

Comment

Comment on “Rotten Kids, Purity, and Perfection”

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After reading Cornes and Silva (1999), one gets the impression that for an important class of problems, in which kids privately provide a public good, the Rotten Kid theorem applies under general preferences, including nontransferable utilities. Technically, the authors show that whenever the solution to the kids problem is interior, it always satisfies the first-order conditions for efficiency. In other words, *if* in equilibrium all kids provide positive amounts of the public good, then they will, as a group, provide the total amount of public good desired by the parent.

Unfortunately, although this statement is certainly correct, its scope is much more restricted than suggested by the authors. In this note, we show that, contrarily to their claims, the solution to the kids problem is almost never interior. Almost always, in the game Cornes and Silver consider, the first-order conditions characterizing each kid's locally optimal strategy are incompatible, so that a corner solution must obtain. Interior solutions exist only in very particular cases, among which pure symmetry is probably the most interesting.

We first present the general argument and then discuss two specific examples.

I. Rotten Kids Are Generically Rotten

A. The Model

We first start by restating the argument of Cornes and Silva (1999). For simplicity we display the argument for the case of two kids; the extension to N kids is immediate.

Let $U^i(x^i, Q)$ denote kid i 's utility, as a function of his private consumption x^i and the level Q of public goods. Also, let $W(U^1, U^2)$ denote the patriarch's welfare index. First-order conditions for efficiency amount, in this context, to the well-known Lindahl-Bowen-Samuelson conditions:

$$\frac{\partial U^1/\partial Q}{\partial U^1/\partial x^1} + \frac{\partial U^2/\partial Q}{\partial U^2/\partial x^2} = 1$$

since prices have been normalized to one.

We now consider the game the rotten kids play. At stage 2, taking Q as given, the patriarch redistributes the private commodity across children. The resulting allocation solves

$$\max_{x^i} W(U^1(x^1, Q), U^2(x^2, Q))$$

under the budget constraint

$$x^1 + x^2 = Y - Q.$$

Let $(x^1(Q), x^2(Q))$ denote the optimal allocation. It is crucial to note that the x^i depend only on the total quantity of public good, Q , and not on individual contributions. More precisely, under the assumption that each kid's private consumption is positive at the optimum, $(x^1(Q), x^2(Q))$ must satisfy the first-order condition

$$\frac{\partial W(U^1, U^2)}{\partial U^1} \frac{\partial U^1(x^1, Q)}{\partial x^1} = \frac{\partial W(U^1, U^2)}{\partial U^2} \frac{\partial U^2(Y - Q - x^1, Q)}{\partial x^2}. \quad (1)$$

At stage 1, each kid chooses his individual contribution to the public good, q^i , taking as given the redistribution scheme $(x^1(Q), x^2(Q))$. The q^i must hence solve

$$\max_{q^i} U^i(x^i(q^1 + q^2), q^1 + q^2),$$

where q^j is taken as given for $j \neq i$.

Assume, now, that for *both* kids the corresponding solution is interior. Then first-order conditions give, respectively,

$$F^1(Q) \equiv \frac{\partial U^1(x^1(Q), Q)}{\partial x^1} \frac{\partial x^1(Q)}{\partial Q} + \frac{\partial U^1(x^1(Q), Q)}{\partial Q} = 0 \quad (2)$$

and

$$F^2(Q) \equiv \frac{\partial U^2(x^2(Q), Q)}{\partial x^2} \frac{\partial x^2(Q)}{\partial Q} + \frac{\partial U^2(x^2(Q), Q)}{\partial Q} = 0. \quad (3)$$

The key remark here is that each kid's first-order condition can be expressed as a function of *one* variable only, namely the *total* provision of public good Q . In other words, first-order conditions lead to two equations with only one unknown. With arbitrary preferences, there is no reason to expect these equations to be compatible.¹ Only in very particular cases (such as the symmetric one described below) do the equations admit the same solution. In general, the two equations are not compatible, and no interior solution can exist. Then one of the kids (at least) chooses not to provide the public good in equilibrium,² and the efficiency result does not obtain.³

B. Symmetric Rotten Kids

One obvious situation in which equations (2) and (3) are compatible is the symmetric case. Indeed, assume that (1) utilities are identical:

$$U^1(x, Q) = U^2(x, Q) \equiv U(x, Q) \quad \forall x, Q$$

and (2) W is symmetric:

$$W(U^1, U^2) = W(U^2, U^1) \quad \forall U^1, U^2.$$

¹ When U^1 and U^2 are arbitrary, the equations are *algebraically independent*, so a common solution must be nongeneric. A fully precise proof of this claim would, however, require some transversality argument, a tool that is clearly excessive in this context.

² Another possibility would be that there exists no pure strategy equilibrium. However, one can readily find mild assumptions under which a pure strategy equilibrium exists. Specifically, assume that $F^i(0) > 0$ and $F^i(Q) < 0$ for Q "large enough." Then both F^1 and F^2 have positive zeros. Let \bar{Q} be the largest of these zeros. Then $F^i(\bar{Q}) = 0$ whereas $F^j(\bar{Q}) < 0$ for $j \neq i$; also, $F^i(\bar{Q}) < 0$ generically. An equilibrium is such that i provides \bar{Q} whereas j provides nothing.

³ In addition, even when the equations are compatible, problems may arise from the second-order conditions of the kids problem. Indeed, even with concave U^1 and U^2 , the response functions $x^i(Q)$ will not be both concave in general (remember that the sum is linear from the budget constraint). We do not pursue this argument here since compatibility of first-order conditions is nongeneric in any case.

Then there always exists an interior, equalitarian solution such that

$$x^1(Q) = x^2(Q) \equiv x(Q) = \frac{Y - Q}{2},$$

and both (2) and (3) boil down to one single equation, namely,

$$\frac{1}{2} \frac{\partial U(x(Q), Q)}{\partial x} = \frac{\partial U(x(Q), Q)}{\partial Q}.$$

Obviously, compatibility is no longer an issue here. We conclude that Cornes and Silva's result applies in the particular case of symmetric agents.

II. Two Explicit Examples

A. Example 1

To illustrate the previous argument, assume that preferences are given by

$$U^i(x^i, Q) = \log(x^i) + \alpha_i f(Q)$$

(where f is a concave, strictly increasing function), and the patriarch's welfare function is

$$W(U^1, U^2) = \lambda^1 U^1 + \lambda^2 U^2.$$

The outcome of the second-stage game is characterized by

$$\frac{\lambda^1}{x^1} = \frac{\lambda^2}{x^2} \Leftrightarrow x^2 = \frac{\lambda^2}{\lambda^1} x^1. \quad (1')$$

From the budget constraint, one gets that

$$x^i = \frac{\lambda^i}{\lambda^1 + \lambda^2} (Y - Q).$$

The first-order conditions of the first-stage problem are necessary and sufficient and give

$$\frac{1}{\alpha_1} = (Y - Q) f'(Q) \quad (2')$$

and

$$\frac{1}{\alpha_2} = (Y - Q) f'(Q). \quad (3')$$

These equations are incompatible unless $\alpha_1 = \alpha_2$.

This necessary property does not hold generically. The set of param-

eters (α_1, α_2) for which it is fulfilled is in fact a one-dimensional submanifold of \mathbb{R}^2 ; as such, its complement is an open, dense set of \mathbb{R}^2 .

In this example, compatibility obtains only when preferences are identical across agents. In that case, interestingly, the example belongs to the quasi-linear family since

$$u^i(x^i, Q) = \exp [U^i(x^i, Q)] = \exp [\alpha_i f(Q)] \cdot x^i.$$

B. Example 2

From the previous example, one may get the impression that Cornes and Silva's result ultimately boils down to a particular case of Bergstrom's quasi linearity proposition. This is not the case, as illustrated by the second example. Here, preferences are given by

$$U^i(x^i, Q) = -\frac{1}{\mu} \exp(-\mu x^i) + \alpha_i f(Q)$$

(where f is a concave, strictly increasing function), and the patriarch's welfare function is, as above,

$$W(U^1, U^2) = \lambda^1 U^1 + \lambda^2 U^2.$$

Note that kids' preferences are *not* quasi-linear, even with identical α_i 's. Now, the outcome of the second-stage game is characterized by

$$\lambda^1 \exp(-\mu x^1) = \lambda^2 \exp(-\mu x^2) \equiv \gamma, \quad (1'')$$

which implies that $dx^i(Q)/dQ = \frac{1}{2}$.

The first-order conditions of the first-stage problem are necessary and sufficient and give

$$(\alpha^1 \lambda^1)^{-1} = 2\gamma^{-1} f'(Q) \quad (2'')$$

and

$$(\alpha^2 \lambda^2)^{-1} = 2\gamma^{-1} f'(Q). \quad (3'')$$

These equations are incompatible unless $\alpha^1 \lambda^1 = \alpha^2 \lambda^2$.

Again, this necessary property does not hold generically. After we impose a normalization on the λ weights, the parameter space is isomorphic to \mathbb{R}^3 . The set of parameters for which the condition is fulfilled is a submanifold of \mathbb{R}^3 of codimension 1—hence of dimension 2. It includes the symmetric case (i.e., $\alpha^1 = \alpha^2$ and $\lambda^1 = \lambda^2$), as well as a continuum of nonsymmetric, non-quasi-linear models in which the efficiency conclusion applies. But these models are nongeneric; the set of parameters for which no interior solution exists is an open, dense set of \mathbb{R}^3 .

III. Conclusion

A puzzling aspect of Cornes and Silva's paper was that it apparently contradicted another result by Bergstrom (1989), stating that the generalized quasi-linear form was necessary for the Rotten Kid theorem to apply, whatever W . Our note provides an explanation for this discrepancy. Cornes and Silva's argument, although formally correct, applies only to interior solutions, whereas the Rotten Kid game is such that solutions are not interior except for nongeneric W cases. It remains that the Rotten Kid theorem holds in the public-good case provided that (i) kids have identical preferences and (ii) the patriarch treats the kids in a purely symmetric way. This result, albeit less general than it initially appeared, still constitutes an insightful contribution to the understanding of these problems. Whether it can be generalized to other settings is still an open question.

References

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