Ellison et al. (2004) model of competing auctions, Krugman's (1991a) model of Marshallian labor market competition, and a two-population version of Pagano's (1989) model of competing securities markets. They show that it allows the approximation of the incentive constraints at allocations with the same seller–buyer ratios in each market, and use this to prove the following result.

Theorem 0. *(See Ellison and Fudenberg, 2003.) Assume Assumption* 1*, and set*

$$r^*(\gamma) = \max\left(\left|\frac{2G_s(\gamma)}{-F_s'(\gamma)} + 1\right|, \left|\frac{2G_b(\gamma)}{\gamma F_b'(\gamma)} + 1\right|\right)$$

[5] Although the formal model we present is one of Nash equilibria in a one-stage game, the intended interpretation is that we are analyzing first-stage choices in the subgame-perfect equilibria of a two-stage game, where the payoff functions that this paper treats as primitives are computed as the Nash equilibria of the game at each market. These payoff functions will typically only be defined for integer numbers of agents; our assumptions are then on any extension of these functions to the reals.

and

$$\alpha^*(\gamma) = \max\{0, 1/2 - 1/2r^*(\gamma)\}.$$

Then for any $\varepsilon > 0$, there exists a $\underline{B}$ such that for any integer $B > \underline{B}$ and any integer S with $\gamma \equiv S/B \in \Gamma$, the model with B buyers and S sellers has a quasi-equilibrium with $B_1 = \alpha_1 B$ buyers in market 1, for every $\alpha_1 \in [\alpha^*(\gamma) + \varepsilon, 1 - \alpha^*(\gamma) - \varepsilon]$.

The main focus of the paper will be on the exact equilibria, but as a preliminary step we provide a sharper characterization of the set of quasi-equilibria. The necessary condition we develop here may be of interest in its own right, and it will also be useful in our subsequent results.

Theorem 1. *Assume Assumption 1. Define $\Gamma^\varepsilon = [\underline{\gamma} + \varepsilon, \bar{\gamma} - \varepsilon]$, and set*

$$T(\gamma) = \frac{G_s(\gamma)}{F_s'(\gamma)} - \frac{G_b(\gamma)}{F_b'(\gamma)},$$

and

$$\alpha^{**}(\gamma) = \max\left\{0, \frac{1}{2} - \frac{\gamma + 1}{2|1 + \gamma - 2T(\gamma)|}\right\}.$$

Then for all $\varepsilon > 0$ there exists $\underline{B}$ such that for any integer $B > \underline{B}$ and any integer S with $\gamma \equiv S/B \in \Gamma^\varepsilon$:

1. *The model with B buyers and S sellers has a quasi-equilibrium with $\alpha_1 B$ buyers in market 1, for every $\alpha_1 \in [\alpha^{**}(\gamma) + \varepsilon, 1 - \alpha^{**}(\gamma) - \varepsilon]$.*
2. *The model with B buyers and S sellers has no quasi-equilibria with $B_1 = \alpha_1 B$ buyers and S_1 sellers in market 1, B_2 buyers and S_2 sellers in market 2, provided $B_1/S_1, B_2/S_2 \in \Gamma^\varepsilon$ and $\alpha_1 \in [\varepsilon, \alpha^{**}(\gamma) - \varepsilon] \cup [1 - \alpha^*(\gamma) + \varepsilon, 1 - \varepsilon]$.*

Remark. The difference between Theorems 0 and 1 is that the proof of Theorem 0 constructed quasi-equilibria with exactly equal seller–buyer ratios in each market, while Theorem 1 works with the full set of allocations. Since the proof of the theorem uses the approximation of the utility functions provided by Assumption 1, it is interesting to note that it gives exactly the range of solutions that Ellison et al. (2004) obtained for the exact utility functions that arise with uniformly distributed buyer values in their auction model. Note that the necessity result in 2. requires two extra conditions. The proposition only applies to equilibria with $B_1/S_1, B_2/S_2 \in \Gamma^\varepsilon$ because Assumption 1 imposes no restrictions at all on payoffs for $\gamma \notin \Gamma$.[6] We restrict attention to $\alpha_1 \notin \{0, 1\}$ because the models often have equilibria with all activity in a single market. The further restriction to $\alpha_1 \in [\varepsilon, 1 - \varepsilon]$ ensures that B_1 and B_2 are both large, which is necessary because Assumption 1 only characterizes payoffs in the large B limit.

The proof of Theorem 1 uses a lemma which notes that the $S1$, $S2$, $B1$, and $B2$ constraints can be rewritten in a fairly simple form when the number of buyers is large.

Lemma 1. *For all $\varepsilon > 0$, there is a $\underline{B}$ such that if $\underline{B} < B_1, B_2$ and $\gamma \equiv S/B \in \Gamma^\varepsilon$, then for $\alpha_1 \equiv B_1/B$ we have*

1. *If*

$$(S_1 - \gamma B_1) - \frac{G_s(\gamma)(1 - 2\alpha_1)}{F_s'(\gamma)} \in (\alpha_1 - 1 + \varepsilon, \alpha_1 - \varepsilon) \tag{2}$$

and

$$(S_1 - \gamma B_1) - \frac{G_b(\gamma)(1 - 2\alpha_1)}{F_b'(\gamma)} \in \left(-\alpha_1\gamma + \varepsilon, (1 - \alpha_1)\gamma - \varepsilon\right) \tag{3}$$

[6] Alternatively, we could have allowed $\Gamma = (0, \infty)$ and weakened Assumption 1 to require that uniformity hold over compact subsets of Γ, rather than all of Γ. Then Theorem 3 remains true, and Theorem 1 remains true provided we define $\Gamma_\varepsilon = [\varepsilon, 1/\varepsilon]$. Theorem 2, however, does not generalize in this way.

then there is a quasi-equilibrium with B_1 buyers and S_1 sellers in market 1.

2. *If $S_1/B_1 \in \Gamma^\varepsilon$, $S_2/B_2 \in \Gamma^\varepsilon$, and*

$$(S_1 - \gamma B_1) - \frac{G_s(\gamma)(1-2\alpha_1)}{F_s'(\gamma)} \notin [\alpha_1 - 1 - \varepsilon, \alpha_1 + \varepsilon],$$

or

$$(S_1 - \gamma B_1) - \frac{G_b({}^\circ\gamma)(1-2\alpha_1)}{F_b'(\gamma)} \notin \big[-\alpha_1\gamma - \varepsilon, (1-\alpha_1)\gamma + \varepsilon\big],$$

then there is no quasi-equilibrium with B_1 buyers and S_1 sellers in market 1.

The proof of Lemma 1 is presented in Appendix A.

Proof of Theorem 1. Lemma 1 implies that if B is large and $S/B \in \Gamma^\varepsilon$, then the model with S sellers and B buyers has a quasi-equilibrium with $B_1 = \alpha_1 B$ buyers in market 1 for all α_1 for which there is a solution to the equations

$$(S_1 - \gamma B_1) - \frac{G_s(\gamma)(1-2\alpha_1)}{F_s'(\gamma)} \in (\alpha_1 - 1 + \varepsilon, \alpha_1 - \varepsilon), \quad \text{and}$$
$$(S_1 - \gamma B_1) - \frac{G_b(\gamma)(1-2\alpha_1)}{F_b'(\gamma)} \in \big(-\alpha_1\gamma + \varepsilon, (1-\alpha_1)\gamma - \varepsilon\big).$$

We can find an S_1 that satisfies both conditions if and only if the intervals $(\alpha_1 - 1 + (\frac{G_s(\gamma)(1-2\alpha_1)}{F_s'(\gamma)} - \frac{G_b(\gamma)(1-2\alpha_1)}{F_b'(\gamma)}) + \varepsilon$, $\alpha_1 + (\frac{G_s(\gamma)(1-2\alpha_1)}{F_s'(\gamma)} - \frac{G_b(\gamma)(1-2\alpha_1)}{F_b'(\gamma)}) - \varepsilon)$ and $(-\alpha_1\gamma + \varepsilon, (1-\alpha_1)\gamma - \varepsilon)$ intersect.

Given our definition $T(\gamma) = \frac{G_s(\gamma)}{F_s'(\gamma)} - \frac{G_b(\gamma)}{F_b'(\gamma)}$, this is equivalent to

$$\big(\alpha_1 - 1 + \varepsilon + T(\gamma)(1-2\alpha_1), \alpha_1 - \varepsilon + T(\gamma)(1-2\alpha_1)\big) \cap \big(-\alpha_1\gamma + \varepsilon, (1-\alpha_1)\gamma - \varepsilon\big) \neq \emptyset.$$

The two intervals in this equation are of length $1 - 2\varepsilon$ and $\gamma - 2\varepsilon$, respectively, and they are both centered at 0 when $\alpha_1 = \frac{1}{2}$. The first interval moves linearly with slope $1 - 2T(\gamma)$ in response to changes in α_1, while the second interval moves linearly with slope $-\gamma$, so the movement of the first interval, relative to the second, is linear with slope $1 + \gamma - 2T(\gamma)$ in response to changes in α_1. The two intervals continue to intersect as long as the magnitude of the relative movement is less than $(1+\gamma)/2 - 2\varepsilon$, which is true whenever $|\alpha_1 - \frac{1}{2}| < \frac{(1+\gamma)/2 - 2\varepsilon}{|1+\gamma-2T(\gamma)|}$, i.e. when

$$\alpha_1 \in \left(\frac{1}{2} - \frac{1+\gamma-4\varepsilon}{2|1+\gamma-2T(\gamma)|}, \frac{1}{2} + \frac{1+\gamma-4\varepsilon}{2|1+\gamma-2T(\gamma)|}\right). \tag{4}$$

Going from this expression to the simpler statement given in part 1 of the theorem would be a simple relabeling of the ε, but for the fact that $|1 + \gamma - 2T(\gamma)|$ term in the denominator is not bounded below. This can be dealt with without much more work by dividing the γ's into two cases.

Partition $\Gamma^\varepsilon = \Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon$, where

$$\Gamma_1^\varepsilon = \left\{\gamma \in \Gamma^\varepsilon\colon\ \frac{1+\gamma}{2|1+\gamma-2T(\gamma)|} > \frac{2}{3}\right\}.$$

For $\gamma \in \Gamma_1^\varepsilon$ note that for any $\varepsilon < 1/16$ we have $\frac{1+\gamma-4\varepsilon}{1+\gamma} > \frac{3}{4}$. Hence

$$\frac{1+\gamma-4\varepsilon}{2|1+\gamma-2T(\gamma)|} > \frac{3}{4}\frac{1+\gamma}{2|1+\gamma-2T(\gamma)|} > \frac{3}{4}\frac{2}{3} = \frac{1}{2},$$

which implies that the interval in (4) is $(0, 1)$. Hence, if we are given any $\varepsilon > 0$, we can choose $\underline{B}$ to be equal to $1/\varepsilon$ times the value $\underline{B}$ (call it $\underline{B}_{1/16}$) which makes the conclusion of Lemma 1 true for $\varepsilon = 1/16$. For $B > \underline{B}$ and $\alpha \in (\varepsilon, 1-\varepsilon)$ we have $\alpha_1 B > \underline{B}_{1/16}$, $(1-\alpha_1)B > \underline{B}_{1/16}$, and that α_1 is in the interval identified in (4). Hence, there is a quasi-equilibrium with $B_1 = \alpha_1 B$.

For $\gamma \in \Gamma_2^\varepsilon$ we have $|1 + \gamma - 2T(\gamma)| > 2(1+\gamma)/3 > 2/3$, so we can just do a simple relabeling of ε: Given any $\varepsilon' > 0$, pick $\underline{B}$ such that the conclusion of Lemma 1 is true for $\varepsilon = \varepsilon'/3$ whenever $B > (\alpha^{**}(\gamma) + \varepsilon')\underline{B}$. Then for

$B > \underline{B}$, we have quasi-equilibria for all α_1 in the interval identified in (4), which contains $(\alpha^{**}(\gamma) + \varepsilon', 1 - \alpha^{**}(\gamma) - \varepsilon')$ when $\varepsilon = \varepsilon'/3$ and $|1 + \gamma - 2T(\gamma)| > 2/3$.

The proof of part 2. of the theorem uses part 2. of Lemma 1 in a nearly identical way. As long as B_1 and B_2 are both large we know that there is no quasi-equilibrium with $S_1/B_1, S_2/B_2 \in \Gamma^\varepsilon$ if

$$\alpha_1 \notin \left(\frac{1}{2} - \frac{1+\gamma+4\varepsilon}{2|1+\gamma-2T(\gamma)|}, \frac{1}{2} + \frac{1+\gamma+4\varepsilon}{2|1+\gamma-2T(\gamma)|}\right).$$

For $\gamma \in \Gamma_2^\varepsilon$, the conclusion of part 2. of the theorem follows from a relabeling of the ε's. For $\gamma \in \Gamma_1^\varepsilon$, the conclusion of part 2. is completely vacuous. □

Theorem 1 on its own says nothing about the set of full equilibria of the game. To fix ideas, we reproduce (as Fig. 1) two illustrations from Ellison et al. (2004) that show the four constraints in an auction model with uniformly

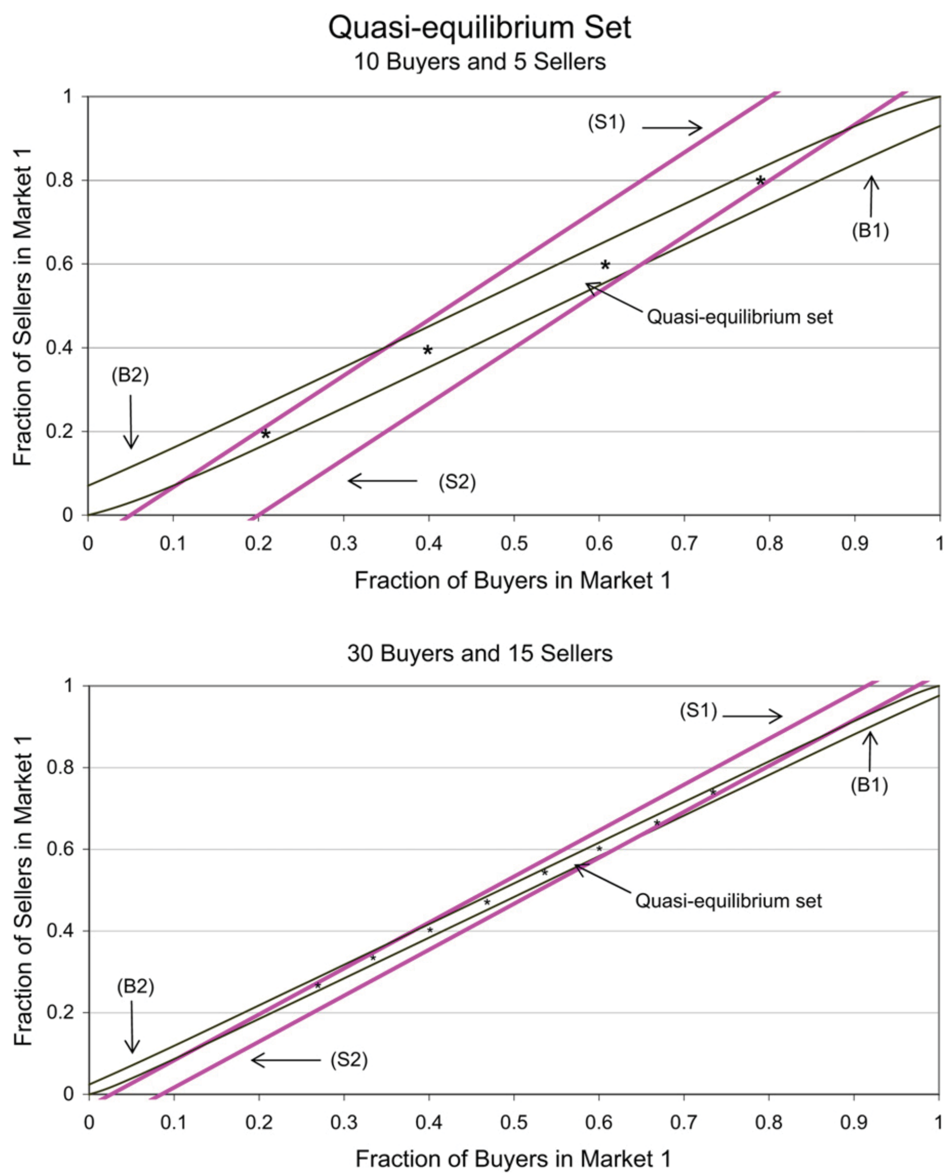

Fig. 1. The quasi-equilibrium and equilibrium sets in the Ellison et al. (2004) competing auction model.

distributed buyer values.[7] Here the lines S1, S2, B1, B2 correspond to the points where the corresponding incentive constraints hold with equality. That is, along S1, each seller that is currently choosing market 1 is just indifferent between staying in market 1 or switching to market 2. Since sellers prefer there to be fewer other sellers, the points above this line are not consistent with equilibrium. Reasoning in this way about each of the incentive constraints, we see that the quasi-equilibrium set is the diamond-shaped region in the center of the figure. The exact equilibria, marked with stars, are the integer grid points that lie in the quasi-equilibrium set.

The top panel corresponds to a model with ten buyers and five sellers. In equilibrium the smaller market can have two buyers and one seller, or four buyers and two sellers; there is no equilibrium with three or five buyers in the smaller market. With three buyers in the smaller market, for example, there are a range of values of near one-and-a-half that satisfy the quasi-equilibrium conditions, but none of them satisfy the integer constraints. The bottom panel gives an idea of what happens as markets grows to thirty buyers and fifteen sellers. The quasi-equilibrium region looks much "flatter" because the impact of one agent moving from one market to another is smaller, so the utilities in the two markets need to be closer together in order to discourage agents whose equilibrium utility is lower from switching markets. (Lemma 4 makes a related observation: If (S_1, S_2, B_1, B_2) is a quasi-equilibrium, then the seller–buyer ratio in market 1 must equal the overall ratio S/B plus a term that goes to 0 at rate $O(1/B_1)$.) The exact equilibria are again marked with stars.

This leads us to the current question: when will there be exact equilibria? As the economy grows, the quasi-equilibrium set gets "narrower," suggesting it might be harder to find an equilibrium at a given point, but at the same time there are more candidate integers in a given range of relative sizes, so the answer may not be obvious.

3. An example with complete tipping

The main result of Ellison and Fudenberg (2003) is that models satisfying Assumption 1 always have a large plateau of quasi-equilibria with two active markets when the number of agents is large. In this section, we show that the result does not always extend to a true equilibrium analysis. In particular, we provide a class of sequences of economies in which the increasing returns satisfy Assumption 1, but which have *no* pure strategy equilibria other than the completely "tipped" equilibria with all activity in one market.

Example 1. Let u_s and u_b be utility functions satisfying Assumption 1. Suppose that $u_s(\gamma B, B)$ is increasing in B and decreasing in γ, and that $u_b(\gamma B, B)$ is increasing in B and γ. Then, regardless of the size of B, a model with B buyers and $S = B + 1$ sellers has no pure strategy Nash equilibrium with two active markets.

Proof. Suppose to the contrary that there is an equilibrium with $0 < B_1 < B$ and $0 < S_1 < S$ sellers in market 1, and $B_2 = B - B_1$ buyers and $S_2 = S - S_1$ sellers in market 2, and assume without loss of generality that $B_1 \leqslant B_2$.

We proceed to rule out all possible relationships between S_1 and B_1

- If $S_1 < B_1$, then a buyer in market 1 will defect. After the defection market 2 will be larger (have more buyers) and have a more favorable seller–buyer ratio for buyers.
- If $S_1 = B_1$, then a buyer in market 1 will defect. Market 2 is larger after the defection. Also, $S_1 = B_1$ implies $S_2 = B_2 + 1$, so after the defection the ratio of sellers to buyers in market 2 becomes one, the same as in market 1. Hence, the buyer is better off after the defection
- If $S_1 = B_1 + 1$ and $B_1 = B_2$, then a buyer in market 2 will defect. $S_2 = B_2$ so the situation is exactly as in the case above, but with the labels reversed.
- If $S_1 = B_1 + 1$ and $B_1 < B_2$, then a seller in market 1 will defect. Market 2 is larger, and even after the defection its seller–buyer ratio is lower: $(S_2 + 1)/B_2 = (B_2 + 1)/B_2 < (B_1 + 1)/B_1 = S_1/B_1$.
- If $S_1 > B_1 + 1$, then a seller in market 1 will defect. Market 2 is at least as large, and even after the deviation it will have a lower seller–buyer ratio. □

[7] In Ellison et al. (2004), each seller has a single unit to sell, and reservation value 0 for keeping the item; each buyer wants to purchase a single unit, and buyer values are determined after location choice. Equilibrium thus payoffs in each market are determined by a uniform-price auction.

In some sense the assumption that $S = B + 1$ in the example is a worst case for the existence of equilibria. Equilibrium requires that the seller–buyer ratios be very close in the two markets. When market 1 and market 2 are about the same size, the fact that we have to put the one extra seller somewhere means we have to have about one-half too few sellers in one market and about one-half too many sellers in the other. The "market impact" of moving to the large market is one seller or one buyer, so it doesn't outweigh this. This suggests that finding equilibria may be easier when the seller–buyer is bounded away from 1, as then the impact of the "rounding error" is less severe. More generally, one way of looking at the example is that the size of the set of equilibria will depend on how closely the overall seller–buyer ratio can be approximated in the two markets when we are restricted to integer allocations.

4. A sufficient condition for a plateau of equilibria in large finite economies

In this section we provide sufficient conditions for the existence of a plateau of true equilibria. The sufficient conditions illustrate that the nonexistence example was special. In the example, the seller–buyer ratio is very close to one, but not equal to one. The main proposition of this section shows that if the ratio of the number of agents on the two sides of the market is not almost exactly an integer (and the number of agents is large), then there is a nondegenerate plateau of untipped equilibria. It also shows that within the equilibrium plateau the equilibria are not too far apart.

Let $\mathbf{N}$ be the set of non-negative integers. The main result of this section is

Theorem 2. *Fix functions u_b and u_s satisfying Assumption* 1. *There are constants $\underline{\alpha}, k_1, k_2, k_3, k_4$ with $\underline{\alpha} < 1/2$ such that for any $\varepsilon > 0$, any $\alpha \in [\underline{\alpha}, 1 - \underline{\alpha}]$, and any positive integers B and S satisfying*

(i) $S/B \in \Gamma$,
(ii) $|B/S - n| > \varepsilon$ *and* $|S/B - n| > \varepsilon$ *for all* $n \in \mathbf{N}$, *and*
(iii) $B + S > k_1 + k_2/\varepsilon$,

the model with B buyers and S sellers has an equilibrium with B_1 buyers and S_1 sellers in market 1 for some B_1, S_1 with $|B_1/B - \alpha| < (k_3 + k_4/\varepsilon)/B$ and $|S_1/S - \alpha| < (k_3 + k_4/\varepsilon)/S$.

The proof includes more information on the constants k_1, k_2, k_3, and k_4, including explicit formulas for some of them. The following corollary is a simpler statement about the existence of an equilibrium plateau.

Corollary 1. *Fix functions u_b and u_s satisfying Assumption* 1. *Then there is $\underline{\alpha} < 1/2$ such that for any $\varepsilon > 0$ and any $\delta > 0$ there is an M such that if*

(i) $S/B \in \Gamma$,
(ii) $|B/S - n| > \varepsilon$ *and* $|S/B - n| > \varepsilon$ *for all* $n \in \mathbf{N}$, *and*
(iii) $B + S > M$,

then for every $\alpha \in [\underline{\alpha}, 1 - \underline{\alpha}]$, the model with B buyers and S sellers has an equilibrium with B_1 buyers and S_1 sellers in market 1 for some B_1, S_1 with $|B_1/B - \alpha| < \delta$ and $|S_1/S - \alpha| < \delta$.

The intuition for this result starts from the observation that not only are the quasi-equilibrium constraints satisfied at the allocation $(S/2, S/2, B/2, B/2)$, they are satisfied with enough slack that the $S1$ and $S2$ constraints are also satisfied at $(S/2 - 1/2, S/2 + 1/2, B/2, B/2)$.[8] Hence, when B is even, there is always an integer S_1 for which the $S1$ and $S2$ constraints are both satisfied at $(S_1, S - S_1, B/2, B/2)$.

To show that the $S1$ and $S2$ constraints can be satisfied for some integers S_1, B_1 near any given B_1' that is close to $B/2$ one can argue as follows. Using continuity $(S_1, S - S_1, B_1, B - B_1)$ will satisfy the $S1$ and $S2$ constraints if $S_1 \in [\gamma B_1 - (1/2 - \varepsilon/2), \gamma B_1 + (1/2 - \varepsilon/2)]$ and B_1 is within some distance of $B/2$. Suppose B_1' is within this

[8] At this allocation a seller who moves to market 1 will make it identical to market 2, so it is no longer better for sellers.

distance of $B/2$. Because $\varepsilon > 0$, there may be no S_1 for which $(S_1, S - S_1, B_1', B - B_1')$ satisfies the $S1$ and $S2$ constraints. However, this can only happen if $\gamma B_1'$ is within $\varepsilon/2$ of being one half more than an integer. If this is true, consider a split with $B_1'' = B_1' - 1$ buyers in market 1. There must be an integer split satisfying the $S1$ and $S2$ constraints with $B_1 = B_1''$ unless $\gamma B_1'' = \gamma(B_1' - 1) = \gamma B_1' - \gamma$ is also within $\varepsilon/2$ of being one-half more than an integer. It is impossible for $\gamma B_1'$ and $\gamma B_1' - \gamma$ to both be this close to one-half more than an integer unless γ itself is within ε of being an integer. Hence, we know there is an integer allocation for B_1 very close to B_1' that satisfies the $S1$ and $S2$ constraints.

The argument above only deals with the $S1$ and $S2$ constraints. The full proof is a bit more complicated because it also deals with the $B1$ and $B2$ constraints (which are only satisfied for a narrower range of S_1).

Proof of Theorem 2. We proceed in two main steps. The first step is to show that the quasi-equilibrium set contains a parallelogram in B-S space that includes all points on the line segment defined by $B_1/B = S_1/S$ and $B_1/B \in [\underline{\alpha}, 1-\underline{\alpha}]$ and that is not "too thin." The second step is to show that every point in the parallelogram is within a specified distance of an integer point that is also within the parallelogram. The integer points are true equilibria.

To simplify the exposition we will prove the result only for pairs with $S \leqslant B$. "Sellers" and "buyers" are completely symmetric in the model, so the argument for $B \leqslant S$ would be identical.

Lemma 2. *There exist $\underline{\alpha}$ and $\underline{N}$ with $\underline{\alpha} < 1/2$ such that for all B and S with $B + S > \underline{N}$ and $\gamma \equiv S/B \in \Gamma$, it is a quasi-equilibrium of the model with B buyers and S sellers to have B_1 buyers and S_1 sellers in market 1 for any B_1 and S_1 with and any B_1, S_1 with $B_1/B \in [\underline{\alpha}, 1 - \underline{\alpha}]$ and $|S_1/S - B_1/B| \leqslant 1/4B$.*

Remark. When we graph the space of allocations as in Figs. 1 and 2, with B_1/B on the x-axis and S_1/S, on the y-axis, the lemma says that the quasi-equilibrium set contains the parallelogram bounded by $(\underline{\alpha}, \underline{\alpha} \pm 1/4B)$ and $(1 - \underline{\alpha}, 1 - \underline{\alpha} \pm 1/4B)$.

Proof of Lemma 2. We will show that buyers in market 1 do not wish to switch to market 2, i.e. that the $B1$ constraint is satisfied. The argument for the $B2$ constraint is identical. The arguments for the $S1$ and $S2$ constraints are very similar.

We begin with the "hardest" case for the $B1$ constraint: when $S_1 = (\alpha - 1/4B)S$ and $B_1 = \alpha B$. In this case we need to show that

$$u_b\big((\alpha - 1/4B)S, \alpha B\big) \geqslant u_b\big((1 - \alpha + 1/4)S, (1 - \alpha)B + 1\big).$$

Given Assumption 1, this will be satisfied whenever $B + S$ is greater than some $\underline{N}$ if

$$F_b\left(\gamma\left(1 - \frac{1}{4\alpha B}\right)\right) - G_b\left(\gamma\left(1 - \frac{1}{4\alpha B}\right)\right)\frac{1}{\alpha B}$$
$$> F_b\left(\gamma\left(1 - \frac{3}{4(1-\alpha)B}\right)\right) - G_b\left(\gamma\left(1 - \frac{3}{4(1-\alpha)B}\right)\right)\frac{1}{(1-\alpha)B}$$

whenever B is greater than some $\underline{B}$. (The restriction to $S/B \in [\underline{\gamma}, \overline{\gamma}] \subset (0, \infty)$ implies that B is always large when $B + S$ is large.) There exists $\underline{B}$ so that this is true for all $B > \underline{B}$ if for all $\gamma \in \Gamma$ and all $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ we have

$$\left(-\frac{1}{4\alpha} + \frac{3}{4(1-\alpha)}\right)\gamma F_b'(\gamma) > \left(\frac{1}{\alpha} - \frac{1}{1-\alpha}\right)G_b(\gamma),$$

which is equivalent to

$$\frac{\alpha - 1/4}{1 - 2\alpha} > \frac{G_b(\gamma)}{\gamma F_b'(\gamma)}.$$

We can choose $\underline{\alpha} \in (1/4, 1/2)$ so that this is true uniformly over γ. Note that for the same $\underline{\alpha}$ the equation we would have derived had we started with any larger S_1 will also be satisfied due to the monotonicity of F_b. Hence, having chosen an $\underline{\alpha}$ satisfying this equation we can then choose $\underline{B}$ so that $B1$ is satisfied for all B_1 and S_1 in the parallelogram provided that $B > \underline{B}$, which completes the proof of Lemma 2. □

Given the result of Lemma 2 it suffices to prove the following lemma about parallelograms containing grid points. Lemma 3 concludes that there exist integers B_1 and S_1 satisfying several properties. The first two of these are that $(B_1/B, S_1/S)$ belongs to the parallelogram described in Lemma 2, which implies that it is a true equilibrium to have B_1 buyers and S_1 sellers in market 1. The last two indicate we can find such equilibria close to every point of the plateau identified in the theorem.

Lemma 3. *Assume* $\underline{\alpha} < 1/2$. *Then there are constants* k_1, k_2, k_3, k_4 *such that for any* $\varepsilon > 0$, *any* $\alpha \in [\underline{\alpha}, 1-\underline{\alpha}]$ *and any* S, B *satisfying*

(i) $S \leqslant B$,
(ii) $S/B \in \Gamma$,
(iii) $|B/S - n| > \varepsilon$ *for all* $n \in N$, *and*
(iv) $S > k_1 + k_2/\varepsilon$,

there exist integers S_1 *and* B_1 *such that* $B_1/B \in [\underline{\alpha}, 1-\underline{\alpha}]$, $|S_1/S - B_1/B| < 1/4B$, $|B_1/B - \alpha| < (k_3 + k_4/\varepsilon)(1/B)$, *and* $|S_1/S - \alpha| < (k_3 + k_4/\varepsilon)(1/S)$.

Proof of Lemma 3. Fix some $\alpha \in [\underline{\alpha}, 1-\underline{\alpha}]$. We assume here that $\alpha > 1/2$. (A symmetric argument applies for $\alpha < \frac{1}{2}$.)

Let B_1^0 be the integer closest to αB in the set of integers B' such that $B'/B \in (\underline{\alpha}, 1-\underline{\alpha})$. (This set will be non-empty for k_1 sufficiently large given the assumption that $S/B \in \Gamma$.)

We now define a sequence of integer pairs (S_1^k, B_1^k) by repeatedly reducing S_1 by one and B_1 by an integer close to B/S. We will choose (S_1, B_1) to be the first point in the sequence that satisfies $|S_1^k/S - B_1^k/B| < 1/4B$. The key observation is that this always occurs for some k with $k \leqslant 4 + 1/\varepsilon$.

Let $S_1^1 = \lfloor (B_1^0/B)S \rfloor$. Note that $S_1^1/S \leqslant B_1^0/B$. Let B_1^1 be the largest integer with $B_1^1/B \leqslant S_1^1/S + 1/4B$.

If $S_1^1/S < B_1^1/B + 1/4B$, then choose we will choose $S_1 = S_1^1$ and $B_1 = B_1^1$. Note in this case that S_1 and B_1 satisfy $|S_1/S - B_1/B| < 1/4B$.

Otherwise, if for some $k \geqslant 1$ we have $S_1^k/S \geqslant B_1^k/B + 1/4B$, then define S_1^{k+1} and B_1^{k+1} by $S_1^{k+1} = S_1^k - 1$, and

$$B_1^{k+1} = \begin{cases} B_1^k - \lfloor B/S \rfloor & \text{if } B/S - \lfloor B/S \rfloor \leqslant S_1^k/S - B_1^k/B + 1/4B \\ B_1^k - \lfloor B/S \rfloor - 1 & \text{if } B/S - \lfloor B/S \rfloor > S_1^k/S - B_1^k/B + 1/4B \end{cases}.$$

We then choose $(S_1, B_1) = (S_1^k, B_1^k)$ where k is the smallest index for which $|S_1^k/S - B_1^k/B| \leqslant 1/4B$.

Claim: This choice of (S_1, B_1) is well-defined and the index k satisfies $k \leqslant 3 + 1/\varepsilon$.

To show this, we write x for $B/S - \lfloor B/S \rfloor$ and define $a^k \equiv B(S_1^k/S - B_1^k/B - 1/4B)$. Note that (S_1^k, B_1^k) are the desired (S_1, B_1) if $a^k \in (-1/2, 0]$.

The definition of B_1^1 as the largest integer satisfying $B_1^1/B \leqslant S_1^1/S + 1/4B$ implies that $(B_1^1 + 1)/B > S_1^1/S + 1/4B$, $a^1 = B(S_1^1/S - B_1^1/B - 1/4B) \in [-1/2, 1/2)$.

If (S_1^1, B_1^1) is not the desired pair of integers, then $a^1 \in (0, 1/2)$. For $a^k \in (0, 1/2)$ the expressions for S_1^{k+1}, B_1^{k+1} can be rewritten as

$$S_1^{k+1} = S_1^k - 1,$$
$$B_1^{k+1} = \begin{cases} B_1^k - \lfloor B/S \rfloor & \text{if } x \leqslant a^k + 1/2 \\ B_1^k - \lfloor B/S \rfloor - 1 & \text{if } x > a^k + 1/2 \end{cases}.$$

Hence,

$$a^{k+1} = \begin{cases} B\left(\frac{S_1^{k+1}}{S} - \frac{B_1^{k+1}}{B} - \frac{1}{4B}\right) = B\left(\frac{S_1^k}{S} - \frac{1}{S} - \frac{B_1^k}{B} - \lfloor B/S \rfloor - \frac{1}{4B}\right) = a^k - x & \text{if } x \leqslant a^k + 1/2 \\ a^k + (1-x) & \text{if } x > a^k + 1/2 \end{cases}.$$

We complete the proof of our claim by considering two cases:

Case 1: $x \leqslant 1/2$.

In this case the dynamics are $a^{k+1} = a^k - x$ whenever $a^k > 0$. Hence we have $a^k \in [-1/2, 0)$ for $k = 1 + \lceil \frac{a^1 - x}{x} \rceil < 2 + 1/x \leqslant 1 + 2/\varepsilon$.

Case 2: $x > 1/2$.

If $a^1 \in [-1/2, 0]$, then (S_1^1, B_1^1) has the desired property. If $a^1 \in [x - 1/2, 1/2)$, then $a^2 = a^1 - x \in [-1/2, 0]$, so (S_1^2, B_1^2) has the desired property. Finally, if $a^1 \in (0, x - 1/2)$, then $a^2 = a^1 + 1 - x$, and $a^{j+1} = a^j + 1 - x$ for all j with $a^j \in (0, x - 1/2)$. Hence, $a^{1 + \lceil (x - 1/2 - a^1)/(1-x) \rceil} \in (x - 1/2, 1/2)$. Hence (S^k, B^k) has the desired property for $k = 2 + \lceil (x - 1/2 - a^1)/(1-x) \rceil \leqslant 2 + \lceil 1/2(1-x) \rceil \leqslant 3 + 1/2\varepsilon$.

The statement of the Lemma requires that (S_1, B_1) have four properties (which hold uniformly given some choice of k_1, k_2, k_3, and k_4).

The first of these was that $B_1/B \in [\underline{\alpha}, 1 - \underline{\alpha}]$. It is immediate from out construction that $B_1 \in [B_1^0 - k\lceil B/S \rceil, B_1^0$ for $k \leqslant 3 + 1/\varepsilon$. The fact that $B_1^0/B \leqslant 1 - \underline{\alpha}$ implies $B_1/B \leqslant 1 - \underline{\alpha}$. Given that $\alpha \geqslant 1/2$ we have $B_1/B \geqslant 1/2 - 1/B - (3 + 1/\varepsilon)(1/S + 1/B)$. This is greater than $\underline{\alpha}$ when $B > k_1 + k_2/\varepsilon$ given an appropriate choice of k_1 and k_2.

The second of these was that $|S_1/S - B_1/B| < 1/4B$. We chose (S_1, B_1) so that this holds.

The third is that $|B_1/B - \alpha B| < (k_3 + k_4/\varepsilon)(1/B)$. That we can choose k_3 and k_4 so that this is true again follows easily from $B_1 \in [B_1^0 - k\lceil B/S \rceil, B_1^0$ for $k \leqslant 3 + 1/\varepsilon$. The upper bound gives $B_1/B \leqslant B_1^0/B \leqslant \lceil \alpha B/B \rceil < \alpha + (1/B)$. The lower bound gives $B_1/B < \alpha - 1/B - (3 + 1/\varepsilon)(\lceil B/S \rceil)$. Using $\lceil B/S \rceil < 2(B/S)$, this is greater than $\underline{\alpha} - (k_3 + k_4/\varepsilon)(1/B)$ for $k_3 = 7$ and $k_4 = 2$.

The argument for the fourth condition is similar, but simpler because $S_1^k = S_1^1 - (k-1)$.

This finishes the proof of Lemma 3, and thus proves the theorem. □

Fig. 2 provides some intuition for Lemma 3. It shows a magnified view of the parallelogram $\{(x, y)\colon x \in [\underline{\alpha}, 1 - \underline{\alpha}], |y - x| < 1/4B\}$ passing through the lattice $\{(x, y)\colon \exists B_1, S_1 \text{ with } x = B_1/B, y = S_1/S\}$. The parallelogram is half as wide as the gap between adjacent x's, so for a given value of y the parallelogram need not intersect any grid points, e.g. in the top row of the grid the parallelogram passes between $(B_1^1/B, S_1^1/S)$ and $((B_1^1 + 1)/B, S_1^1/S)$. Because B/S is not an integer, the parallelogram is shifted relative to the grid at different y values. The figure has $B/S = 1.3$, so the intersection shifts to the left by 1.3 grid points when y is reduced to $(S_1^1 - 1)/S$. This implies that the parallelogram will intersect the grid at least once in every three adjacent rows. When B/S is ε away from an integer, the number of consecutive rows that can have no intersection is larger, but still bounded above by $1 + 1/2\varepsilon$.

Our theorem shows that we get a plateau of true equilibria whenever the seller–buyer ratio (and the buyer–seller ratio) is not too close to an integer. The example of the previous section was one where S_n/B_n was becoming closer and closer to one as n increased. It is closeness to an integer, not being exactly equal to an integer, that can lead to nonexistence. If the buyer–seller ratio is exactly equal to an integer, then an analogue to Theorem 2 would hold, and

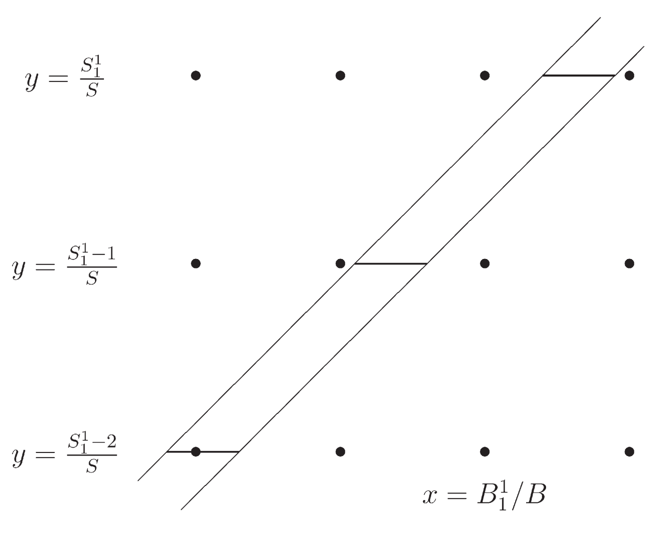

Fig. 2. Illustration of Lemma 3.

in fact we would obtain an even stronger result: there would be an exact equilibrium for every S_1 in the specified range.[9] We should also note that having the buyer–seller ratio close to an integer does not always lead to nonexistence of split-market equilibria. For example, there will be a plateau of equilibria even when $S_n/B_n \to 1$ provided that $S_n + B_n$ is even for all n, since that permits us to find an integer pair in the parallelogram of quasi-equilibria.

5. The density of the set of equilibria

In this section we provide a characterization of the equilibrium set for large finite economies. Roughly, we show that in most large economies the equilibrium set fills out a wide plateau and we specify the density of equilibria within the plateau. Moreover, there exists an $\alpha^{**} < \frac{1}{2}$ depending on γ such that the density is as large as possible when the fraction of agents in the smaller market is at least α^{**}. If there are fewer sellers than buyers, then essentially all integers S_1 in $(S\alpha^{**}, S(1-\alpha^{**}))$ there is an equilibrium with S_1 sellers in market 1. If there are fewer buyers than sellers, then for essentially all integers B_1 in $(B\alpha^{**}(\gamma), B(1-\alpha^{**}(\gamma))$, there is an equilibrium with B_1 sellers in market 1. More precisely, our theorem provides two results of this type. The first shows that given any sequence of economies $\{B_n, S_n\}$ with S_n/B_n converging to an irrational limit, the width of the equilibrium plateau and the density of equilibria within it converge to the functions we identify as $n \to \infty$. The second shows that if a large economy is selected at random by choosing the buyer–seller ratio to approximate a random draw from a nonatomic distribution then the probability that the equilibrium set will be within any ε of the limiting characterization converges to one as the number of buyers increases.

Although the proof uses nonstandard analysis, the statement of Theorem 3 is completely standard and can be understood without any knowledge of nonstandard analysis. The statements of the lemmas do involve nonstandard analysis. Before each of the nonstandard statements, we give the intuition behind the statement, to assist readers unfamiliar with nonstandard analysis. Similarly, before each of the proofs, we give an outline of the intuition behind the nonstandard proof.

In the following definition, $\mathcal{N}(B, S)$ is essentially the set of B_1 such that in the model with B buyers and S sellers, there is an equilibrium with B_1 buyers in market 1.[10]

Definition 1. Let $\mathcal{N}(B, S)$ denote the set of all $B_1 \in \{0, 1, \ldots, B\}$ such that there exists $S_1 \in \{0, 1, \ldots, S\}$ such that

1. the market with B buyers and S sellers has a Nash equilibrium with B_1 buyers and S_1 sellers in market 1; and
2. if we define $S_2 = S - S_1$ and $B_2 = B - B_1$, then

$$\left\{\frac{S_1}{B_1}, \frac{S_2}{B_2}, \frac{S_1+1}{B_1}, \frac{S_2+1}{B_2}, \frac{S_1}{B_1+1}, \frac{S_2}{B_2+1}\right\} \subset \Gamma.$$

The following definition specifies, for each γ, a piecewise linear function of α which is strictly positive on an open interval containing $\alpha = \frac{1}{2}$. We shall see that the density of the set $\mathcal{N}(B, S)$ converges to this piecewise linear function.

Definition 2. Let

$$T(\gamma) = \frac{G_s(\gamma)}{F_s'(\gamma)} - \frac{G_b(\gamma)}{F_b'(\gamma)},$$

$$\alpha^*(\gamma) = \max\left\{0, \frac{1}{2} - \frac{\gamma+1}{2|1+\gamma-2T(\gamma)|}\right\},$$

$$\alpha^{**}(\gamma) = \max\left\{0, \frac{1}{2} - \frac{|\gamma-1|}{2|1+\gamma-2T(\gamma)|}\right\},$$

[9] This is true if $S \leqslant B$. When $B \leqslant S$ there would be an exact equilibrium for every B_1 in the range.

[10] $\mathcal{N}(B, S)$ is actually a slightly smaller set than the one just described, because of condition 2. If $\frac{S_1}{B_1} \notin \Gamma$, or $\frac{S_2}{B_2} \notin \Gamma$, or a defection by one player could take the ratio of the sellers to buyers in that player's new market outside of Γ, then Assumption 1 gives us no control over payoffs, and hence makes it impossible to prove that a given pair (B_1, S_1) is not an equilibrium. Because the set we identify is no larger than the set of equilibria, the density of the set of all equilibria, including equilibria with ratios outside Γ, will be at least as great as indicated in Theorem 3.

$$H(\alpha,\gamma)=\begin{cases}0 & \text{if } \alpha\in[0,\alpha^*(\gamma)]\\ \frac{(\alpha-\alpha^*(\gamma))\min\{1,\gamma\}}{\alpha^{**}(\gamma)-\alpha^*(\gamma)} & \text{if } \alpha\in(\alpha^*(\gamma),\alpha^{**}(\gamma))\\ \min\{1,\gamma\} & \text{if } \alpha\in[\alpha^{**}(\gamma),\ 1-\alpha^{**}(\gamma)]\\ \frac{(1-\alpha-\alpha^*(\gamma))\min\{1,\gamma\}}{\alpha^{**}(\gamma)-\alpha^*(\gamma)} & \text{if } \alpha\in(1-\alpha^{**}(\gamma),1-\alpha^*(\gamma))\\ 0 & \text{if } \alpha\in[1-\alpha^{**}(\gamma),1]\end{cases}. \tag{5}$$

Our theorem characterizes the local density of equilibrium throughout the equilibrium plateau. To simplify the statement of the theorem we first define this object.

Definition 3. Given an economy $\{B,S\}$ and $\varepsilon>0$ with $B\varepsilon>1$, the ε-local density of equilibria at α in the economy is

$$H_\varepsilon(B,S,\alpha)\equiv \text{Prob}\big\{B_1\in\mathcal{N}(B,S):\ B_1\in\mathbf{N},\ |B_1/B-\alpha|<\varepsilon\big\},$$

where the probability is with respect to a uniform distribution over all integers B_1 satisfying the constraint $|B_1/B-\alpha|<\varepsilon$.

Theorem 3. *Suppose Assumption* 1 *holds,* $B_n\in\mathbf{N}$*,* $B_n\to\infty$*,* $\varepsilon_n>0$*,* $\varepsilon_n\to 0$*, and* $\varepsilon_n B_n\to\infty$*.*

1. *If* S_n *is a sequence of positive integers with* $\frac{S_n}{B_n}\to\gamma\in(\underline{\gamma},\bar{\gamma})\setminus\mathbf{Q}$*, then*

$$\sup_{\alpha\in[0,1]}\big|H_{\varepsilon_n}(B_n,S_n,\alpha)-H(\alpha,\gamma)\big|\to 0\quad \textit{as } n\to\infty.$$

2. *Suppose that* γ_n *is a sequence of independent identically distributed random variables whose common distribution is absolutely continuous with respect to Lebesgue measure and has support contained in* Γ *and* $S_n=\lfloor\gamma_n B_n\rfloor$*. Then,*

$$\sup_{\alpha\in[0,1]}\big|H_{\varepsilon_n}(B_n,S_n,\alpha)-H(\alpha,\gamma_n)\big|\to 0\quad \textit{in probability as } n\to\infty$$

i.e. for every $\delta>0$*, there exists* $N\in\mathbf{N}$ *such that for every* $n>N$*,*

$$P\Big(\sup_{\alpha\in[0,1]}\big|H_{\varepsilon_n}(B_n,S_n,\alpha)-H(\alpha,\gamma_n)\big|>\delta\Big)<\delta.$$

Fig. 3 contains a graph of the function $H(\alpha,\gamma)$ for the application considered in Fig. 1: the Ellison et al. (2004) competing auction model with $S/B=\frac{1}{2}$. In this case, we find $\alpha^*(\gamma)=1/8$, $\alpha^{**}(\gamma)=3/8$, and $\text{Min}(1,\gamma)=1/2$. Hence, when the number of buyers and sellers is large the equilibrium set includes splits with B_1/B ranging from

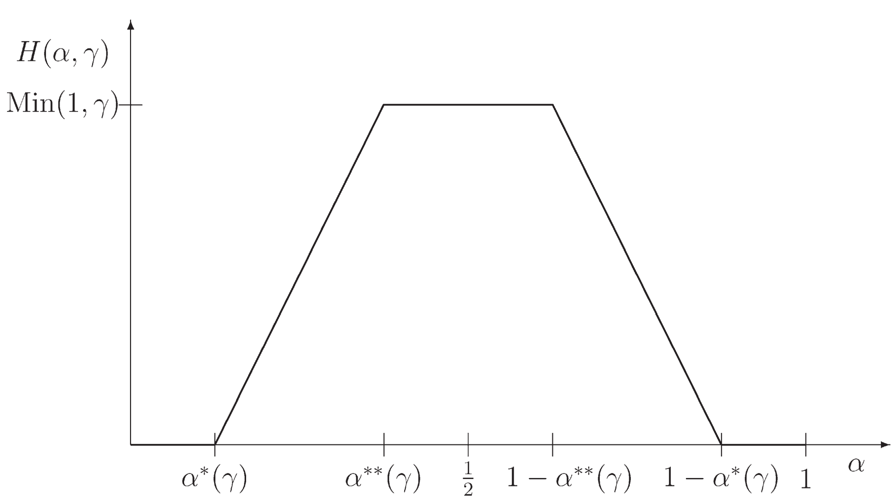

Fig. 3. The density-of-equilibria function $H(\alpha,\gamma)$.

about 1/8 to 7/8. The equilibrium set reaches a maximum density of 1/2 for values of α between 3/8 and 5/8. This means that there are true equilibria for about half of the B_1 in this range. There are only half as many sellers as buyers, so this means that there is an equilibrium for almost all S_1 between $\frac{3}{8}S$ and $\frac{5}{8}S$.

Before proving the theorem we first need some notation and two lemmas. We will suppose throughout that $B_1, B_2, S_1, S_2 \in \mathbf{N}$, $B_1 + B_2 = B$, $S_1 + S_2 = S$, $\gamma_1 = S_1/B_1$, $\gamma_2 = S_2/B_2$, $\alpha_1 = B_1/B$. The proofs of the lemmas and theorem make use of nonstandard analysis. Let ${}^*\mathbf{N}$ be the set of nonstandard integers, so that every $n \in {}^*\mathbf{N} \setminus \mathbf{N}$ is an infinite nonstandard natural number. Given any finite nonstandard number α, let ${}^\circ\alpha$ denote the standard part of α, the unique standard real number infinitely close to α. We write $x \simeq y$ to mean that $x - y$ is infinitesimal.

Our first lemma, Lemma 4, puts an upper bound on the equilibrium set: roughly it says that all equilibria must have $S_1/B_1 \approx S/B$ if B is large. More formally, it is equivalent to the following standard statement: Suppose we consider a sequence of models with S_n sellers and B_n buyers, with $S_n \to \infty$ and $B_n \to \infty$. Suppose $S_{n1}, B_{n1} \in \mathbf{N}$, $S_{n1} \leqslant S_n$, $B_{n1} \leqslant B_n$. Suppose there exists $\varepsilon > 0$ such that $S_n/B_n \in [\underline{\gamma} + \varepsilon, \bar{\gamma} - \varepsilon]$ and $B_{n1}/B_n \in [\varepsilon, 1 - \varepsilon]$ for all n. Write γ_n for S_n/B_n and γ_{n1} for S_{n1}/B_{n1}. Then, if $B_n|\gamma_{n1} - \gamma_n| \to \infty$, there exists N such that for all $n > N$, the model with S_n sellers and B_n buyers has no equilibrium with B_{n1} buyers and S_{n1} sellers in market 1.

For the proof, consider a seller located in the market with a higher ratio of sellers to buyers, which we may assume without loss of generality to be market 1. Since $F_s'(\gamma) < 0$ for all $\gamma \in \Gamma$ and Γ is compact, $F_s'(\gamma)$ is uniformly bounded away from zero. By the Mean Value Theorem, the benefit to the seller in switching to market 2 contains a term which is bounded below by $(\min_{\gamma \in \Gamma} |F_s'(\gamma)|)(\gamma_{n2} - \gamma_{n1}) > 0$; this term dominates the other terms because $B_n|\gamma_{n1} - \gamma_n| \to \infty$, and it follows that the seller would benefit from switching markets.

Lemma 4. *Suppose $S, B \in {}^*\mathbf{N} \setminus \mathbf{N}$, ${}^\circ\gamma \in (\underline{\gamma}, \bar{\gamma})$, and ${}^\circ\alpha_1 \in (0, 1)$, Then if $\gamma_1 \neq \gamma + O(\frac{1}{B_1})$, there is no equilibrium with B_1 buyers and S_1 sellers in market 1.*

Proof of Lemma 4. We may assume without loss of generality that $\gamma_1 > \gamma_2$. Consider a seller contemplating switching from market 1 to market 2. The change in the seller's payoff resulting from the switch is

$$\begin{aligned} &F_s\left(\gamma_2 + \frac{1}{B_2}\right) - \frac{G_s(\gamma_2 + \frac{1}{B_2})}{B_2} - F_s(\gamma_1) + \frac{G_s(\gamma_1)}{B_1} + o\left(\frac{1}{B_1}\right) \\ &\quad \geqslant \max\left\{F_s'(\gamma') \colon \gamma' \in \Gamma\right\}\left(\gamma_2 - \gamma_1 + \frac{1}{B_2}\right) + O\left(\frac{1}{B_1}\right) \\ &\quad > 0 \end{aligned}$$

provided ${}^\circ\gamma_1 \in (\underline{\gamma}, \bar{\gamma})$ and ${}^\circ\gamma_2 \in (\underline{\gamma}, \bar{\gamma})$, which shows there is no equilibrium with B_1 buyers and S_1 sellers in market 1. □

The previous lemma formalizes the observation that the "diamond" of quasi-equilibria becomes flatter as the economy grows so that splits of agents between the markets with $\gamma_{n1} \not\approx \gamma_n$ will not give equilibria. The next step is to characterize which splits with $\gamma_{n1} \approx \gamma_n$ are indeed compatible with the incentive constraints. Ellison and Fudenberg (2003) gave a sufficient condition for incentive compatibility that assumed $\gamma_1 = \gamma_2 = \gamma$. The next lemma improves on it by considering all γ_1 that are consistent with the previous lemma.

The following nonstandard lemma is equivalent to the following, rather complicated, standard statement: Suppose we consider a sequence of models with S_n sellers and B_n buyers, with $S_n \to \infty$ and $B_n \to \infty$. Suppose $S_{n1}, B_{n1} \in \mathbf{N}$, $S_{n1} \leqslant S_n$, $B_{n1} \leqslant B_n$. Write γ_n for S_n/B_n, γ_{ni} for S_{ni}/B_{n1}, and $\hat{\alpha}_{ni}$ for B_{ni}/B_n. Suppose there exists $\varepsilon > 0$ such that $\gamma_n \in [\underline{\gamma} + \varepsilon, \bar{\gamma} - \varepsilon]$ and $\hat{\alpha}_{n1} \in [\varepsilon, 1 - \varepsilon]$ for all n. Part 1 says that if the sequence $\{B_n|\gamma_{n1} - \gamma_n|\}$ is uniformly bounded and $\delta > 0$, then for every subsequence n_m such that S_{n_m}/B_{n_m} converges to a limit γ, there exists N such that whenever

$$(S_{n_m 1} - \gamma B_{n_m 1}) - \frac{G_s(\gamma)(1 - 2\hat{\alpha}_{n_m 1})}{F_s'(\gamma)} \in (\hat{\alpha}_{n_m 1} - 1 + \delta, \hat{\alpha}_{n_m 1} - \delta)$$

and

$$(S_{n_m 1} - \gamma B_{n_m 1}) - \frac{G_b(\gamma)(1 - 2\hat{\alpha}_{n_m 1})}{F_b'(\gamma)} \in \bigl(-(\hat{\alpha}_{n_m 1})\gamma + \delta, (1 - \hat{\alpha}_{n_m 1})\gamma - \delta\bigr)$$

and $n_m > N$, then there is an equilibrium with $B_{n_m 1}$ buyers and $S_{n_m 1}$ sellers in market 1. Part 2 says suppose that for all n, $\gamma_{n1} \in [\underline{\gamma} + \varepsilon, \bar{\gamma} - \varepsilon]$ and $\gamma_{n2} \in [\underline{\gamma} + \varepsilon, \bar{\gamma} - \varepsilon]$. Then for every $\delta > 0$ and every subsequence n_m such that S_{n_m}/B_{n_m} converges to a limit γ, there exists N such that whenever

$$(S_{n_m 1} - \gamma B_{n_m 1}) - \frac{G_s(\gamma)(1 - 2\hat{\alpha}_{n_m 1})}{F_s'(\gamma)} \notin (\hat{\alpha}_{n_m 1} - 1 - \delta, \hat{\alpha}_{n_m 1} + \delta)$$

or

$$(S_{n_m 1} - \gamma B_{n_m 1}) - \frac{G_b(\gamma)(1 - 2\hat{\alpha}_{n_m 1})}{F_b'(\gamma)} \notin \big(-(\hat{\alpha}_{n_m 1})\gamma - \delta, (1 - \hat{\alpha}_{n_m 1})\gamma + \delta\big)$$

and $n_m > N$, then there is no equilibrium with $B_{n_m 1}$ buyers and $S_{n_m 1}$ sellers in market 1. The nonstandard proof of Lemma 5 is a straightforward application of Lemma 1. The complication in the standard statement comes from the fact that N depends on both δ and the subsequence n_m, and a standard proof would have to use the convergence of $\gamma_n = S_n/B_n$ to γ to change γ_n to γ in the statement.

Lemma 5. *Suppose* $S, B \in {}^*\mathbf{N} \setminus \mathbf{N}$, ${}^\circ\gamma \in (\underline{\gamma}, \bar{\gamma})$, *and* ${}^\circ\alpha_1 \in (0, 1)$. *Then*

1. *If* $\gamma_1 = \gamma + O(\frac{1}{B_1})$,
$$ {}^\circ(S_1 - \gamma B_1) - \frac{G_s({}^\circ\gamma)(1 - 2{}^\circ\alpha_1)}{F_s'({}^\circ\gamma)} \in \big({}^\circ\alpha_1 - 1, {}^\circ\alpha_1\big) \tag{6}$$
and
$$ {}^\circ(S_1 - \gamma B_1) - \frac{G_b({}^\circ\gamma)(1 - 2{}^\circ\alpha_1)}{F_b'({}^\circ\gamma)} \in \big(-{}^\circ\alpha_1 {}^\circ\gamma, \big(1 - {}^\circ\alpha_1\big){}^\circ\gamma\big) \tag{7}$$
then there is an equilibrium with B_1 *buyers and* S_1 *sellers in market 1.*
2. *Suppose*
$$ {}^\circ\left(\frac{S_1}{B_1}\right) \in (\underline{\gamma}, \bar{\gamma}) \quad \textit{and} \quad {}^\circ\left(\frac{S_2}{B_2}\right) \in (\underline{\gamma}, \bar{\gamma}).$$
If
$$ {}^\circ(S_1 - \gamma B_1) - \frac{G_s({}^\circ\gamma)(1 - 2{}^\circ\alpha_1)}{F_s'({}^\circ\gamma)} \notin \big[{}^\circ\alpha_1 - 1, {}^\circ\alpha_1\big]$$
or
$$ {}^\circ(S_1 - \gamma B_1) - \frac{G_b({}^\circ\gamma)(1 - 2{}^\circ\alpha_1)}{F_b'({}^\circ\gamma)} \notin \big[-{}^\circ\alpha_1 {}^\circ\gamma, \big(1 - {}^\circ\alpha_1\big){}^\circ\gamma\big]$$
then there is no equilibrium with B_1 *buyers and* S_1 *sellers in market 1.*

Proof of Lemma 5. For part 1, we may find $\varepsilon \in \mathbf{R}$, $\varepsilon > 0$ such that

$$ {}^\circ\gamma \in (\underline{\gamma} + \varepsilon, \bar{\gamma} - \varepsilon),$$
$$ {}^\circ(S_1 - \gamma B_1) - \frac{G_s({}^\circ\gamma)(1 - 2{}^\circ\alpha_1)}{F_s'({}^\circ\gamma)} \in \big({}^\circ\alpha_1 - 1 + \varepsilon, {}^\circ\alpha_1 - \varepsilon\big),$$
$$ {}^\circ(S_1 - \gamma B_1) - \frac{G_b({}^\circ\gamma)(1 - 2{}^\circ\alpha_1)}{F_b'({}^\circ\gamma)} \in \big(-{}^\circ\alpha_1 {}^\circ\gamma + \varepsilon, \big(1 - {}^\circ\alpha_1\big){}^\circ\gamma - \varepsilon\big).$$

Find $\underline{B} \in \mathbf{N}$ satisfying the conclusion of Lemma 1. Since G_s, F_s', G_b, F_b' are continuous,

$$\gamma \in {}^*(\underline{\gamma} + \varepsilon, \bar{\gamma} - \varepsilon) = {}^*\Gamma_\varepsilon,$$
$$(S_1 - \gamma B_1) - \frac{G_s(\gamma)(1 - 2\alpha_1)}{F_s'(\gamma)} \in {}^*(\alpha_1 - 1 + \varepsilon, \alpha_1 - \varepsilon),$$
$$(S_1 - \gamma B_1) - \frac{G_b(\gamma)(1 - 2\alpha_1)}{F_b'(\gamma)} \in {}^*\big(-\alpha_1\gamma + \varepsilon, (1 - \alpha_1)\gamma - \varepsilon\big).$$

Since ${}^\circ\alpha_1 \in (0,1)$, $B_1, B_2 \in {}^*\mathbf{N}\setminus\mathbf{N}$, so $B_1, B_2 > \underline{B}$. Thus, we have verified the transfer of the hypotheses of Lemma 1. By the Transfer Principle, the transfer of Lemma 1 holds, so the conclusion of the transfer of Lemma 1 holds, but this just says there is a quasi-equilibrium with B_1 buyers and S_1 sellers in market 1. Since $B_1, B_2, S_1, S_2 \in {}^*\mathbf{N}$, this quasi-equilibrium is an equilibrium. The proof of part 2 is analogous. □

Remark. The previous lemma characterizes the allocations of agents to each market that are consistent with the incentive constraints. Since the result uses the approximation of the utility functions provided by Assumption 1, it is interesting to note that it gives exactly the range of solutions that Ellison et al. (2004) obtained for the exact utility functions that arise with uniformly distributed buyer values in their auction model.

The statement of Theorem 3 is standard, but the proof is not; a standard proof would necessarily involve very complex ε, δ arguments. Here, we outline the main ideas. Lemma 4 shows that, in the limit, an equilibrium cannot exist unless $\gamma_{n1} = S_{n1}/B_{n1}$ is within $O(1/B_{n1})$ of $\gamma_n = S_n/B_n$. Lemma 5 shows that if $\gamma_{n1} = S_{n1}/B_{n1}$ is within $O(1/B_{n1})$, then in the limit whether or not an equilibrium exists is determined by the relationship between $S_{n1} - \gamma_n B_{n1}$ and $\alpha_{n1} = B_{n1}/B_n$. Recall that in characterizing the quasi-equilibrium set we defined

$$T(\gamma) = \frac{G_s(\gamma)}{F_s'(\gamma)} - \frac{G_b(\gamma)}{F_b'(\gamma)}.$$

The two conditions for existence of an equilibrium in Lemma 5 together imply that for large n $(S_{n1}, S_{n2}, B_{n1}, B_{n2})$ will be an equilibrium if $S_{n1} - \gamma_n B_{n1}$ lies in an interval very close to the interval

$$\big[(\alpha_{n1} - 1, \alpha_{n1}) + T(\gamma_n)(1 - 2\alpha_{n1})\big] \cap \big(-\alpha_{n1}\gamma_n, (1 - \alpha_{n1})\gamma_n\big). \tag{8}$$

A little algebra shows that the intersection of the two intervals on the RHS of Eq. (8) has length $H(\alpha_{n1}, \gamma_n)$.

There is an equilibrium for a given B_{n1} if and only if there is an integer S_{n1} for which $S_{n1} - \gamma_n B_{n1}$ falls into this interval. The ε_n equilibrium density at α is the fraction of B_{n1}'s with $|B_{n1}/B_n - \alpha| < \varepsilon$ for which such an S_{n1} exists.

Let $r : \mathbf{R} \to [0,1]$ denote the remainder function $r(x) = x - \lfloor x \rfloor$. Let $K_n = \lfloor \varepsilon_n B_n \rfloor$, $\mathbf{K}_n = \{-K_n, -K_n + 1, \ldots, K_n\}$. Given $k \in \mathbf{K}_n$, let $B_{nk} = \lfloor \alpha B_n \rfloor + k$ and $\alpha_{nk} = B_{nk}/B_n$. These are the possible choices of B_{n1} with B_{n1}/B close to α. Since the interval $[\alpha_{nk} - 1, \alpha_{nk})$ has length 1, for each k there is exactly one integer S_{nk} such that

$$S_{nk} - \gamma_n B_{nk} - \frac{G_s(\gamma_n)(1 - 2\alpha_{nk})}{F_s'(\gamma_n)} \in [\alpha_{nk} - 1, \alpha_{nk}). \tag{9}$$

Equilibrium will exist for a given $k \in \mathbf{K}_n$ if and only if, apart from a small δ, Eq. (8) is satisfied for B_{nk} and the unique S_{nk} that satisfies Eq. (9). Some algebra reduces this to a condition on $r(\gamma_n B_{nk})$. Since $B_{n(k+1)} = B_{nk} + 1$, $r(\gamma_n B_{n(k+1)k}) = r(\gamma_n B_{nk}) + \gamma_n$ (modulo 1). By assumption in part 1, $\gamma_n \to \gamma$ for some irrational $\gamma \in (\underline{\gamma}, \bar{\gamma})$. Since γ is irrational, it is well known, and not difficult to show, that the proportion of $k \in \mathbf{K}_n$ such that $r(\gamma B_{nk})$ lies in any given interval converges to the length of that interval; see for example Exercise VII.8.9 in Dunford and Schwartz (1957).[11] An ε, δ argument on the rate of convergence in the Ergodic Theorem and the rate at which irrational numbers can be approximated by rational numbers of a given denominator would allow one to show that the same conclusion holds if we replace γ with γ_n, which proves part 1.

The result in part 2 can be derived from the result in part 1 by a standard argument. To see this, let $Y_n(x) = \sup_\alpha |H_{\varepsilon_n}(B_n, \lfloor x B_n \rfloor, \alpha) - H(\alpha, x)|$ for $x \in \Gamma$. The conclusion of part 1 says that $\lim Y_n(x) = 0$ whenever x is irrational, so $Y_n(x) \to 0$ almost everywhere, which implies that $Y_n(x)$ converges to zero in measure because Γ is compact. Since the common distribution of the random variables γ_n is absolutely continuous with respect to Lebesgue measure, $\sup_\alpha |H_{\varepsilon_n}(B_n, \lfloor \gamma_n B_n \rfloor, \alpha) - H(\alpha, \gamma_n)|$ converges to zero in probability.

Proof of Theorem 3. Consider three sequences S_n, B_n and ε_n satisfying the hypotheses of the theorem. Fix any $n \in {}^*\mathbf{N}\setminus\mathbf{N}$. Let $S = S_n$, $B = B_n$, $\hat{\gamma} = \frac{S_n}{B_n}$, $\varepsilon = \varepsilon_n$, $K = \lfloor \varepsilon_n B_n \rfloor$ and $\mathbf{K} = \{-K, -K+1, \ldots, -1, 0, 1, \ldots, K-1, K\}$.

[11] This is a very special case of the Ergodic Theorem. When γ is irrational, translation by γ, modulo 1, is an ergodic, measure-preserving transformation from $[0,1)$ to $[0,1)$, i.e. if a measurable set V satisfies $V + \gamma = V$ (modulo 1), then the measure of V is either zero or one, and for every measurable set $W \subset [0,1)$, $W - \gamma$ (modulo 1) has the same measure as W. Note that if γ is rational, then $\gamma = a/b$ where a and b are integers with no common factor. Then translation by γ (modulo 1) is not ergodic because if $W = [0, 1/2b) \cup [1/b, 3/2b) \cup \cdots \cup [1 - 1/b, 1 - 1/2b)$, then $W + \gamma = W$ (modulo 1); since the measure of W is 1/2, translation by γ (modulo 1) is not ergodic.

Notice that $B \in {}^*\mathbf{N} \setminus \mathbf{N}$, so $S \geqslant \underline{\gamma} B \in {}^*\mathbf{N} \setminus \mathbf{N}$. Notice also that in part 1, ${}^\circ\hat{\gamma} = \gamma \in (\underline{\gamma}, \bar{\gamma}) \setminus \mathbf{Q}$; in part 2, notice that ${}^\circ\hat{\gamma} \in (\underline{\gamma}, \bar{\gamma}) \setminus \mathbf{Q}$ with Loeb probability one. Thus, we assume for the moment that ${}^\circ\hat{\gamma} \in (\underline{\gamma}, \bar{\gamma}) \setminus \mathbf{Q}$ and study the consequences of that assumption.

For $\alpha \in {}^*[0,1]$, let $L(\alpha)$ denote the Lebesgue measure of the intersection

$$\big[(\alpha - 1, \alpha) + T(\hat{\gamma})(1 - 2\alpha)\big] \cap \big(-\alpha\hat{\gamma}, (1-\alpha)\hat{\gamma}\big).$$

Notice that L is piecewise linear in α, and that $L(\frac{1}{2}) = \min\{1, \hat{\gamma}\}$. The two intervals $(\alpha - 1, \alpha) + T(\hat{\gamma})(1-2\alpha)$ and $(-\alpha\hat{\gamma}, (1-\alpha)\hat{\gamma})$ are of length 1 and $\hat{\gamma}$, and they are both centered at 0 when $\alpha_1 = \frac{1}{2}$. The first interval moves linearly with slope $1 - 2T(\hat{\gamma})$ in response to changes in α, while the second interval moves linearly with slope $-\hat{\gamma}$, so the movement of the first interval, relative to the second, is linear with slope $1 + \hat{\gamma} - 2T(\hat{\gamma})$ in response to changes in α. The two intervals cease to be nested when the magnitude of the relative movement equals $\frac{|1-\hat{\gamma}|}{2}$, i.e. when

$$\alpha = \frac{1}{2} \pm \frac{|1-\hat{\gamma}|}{2|1+\hat{\gamma} - 2T(\hat{\gamma})|}.$$

The two intervals cease to intersect when the magnitude of the relative movement equals $\frac{1+\hat{\gamma}}{2}$, i.e. when

$$\alpha = \frac{1}{2} \pm \frac{1+\hat{\gamma}}{2|1+\hat{\gamma} - 2T(\hat{\gamma})|}.$$

Therefore,

$$L(\alpha) = {}^*H(\alpha, \hat{\gamma})$$

for all $\alpha \in {}^*[0,1]$.

Fix $\alpha \in {}^*[0,1]$. Given $k \in \mathbf{K}$, let

$$B_k = \lfloor \alpha B \rfloor + k \text{ and } \alpha_k = \frac{B_k}{B}.$$

Let $r : \mathbf{R} \to [0,1]$ denote the remainder function $r(x) = x - \lfloor x \rfloor$. Since the interval $[\alpha_k - 1, \alpha_k)$ has length 1, there is exactly one S_k such that

$$S_k - \hat{\gamma} B_{\alpha k} - \frac{G_s(\hat{\gamma})(1-2\alpha_k)}{F_s'(\hat{\gamma})} \in [\alpha_k - 1, \alpha_k)$$

and for this S_k, we have

$$\begin{aligned}
&S_k - \hat{\gamma} B_k - \frac{G_s(\hat{\gamma})(1-2\alpha_k)}{F_s'(\hat{\gamma})} - \alpha_k + 1 \\
&\quad = -{}^*r\left(\hat{\gamma} B_k - \frac{G_s(\hat{\gamma})(1-2\alpha_k)}{F_s'(\hat{\gamma})} - \alpha_k + 1\right) \\
&\quad \simeq -{}^*r\left(\hat{\gamma} B_k - \frac{G_s(\gamma)(1-2\alpha)}{F_s'(\gamma)} - \alpha + 1\right) \\
&S_k - \hat{\gamma} B_k - \frac{G_b(\hat{\gamma})(1-2\alpha_k)}{F_b'(\hat{\gamma})} + \alpha_k\hat{\gamma} \\
&\quad = S_k - \hat{\gamma} B_k - \frac{G_s(\hat{\gamma})(1-2\alpha_k)}{F_s'(\hat{\gamma})} - \alpha_k + 1 + T(\hat{\gamma})(1-2\alpha_k) + \alpha_k(1+\hat{\gamma}) - 1 \\
&\quad = -{}^*r\left(\hat{\gamma} B_k - \frac{G_s(\hat{\gamma})(1-2\alpha_k)}{F_s'(\hat{\gamma})} - \alpha_k + 1\right) + T(\hat{\gamma})(1-2\alpha_k) + \alpha_k(1+\hat{\gamma}) - 1 \\
&\quad \simeq -{}^*r\left(\hat{\gamma} B_k - \frac{G_s(\gamma)(1-2\alpha)}{F_s'(\gamma)} - \alpha + 1\right) + T(\gamma)(1-2\alpha) + \alpha(1+\gamma) - 1.
\end{aligned}$$

For $k_0 \in \mathbf{K}$ and $k \in \mathbf{N}$,

$$\hat{\gamma} B_{k_0+k} = \hat{\gamma} B + \hat{\gamma}(k_0 + k) = \hat{\gamma} B_{k_0} + \hat{\gamma} k \simeq \hat{\gamma} B_{k_0} + \gamma k$$

so

$$\left\{K_1 \in {}^*\mathbf{N}\colon \forall_{k_0\in\mathbf{K}}\forall_{k\in\{-K_1,-K_1+1,\dots,K_1-1,K_1\}}|\hat{\gamma}B_{k_0+k} - \hat{\gamma}B_{k_0} + \gamma k| < \frac{1}{K_1},\ K_1^2 < K\right\} \supseteq \mathbf{N}.$$

Since the set just described is internal, by the Spillover Principle, it contains an element of ${}^*\mathbf{N} \setminus \mathbf{N}$. Thus, we may pick $K_1 \in {}^*\mathbf{N} \setminus \mathbf{N}$, $\frac{K_1}{K} \simeq 0$, such that

$$\hat{\gamma}B_{k_0+k_1} \simeq \hat{\gamma}B_{k_0} + \gamma k_1$$

for $k \in \mathbf{K}_1 = \{-K_1, -K_1+1, \dots, -1, 0, 1, \dots, K_1-1, K_1\}$ and $k_0 \in \mathbf{K}$.

Since ${}^\circ\hat{\gamma}$ is irrational, Exercise VII.8.9 in Dunford and Schwartz (1957)[12] implies that for any interval $(a,b) \subset [0,1)$ and any $k_0 \in \mathbf{K}$,

$$\frac{|\{k \in \mathbf{K}_1\colon {}^{\circ *}r(\hat{\gamma}B_{k_0+k}) \in (a,b)\}|}{2K_1+1} \simeq b-a$$

hence

$$\begin{aligned}&\frac{|\{k \in k_0+\mathbf{K}_1\colon\ B_k \in \mathcal{N}(B,S)\}|}{2K_1+1}\\ &\quad\simeq \frac{|\{k \in k_0+\mathbf{K}_1\colon {}^\circ(S_k - \hat{\gamma}B_k - \frac{G_b(\gamma)(1-2\alpha)}{F_b'(\gamma)} + \alpha\gamma) \in (0,\gamma)\}|}{2K_1+1}\\ &\quad\simeq L(\alpha) = {}^*H(\alpha,\hat{\gamma}) \simeq H\big({}^\circ\alpha, {}^\circ\hat{\gamma}\big).\end{aligned}$$

We may write $\mathbf{K}$ as a union of sets of the form $k_0 + \mathbf{K}_1$, so

$${}^*H_\varepsilon(B,S,\alpha) = \frac{|\{k \in \mathbf{K}\colon\ B_k \in \mathcal{N}(B,S)\}|}{2K+1} \simeq H\big({}^\circ\alpha, {}^\circ\hat{\gamma}\big).$$

Now, we let α vary over ${}^*[0,1]$. We have shown that for all $\alpha \in {}^*[0,1]$,

$$\begin{aligned}&\big|{}^*H_\varepsilon(B,S,\alpha) - {}^*H(\alpha,\hat{\gamma})\big|\\ &\quad\leqslant \big|{}^*H_\varepsilon(B,S,\alpha) - H\big({}^\circ\alpha,{}^\circ\hat{\gamma}\big)\big| + \big|H\big({}^\circ\alpha,{}^\circ\hat{\gamma}\big) - {}^*H(\alpha,\hat{\gamma})\big|\\ &\quad\simeq 0\end{aligned}$$

since H is continuous.

In part 1, for every $n \in {}^*\mathbf{N} \setminus \mathbf{N}$ we have $\hat{\gamma} \simeq \gamma$ and $\varepsilon_n = \varepsilon$, so since H is continuous,

$$\sup_{\alpha\in{}^*[0,1]} \big|{}^*H_{\varepsilon_n}(B_n,S_n,\alpha) - {}^*H(\alpha,\gamma)\big| \simeq 0$$

so

$$\sup_{\alpha\in[0,1]} \big|H_{\varepsilon_n}(B_n,S_n,\alpha) - H(\alpha,\gamma)\big| \to 0$$

which is the conclusion of part 1.

In part 2, note that for every $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, $\hat{\gamma} = S_n/B_n = \lfloor\gamma_n B_n\rfloor/B_n \simeq \gamma_n$, so since H is continuous, we have on a set of Loeb probability one,

$$\sup_{\alpha\in{}^*[0,1]} \big|{}^*H_{\varepsilon_n}(B_n,S_n,\alpha) - {}^*H(\alpha,\gamma_n)\big| \simeq 0$$

so for every $\delta \in \mathbf{R}$, $\delta > 0$,

$${}^*P\Big(\sup_{\alpha\in{}^*[0,1]} \big|{}^*H_{\varepsilon_n}(B_n,S_n,\alpha) - {}^*H(\alpha,\gamma_n)\big| > \delta\Big) < \delta.$$

[12] As noted above, this is a very special case of the Ergodic Theorem.

The set of all n for which this statement is true is internal and includes all $n \in {}^*\mathbf{N} \setminus \mathbf{N}$, so by the Spillover Principle, there exists $N \in \mathbf{N}$ such that

$$n > N \quad \Rightarrow \quad \Big(\sup_{\alpha\in[0,1]} \big|H_{\varepsilon_n}(B_n, S_n, \alpha) - H(\alpha, \gamma_n)\big| > \delta\Big) < \delta$$

which is the conclusion of part 2. □

6. Conclusion

In much of the economics literature it is customary to model a large population using a continuum of agents. Ellison and Fudenberg (2003) argued that in models of two-sided location choice, it can be difficult to identify the equilibria with a large finite number of agents by working with the continuum limit. In particular, they noted that a broad range of outcomes can be consistent with the no-deviation constraints in some situations that might be regarded as having only "fully tipped" outcomes when analyzed with a continuum-of-agents approximation.[13] Ellison and Fudenberg studied only the incentive constraints, and ignored the constraint that the number of agents of each type in each market must be an integer. In this paper, we have extended their analysis to fully take discreteness into account. We find that in the generic case the equilibrium set fills out an interval strictly containing the "quasi-equilibrium" set described in Ellison and Fudenberg (2003), but that there are non-generic cases where the equilibrium set is much smaller. In addition to extending the earlier result, this paper bolsters the message that taking discreteness into account can be important even in very large economies. One suggestion for future research would be to examine the implications of the discreteness of agents in other classes of models. Another would be to examine equilibrium selection in models of the type discussed here. We know that these models have a dense plateau of equilibria. It would be interesting to know more about whether good evolutionary arguments can be made to regard some of the equilibria as more likely than others.

Appendix A

Proof of Lemma 1. Proof of 1. We need to consider the four incentive compatibility constraints $S1$, $S2$, $B1$, and $B2$. First, consider a seller contemplating switching from market 1 to market 2. The change in the seller's payoff resulting from the switch is approximated to order $o(1/\underline{B})$ by $\Delta_{S1} \equiv F_s(\gamma_2 + \frac{1}{B_2}) - \frac{G_s(\gamma_2+\frac{1}{B_2})}{B_2} - F_s(\gamma_1) + \frac{G_s(\gamma_1)}{B_1}$. The $S1$ constraint is satisfied for B_1 and B_2 large if Δ_{S1} is negative and bounded away from zero by a term that is $O(1/\underline{B})$ (or larger) when B_1 and B_2 are large.

Condition (2) implies that γ_1 and γ_2 are both equal to $\gamma + O(1/\underline{B})$. Thus we can approximate the values of F_s and G_s in the above expression using their values at γ and derivatives, yielding

$$\begin{aligned}
\Delta_{S1} &= \left(F_s(\gamma) + F_s'(\gamma)\left(\gamma_2 - \gamma + \frac{1}{B_2}\right) - F_s(\gamma) - F_s'(\gamma)(\gamma_1 - \gamma)\right) \\
&\quad - \left(\frac{G_s(\gamma) + (\gamma_2 - \gamma + 1/B_2)G_s'(\gamma)}{B_2} - \frac{G_s(\gamma) + (\gamma_1 - \gamma)G_s'(\gamma)}{B_1}\right) \\
&\quad + o\left(\gamma_2 - \gamma_1 + \frac{1}{B_2}\right) + o(\gamma_1 - \gamma) + o\left(\frac{1}{B_2}\right) + o\left(\frac{1}{B_1}\right) \\
&= F_s'(\gamma)\left(\gamma_2 - \gamma_1 + \frac{1}{B_2}\right) + G_s(\gamma)\left(\frac{1}{B_1} - \frac{1}{B_2}\right) + o\left(\frac{1}{\underline{B}}\right) \\
&= F_s'(\gamma)\left(\frac{\gamma - \gamma_1}{1 - \alpha_1} + \frac{\alpha_1}{(1 - \alpha_1)B_1}\right) + G_s(\gamma)\left(\frac{1 - 2\alpha_1}{(1 - \alpha_1)B_1}\right) + o\left(\frac{1}{\underline{B}}\right).
\end{aligned}$$

Condition (2) implies that $S_1 - \gamma B_1 - \frac{G_s(\gamma)(1-2\alpha_1)}{F_s'(\gamma)} < \alpha_1 - \varepsilon$. Thus

[13] In the continuum model, any ratio of market sizes is consistent with a two-market equilibrium so long as the buyer–seller ratios are equal and both markets have a continuum of agents.

$$\gamma_1-\gamma-\frac{G_s(\gamma)(1-2\alpha_1)}{F_s'(\gamma)B_1}<\frac{\alpha_1}{B_1}-\frac{\varepsilon}{B_1},$$
$$-\frac{G_s(\gamma)(1-2\alpha_1)}{F_s'(\gamma)(1-\alpha_1)B_1}-\left(\frac{\gamma-\gamma_1}{1-\alpha_1}+\frac{\alpha_1}{(1-\alpha_1)B_1}\right)<-\frac{\varepsilon}{(1-\alpha_1)B_1},$$
$$G_s(\gamma)\left(\frac{1-2\alpha_1}{(1-\alpha_1)B_1}\right)+F_s'(\gamma)\left(\frac{\gamma-\gamma_1}{1-\alpha_1}+\frac{\alpha_1}{(1-\alpha_1)B_1}\right)<+\frac{\varepsilon F_s'(\gamma)}{(1-\alpha_1)B_1}.$$

The last equation implies that Δ_{S1} is negative and bounded away from zero to the desired order, which implies that the S1 constraint is satisfied for $\underline{B}$ sufficiently large.

Now, consider a seller contemplating switching from market 2 to market 1. By symmetry, the $S2$ constraint is satisfied for large enough $\underline{B}$ if

$$\gamma_2-\gamma-\frac{G_s(\gamma)(1-2\alpha_2)}{F_s'(\gamma)B_2}<\frac{\alpha_2}{B_2}-\varepsilon\frac{1}{B_2},$$
$$\frac{\alpha_1}{1-\alpha_1}(\gamma-\gamma_1)-\alpha_1\frac{G_s(\gamma)(2\alpha_1-1)}{(1-\alpha_1)F_s'(\gamma)B_1}<\frac{\alpha_1}{B_1}-\varepsilon\frac{\alpha_1}{(1-\alpha_1)B_1},$$
$$\gamma-\gamma_1-\frac{G_s(\gamma)(2\alpha_1-1)}{F_s'(\gamma)B_1}<\frac{1-\alpha_1}{B_1}-\frac{\varepsilon}{B_1},$$
$$S_1-\gamma B_1-\frac{G_s(\gamma)(1-2\alpha_1)}{F_s'(\gamma)}>\alpha_1-1+\varepsilon.$$

Thus, both the incentive compatibility constraints for sellers are satisfied if

$$(S_1-\gamma B_1)-\frac{G_s(\gamma)(1-2\alpha_1)}{F_s'(\gamma)}\in(\alpha_1-1+\varepsilon,\alpha_1-\varepsilon),$$

that is, when (2) holds.

The calculations to show that $B1$ and $B2$ are satisfied if (3) holds and $\underline{B}$ is large are similar. The change in the buyer's payoff resulting from a switch from market 1 to market 2 is

$$\begin{aligned}
&F_b\left(\frac{B_2}{B_2+1}\gamma_2\right)-\frac{G_b(\frac{B_2}{B_2+1}\gamma_2)}{B_2+1}-F_b(\gamma_1)+\frac{G_b(\gamma_1)}{B_1}+o\left(\frac{1}{B_1}\right)\\
&\quad=F_b'(\gamma)\left(\gamma_2-\gamma_1-\frac{\gamma_2}{B_2+1}\right)+o\left(\gamma_2-\gamma_1-\frac{\gamma_2}{B_2+1}\right)\\
&\qquad+G_b(\gamma)\left(\frac{1}{B_1}-\frac{1}{B_2+1}\right)+o\left(\frac{1}{B_2+1}\right)+o\left(\frac{1}{B_1}\right)\\
&\quad=F_b'(\gamma)\left(\frac{\gamma-\gamma_1}{1-\alpha_1}-\frac{\alpha_1\gamma_2}{(1-\alpha_1)B_1}\right)+G_b(\gamma)\left(\frac{1-2\alpha_1}{(1-\alpha_1)B_1}\right)+o\left(\frac{1}{\underline{B}}\right).
\end{aligned}$$

The $B1$ constraint requires that this be less than or equal to zero. Condition (3) implies $(S_1-\gamma B_1)-\frac{G_b(\gamma)(1-2\alpha_1)}{F_b'(\gamma)}>-\alpha_1\gamma+\varepsilon$. This implies

$$\frac{\gamma-\gamma_1}{1-\alpha_1}-\frac{\alpha_1\gamma}{(1-\alpha_1)B_1}+\frac{G_b(\gamma)(1-2\alpha_1)}{F_b'(\gamma)(1-\alpha_1)B_1}<-\frac{\varepsilon}{B_1(1-\alpha_1)},$$
$$F_b'(\gamma)\left(\frac{\gamma-\gamma_1}{1-\alpha_1}-\frac{\alpha_1\gamma_2}{(1-\alpha_1)B_1}\right)+G_b(\gamma)\left(\frac{1-2\alpha_1}{(1-\alpha_1)B_1}\right)<-\frac{\varepsilon}{B_1(1-\alpha_1)}$$

so there is a $\underline{B}$ such that $B1$ is satisfied for all sufficiently large $\underline{B}$. A symmetric argument applied to the incentive compatibility constraint for buyers in market 2 shows that condition (3) implies that the incentive constraint for buyers is satisfied for large $\underline{B}$. This proves claim 1.

The hypotheses of Claim 1. imply that $\gamma_1,\gamma_2\in\Gamma^{\varepsilon}$; this is what allowed us to apply the payoff approximation in Assumption 1. In Claim 2., we assume $\gamma_1,\gamma_2\in\Gamma^{\varepsilon}$ directly; the above arguments now show that there is not a quasi-equilibrium when $(S_1-\gamma B_1)-\frac{G_s(\gamma)(1-2\alpha_1)}{F_s'(\gamma)}\notin[\alpha_1-1-\varepsilon,\alpha_1+\varepsilon]$ or $(S_1-\gamma B_1)-\frac{G_b(\gamma)(1-2\alpha_1)}{F_b'(\gamma)}\notin[-\alpha_1\gamma-\varepsilon,(1-\alpha_1)\gamma+\varepsilon]$. □

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